Automatica 45 (2009) 1929–1936

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Mean square average-consensus under measurement noises and fixed topologies: Necessary and sufficient conditions☆

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ARTICLE INFO

Article history:
Received 24 August 2007
Received in revised form 29 November 2008
Accepted 17 April 2009
Available online 21 June 2009

Keywords:
Multi-agent systems
Average-consensus
Distributed coordination
Distributed estimation
Stochastic systems

ABSTRACT

In this paper, average-consensus control is considered for networks of continuous-time integrator agents under fixed and directed topologies. The control input of each agent can only use its local state and the states of its neighbors corrupted by white noises. To attenuate the measurement noises, time-varying consensus gains are introduced in the consensus protocol. By combining the tools of algebraic graph theory and stochastic analysis, the convergence of these kinds of protocols is analyzed. Firstly, for noise-free cases, necessary and sufficient conditions are given on the network topology and consensus gains to achieve average-consensus. Secondly, for the cases with measurement noises, necessary and sufficient conditions are given on the consensus gains to achieve asymptotic unbiased mean square average-consensus. It is shown that under the protocol designed, all agents' states converge to a common Gaussian random variable, whose mathematical expectation is just the average of the initial states.

1. Introduction

In recent years, distributed coordination for multi-agent systems has attracted more and more researchers in the control community. For distributed coordination, it is a fundamental requirement that the whole group can achieve consensus on the shared data only through local communications. Consensus control generally means to design a network protocol, such that as time goes on, all agents asymptotically reach an agreement on their states. For some consensus problems, the common value to which the states converge is also required to be the average of the initial states of the group. It is often called average-consensus (Olfati-Saber, Fax, & Murray, 2007).

Average-consensus has wide applications in various areas such as formation control (Sinha & Ghose, 2006), distributed filtering (Olfati-Saber & Shamma, 2005) and distributed computation (Lynch, 1996). Olfati-Saber and Murray (2004) considered the average-consensus control for first-order integrator networks under fixed and switching topologies. They proved that under the fixed topology, if the network is a strongly connected balanced digraph, then the linear time-invariant protocol can ensure average-consensus. Kingston and Beard (2006) extended the results of Olfati-Saber and Murray (2004) to the discrete-time models. In addition to the above works, some researchers also considered the topologies of random graphs (Hatano & Mesbahi, 2005) or control design based on individual performance optimization (Li & Zhang, 2008).

Most researches in the above literature assume that each agent measures its neighbors' states accurately. However, real networks are often in uncertain communication environments. Recently, consensus problems with random measurement noises have attracted the attention of some researchers (Carli, Fagnani, Speranzon, & Zampieri, 2008; Huang & Manton, 2009; Ren, Beard, & Kingston, 2005). However, for average-consensus problems with random measurement noises, there is still lack of good result comparable with those obtained in the noise-free cases, even if the network topology is time-invariant. Ren et al. (2005) introduced time-varying consensus gains and designed consensus protocols based on a Kalman filter structure. They proved that, when there is no noise, the protocols designed can ensure consensus to be achieved asymptotically. Huang and Manton (2009) introduced decreasing consensus gains $a(k)$ (where k is the discrete-time instant) to attenuate the measurement noises. They proved that, if the network topology is a strongly connected circulant graph, and $a(k) = O(1/k^{\gamma})$, $\gamma \in (0.5, 1]$, then the static mean square error between the individual state and the average of the initial states of all agents is in the same order as the variance of the measurement noises; if the network topology is an undirected graph and $a(k)$ satisfies the condition on the step size in classic stochastic approximation, then mean square weak consensus can be achieved.

☆ This paper was presented at the 17th IFAC World Congress, 6–11 July, 2008, Seoul, Korea. This paper was recommended for publication in revised form by Associate Editor Giuseppe De Nicolao under the direction of Editor Ian R. Petersen.

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doi:10.1016/j.automatica.2009.04.017

In this paper, we consider the average-consensus control for networks of continuous-time integrator agents under fixed and directed topologies. The control input of each agent can only use its local state and the states of its neighbors corrupted by white noises. Inspired by Huang and Manton (2009) and Ren et al. (2005), we also use a time-varying consensus gain $a(t)$ in our network protocol to attenuate the measurement noises.

Different from Huang and Manton (2009), here we consider continuous-time models. Consensus control of continuous-time first-order integrator networks has been widely studied for the noise-free case (Moreau & Belgium, 2004; Olfati-Saber & Murray, 2004). Continuous-time models also have wide applications in other fields of the research on cooperative control, such as formation control (Sinha & Ghose, 2006), swarming (Gazi & Passino, 2003) and flocking problems (Olfati-Saber, 2006). Since for many scenarios the robot dynamics can be modeled as continuous-time first-order or second-order integrators under feedback linearization, many cooperative control laws are designed based on the continuous-time consensus strategy (Sinha & Ghose, 2006). Continuous-time first-order integrator models are also adopted in distributed estimation (Olfati-Saber & Shamma, 2005; Ren et al., 2005) and synchronization of coupled oscillators (Preciado & Verghese, 2005). Our methodology could be potentially applied to designing and analyzing distributed estimation and control strategies under uncertain environment.

Due to the measurement noises and the time-varying consensus gains $a(t)$, under the protocol designed, the closed-loop system is a time-varying stochastic differential equation. The state matrix of the equation is a time-varying Laplacian matrix of a digraph. Different from the cases of undirected and circulant graphs, these kinds of state matrices are neither symmetric nor diagonalizable, which results in difficulties for convergence analysis. We combine stochastic analysis and algebraic graph theory together, by introducing the concept and tools of symmetrized graph (Olfati-Saber & Murray, 2004) in stochastic Lyapunov analysis, and dealing with the Itô term by the stopping time truncation method. For the protocol designed, firstly, we prove that, a balanced digraph containing a spanning tree (a strongly connected and balanced digraph) is the weakest condition on the topology to ensure average-consensus. Then, under these kinds of network topologies, we give a necessary and sufficient condition on the consensus gains to ensure asymptotic unbiased mean square average-consensus. This condition consists of two parts, one, $\int_0^\infty a(t)\mathrm{d}t = \infty$, called convergence condition, is to make all agents' states reach an agreement with a proper rate; and the other, $\int_0^\infty a^2(t)\mathrm{d}t < \infty$, called robustness condition, is to make the static error of the closed-loop system finite regardless of measurement noises, that is, to make the consensus protocol robust against measurement noises. We prove that under the protocol designed, the state of each agent converges in mean square to a common Gaussian random variable, whose mathematical expectation is just the average of the initial states. The analytic expression of the variance is also given.

The remainder of this paper is organized as follows. In Section 2, some concepts in graph theory are described, and the problem to be investigated is formulated. In Section 3, for noise-free cases, a necessary and sufficient condition is given on the network topology and the consensus gains to achieve average-consensus. For the cases with measurement noises, necessary and sufficient conditions are given on the consensus gains to achieve asymptotic unbiased mean square average-consensus. In Section 4, two numerical examples are given to illustrate our results. In Section 5, some concluding remarks and future research topics are discussed.

The following notations will be used throughout this paper: $\mathbf{1}$ denotes a column vector with all ones. I_m denotes the $m \times m$-dimensional identity matrix. For a given set S, χ_S denotes its indicator function; $|S|$ denotes its number of elements. For a given vector or matrix A, A^{T} denotes its transpose, and $\|A\|$ denotes its Frobenius norm. For a given square matrix A, $\rho(A)$ denotes its spectral radius, and $\mathrm{tr}(A)$ denotes its trace. For a given random variable X, $E(X)$ denotes its mathematical expectation; $Var(X)$ denotes its variance. For any given real numbers a and b, $a \wedge b$ denotes $\min\{a, b\}$. For a family of random variables (r.v.s) $\{\xi_\lambda, \lambda \in \Lambda\}$, $\sigma(\xi_\lambda, \lambda \in \Lambda)$ denotes the σ-algebra $\sigma(\{\xi_\lambda \in B\}, B \in \mathcal{B}, \lambda \in \Lambda)$, where $\mathcal{B}$ denotes the one-dimensional Borel sets. For a σ-algebra $\mathcal{F}$ and a r.v. ξ, we say that ξ is adapted to $\mathcal{F}$, if ξ is $\mathcal{F}$ measurable.

2. Problem formulation

2.1. Concepts in graph theory

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ be a weighted digraph, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of nodes, node i represents the ith agent; $\mathcal{E}$ is the set of edges, and an edge in $\mathcal{G}$ is denoted by an ordered pair (j, i). $(j, i) \in \mathcal{E}$ if and only if the jth agent can send information to the ith agent directly. In this case, j is called the parent of i, and i is called the child of j. The neighborhood of the ith agent is denoted by $N_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$, which is the set of all parents of i. Node i is called a source, if it has no parent but only children. Node i is called an isolated node, if it has neither parent nor child. Denote the sets of all sources and isolated nodes in $\mathcal{V}$ by $\mathcal{V}_s \triangleq \{j \in \mathcal{V} \mid |N_j| = 0\}$. To avoid the trivial cases, $|\mathcal{V} - \mathcal{V}_s| > 0$ is always assumed in this paper.

$\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is called the weighted adjacency matrix of $\mathcal{G}$. For any $i, j \in \mathcal{V}$, $a_{ij} \geq 0$, and $a_{ij} > 0 \Leftrightarrow j \in N_i$. $deg_{in}(i) = \sum_{j=1}^N a_{ij}$ is called the in-degree of i; $deg_{out}(i) = \sum_{j=1}^N a_{ji}$ is called the out-degree of i; $L_{\mathcal{G}} = \mathcal{D} - \mathcal{A}$ is called the Laplacian matrix of $\mathcal{G}$, where $\mathcal{D} = diag(deg_{in}(1), \ldots, deg_{in}(N))$. $L_{\mathcal{G}}$ has at least one zero eigenvalue, $L_{\mathcal{G}}\mathbf{1} = 0$ and all nonzero eigenvalues have nonnegative real parts (Merris, 1994).

$\mathcal{G}$ is called a balanced digraph, if $deg_{in}(i) = deg_{out}(i)$, $i = 1, 2, \ldots, N$. $\mathcal{G}$ is called an undirected graph, if $\mathcal{A}$ is a symmetric matrix. It is easily shown that an undirected graph must be a balanced digraph and $\mathcal{G}$ is a balanced digraph if and only if $\mathbf{1}^{\mathrm{T}}L_{\mathcal{G}} = 0$.

A sequence (i_1, i_2), (i_2, i_3), …, (i_{k-1}, i_k) of edges is called a directed path from node i_1 to node i_k. $\mathcal{G}$ is called a strongly connected digraph, if for any $i, j \in \mathcal{V}$, there is a directed path from i to j. A directed tree is a digraph, where every node except the root has exactly one parent and the root is a source. A spanning tree of $\mathcal{G}$ is a directed tree whose node set is $\mathcal{V}$ and whose edge set is a subset of $\mathcal{E}$. If $\mathcal{G}$ is a strongly connected digraph, then it must contain a spanning tree. Generally speaking, containing a spanning tree does not imply strong connectivity, however, for a balanced digraph, containing a spanning tree implies being strongly connected. Below is a fundamental property of Laplacian matrices:

Lemma 2.1 (*Godsil & Royle, 2001*). *If $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is an undirected graph, then $L_{\mathcal{G}}$ is a symmetric matrix, and has N real eigenvalues, in an ascending order:*

$$0 = \lambda_1(L_{\mathcal{G}}) \leq \lambda_2(L_{\mathcal{G}}) \leq \cdots \leq \lambda_N(L_{\mathcal{G}}),$$

and

$$\min_{x\neq 0, \mathbf{1}^{\mathrm{T}}x=0} \frac{x^{\mathrm{T}}L_{\mathcal{G}}x}{\|x\|^2} = \lambda_2(L_{\mathcal{G}}),$$

where $\lambda_2(L_{\mathcal{G}})$ is called the algebraic connectivity of $\mathcal{G}$. If $\mathcal{G}$ is strongly connected, then $\lambda_2(L_{\mathcal{G}}) > 0$.

2.2. Consensus protocols

In this paper, we consider the average-consensus control for a network of continuous-time first-order integrator agents with the dynamics

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i(t) \in \mathbb{R}$ is the state of the ith agent, and $u_i(t) \in \mathbb{R}$ is the control input. The initial state $x_i(0)$ is deterministic. Denote $X(t) = [x_1(t), \ldots, x_N(t)]^T$.

The ith agent can receive information from its neighbors:

$$y_{ji}(t) = x_j(t) + \sigma_{ji} n_{ji}(t), \quad j \in N_i, \tag{2}$$

where $y_{ji}(t)$ denotes the measurement of the jth agent's state $x_j(t)$ by the ith agent. $\{n_{ji}(t), i, j = 1, 2, \ldots, N\}$ are independent standard white noises, where $\sigma_{ji} \geq 0$ is the noise intensity. Therefore, the graph $\mathcal{G}$ shows the structure of the information flow in the system (1), called the information flow graph or network topology graph of the system (1). $(\mathcal{G}, X)$ is usually called a dynamic network (Olfati-Saber & Murray, 2004).

We call the group of controls $\mathcal{U} = \{u_i, i = 1, 2 \ldots, N\}$ a measurement-based distributed protocol, if

$$u_i(t) \in \sigma(x_i(s), y_{ji}(s), j \in N_i, 0 \leq s \leq t), \quad \forall t \geq 0, i = 1, 2, \ldots, N.$$

The so-called average-consensus control means to design a distributed protocol for the dynamic network $(\mathcal{G}, X)$, such that the states of all the agents converge towards $\frac{1}{N}\sum_{j=1}^N x_j(0)$, when $t \to \infty$, that is, to compute $\frac{1}{N}\sum_{j=1}^N x_j(0)$ in a distributed way.

Applying the distributed protocol $\mathcal{U}$ to the system (1)–(2), generally speaking, will lead to a stochastic closed-loop system, and $x_i(t)$, $i = 1, 2, \ldots, N$, are all stochastic processes. Below we give the definition of average-consensus protocol in mean square for stochastic systems.

Definition 2.1. A distributed protocol $\mathcal{U}$ is called an asymptotic unbiased mean square average-consensus protocol if it makes the system (1)–(2) have the following properties: for any given $X(0) \in \mathbb{R}^n$, there is a random variable x^*, such that $E(x^*) = \frac{1}{N}\sum_{j=1}^N x_j(0)$, $Var(x^*) < \infty$, and

$$\lim_{t\to\infty} E(x_i(t) - x^*)^2 = 0, \quad i = 1, 2, \ldots, N.$$

Remark 1. Here the term "asymptotic unbiased" is borrowed from mathematical statistics, since the average-consensus can be viewed as a distributed estimation problem for the group decision value $\frac{1}{N}\sum_{j=1}^N x_j(0)$. If $\mathcal{U}$ is an asymptotic unbiased mean square average-consensus protocol, then $x_i(t)$ is the asymptotic unbiased estimate for $\frac{1}{N}\sum_{j=1}^N x_j(0)$, that is,

$$\lim_{t\to\infty} E[x_i(t)] = \frac{1}{N}\sum_{j=1}^N x_j(0), \quad i = 1, 2, \ldots, N.$$

If there is no measurement noise, and $\mathcal{U}$ is an asymptotic unbiased mean square average-consensus protocol, then $Var(x^*) = 0$, that is, $x^* = \frac{1}{N}\sum_{j=1}^N x_j(0)$. In this case, Definition 2.1 is equivalent to the definition of average-consensus protocol for deterministic systems in Olfati-Saber and Murray (2004).

For the dynamic network $(\mathcal{G}, X)$, we propose the distributed protocol as

$$u_i(t) = \begin{cases} 0, & i \in \mathcal{V}_s, \\ a(t) \sum_{j\in N_i} a_{ij}(y_{ji}(t) - x_i(t)), & i \in \mathcal{V} - \mathcal{V}_s, \end{cases} \quad \forall t \geq 0, \tag{3}$$

where $a(\cdot) : [0, \infty) \to (0, \infty)$ is piecewise continuous, called consensus-gain function.

Remark 2. The weighted digraph is a widely used model to describe the communication network in cooperative control systems (Ballal & Lewis, 2008; Olfati-Saber & Murray, 2004). There are many ways to choose the weights a_{ij}. In Jadbabaie, Lin, and Morse (2003), the *nearest neighbor rule* gives $a_{ij} = \frac{1}{1+|N_i|}$, $j \in N_i$, $i = 1, 2, \ldots, N$. For undirected networks, there are two popular rules (Boyd, Diaconis, & Xiao, 2004):

- *Maximum-degree weights*:

$$a_{ij} = a_{ji} = \frac{1}{\max_{1\leq l\leq N} |N_l|}, \quad (i, j) \in \mathcal{E}.$$

- *Metropolis weights*:

$$a_{ij} = a_{ji} = \frac{1}{1 + \max\{|N_i|, |N_j|\}}, \quad (i, j) \in \mathcal{E}.$$

These rules can ensure fast convergence to consensus. Boyd et al. (2004) also considered how to choose weights to ensure fastest consensus. In this paper, besides the convergence rate, the impact of the measurement noises may also be considered for choosing the weights. For example, if the noise intensity σ_{ji} is large, then the measurement y_{ji} of x_j is untrustworthy. In this case, we may choose a smaller weight a_{ij}.

In this paper, we will prove that under mild conditions, the control law (3) is an asymptotic unbiased mean square average-consensus protocol.

3. Convergence analysis

Denote the ith row of the matrix $\mathcal{A}$ by α_i, and $\Sigma_i = diag(\sigma_{1i}, \ldots, \sigma_{Ni})$, $i = 1, 2, \ldots, N$, where $\sigma_{ji} = 0, j \notin N_i$. $\Sigma = diag(\alpha_1^T \Sigma_1, \ldots, \alpha_N^T \Sigma_N)$ is an $N \times N^2$-dimensional block diagonal matrix. $n_i(t) = [n_{1i}(t), \ldots, n_{Ni}(t)]^T$, $\eta(t) = [n_1^T(t), \ldots, n_N^T(t)]^T$. Substituting the protocol (3) into the system (1) leads to

$$\frac{dX(t)}{dt} = [-a(t)L_{\mathcal{G}} X(t)] + a(t)\Sigma\eta(t).$$

It is a system driven by an N^2-dimensional standard white noise, which can be written in the form of the Itô stochastic differential equation

$$dX(t) = [-a(t)L_{\mathcal{G}} X(t)]dt + a(t)\Sigma dW(t), \tag{4}$$

where $W(t) = [W_{11}(t), \ldots, W_{N1}(t), \ldots, W_{NN}(t)]^T$ is an N^2-dimensional standard Brownian motion.

To get the main results, we need the following assumptions:

(A1) $\mathcal{G}$ is a balanced digraph.
(A2) $\mathcal{G}$ contains a spanning tree.
(A3) Convergence condition: $\int_0^\infty a(s)ds = \infty$.
(A4) Robustness condition: $\int_0^\infty a^2(s)ds < \infty$.

Remark 3. It can be seen that if there are constants $\beta_1 \leq 1$, $\beta_2 > -0.5$, $\gamma_1 \leq 1$, $\gamma_2 > 0.5$, $C_1 > 0$, $C_2 > 0$, such that $\frac{C_1}{t^{\gamma_1}[\log(t)]^{\beta_1}} \leq a(t) \leq \frac{C_2[\log(t)]^{\beta_2}}{t^{\gamma_2}}$, $t \to \infty$, then (A3)–(A4) hold.

Firstly, we will show what kind of conditions on the network topology is needed to ensure average-consensus under the protocol (3) for the noise-free case. For simplicity of problem formulation, we introduce the following assumption:

(A5) In the dynamic network $(\mathcal{G}, X)$, there is an edge $(j, i) \in \mathcal{E}$ such that $\sigma_{ji} > 0$.

The intuitive meaning of Assumption (A5) is that, there is at least one noisy communication channel in the dynamic network. The negative proposition of (A5) is given by

(A5′) In the dynamic network $(\mathcal{G}, X)$, for any $(j, i) \in \mathcal{E}$, we have $\sigma_{ji} = 0$.

When (A5′) holds, the dynamic network degenerates to the noise-free case, and the protocol (3) can be written as

$$u_i(t) = \begin{cases} 0, & i \in \mathcal{V}_s, \\ a(t) \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)), & i \in \mathcal{V} - \mathcal{V}_s. \end{cases} \quad (5)$$

Denote $J = \frac{1}{N}\mathbf{1}\mathbf{1}^T$, $\widehat{L}_{\mathcal{G}} = \frac{L_{\mathcal{G}} + L_{\mathcal{G}}^T}{2}$, $\delta(t) = X(t) - JX(t)$, and $V(t) = \delta^T(t)\delta(t)$. $\delta(t)$ is called the consensus error, and $V(t) = \frac{1}{N}\sum_{1 \le i < j \le N}(x_i(t) - x_j(t))^2$ is the energy function of consensus error (Olfati-Saber & Murray, 2004). Then, we have the following theorem.

Theorem 3.1. *Apply the protocol* (3) *to the system* (1)–(2) *and suppose that Assumption* (A5′) *holds. Then,*

$$\lim_{t\to\infty} \|X(t) - JX(0)\| = 0, \quad \forall X(0) \in \mathbb{R}^N, \quad (6)$$

if and only if (A1)–(A3) *hold.*

Proof. Sufficiency. It can be seen that Assumption (A5′) implies $\Sigma = 0$, and from Assumption (A1), it is known that $JL_{\mathcal{G}} = 0$. This together with (4) gives

$$\frac{\mathrm{d}JX(t)}{\mathrm{d}t} = 0, \quad (7)$$

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = -2a(t)\delta^T(t)L_{\mathcal{G}}\delta(t) = -2a(t)\delta^T(t)\widehat{L}_{\mathcal{G}}\delta(t). \quad (8)$$

From (A1), $\widehat{L}_{\mathcal{G}}$ is the Laplacian matrix of $\widehat{\mathcal{G}}$, which is the symmetrized graph of $\mathcal{G}$ (see Definition 2 and Theorem 7 of Olfati-Saber and Murray (2004)). Noticing that $\widehat{\mathcal{G}}$ is strongly connected, by Lemma 2.1, we have $\delta^T(t)\widehat{L}_{\mathcal{G}}\delta(t) \ge \lambda_2(\widehat{L}_{\mathcal{G}})V(t)$, and $\lambda_2(\widehat{L}_{\mathcal{G}}) > 0$. This together with (8) leads to

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le -2\lambda_2(\widehat{L}_{\mathcal{G}})a(t)V(t).$$

This together with the comparison theorem (Michel & Miller, 1977) gives

$$V(t) \le V(0)\exp\left\{-2\lambda_2(\widehat{L}_{\mathcal{G}})\int_0^t a(s)\mathrm{d}s\right\}. \quad (9)$$

Noticing that $\lambda_2(\widehat{L}_{\mathcal{G}}) > 0$, from (9) and Assumption (A3), we have

$$\lim_{t\to\infty} \|\delta(t)\| = 0. \quad (10)$$

From (7), it can be seen that $JX(t) \equiv JX(0)$, $\forall X(0) \in \mathbb{R}^n$, which together with (10) implies (6).

Necessity.

Step 1. We will prove the necessity of (A1). To this end, we need only to prove that: if $\mathcal{G}$ is not a balanced digraph, then (6) does not hold. Suppose that $\mathcal{G}$ is not balanced. Since $L_{\mathcal{G}}$ has a zero eigenvalue, there is an N-dimensional vector α, such that $\alpha^T\mathbf{1} = 1$, and $\alpha^T L_{\mathcal{G}} = 0$. Since $\mathcal{G}$ is not balanced, $\alpha \ne \frac{1}{N}\mathbf{1}$. This together with (4) and $\Sigma = 0$ leads to $\frac{\mathrm{d}\alpha^T X(t)}{\mathrm{d}t} = 0$. Therefore,

$$\alpha^T X(t) \equiv \alpha^T X(0), \quad \forall X(0) \in \mathbb{R}^n. \quad (11)$$

If (6) was correct, then we would have

$$\lim_{t\to\infty} \alpha^T X(t) = \alpha^T JX(0) = \frac{1}{N}\mathbf{1}^T X(0), \quad \forall X(0) \in \mathbb{R}^n.$$

This together with (11) would give $\alpha = \frac{1}{N}\mathbf{1}$, which contradicts $\alpha \ne \frac{1}{N}\mathbf{1}$. Hence, (6) does not hold.

Step 2. We will prove the necessity of (A2). To this end, we need only to prove that: if $\mathcal{G}$ does not contain a spanning tree, then (6) does not hold. Consider the case where $\mathcal{G}$ does not contain a spanning tree. In this case, there are only three situations (Ren & Beard, 2005):

(I) There is at least one isolated node i_0 in $\mathcal{G}$. Applying protocol (5), we get the closed-loop system described by

$$\frac{\mathrm{d}x_{i_0}(t)}{\mathrm{d}t} = 0,$$

$$\frac{\mathrm{d}\widetilde{X}(t)}{\mathrm{d}t} = -a(t)\widetilde{\mathcal{L}}\widetilde{X}(t),$$

where $\widetilde{X}(t) = [x_1(t), \ldots, x_{i_0-1}(t), x_{i_0+1}(t), \ldots, x_N(t)]^T$, and $\widetilde{\mathcal{L}}$ is the Laplacian matrix of the subgraph formed by deleting i_0 from $\mathcal{G}$. Take $x_{i_0}(0) = 0$, $x_j(0) = 1$, $\forall j \ne i_0$. Then, by $x_{i_0}(0) = 0$ we have $x_{i_0}(t) \equiv 0$. From $\widetilde{\mathcal{L}}\mathbf{1} = 0$, it is known that $\widetilde{X}(t) \equiv \mathbf{1}$ is the equilibrium solution of $\frac{\mathrm{d}\widetilde{X}(t)}{\mathrm{d}t} = -a(t)\widetilde{\mathcal{L}}\widetilde{X}(t)$ with $\widetilde{X}(0) = \mathbf{1}$. Noticing that $a(t)$ is piecewise continuous, from the uniqueness of the solutions of linear time-varying differential equations, one can get that $x_j(t) \equiv 1, j \ne i_0$. Thus, (6) does not hold.

(II) There is no isolated node but at least two sources i_1, i_2 in $\mathcal{G}$. Taking $x_{i_1}(0) = 0$, $x_{i_2}(0) = 1$, and applying protocol (5), similar to the proof in (I), we have $x_{i_1}(t) \equiv 0 \ne 1 \equiv x_{i_2}(t)$. Thus, (6) does not hold.

(III) There is no isolated node and at most one source in $\mathcal{G}$, but $\mathcal{G}$ can be divided into two separated subgraphs $\mathcal{G}_1 = \{\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1\}$ and $\mathcal{G}_2 = \{\mathcal{V}_2, \mathcal{E}_2, \mathcal{A}_2\}$, satisfying $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \Phi$, $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ and $\mathcal{E}_1 \cap \mathcal{E}_2 = \Phi$. Without loss of generality, suppose that $\mathcal{V}_1 = \{1, 2, \ldots, |\mathcal{V}_1|\}$, $\mathcal{V}_2 = \{|\mathcal{V}_1| + 1, \ldots, |\mathcal{V}_1| + |\mathcal{V}_2|\}$, and $\mathcal{A} = diag(\mathcal{A}_1, \mathcal{A}_2)$ is a block diagonal matrix. Applying protocol (5), the closed-loop system is given by

$$\frac{\mathrm{d}X_1(t)}{\mathrm{d}t} = -a(t)\widetilde{\mathcal{L}}_1 X_1(t),$$

$$\frac{\mathrm{d}X_2(t)}{\mathrm{d}t} = -a(t)\widetilde{\mathcal{L}}_2 X_2(t),$$

where $X_1(t)$ and $X_2(t)$ are the states of agents in $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively, $\widetilde{\mathcal{L}}_1$ and $\widetilde{\mathcal{L}}_2$ are the Laplacian matrices of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. Take $x_i(0) = 0$, $\forall i \in \mathcal{V}_1$, and $x_j(0) = 1$, $\forall j \in \mathcal{V}_2$. Then, similar to the proof in (I), by the uniqueness of the solutions of linear time-varying differential equations, we have $x_i(t) \equiv 0 \ne 1 \equiv x_j(t)$, $\forall i \in \mathcal{V}_1$, $\forall j \in \mathcal{V}_2$. Thus, (6) does not hold.

Step 3. We prove the necessity of (A3). To do so, we need only to prove that: if $\int_0^\infty a(t)\mathrm{d}t < \infty$, then (6) does not hold. Let $\widetilde{V}(t) = (X(t) - JX(0))^T(X(t) - JX(0))$.

Noticing that $\widehat{L}_{\mathcal{G}}$ is a real symmetric matrix, similar to (8), we have

$$\frac{\mathrm{d}\widetilde{V}(t)}{\mathrm{d}t} = -2a(t)(X(t) - JX(0))^T\widehat{L}_{\mathcal{G}}(X(t) - JX(0)) \ge -2\rho(\widehat{L}_{\mathcal{G}})a(t)\widetilde{V}(t).$$

By this and Gronwall inequality (Gronwall, 1919), we get

$$\widetilde{V}(t) \ge \widetilde{V}(0)\exp\left\{-2\rho(\widehat{L}_{\mathcal{G}})\int_0^t a(s)\mathrm{d}s\right\},$$

which implies that for any $\delta(0) \ne 0$,

$$\liminf_{t\to\infty} \widetilde{V}(t) \ge \widetilde{V}(0)\exp\left\{-2\rho(\widehat{L}_{\mathcal{G}})\int_0^\infty a(s)\mathrm{d}s\right\} > 0.$$

Thus, (6) does not hold. □

Remark 4. From Theorem 3.1, it can be seen that for the fixed topology case, when there is no measurement noise, Assumptions (A1) and (A2) are the weakest conditions on the network topology for the protocol (3) to achieve average-consensus. Containing a spanning tree ensures that different agents may asymptotically agree on their states; while the balance of the digraph is to make the centroid of the states to be a constant, such that the final group decision value is the average of the initial states.

From the proof of Theorem 3.1, it can be seen that Assumption (A3) is to ensure that the consensus error converges to zero with a certain rate. In fact, when $a(t) \equiv 1$, the protocol (5) degenerates to the time-invariant protocol (A.1) in Olfati-Saber and Murray (2004) where (A3) holds naturally, and the consensus error converges to zero exponentially. Therefore, we call (A3) the convergence condition on consensus gains.

Remark 5. Different from Olfati-Saber and Murray (2004), here we use Assumption (A2) rather than the assumption "$\mathcal{G}$ is strongly connected". At a first glance, a balanced digraph containing a spanning tree implies strong connectivity and there is no need to refer to the concept of spanning tree. However, since Theorem 3.1 is to give necessary and sufficient conditions for the protocol (3) to ensure average-consensus, the conditions given ought to be as weak as possible. Containing a spanning tree is weaker than being strongly connected and is the weakest condition considered on the network topology to ensure consensus. Moreover, by introducing the concept of containing a spanning tree, the proof of the necessity part of Theorem 3.1 is clear, where Steps 1, 2 and 3 are for the necessity of (A1), (A2) and (A3), respectively.

Remark 6. Substituting the protocol (5) into the system (1), we get the closed-loop system for the noise-free case:

$$\dot{X}(t) = -L_{\mathcal{G}(t)}X(t), \quad t \geq 0, \tag{12}$$

where $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}, a(t)\mathcal{A}\}$ is a digraph with the time-varying weighted adjacency matrix $a(t)\mathcal{A}$. The system (12) can be regarded as a special case of a kind of time-varying system described by

$$\dot{X}(t) = -L_{\mathcal{G}(t)}X(t), \quad t \geq 0, \tag{13}$$

where $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t)\}$ is a digraph with time-varying topologies.

The convergence properties of the system (13) and its discrete-time version have been widely studied (Moreau, 2005; Moreau & Belgium, 2004; Tsitsiklis, Bertsekas, & Athans, 1986). Moreau and Belgium (2004) gave a sufficient condition to guarantee that all the state components of the system (13) converge to a common value as time goes on. From Theorem 3.1, it can be seen that, the condition given by Moreau and Belgium (2004) is not necessary. It is easy to verify that if (A1) is satisfied, and $a(t) = \frac{1}{t+1}$, then by Theorem 3.1, all the state components of (12) converge to $\frac{1}{N}\sum_{j=1}^{N} x_j(0)$, but the condition given by Moreau and Belgium (2004) is not satisfied (see Theorem 1 of Moreau and Belgium (2004)).

Below we will prove that under Assumptions (A1)–(A4), the control law (3) is an asymptotic unbiased mean square average-consensus protocol, which needs the following lemma.

Lemma 3.1. *Applying the protocol* (3) *to the system* (1)–(2), *if Assumption* (A1) *holds, then*

$$E\int_{t_0}^{t} a(s)\delta^{\mathrm{T}}(s)(I-J)\Sigma \mathrm{d}W(s) = 0, \quad \forall t \geq t_0. \tag{14}$$

Proof. By (4) and (A1), we have

$$\begin{aligned} \mathrm{d}\delta(t) &= [-a(t)L_{\mathcal{G}}X(t)]\mathrm{d}t + a(t)(I-J)\Sigma \mathrm{d}W(t) \\ &= -a(t)L_{\mathcal{G}}\delta(t)\mathrm{d}t + a(t)(I-J)\Sigma \mathrm{d}W(t). \end{aligned}$$

Furthermore, by (A1), Lemma 2.1 and the Itô formula, we have

$$\begin{aligned} \mathrm{d}V(t) &= [-2a(t)\delta^{\mathrm{T}}(t)\widehat{L}_{\mathcal{G}}\delta(t) + a^2(t)C_0]\mathrm{d}t \\ &\quad + 2a(t)\delta^{\mathrm{T}}(t)(I-J)\Sigma \mathrm{d}W(t) \\ &\leq [-2\lambda_2(\widehat{L}_{\mathcal{G}})a(t)V(t) + a^2(t)C_0]\mathrm{d}t \\ &\quad + 2a(t)\delta^{\mathrm{T}}(t)(I-J)\Sigma \mathrm{d}W(t), \end{aligned} \tag{15}$$

where

$$C_0 = \mathrm{tr}((I-J)^2\Sigma\Sigma^{\mathrm{T}}). \tag{16}$$

For any given $t_0 \geq 0, T \geq t_0$, let

$$\tau_K^{t_0,T} = \begin{cases} \inf\{t \geq t_0 : \|\delta(t)\| \geq K\}, \\ \quad \text{if } \|\delta(t)\| \geq K \quad \text{for some } t \in [t_0, T]; \\ T, \quad \text{otherwise}. \end{cases}$$

Then, by (15) we have

$$\begin{aligned} &E\left[V(t \wedge \tau_K^{t_0,T})\chi_{\{t \leq \tau_K^{t_0,T}\}}\right] - E[V(t_0)] \\ &\leq -2\lambda_2(\widehat{L}_{\mathcal{G}})\int_{t_0}^{t} a(s)EV\left(s \wedge \tau_K^{t_0,T}\right)\chi_{\{s \leq \tau_K^{t_0,T}\}}\mathrm{d}s \\ &\quad + C_0\int_{t_0}^{t} a^2(s)\mathrm{d}s \\ &\leq C_0\int_0^{T} a^2(s)\mathrm{d}s, \quad \forall t \in [t_0, T]. \end{aligned}$$

This implies that there is a constant $C_{t_0,T} < \infty$ such that

$$E\left[V\left(t \wedge \tau_K^{t_0,T}\right)\chi_{\{t \leq \tau_K^{t_0,T}\}}\right] \leq C_{t_0,T}, \quad \forall t \in [t_0, T].$$

Noticing that $\lim_{K\to\infty} t \wedge \tau_K^{t_0,T} = t$ a.s., $\forall t \in [t_0, T]$, by Fatou lemma (Chow & Teicher, 1997), we have

$$\sup_{t_0 \leq t \leq T} E(V(t)) \leq C_{t_0,T}.$$

Thus,

$$E\left[\int_{t_0}^{t} a^2(s)V(s)\mathrm{d}s\right] \leq \sup_{t_0 \leq s \leq t} E(V(t))\int_0^{\mathrm{T}} a^2(s)\mathrm{d}s < \infty, \quad \forall t \geq t_0 \geq 0.$$

Furthermore, noticing that

$$E\left[\int_{t_0}^{t} a^2(s)\|\delta^{\mathrm{T}}(s)(I-J)\Sigma\|^2\mathrm{d}s\right] \leq C_0E\left[\int_{t_0}^{t} a^2(s)V(s)\mathrm{d}s\right], \quad \forall t \geq t_0 \geq 0,$$

by the property of Itô integral (Friedman, 1975), we have (14). □

Theorem 3.2. *Applying the protocol* (3) *to the system* (1)–(2), *if Assumptions* (A1)–(A4) *hold, then*

$$\lim_{t\to\infty} E[V(t)] = 0. \tag{17}$$

Proof. By (14) and (15) we have

$$E[V(t)] - E[V(t_0)] \leq -2\lambda_2(\widehat{L}_{\mathcal{G}})\int_{t_0}^{t} a(s)E[V(s)]\mathrm{d}s + C_0\int_{t_0}^{t} a^2(s)\mathrm{d}s, \quad \forall\, t \geq t_0 \geq 0. \tag{18}$$

Denote $I_1(t) = \int_0^t \exp\{-2\lambda_2(\widehat{L}_{\mathcal{G}}) \int_s^t a(u)\mathrm{d}u\} a^2(s)\mathrm{d}s$ and $I_2(t) = V(0)\exp\{-2\lambda_2(\widehat{L}_{\mathcal{G}}) \int_0^t a(s)\mathrm{d}s\}$. Then, by (18) and the comparison theorem (Michel & Miller, 1977) we have

$$E[V(t)] \le C_0 I_1(t) + I_2(t). \tag{19}$$

From (A3) and $\lambda_2(\widehat{L}_{\mathcal{G}}) > 0$, one can get $\lim_{t\to\infty} I_2(t) = 0$. Thus, to prove (17), we need only to prove $\lim_{t\to\infty} I_1(t) = 0$. In fact, for any given $\epsilon > 0$, by Assumption (A4), there is $s_0 > 0$ such that $\int_{s_0}^{\infty} a^2(s)\mathrm{d}s < \epsilon$. Therefore,

$$\begin{aligned} I_1(t) &= \int_0^{s_0} \exp\left\{-2\lambda_2(\widehat{L}_{\mathcal{G}}) \int_s^t a(u)\mathrm{d}u\right\} a^2(s)\mathrm{d}s \\ &\quad + \int_{s_0}^t \exp\left\{-2\lambda_2(\widehat{L}_{\mathcal{G}}) \int_s^t a(u)\mathrm{d}u\right\} a^2(s)\mathrm{d}s \\ &\le \exp\left\{-2\lambda_2(\widehat{L}_{\mathcal{G}}) \int_{s_0}^t a(u)\mathrm{d}u\right\} \int_0^{s_0} a^2(s)\mathrm{d}s + \int_{s_0}^t a^2(s)\mathrm{d}s \\ &\le \exp\left\{-2\lambda_2(\widehat{L}_{\mathcal{G}}) \int_{s_0}^t a(u)\mathrm{d}u\right\} \int_0^{\infty} a^2(s)\mathrm{d}s + \int_{s_0}^{\infty} a^2(s)\mathrm{d}s \\ &\le \exp\left\{-2\lambda_2(\widehat{L}_{\mathcal{G}}) \int_{s_0}^t a(u)\mathrm{d}u\right\} \int_0^{\infty} a^2(s)\mathrm{d}s \\ &\quad + \epsilon, \quad \forall t \ge s_0. \end{aligned} \tag{20}$$

From (A3) and $\lambda_2(\widehat{L}_{\mathcal{G}}) > 0$ it follows that

$$\lim_{t\to\infty} \exp\left\{-2\lambda_2(\widehat{L}_{\mathcal{G}}) \int_{s_0}^t a(u)\mathrm{d}u\right\} = 0.$$

Thus, by the arbitrariness of ϵ and (20), we have $\lim_{t\to\infty} I_1(t) = 0$. Hence, (17) holds. □

Theorem 3.3. *Applying the protocol (3) to the system (1)–(2), if Assumptions (A1)– (A4) hold, then*

$$\lim_{t\to\infty} \max_{1\le i\le N} E(x_i(t) - x^*)^2 = 0,$$

where x^* is a Gaussian random variable whose mathematical expectation is $\frac{1}{N}\sum_{j=1}^N x_j(0)$, and variance is $\frac{\sum_{j=1}^N \sum_{j\in N_i} \sigma_{ji}^2 a_{ij}^2}{N^2} \int_0^\infty a^2(s)\mathrm{d}s$, that is, (3) is an asymptotic unbiased mean square average-consensus protocol.

Proof. By (4) and (A1), we have

$$\mathrm{d}\left(\frac{1}{N}\sum_{j=1}^N x_j(t)\right) = a(t)\frac{1}{N}\mathbf{1}^{\mathrm{T}}\Sigma \mathrm{d}W(t),$$

or equivalently,

$$\frac{1}{N}\sum_{j=1}^N x_j(t) = \frac{1}{N}\sum_{j=1}^N x_j(0) + \frac{\mathbf{1}^{\mathrm{T}}\Sigma}{N}\int_0^t a(s)\mathrm{d}W(s). \tag{21}$$

From (A4) and the definition of the Itô integral (Friedman, 1975), we know that $\int_0^\infty a(s)\mathrm{d}W(s)$ is well defined. Let

$$x^* = \frac{1}{N}\sum_{j=1}^N x_j(0) + \frac{1}{N}\mathbf{1}^{\mathrm{T}}\Sigma\int_0^\infty a(s)\mathrm{d}W(s).$$

Then, by (21) and the Itô isometry, we have

$$\begin{aligned} &\lim_{t\to\infty} E\left(\frac{1}{N}\sum_{j=1}^N x_j(t) - x^*\right)^2 \\ &= \lim_{t\to\infty} E\left(\frac{1}{N}\mathbf{1}^{\mathrm{T}}\Sigma\int_t^\infty a(s)\mathrm{d}W(s)\right)^2 \\ &= \frac{\mathrm{tr}(\Sigma\Sigma^{\mathrm{T}})}{N^2}\int_t^\infty a^2(s)\mathrm{d}s = o(1), \quad t\to\infty. \end{aligned} \tag{22}$$

Notice that

$$E[x^*] = \frac{1}{N}\sum_{j=1}^N x_j(0),$$

$$\begin{aligned} Var(x^*) &= E\left(\frac{1}{N}\mathbf{1}^{\mathrm{T}}\Sigma\int_0^\infty a(s)\mathrm{d}W(s)\right)^2 \\ &= \frac{\sum_{j=1}^N\sum_{j\in N_i}\sigma_{ji}^2 a_{ij}^2}{N^2}\int_0^\infty a^2(s)\mathrm{d}s. \end{aligned}$$

Then, from (22) and Theorem 3.2, the conclusion of Theorem 3.3 holds. □

It is shown by Theorem 3.3 that, if the network topology is a balanced digraph containing a spanning tree, then (A3)–(A4) are sufficient conditions to achieve asymptotic unbiased mean square average-consensus. From the following Theorem 3.4, it can be seen that, when there exist measurement noises, (A3)–(A4) are also necessary. To see this, we need the following lemma

Lemma 3.2. *Applying the protocol (3) to the system (1)–(2), if Assumption (A1) holds, then for the constant C_0 given by (16) and all $t \ge t_0 \ge 0$,*

$$\begin{aligned} E[V(t)] &\ge E[V(t_0)]\exp\left\{-2\lambda_N(\widehat{L}_{\mathcal{G}})\int_{t_0}^t a(s)\mathrm{d}s\right\} \\ &\quad + C_0\int_{t_0}^t \exp\left\{-2\lambda_N(\widehat{L}_{\mathcal{G}})\int_s^t a(u)\mathrm{d}u\right\} a^2(s)\mathrm{d}s. \end{aligned}$$

Proof. Noticing that $\delta^{\mathrm{T}}(t)\widehat{L}_{\mathcal{G}}\delta(t) \le \lambda_N(\widehat{L}_{\mathcal{G}})V(t)$, by Lemma 3.1, similar to (18), we have that for all $t \ge t_0 \ge 0$,

$$\begin{aligned} &E[V(t)] - E[V(t_0)] \\ &\ge -2\lambda_N(\widehat{L}_{\mathcal{G}})\int_{t_0}^t a(s)E[V(s)]\mathrm{d}s + C_0\int_{t_0}^t a^2(s)\mathrm{d}s. \end{aligned}$$

This together with the comparison theorem gives the result. □

Theorem 3.4. *Apply the protocol (3) to the system (1)–(2) and suppose that Assumptions (A1), (A2) and (A5) hold. Then, (3) is an asymptotic unbiased mean square average-consensus protocol if and only if (A3)–(A4) hold.*

Proof. From Theorem 3.3, the sufficiency is obvious. Below we need only prove the necessity.

We use the reduction to absurdity. Firstly, we prove the necessity of (A3). If (A3) was not true, then from Lemma 3.2 we would have that for any $\delta(0) \ne 0$,

$$\liminf_{t\to\infty} E[V(t)] \ge \liminf_{t\to\infty} V(0)\exp\left\{-2\lambda_N(\widehat{L}_{\mathcal{G}})\int_0^t a(s)\mathrm{d}s\right\} > 0.$$

This contradicts the fact that $x_i(t)$, $i = 1, 2, \ldots, N$ converge to a common random variable in mean square. Hence, (A3) is necessary.

Below we prove the necessity of (A4). If (A4) did not hold, noticing that $x_i(t)$, $i = 1, 2, \ldots, N$, converge in mean square to a common random variable with finite second-order moment, then by (21) we would see that as $t \to \infty$, $\frac{1}{N}\mathbf{1}^{\mathrm{T}}\Sigma\int_0^t a(s)\mathrm{d}W(s)$ converges in mean square to a random variable x_w with finite second-order moment. Thus, by Corollary 4.2.5 in Chow and Teicher (1997), we would have

$$\lim_{t\to\infty} E\left(\frac{1}{N}\mathbf{1}^{\mathrm{T}}\Sigma\int_0^t a(s)\mathrm{d}W(s)\right)^2 = E(x_w)^2 < \infty. \tag{23}$$

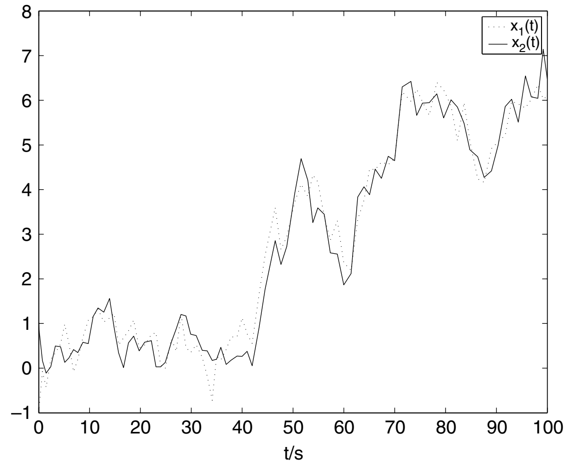

Fig. 1. Curves of states of Example 1.

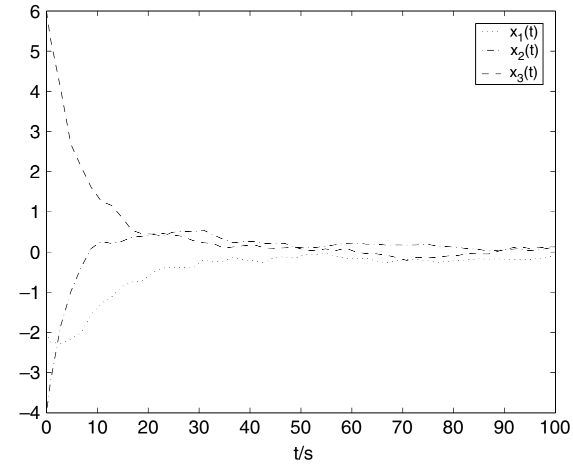

Fig. 2. Curves of states of Example 2.

On the other hand, from (A5), it is known that $\mathrm{tr}(\Sigma\Sigma^T) > 0$. Hence, by the Itô isometry, we have

$$\lim_{t\to\infty} E\left(\frac{1}{N}\mathbf{1}^T\Sigma\int_0^t a(s)\mathrm{d}W(s)\right)^2 = \lim_{t\to\infty}\frac{\mathrm{tr}(\Sigma\Sigma^T)}{N^2}\int_0^t a^2(s)\mathrm{d}s = \infty.$$

This contradicts (23). Therefore, (A4) is necessary, too. □

Remark 7. Combining Theorems 3.3 and 3.4, one can see the important role played by (A4). When there is no measurement noise, to achieve average-consensus, it is only required that the consensus gains satisfy the convergence condition (A3). However, in the noisy environment, from (21) one can see that the state average of the closed-loop system is not a constant any more, and (A3) itself is no longer sufficient. (A4) ensures that as time goes on, the state average of the closed-loop system converges in mean square. Theorems 3.3 and 3.4 also tell us that the time-invariant protocol (A.1) proposed by Olfati-Saber and Murray (2004) is not robust with respect to Gaussian noises. The purpose of the introduction of time-varying consensus gains and Assumption (A4) is just to attenuate the measurement noises, such that the consensus protocol is robust with respect to measurement noises. We call (A4) the robustness condition on consensus gains.

4. Numerical examples

Example 1. In this example we investigate the necessity of (A4) when there are measurement noises by a two-agent interacting system with the topology graph $\mathcal{G}_1 = \{1, 2, \{(1, 2), (2, 1)\}, \mathcal{A}_1 = [a_{ij}]_{2\times 2}\}$, where $a_{11} = a_{22} = 0$, $a_{12} = a_{21} = 1$. The intensity of the measurement noises $\sigma_{21} = \sigma_{12} = 1$, and the initial states of the agents are $x_1(0) = 1$ and $x_2(0) = -1$, respectively. The consensus-gain function $a(t)$ is taken as $a(t) \equiv 1$, $\forall t \geq 0$. In this case, Assumptions (A1), (A2) and (A3) hold, but (A4) does not hold. Under the control of protocol (3), the states of the closed-loop system are shown in Fig. 1. It can be seen that the closed-loop system is divergent.

Example 2. Consider a dynamic network of three agents with the topology graph $\mathcal{G}_2 = \{1, 2, 3, \{(1, 2), (2, 3), (3, 1)\}, \mathcal{A}_2 = [a_{ij}]_{3\times 3}\}$, where $a_{13} = a_{32} = a_{21} = 1$, $a_{11} = a_{12} = a_{22} = a_{23} = a_{31} = a_{33} = 0$. The intensity of the measurement noises $\sigma_{12} = \sigma_{23} = \sigma_{31} = 1$. The initial states of agents are given by $x_1(0) = -2$, $x_2(0) = -4$ and $x_3(0) = 6$, respectively. The consensus-gain function $a(t)$ is taken as $a(t) = \frac{\log(t+2)}{t+2}$, $t \geq 0$. Under the control of the protocol (3), the states of the closed-loop system are shown in Fig. 2. It can be seen that as time goes on, the states of the group asymptotically achieve consensus, and approach the average of the initial states of all agents.

5. Concluding remarks

In this paper, the average-consensus control has been considered for networks of first-order integrator agents under fixed and directed topologies. The control input of each agent can only use its local state and the states of its neighbors corrupted by white noises. Though the network topology is time-invariant, due to the measurement noises, the convergence of the closed-loop system cannot be ensured by using only the time-invariant protocol proposed by Olfati-Saber and Murray (2004). Thus, time-varying consensus gains are used in the consensus protocol. The concept and tools of symmetrized matrices are used in the stochastic Lyapunov analysis. Firstly, it is proved that a balanced graph containing a spanning tree is the weakest condition on the network topology to ensure average-consensus for these kinds of protocols. Then, a necessary and sufficient condition is given on the consensus gains to ensure asymptotic unbiased mean square average-consensus. It is proved that under the protocol designed, the state of each agent converges in mean square to a common Gaussian random variable, whose mathematical expectation of the random variable is just the average of the initial states.

For future research, it is an issue worth investigating how to choose consensus gains properly to ensure almost sure consensus and how to characterize the class of control laws that guarantee average-consensus under measurement noises. In addition, consensus problems under measurement noises with leaders are also valuable for some applied scenarios.

Acknowledgements

The research of Tao Li and Ji-Feng Zhang was supported by the National Natural Science Foundation of China under grants 60821091 and 60674038.

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