Asymptotically Optimal Decentralized Control for Large Population Stochastic Multiagent Systems

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***Abstract*—The interaction of interest-coupled decision-makers and the uncertainty of individual behavior are prominent characteristics of multiagent systems (MAS). How to break through the framework of conventional control theory, which aims at single decision-maker and single decision objective, and to extend the methodology and tools in the stochastic adaptive control theory to analyze MAS are of great significance. In this paper, a preliminary exploration is made in this direction, and the decentralized control problem is considered for large population stochastic MAS with coupled cost functions. Different from the deterministic discounted costs in the existing differential game models, a time-averaged stochastic cost function is adopted for each agent. The decentralized control law is constructed based on the state aggregation method and tracking-like quadratic optimal control. By using probability limit theory, the stability and optimality of the closed-loop system are analyzed. The main contributions of this paper include the following points. 1) The concepts of asymptotic Nash-equilibrium in probability and almost surely, respectively, are introduced and the relationship between these concepts is illuminated, which provide necessary tools for analyzing the optimality of the decentralized control laws. 2) The closed-loop system is shown to be *almost surely uniformly stable*, and bounded independently of the number of agents N. 3) The population state average (PSA) is shown to converge to the infinite population mean (IPM) trajectory in the sense of both $\mathcal{L}_2$-norm and time average almost surely, as N increases to infinity. 4) The decentralized control law is designed and shown to be *almost surely* asymptotically optimal; the cost of each agent based on local measurements converges to that based on global measurements *almost surely*, as N increases to infinity.**

***Index Terms*—Asymptotic Nash equilibrium, decentralized control, multiagent systems, stochastic cost function, stochastic differential game.**

I. Introduction

A. Motivations and Issues

IN RECENT years, analysis and control design for multiagent systems (MAS) have become very popular in the control community, forming an active area in the study of complex systems. A typical kind of MAS, which we called large population stochastic multiagent systems (LPSMAS), is focused on in this paper. There are many practical examples for LPSMAS in engineering, biological, social and economic systems, such as wireless sensor networks ([1]), very large scale robotics ([2]); swarm and flocking phenomenon in biological systems ([3], [4]); evacuation of large crowd in emergency ([5]), sharing and competing for resources on the Internet ([6]), and so on.

Agents in LPSMAS are autonomous and interacting. Autonomy refers to that each agent is a unity of plant, sensor, and controller, with (relatively independent) its own performance criteria or cost function. Agent to agent interaction is usually due to the coupling of their dynamics or cost functions, which leads to conflicting objectives or consistent emergence *population behavior*.

Agents in LPSMAS have self-governed but limited capability of sensing, computing (decision-making) and communicating, leading to the decentralized control for the whole system. Namely, decision-making of each agent can only depend on its local state, or, under certain circumstances, include those of others in its sensing neighborhood. While, in conventional control systems, control laws are always constructed based upon the overall states of the plants. So, from this point of view, conventional control systems can be viewed as single-agent systems.

Agents in LPSMAS behave randomly and cohesively. Randomness consists in that the dynamics of agents are always influenced by some random noises and so evolve uncertainly. Cohesiveness is particularly prominent in biological and social systems, which means that every agent has the tendency of approaching the average state of the population, reflecting the relationship between the microscopic agents' behaviors and the macroscopic population behavior. Driven by random noises, individual agent behaves as a stochastic process, but due to cohesiveness, the whole system often takes on some deterministic pattern, reflecting the emergence characteristics of complex systems.

As mentioned above, autonomy and interaction of agents give rise to the dynamic game-theory modeling for LPSMAS; stochastic systems theory is a powerful tool to describe the uncertainty of agents' behaviors; the control design for LPSMAS is required to be put under the framework of decentralized control systems. So, the problem considered is actually a large scale distributed stochastic game involving numerous agents. Just as stabilization and optimization are two fundamental issues for single-agent systems, for LPSMAS we are also concerned with how to construct decentralized control laws to make closed-loop systems stable and performances of agents optimized.

Different from conventional control systems, for LPSMAS, we are not only concerned with the property of evolution in time scale but also that of variation in "space" scale, here variation in "space" scale refers to variation of the number of agents N in the

Manuscript received October 11, 2006; revised September 28, 2007. Current version published September 10, 2008. This work was supported by the National Natural Foundation of China under Grants 60221301 and 60674038. Recommended for publication by Associate Editor J. P. Hespanha.

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Digital Object Identifier 10.1109/TAC.2008.929370

system, especially the asymptotic property of the whole system as $N \to \infty$, such as whether the closed-loop stability and optimality can be retained, whether the control law is asymptotically optimal. Concerning the asymptotic property is an important reason why we use LPSMAS instead of MAS for our problem formulation. "Large population" embodies our angle of view, which is a dynamic concept, different from the one "large scale." The MAS in the above-mentioned examples can be taken as LPSMAS to study, but there does be some MAS that cannot be viewed as LPSMAS, for example, the oligopoly models in game theory.

B. State of the Art for MAS and Main Contributions of the Paper

There has been much quantitative analysis on MAS, mainly concentrated on system stability and optimality. The models wherein used can be divided into two categories: deterministic and stochastic models. [7]–[12] used deterministic model to investigate the temporal evolvement of MAS in swarm, network consensus problem, coordination control and formation control, and analyzed the closed-loop stability for the case of fixed agent number. In swarm systems, stability usually means that agents' states retain bounded as time goes on, called cohesive stability ([7], [8]). While in the problems such as network consensus, coordination control, and formation control, stability usually means agents' states asymptotically approach certain configurations as time goes on, called consensus stability ([9]–[12]). Both of these stability concepts are presented under the condition of fixed number of agents N, while, in many MAS, stability conditions are closely related to N ([13]–[15]). So, it is necessary to study whether the closed-loop stability is still retained when N is changed, that is, whether the closed-loop system is uniformly stable with respect to N.

The frameworks of optimization analysis for MAS can be divided into game-theory-based and nongame-theory-based. Under the nongame-theory-based framework, a centralized cost function is used to characterize the system performance, and the objective of the control design is usually to draw the states of the closed-loop system to the minimizer of the cost. The research in [7] and [12] indicate that many swarm, flocking algorithms can be viewed as optimization processes for a *collective potential function*, which is in fact the centralized cost function. Under the game-theory-based framework, each agent has its own cost function, all of which comprise a cost group, and the objective of the control design is usually to obtain equilibrium strategies with respect to the cost group. [16] gave a good survey for noncooperative game models, which are widely used in the flow control and routing of networks ([17]–[19]).

Concerning about the optimal control of MAS with stochastic dynamic game models, roughly speaking, there are two classes of cost functions: stochastic and deterministic, according to whether or not depending on sample paths of the underlying probability space. Stochastic cost functions have clearer physical interpretations and practical meanings, however, up to now, are limited to be used in finite strategy games ([20]–[22]). While, most of the existing research on infinite strategy stochastic games are restricted to deterministic costs (for instance, in the mathematical expectation form). [23] considered discrete time, discrete state, coupled Markov game applied to power control for CDMA communication network. [24]–[26] studied the LQG games with scalar agent models and deterministic discounted cost functions.

In this paper, the decentralized control for LPSMAS with coupled stochastic cost functions is considered. The state aggregation method ([25], [26]) is used to design the control law. The closed-loop system is shown *almost surely* asymptotically optimal in the sense of Nash equilibrium and *almost surely uniformly* stable with respect to N, that is the closed-loop stability is retained even if the number of agents becomes arbitrarily large. Compared with the existing work, the paper is characterized by the following points. 1) Agent models are multivariabled, which are not limited to be position or velocity. 2) Not only the temporal evolvement but also the asymptotic property of the system as $N \to \infty$ are investigated. It is proved that the closed-loop system is almost surely uniformly stable in the sense of time average, that is, the closed-loop stability will not be destroyed in case of proliferation of agents. 3) Stochastic time-averaged cost functions are optimized, which have a clearer and intuitive physical meaning than expectation-type deterministic cost functions, but, in the stochastic case, the costs to be studied are random variables (which are actually functions defined on some probability space) rather than scalars encountered in deterministic cases. So, in this case, the cost minimization essentially involves comparison of functions, which makes the existing concepts of Nash-equilibria for deterministic cost functions not suitable. Thus, to solve the decentralized optimal control problem for LPSMAS with coupled stochastic cost functions, some new concepts such as *asymptotic Nash equilibria in the probabilistic sense* are introduced. To obtain the optimality of stochastic cost functions, the analysis of the property of the closed-loop sample paths seems to be key. This brings difficulty to the convergence analysis of the estimates involved, due to the essential difference between the growth rates of the sample paths and moment paths of diffusion processes. By using the probability limit theory, we obtain some laws of large numbers related to the closed-loop system, with which, show that the decentralized control law constructed is almost surely asymptotically optimal. 4) The laws of large numbers of the closed-loop system are presented to show the convergence of the PSA to the IPM trajectory in the sense of both $\mathcal{L}_2$ norm and time average almost surely, which, in some sense, reflects the resultant characteristics of MAS from microscopic uncertain behaviors to macroscopic deterministic behaviors.

C. Organization of the Paper and Notation

The following notation will be used throughout this paper. $\|\cdot\|$ denotes 2-norm of vectors or F-norm of matrices; $tr(X)$ denotes the trace of a square matrix X; X^T denotes the transpose of a vector or matrix X; for any given appropriate dimensional vector x and symmetric matrix $Q \geq 0, \|x\|_Q = (x^TQx)^{1/2}$; $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the maximum and minimum eigenvalue of symmetric matrices, respectively; χ_A denotes the indicator function of a set A; $\mathcal{C}_n$ denotes the family of all n dimensional continuous vector-valued functions on $[0,\infty)$; $\mathcal{C}_n^b = \{x \mid x \in \mathcal{C}_n, \sup_{t\geq 0}\|x(t)\| < \infty\}$ denotes the family of all bounded functions in $\mathcal{C}_n$. For any $x \in \mathcal{C}_n^b, \|x\|_\infty \triangleq \sup_{t\geq 0}\|x(t)\|$.

The remainder of this paper is organized as follows. In Section II, we present some relevant concepts of Nash-equilibria in the probabilistic sense, and the model and assumptions of cost-coupled LPSMAS. In Section III, we first investigate the tracking-like quadratic optimal control problem, and then, construct the decentralized control law by state aggregation. Main results of this paper are presented in Sections IV–VI. In Section IV, we demonstrate that the closed-loop system is almost surely uniformly stable. In Section V, we present the laws of large numbers of the closed-loop system, with which we analyze the asymptotic optimality of the decentralized control law in Section VI. In Section VII, we use a numerical example to verify our results. In Section VIII, some concluding remarks are included with some further research topics.

II. Preliminary Concepts and Problem Formulation

A. Nash Asymptotically Optimal Decentralized Control

We denote a system of N agents by $\mathbf{S}^N$, and the dynamic equation for the ith agent is given by

$$dx_i^N = \left(f_i^N\left(x_i^N, x_{-i}^N, t\right) + b_i^N u_i^N\right)dt + g(x_i,t)dW_i(t)$$

where $x_i^N \in \mathbb{R}^n$ is its state, $x_{-i}^N \triangleq (x_1^N, \ldots, x_{i-1}^N, x_{i+1}^N, \ldots, x_N^N)$; u_i^N is its control input, $\{(W_i(t), \mathcal{F}_t^i), -\infty < t < \infty, i \geq 1\}$ is a sequence of independent standard Brownian motions with proper dimension on some probability space $(\Omega, \mathcal{F}, P)$.

For convenience, for agent i, we denote the *local-measurement-based admissible control set* by $\mathcal{U}_{l,i}^N$, *global-measurement-based admissible control set* by $\mathcal{U}_{g,i}^N$, admissible control set by $\mathcal{U}_i^N$. In different problems, there may be different interpretation and relationship of $\mathcal{U}_{l,i}^N, \mathcal{U}_{g,i}^N$, and $\mathcal{U}_i^N$. For example

$$\begin{cases}\mathcal{U}_{l,i}^N = \left\{u_i^N \mid u_i^N(t) \text{ is adapted to } \sigma\left(x_i^N(s), s \leq t\right)\right\}\\ \mathcal{U}_{g,i}^N = \left\{u_i^N \mid u_i^N(t) \text{ is adapted to } \sigma\left(x_i^N(s), s \leq t\right.\right.\\ \qquad\qquad \left.\left. 1 \leq i \leq N\right)\right\};\end{cases}$$

$$\begin{cases}\mathcal{U}_{l,i}^N = \left\{u_i^N \mid u_i^N(t) \text{ is adapted to } \sigma\left(x_i^N(s), s \leq t\right)\right.\\ \qquad\qquad \left.\int_0^T \|x_i^N(t)\|^2 dt = O(T) \ a.s.\right\}\\ \mathcal{U}_{g,i}^N = \left\{u_i^N \mid u_i^N(t) \text{ is adapted to } \sigma\left(x_i^N(s), s \leq t\right.\right.\\ \qquad\qquad \left.\left. 1 \leq i \leq N\right), \int_0^T \left\|x_i^N(t)\right\|^2 dt = O(T) \ a.s.\right\}.\end{cases}$$

When $\mathcal{U}_i^N = \mathcal{U}_{g,i}^N$, the corresponding control is called as centralized control; while, when $\mathcal{U}_i^N = \mathcal{U}_{l,i}^N$, as decentralized control.

The so-called optimal decentralized control means that the synthesis of u_i^N can only be based on the local measurement of agent i, that is, $\mathcal{U}_i^N = \mathcal{U}_{l,i}^N$, to minimize the corresponding cost function $J_i^N(u_i^N, u_{-i}^N)$, where $u_{-i}^N = (u_1^N, \ldots, u_{i-1}^N, u_{i+1}^N, \ldots, u_N^N)$. For convenience, we denote a control group of $\mathbf{S}^N$ by $\mathbf{U}^N = \{u_i^N, 1 \leq i \leq N\}$, and its corresponding cost group by $\mathbf{J}^N = \{J_i^N(u_i^N, u_{-i}^N), 1 \leq i \leq N\}$, which is a group of random variables (r.v.s.).

Below we give the definitions of asymptotic Nash-equilibria in the probabilistic sense.

Definition 2.1: A sequence of control groups $\{\mathbf{U}^N = \{u_i^N \in \mathcal{U}_i^N, 1 \leq i \leq N\}, N \geq 1\}$ is called an asymptotic Nash-equilibrium with respect to the corresponding sequence of cost groups $\{\mathbf{J}^N = \{J_i^N, 1 \leq i \leq N\}, N \geq 1\}$, if for any $\epsilon > 0, \delta > 0$, there exists $M > 0$ such that for any $N > M$

$$P\left\{\max_{1\leq i\leq N}\left\{J_i^N(u_i^N, u_{-i}^N) - \inf_{v_i \in \mathcal{U}_{g,i}^N} J_i^N\left(v_i, u_{-i}^N\right)\right\} \geq \epsilon\right\} \leq \delta.$$

Theorem 2.1: A sequence of control groups $\{\mathbf{U}^N = \{u_i^N, 1 \leq i \leq N\}, N \geq 1\}$ is an asymptotic Nash-equilibrium in probability with respect to $\{\mathbf{J}^N = \{J_i^N, 1 \leq i \leq N\}, N \geq 1\}$ if and only if that there exists a sequence of nonnegative r.v.s. $\{\epsilon_N, N \geq 1\}$ on $(\Omega, \mathcal{F}, P)$ satisfying $\epsilon_N \xrightarrow[N\to\infty]{p} 0$ and a measurable set $A \in \mathcal{F}, P(A) = 1$, such that for any $\omega \in A, N \geq 1$, the control group $\mathbf{U}^N = \{u_i^N, 1 \leq i \leq N\}$ is an $\epsilon_N(\omega)$—Nash-equilibrium[1] with respect to $\mathbf{J}^N = \{J_i^N, 1 \leq i \leq N\}$.

Proof: Sufficiency: For any given $\epsilon > 0, \delta > 0$, by $\epsilon_N \xrightarrow[N\to\infty]{p} 0$, there exists $M > 0$ such that $P\{\omega : \epsilon_N(\omega) \geq \epsilon\} \leq \delta, \forall N > M$. Since for any $\omega \in A, N \geq 1$, the control group $\mathbf{U}^N = \{u_i^N, 1 \leq i \leq N\}$ is an $\epsilon_N(\omega)$—Nash-equilibrium with respect to $\mathbf{J}^N = \{J_i^N, 1 \leq i \leq N\}$, we have for any $N \geq 1, J_i^N(u_i^N, u_{-i}^N) \leq \inf_{v_i \in \mathcal{U}_{g,i}^N} J_i^N(v_i, u_{-i}^N) + \epsilon_N, \forall \omega \in A, \forall 1 \leq i \leq N$, which leads to

$$\max_{1\leq i\leq N}\left\{J_i^N\left(u_i^N, u_{-i}^N\right) - \inf_{v_i \in \mathcal{U}_{g,i}^N} J_i^N\left(v_i, u_{-i}^N\right)\right\} \leq \epsilon_N(\omega), \quad \forall\, \omega \in A, \quad \forall\, N \geq 1. \quad (1)$$

Let $B_N^\epsilon = \{\max_{1\leq i\leq N}\{J_i^N(u_i^N, u_{-i}^N) - \inf_{v_i \in \mathcal{U}_{g,i}^N} J_i^N (v_i, u_{-i}^N)\} \geq \epsilon\}$. Then, by (1) we have $B_N^\epsilon \cap A \subseteq \{\omega : \epsilon_N(\omega) \geq \epsilon\}, \forall N \geq 1$. Thus, $P(B_N^\epsilon) = P(B_N^\epsilon \cap A) \leq P\{\omega : \epsilon_N(\omega) \geq \epsilon\} \leq \delta, \quad \forall\, N > M$.

Necessity: If the sequence of control groups $\{\mathbf{U}^N = \{u_i^N, 1 \leq i \leq N\}, N \geq 1\}$ is an asymptotic Nash-equilibrium in probability with respect to $\{\mathbf{J}^N = \{J_i^N, 1 \leq i \leq N\}, N \geq 1\}$, then we can take $A = \Omega$ and

$$\epsilon_N = \max_{1\leq i\leq N}\left\{J_i^N\left(u_i^N, u_{-i}^N\right) - \inf_{v_i \in \mathcal{U}_{g,i}^N} J_i^N\left(v_i, u_{-i}^N\right)\right\}.$$

In this case, we have for all $i = 1, \ldots, N$

$$J_i^N\left(u_i^N, u_{-i}^N\right) \leq \inf_{v_i \in \mathcal{U}_{g,i}^N} J_i^N\left(v_i, u_{-i}^N\right) + \epsilon_N(\omega), \quad \forall\, \omega \in \Omega.$$

That is, for any $\omega \in \Omega$ and $N \geq 1, \{u_i^N, 1 \leq i \leq N\}$ is an $\epsilon_N(\omega)$—Nash-equilibrium with respect to $\{J_i^N, 1 \leq i \leq N\}$. Since for any $\epsilon > 0, \delta > 0$, there exists $M > 0$ such that for any $N > M, P\{\epsilon_N \geq \epsilon\} = P\{\max_{1\leq i\leq N}\{J_i^N(u_i^N, u_{-i}^N) - \inf_{v_i \in \mathcal{U}_{g,i}^N} J_i^N(v_i, u_{-i}^N)\} \geq \epsilon\} \leq \delta$, we have $\epsilon_N \xrightarrow[N\to\infty]{p} 0$. □

Definition 2.2: A sequence of control groups $\{\mathbf{U}^N = \{u_i^N, 1 \leq i \leq N\}, N \geq 1\}$ is called an almost sure asymptotic Nash-equilibrium with respect to the corresponding sequence of cost groups $\{\mathbf{J}^N = \{J_i^N, 1 \leq i \leq N\}, N \geq 1\}$, if there exists a sequence of nonnegative r.v.s. $\{\epsilon_N, N \geq 1\}$ on $(\Omega, \mathcal{F}, P)$, and a measurable set $A \in \mathcal{F}, P(A) = 1$, such that for any $\omega \in A, \lim_{N\to\infty} \epsilon_N = 0$, and the control group

[1]For the definition of ϵ-Nash-equilibrium, the readers are referred to Definition 4.1 of [16].

$\mathbf{U}^N = \{u_i^N, 1 \le i \le N\}$ is an $\epsilon_N(\omega)$—Nash-equilibrium with respect to $\mathbf{J}^N = \{J_i^N, 1 \le i \le N\}$.

Remark 1: Asymptotic Nash-equilibrium in probability means that even if agent i changes its strategy unilaterally among global-measurement-based admissible control set $\mathcal{U}_{g,i}^N$, the probability of its gaining a cost reduction by ϵ may be arbitrary small, provided N is sufficiently large.

Almost sure asymptotic Nash-equilibrium is to say that when N is sufficiently large, the local-measurement-based cost $J_i^N(u_i^N, u_{-i}^N)$ of agent i deviates from that based on global measurements by only a small quantity ϵ_N which is almost surely convergent to zero as $N \to \infty$.

Theorem 2.2: If a sequence of control groups $\{\mathbf{U}^N = \{u_i^N, 1 \le i \le N\}, N \ge 1\}$ is an almost sure asymptotic Nash-equilibrium with respect to the corresponding sequence of cost groups $\{\mathbf{J}^N = \{J_i^N, 1 \le i \le N\}, N \ge 1\}$, then $\{\mathbf{U}^N, N \ge 1\}$ is also an asymptotic Nash-equilibrium in probability with respect to $\{\mathbf{J}^N, N \ge 1\}$.

If the sequence of control groups $\{\mathbf{U}^N = \{u_i^N \in \mathcal{U}_i^N = \mathcal{U}_{l,i}^N, 1 \le i \le N\}, N \ge 1\}$ of the sequence of systems $\{\mathbf{S}^N, N \ge 1\}$ is an almost sure (in probability) asymptotic Nash-equilibrium with respect to $\{\mathbf{J}^N = \{J_i^N, 1 \le i \le N\}, N \ge 1\}$, then we call it almost surely (in probability) asymptotically optimal decentralized control in the sense of Nash-equilibrium.

Remark 2: In static game models, the utility function of agent i is usually denoted by $J_i(u, \eta)$, where $u = (u^1, \dots, u^N)$. u^i is its strategy, belonging to U_i, which is a number set; η is a system parameter. Here, the strategy of agent i is $u_i(t)$, which is a stochastic process on $(\Omega, \mathcal{F}, P), \eta$ is the state of nature $\omega \in \Omega$. In stochastic control problems with deterministic cost functions, the costs are usually mathematical expectation of some stochastic cost functions

$$J(u) = E\int_0^T (g(x,t) + h(u))\,dt.$$

In fact, this kind of cost functions involve all possible states of nature $\omega \in \Omega$. In engineering systems, enumerating all $\omega \in \Omega$ is impossible, but a single long-time experiment is performed to compute the cost function. On the other hand, for the control law designed, we are naturally concerned with the optimality of the closed-loop system for any given $\omega \in \Omega$. We hope to get optimality with large probability, best with probability 1. So, it is of great importance to consider how to optimize stochastic cost functions in the probabilistic sense. There are many achievements on optimizing stochastic cost functions in the stochastic control and optimization for single-agent systems ([27], [28]). Naturally, it is inevitable to introduce the concepts of Nash-equilibria in the probabilistic sense for the optimal control of MAS under the game-theory-based framework.

Remark 3: Nash-equilibrium is a specific form of "optimality" often considered in a noncooperative game, which says that one player cannot reduce its cost by altering his strategy unilaterally ([16]). While in a cooperative game, what we often encounter is Pareto optimality, which says that no other joint strategy can reduce the cost of at least one player, without increasing the cost of the others. In this paper, we will focus on the noncooperative case, and design a decentralized control law to achieve a Nash-equilibrium (asymptotically). For (stochastic) cooperative games, the readers are referred to [29].

B. Problem Formulation

In this paper, we consider the system of agents described by linear dynamics

$$dy_i = (A(\theta_i)y_i + B(\theta_i)u_i + f(t))dt + DdW_i(t) \qquad (2)$$

where $i = 1, 2, \dots; y_i \in \mathbb{R}^n$ is the measurable state of agent $i, u_i \in \mathbb{R}^m$ is its control input, $\{(W_i(t), \mathcal{F}_t^i), -\infty < t < \infty, i \ge 1\}$ is a sequence of independent d dimensional standard Brownian motions on a probability space $(\Omega, \mathcal{F}, P)$; $\theta_i = (\theta_i^1, \dots, \theta_i^{N_\theta}) \in \mathbb{R}^{N_\theta}$ is its dynamic parameter, called *parameter vector*; $A(\cdot) : \mathbb{R}^{N_\theta} \to \mathbb{R}^{n\times n}, B(\cdot) : \mathbb{R}^{N_\theta} \to \mathbb{R}^{n\times m}$ are matrix-valued continuous functions with proper dimensions; D is the $n \times d$ dimensional noise intensity matrix; $f(\cdot) : [0, \infty) \to \mathbb{R}^n \in \mathcal{C}_n^b$, reflecting the impact on agent i by the external environment.

As mentioned before, $\mathbf{S}^N$ denotes the system comprised of the first N dynamical equations of (2). Here, a control group of $\mathbf{S}^N$ is $\mathbf{U}^N = \{u_1, u_2, \dots, u_N\}$, and the cost function of agent i has the coupled quadratic form

$$J_i^N\left(u_i, u_{-i}^N\right) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left\{ \left\| y_i - \frac{\gamma}{N}\sum_{j=1}^N y_j \right\|_Q^2 + \|u_i\|_R^2 \right\} dt \qquad (3)$$

where Q is an $n \times n$ semi-positive definite matrix, R is an $m \times m$ positive definite matrix, $\gamma > 0$. The admissible control set for agent i is described by $\mathcal{U}_i = \mathcal{U}_{l,i}$

$$\mathcal{U}_{l,i} = \left\{ u_i \,|\, u_i(t) \text{ is adapted to } \sigma(y_i(s), s \le t), \; \|y_i(T)\| = o(\sqrt{T}) \; \int_0^T \|y_i(t)\|^2 dt = O(T) \; a.s. \right\}.$$

$$\mathcal{U}_{g,i} = \left\{ u_i \,|\, u_i(t) \text{ is adapted to } \sigma(y_i(s) \; s \le t, 1 \le i \le N), \|y_i(T)\| = o(\sqrt{T}) \; \int_0^T \|y_i(t)\|^2 dt = O(T) \; a.s. \right\}.$$

Remark 4: For static games, Rosen ([30]) gave a condition on the existence of Nash-equilibrium strategies. For stochastic linear quadratic games, Corollary A.1 and Corollary B.2 in [16] show that there is always a Nash-equilibrium strategy in the linear feedback form if the dynamics of agents are noncoupled. However, for this type of strategies, the controller of each agent, generally speaking, will use the states of all the other agents in the real time, and so, is not decentralized. In this paper, we will find a sequence of decentralized control groups $\{\mathbf{U}^N = \{u_j^*, 1 \le j \le N\}, N \ge 1\}$, such that not only is it an almost sure (in probability) asymptotic Nash-equilibrium, but also ensures the closed-loop systems to be uniformly stable in the sense of time average.[2]

The parameter vector θ_i in model (2) has the following property: $\{\theta_i, i \geq 1\}$ are independently sampled from the statistical structure $(\mathbb{R}^{N_\theta}, F(\theta))$, where $F(\cdot)$: $\mathbb{R}^{N_\theta} \to [0,1]$ is a distribution function on the parameter vector space $\mathbb{R}^{N_\theta}$, called prior distribution. We can construct the empirical distribution functions by $\{\theta_i, i \geq 1\}$

$$F_N(\theta) \triangleq \frac{1}{N}\sum_{i=1}^{N} \chi_{\{\theta_i<\theta\}}, \quad N \geq 1 \tag{4}$$

where $\theta = (\theta^1, \ldots, \theta^{N_\theta}), \{\theta_i < \theta\} \triangleq \{\theta_i^1 < \theta^1, \ldots, \theta_i^{N_\theta} < \theta^{N_\theta}\}$.

Remark 5: In model (2), the noise intensity matrix can also continuously depend on the parameter vector. However, this may result in no essential difference in the controller design and closed-loop analysis, and so, for simplicity, here we assume that the noise intensity matrix is the same for all agents.

For the model considered, we have the following assumptions.

A1) The support of $F(\cdot)$ denoted by $\mathcal{M}$ is compact.

A2) $\{F_N, N \geq 1\}$ converge to F weakly, denoted by $F_N(\theta) \xrightarrow[N\to\infty]{p} F(\theta)$.

A3) For any $\theta \in \mathcal{M}, (A(\theta), B(\theta), Q^{1/2})$ is stabilizable and detectable.

A4) $\{y_i(0), i \geq 1\}$ are independent r.v.s. with identical mathematical expectation, and independent of $\sigma(W_i(t), i \geq 1)$; $y_i(0)$ is measurable with respect to $\mathcal{F}_0^i, \sup_{i\geq 1} E\|y_i(0) - y_0\|^4 < \infty$, where $y_0 = Ey_1(0)$.

Remark 6: The assumption of $\{\theta_i, i \geq 1\}$ embodies the parameter distribution of LPSMAS, which is different from conventional control systems. In conventional control systems, to describe a plant, we mainly use the state vector, whose components represent a certain attribute of the plant. All available values of the state vector comprise the state space. For LPSMAS, the distribution information of the parameter space is also valuable. Here, by aid of the concepts of Bayesian statistical inference, we call $F(\theta)$ the prior distribution function on parameter space. When the number of agents N becomes arbitrarily large, $F_N(\theta)$ tends to this given distribution $F(\theta)$. The assumption of the compactness of its support is relatively strong, but it is suitable for many bounded distribution in practical systems, though not for normal, exponential and other unbounded distributions. For example, [31] considered a sensor network applied to environment monitoring, where a large number of micro-senors are scattered randomly on a bounded monitoring area $\mathcal{D}$. The sensed object for the network is a scalar field on this area, and the position $p_i = (x_i, y_i)$ for each sensor is a random variable with uniform distribution $F(x,y)$, all of which comprise a sequence of independent and identical distributed (i.i.d.) r.v.s.

Remark 7: For $\{\theta_i, i \geq 1\}$, we are only concerned about the statistical information, while, omitting the underlying probability space for a succinct description. In fact, we can also view $\{\theta_i, i \geq 1\}$ as a sequence of r.v.s. independent of Brownian motions $\{(W_i(t), \mathcal{F}_t^i), -\infty < t < \infty, i \geq 1\}$: we can define the product probability space $(\Omega, \mathcal{F}, P) \times (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, and on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}), \{\Theta_i, i \geq 1\}$ are i.i.d. r.v.s. with the common distribution $F(\theta)$. Define $\tilde{\Theta}_i(\omega, \tilde{\omega}) = \Theta_i(\tilde{\omega}), (\omega, \tilde{\omega}) \in \Omega \times \tilde{\Omega}, i \geq 1$. Then it is known that $\{\tilde{\Theta}_i, i \geq 1\}$ are i.i.d. with the same distribution as $\{\Theta_i, i \geq 1\}$. Define $\tilde{W}_i(\omega, \tilde{\omega}) = W_i(\omega), \tilde{\mathcal{F}}_t^i = \sigma(\sigma(\tilde{\Theta}_i) \cup \mathcal{F}_t^i), i \geq 1$. Then $\{(\tilde{W}_i(t), \tilde{\mathcal{F}}_t^i), i \geq 1\}$ is a sequence of independent Brownian motions, which is also independent of $\{\tilde{\Theta}_i, i \geq 1\}$. Instead of state equation (2), we may consider the system of agents described by

$$dy_i = (A(\tilde{\Theta}_i)y_i + B(\tilde{\Theta}_i)u_i + f(t))dt + Dd\tilde{W}_i(t). \tag{5}$$

In this paper, we view $\{\theta_i, i \geq 1\}$ as a realization of $\{\tilde{\Theta}_i, i \geq 1\}$, namely, there exists $\tilde{\omega}_0 \in \tilde{\Omega}$ such that $\theta_i = \tilde{\Theta}_i(\omega, \tilde{\omega}_0) = \Theta_i(\tilde{\omega}_0), \forall \omega \in \Omega$. In fact, owing to the independency of $\{(\tilde{W}_i(t), \tilde{\mathcal{F}}_t^i), i \geq 1\}$ and $\{\tilde{\Theta}_i, i \geq 1\}$, there is no essential difference between (2) and (5). Therefore, for simplicity, we will view $\{\theta_i, i \geq 1\}$ as a deterministic sequence when considering the system of agents described by (2).

By the Glivenko-Cantelli theorem ([32]), it is known that

$$\tilde{F}_N(\theta) \xrightarrow[N\to\infty]{w} F(\theta) \quad (\tilde{P} - a.s.)$$

where $\tilde{F}_N(\theta) = \frac{1}{N}\sum_{i=1}^{N} \chi_{\{\Theta_i<\theta\}}$. So, without loss of generality, we assume that $\{\Theta_i(\tilde{\omega}_0), i \geq 1\}$ satisfies

$$\tilde{F}_N(\theta)(\tilde{\omega}_0) \xrightarrow[N\to\infty]{w} F(\theta)$$

that is, Assumption A2) holds.

Remark 8: Assumption A3) is to ensure the closed-loop stability. By the continuity of $A(\cdot), B(\cdot)$ and the compactness of $\mathcal{M}$. It is easy to know the following.

i) For any $\theta \in \mathcal{M}$, Riccati equation

$$P(\theta)A(\theta) + A^T(\theta)P(\theta) - P(\theta)L(\theta)P(\theta) + Q = 0 \tag{6}$$

has a unique semi-positive definite solution $P(\theta)$, where $L(\theta) \triangleq B(\theta)R^{-1}B^T(\theta)$; $L(\cdot)$ and $P(\cdot)$ are both continuous on $\mathcal{M}$.

ii) For any $\theta \in \mathcal{M}$, all the eigenvalues of $G(\theta) \triangleq A(\theta) - B(\theta)R^{-1}B^T(\theta)P(\theta)$ have negative real part; $G(\cdot)$ is continuous over $\mathcal{M}$; there exists $\kappa > 0, \rho > 0$, such that $\|e^{G(\theta)t}\| \leq \kappa e^{-\rho t}, \forall\, t \geq 0$.

iii) For any $\theta \in \mathcal{M}$, Lyapunov equation $S(\theta)G(\theta) + G^T(\theta)S(\theta) = -I$ has a unique positive definite solution $S(\theta)$, with the property that $\sup_{\theta\in\mathcal{M}} \lambda_{\max}(S(\theta)) < \infty, \inf_{\theta\in\mathcal{M}} \lambda_{\min}(S(\theta)) > 0$.

Remark 9: For a more general case, we may consider different expectations of initial states $Ey_i(0)$ of agents. In this case, we may expand the parameter vector θ_i to $\bar{\theta}_i = (\theta_i^T, Ey_i^T(0))^T$, and then introduce $F(\bar{\theta})$ and $F_N(\bar{\theta})$ as the prior and empirical distribution functions, respectively. To avoid notational complication, we assume the same expectation of the initial states of all agents in Assumption A4).

[2]The definition of uniform stability in the sense of time average will be given in Theorem 4.1.

III. Decentralized Control Law Construction

By the model (2) and the cost function (3), it can be seen that we are going to consider a tracking-like quadratic optimal control problem, where the reference signal for agent i is $\gamma(1/N)\sum_{j=1}^N y_j$, which is unknown and cannot be used for constructing the decentralized controller u_i by agent i, since u_i is adapted to $\mathcal{U}_{l,i}$. So, to design the decentralized control law, it is key to get a good approximation for the PSA $(1/N)\sum_{j=1}^N y_j$. Precisely, the designing procedure has the following three steps: 1) solve the tracking-like quadratic optimal control problem with a deterministic reference signal; 2) use the state aggregation method to approximate the PSA by a deterministic signal y^*; 3) construct the decentralized control law by using y^*. The local controller of each agent consists of two parts: one is the feedback of its own state $y_i(t)$ for stabilization, and the other is the filer of y^* for tracking the population behavior. The local controller has the same structure and parameters as those of the tracking-like quadratic optimal control with a deterministic reference signal, but with the reference signal replaced by y^*.

Intuitively, y^* should possess the following property: if every agent views it as the approximation of the PSA, and according to which, makes the optimal decision, then the expectation of the closed-loop PSA ought to approach to y^* when the number of agents N tends to infinity. This kind of methodology for construction of the decentralized control law is called *Nash certainty equivalence principle* ([33]), which has the similar spirit to the well-known certainty equivalence principle adopted in adaptive control, where the unknown parameters are estimated, and the estimates are used as the true parameters to construct the control laws. Based on the Nash certainty equivalence principle, in Subsection III-A, Step 1) is accomplished; then in Subsection III-B, Step 2) and Step 3) are accomplished to realize the decentralized control law design.

A. Infinite Horizon Tracking-Like Quadratic Optimal Control

Consider the system

$$\begin{cases} dx(t) = (Ax(t) + Bu(t) + f^b(t))dt + DdW(t) \\ y(t) = Cx(t) \end{cases} \tag{7}$$

where $x(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}^r$ are system state and output, respectively; $f^b(t)$ is a deterministic signal; $W(t)$ is a d-dimensional standard Brownian motion.

The admissible control set is described by

$$\mathcal{U} = \Big\{ u \mid u(t) \in \sigma(x(0), W(s), s \le t), \quad \|x(T)\| = o(\sqrt{T}), \quad \int_0^T \|x(t)\|^2 dt = O(T)\ a.s.,\ T \to \infty \Big\}.$$

The cost function is described by

$$J(u) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left(\|y(t) - y^d(t)\|_Q^2 + \|u(t)\|_R^2 \right) dt \tag{8}$$

where Q is semi-positive definite, R is positive definite.

Theorem 3.1: If $y^d(t) \in \mathcal{C}_r^b$, $f^b(t) \in \mathcal{C}_n^b$, (A, B) is stabilizable, $(A, (C^TQC)^{1/2})$ is detectable, then for the system (7) and the cost function (8), we have the following results.

i) The algebraic Riccati equation $PA + A^TP - PBR^{-1}B^TP + C^TQC = 0$ has a unique semi-positive definite solution P.

ii) All the eigenvalues of $G \triangleq A - BR^{-1}B^TP$ have negative real part.

iii) The differential equation

$$\dot{\xi}(t) = -G^T\xi(t) + Pf^b(t) - C^TQy^d(t) \tag{9}$$

has a unique solution in $\mathcal{C}_n^b$

$$\xi^*(t) = \int_t^\infty e^{-G^T(t-\tau)}(C^TQy^d(\tau) - Pf^b(\tau))d\tau. \tag{10}$$

iv) The optimal control $u^* \triangleq \arg\inf_{u\in U} J(u)$ is given by

$$u^*(t) = -R^{-1}B^TPx(t) + R^{-1}B^T\xi^*(t) \tag{11}$$

v) The optimal value of the cost function is

$$J(u^*) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \Big(y^{d^T}(t)Qy^d(t) - 2\xi^{*^T}(t)f^b(t) - \xi^{*^T}(t)BR^{-1}B^T\xi^*(t) \Big) dt + trPDD^T\ a.s. \tag{12}$$

The proof of this theorem is put into Appendix B.

B. Infinite Population Mean and Controller Design

In the cost function (3), if $(1/N)\sum_{j=1}^N y_j$ is replaced by some $y^c \in \mathcal{C}_n^b$, then, by Theorem 3.1 we may obtain the optimal control for agent i

$$u_i(t) = -R^{-1}B^T(\theta_i)P_iy_i(t) + R^{-1}B^T(\theta_i)\xi_i(t) \tag{13}$$

where $\xi_i(t) = \xi_\theta(t)|_{\theta=\theta_i}, \xi_\theta(t) = \int_t^\infty e^{-G^T(\theta)(t-\tau)}(Q\gamma y^c(\tau) - P(\theta)f(\tau))d\tau$; $P_i = P(\theta)|_{\theta=\theta_i}$, which is the unique semi-positive definite solution of Riccati (6).

Denote $A_i = A(\theta_i), B_i = B(\theta_i), G_i = G(\theta)|_{\theta=\theta_i}, L_i = L(\theta)|_{\theta=\theta_i}$. Substituting (13) into (2), we have the closed-loop equation for agent i

$$dy_i(t) = (G_iy_i(t) + B_iR^{-1}B_i^T\xi_i(t) + f(t))dt + DdW_i(t).$$

Taking expectation on both sides of this equation, we have

$$\frac{dEy_i(t)}{dt} = G_iEy_i(t) + B_iR^{-1}B_i^T\xi_i(t) + f(t).$$

We are now in a position to approximate the PSA by state aggregation.

First, we construct an auxiliary system

$$\begin{cases} \frac{dEy_\theta(t)}{dt} = G(\theta)Ey_\theta(t) + L(\theta)\xi_\theta(t) + f(t), \\ Ey_\theta(0) = y_0, \\ \xi_\theta(t) = \int_t^\infty e^{-G^T(\theta)(t-\tau)}(Q\gamma y^c(\tau) - P(\theta)f(\tau))d\tau, \\ y^c(t) = \int_{\mathbb{R}^{N_\theta}} Ey_\theta(t)dF(\theta). \end{cases} \tag{14}$$

This describes the limit system of $\mathbf{S}^N$ when $N \to \infty$, which is a continuum of agents, each agent is marked by a parameter vector θ. $F(\theta)$ indicates the distribution of the parameter vectors. By (14) we have

$$Ey_\theta(t) = e^{G(\theta)t}y_0 + \int_0^t e^{G(\theta)(t-\tau)} \times \left(L(\theta)\int_\tau^\infty e^{-G^T(\theta)(\tau-s)} \times (Q\gamma y^c(s) - P(\theta)f(s))ds + f(\tau)\right) d\tau. \quad (15)$$

Define an operator $\mathcal{T}$ on $(\mathcal{C}_n^b, \|\cdot\|_\infty)$

$$(\mathcal{T}x)(t) \triangleq \int_{\mathbb{R}^{N_\theta}} \left\{ \int_0^t e^{G(\theta)(t-\tau)} \times \left(L(\theta)\int_\tau^\infty e^{-G^T(\theta)(\tau-s)} \times (Q\gamma x(s) - P(\theta)f(s))ds + f(\tau)\right) d\tau + e^{G(\theta)t}y_0 \right\} dF(\theta). \quad (16)$$

Then, from Remark 8 and $f \in \mathcal{C}_n^b$, it is easy to know that $\sup_{t\geq 0} \|(\mathcal{T}x)(t)\| < \infty, \forall x \in \mathcal{C}_n^b$. So, $\mathcal{T}$ is indeed an operator on $(\mathcal{C}_n^b, \|\cdot\|_\infty)$.

By virtue of $\mathcal{T}$ and (15), the auxiliary system (14) can be rewritten as

$$y^c = \mathcal{T}y^c. \quad (17)$$

Equation (17) especially embodies the property that the approximation y^c of the PSA should possess: if every agent views y^c as the approximation of $(1/N)\sum_{j=1}^N y_j$, and according to which, makes the optimal decision, then the expectation of the resulted closed-loop PSA should converge to y^c when $N \to \infty$. So, if (17) has a unique solution y^*, then it can be used as the approximation of PSA, and will be called the Infinite Population Mean (IPM) trajectory below. For convenience of the controller design, we make the following assumption.

A5) Equation (17) has a unique solution, denoted by y^*.

Generally speaking, it seems difficult to verify this assumption. However, since it is imposed on the system structure, we can obtain some sufficient conditions which can be verified easily. See, for instance, the condition given in Theorem 3.2 below.

Theorem 3.2: If

$$\|R^{-1}\|\|Q\||\gamma| \int_{\mathbb{R}^{N_\theta}} \|B(\theta)\|^2 \left(\int_0^\infty \|e^{G(\theta)\tau}\|d\tau\right)^2 dF(\theta) < 1$$

then (17) has the unique solution $y^* \in \mathcal{C}_n^b$.

Proof: By the definition of $\mathcal{T}$ it is known that for any $x, y \in \mathcal{C}_n^b, t \in [0, \infty)$

$$(\mathcal{T}x)(t) - (\mathcal{T}y)(t) = \int_{R^{N_\theta}} \left\{ \int_0^t e^{G(\theta)(t-\tau)} \left(L(\theta)\int_\tau^\infty e^{-G^T(\theta)(\tau-s)} \times (Q\gamma(x(s) - y(s)))ds\right) d\tau \right\} dF(\theta).$$

Thus,

$$\|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)\| \leq \|x-y\|_\infty \|R^{-1}\|\|Q\||\gamma|\|B(\theta)\|^2 \times \int_{\mathbb{R}^{N_\theta}} \int_0^t \int_\tau^\infty \|e^{G(\theta)(t-\tau)}\| \times \|e^{-G^T(\theta)(\tau-s)}\| ds d\tau dF(\theta)$$

which leads to

$$\|\mathcal{T}x - \mathcal{T}y\|_\infty \leq \|x-y\|_\infty \|R^{-1}\|\|Q\||\gamma| \int_{\mathbb{R}^{N_\theta}} \|B(\theta)\|^2 \times \left(\int_0^\infty \|e^{G(\theta)\tau}\|d\tau\right)^2 dF(\theta).$$

By the condition of the theorem, $\mathcal{T}$ is a contraction on $(\mathcal{C}_n^b, \|\cdot\|_\infty)$. By the contractive mapping theorem, (17) has a unique solution $y^* \in \mathcal{C}_n^b$. □

Remark 10: For one-dimensional systems, by a direct calculation, it can be verified that the condition of Theorem 3.2 holds if and only if

$$|\gamma| \int \frac{B^2(\theta)Q}{A^2(\theta)R + B^2(\theta)Q} dF(\theta) < 1. \quad (18)$$

If $|\gamma| < 1$, then (18) always holds, regardless of the distribution function $F(\theta)$. If $A(\theta) \equiv A, B(\theta) \equiv B$ and $Q > 0$, then (18) holds if and only if $|\gamma| < 1 + (A^2R)/(B^2Q)$, which, intuitively, means that a large ratio R/Q is likely to make (18) hold.

Under Assumption A5), we can construct the decentralized controller for agent i, denoted by u_i^0

$$u_i^0(t) = -R^{-1}B_i^TP_iy_i(t) + R^{-1}B_i^T\xi_i(t) \quad (19)$$

where

$$\xi_i(t) = \int_t^\infty e^{-G_i^T(t-\tau)}(Q\gamma y^*(\tau) - P_if(\tau))d\tau. \quad (20)$$

By (16) we know that the IPM y^* depends only on the structural information of the system $A(\cdot), B(\cdot), F(\cdot)$, the cost parameters Q, R, and the mathematical expectation of the initial state y_0, independent of agents' states $y_i(t), i = 1, \ldots, N$ in the real time. So, the control law given by (19) indeed depends only on the local measurements of agent i. In addition, if $\mathcal{T}$ is a contraction on $(\mathcal{C}_n^b, \|\cdot\|_\infty)$, then we can construct algorithms for getting approximate solutions of (17), which is similar to the proof of existence and uniqueness of the solution of ordinary differential equations ([34]).

By now, we have accomplished the decentralized control law design based on the tracking-like quadratic optimal control and the Nash certainty equivalence principle. Some questions come out naturally: as the approximation of the PSA, does the IPM have a good property? How to characterize the approach of the IPM to the PSA when $N \to \infty$? Before further analysis of the property of the IPM, we will consider the stability property of the closed-loop system. It can be seen that the closed-loop stability depends only on the boundness of the IPM.

IV. Stability Analysis

Theorem 4.1: (Almost surely uniformly stable in the sense of time average) For system (2), if Assumptions A1), A3)–A5) hold, then the control law (19) and its corresponding closed-loop solution $y_i^0(t)$ satisfies

$$\sup_{N\geq 1}\max_{1\leq i\leq N}\limsup_{T\to\infty}\frac{1}{T}\times\int_0^T\left(\left\|y_i^0(t)\right\|^2+\left\|u_i^0(t)\right\|^2\right)dt<\infty. \tag{21}$$

Proof: Substituting the control law (19) into the state equation (2), we have the closed-loop solution

$$y_i^0(t)=e^{G_it}y_i(0)+\int_0^t e^{G_i(t-\tau)}DdW_i(\tau)+\int_0^t e^{G_i(t-\tau)}(L_i\xi_i(\tau)+f(\tau))\,d\tau. \tag{22}$$

For G_i is stable, we have

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T\|e^{G_it}y_i(0)\|^2dt=0\ a.s. \tag{23}$$

Denote $M_{y^*}=\|y^*\|_\infty, M_f=\|f\|_\infty, M_P=\sup_{\theta\in\mathcal{M}}\|P(\theta)\|, M_B=\sup_{\theta\in\mathcal{M}}\|B(\theta)\|$. Then, by Remark 8 (ii) and (20) we get

$$\|\xi_i\|_\infty\leq(\|Q\|\|\gamma\|M_{y^*}+M_PM_f)\kappa/\rho\triangleq M_\xi. \tag{24}$$

Furthermore,

$$\left\|\int_0^t e^{G_i(t-\tau)}(L_i\xi_i(\tau)+f(\tau))d\tau\right\|\leq\left(\|R^{-1}\|M_B^2M_\xi+M_f\right)\kappa/\rho\triangleq K_1$$

which leads to

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T\left\|\int_0^t e^{G_i(t-\tau)}(L_i\xi_i(\tau)+f(\tau))d\tau\right\|^2dt\leq K_1^2.$$

By Lemma 12.4 of [27] we have

$$\begin{aligned}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\left\|\int_0^t e^{G_i(t-\tau)}DdW_i(\tau)\right\|^2dt&=\int_0^\infty tr\left(e^{G_i(t)}DD^Te^{G_i^Tt}\right)dt\\&\leq\|D\|^2\kappa^2/2\rho\triangleq K_2\ a.s.\end{aligned}$$

Thus, from (22) and (23) it follows that

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T\left\|y_i^0\right\|^2dt\leq K_1^2+K_2<\infty\ a.s. \tag{25}$$

This together with (19) and (24) results in $\limsup_{T\to\infty}(1/T)\int_0^T\|u_i^0\|^2dt\leq\|R^{-1}\|M_BM_P(K_1^2+K_2)+\|R^{-1}\|M_BM_\xi$ a.s. Noticing that K_1,K_2 are independent of i,N, by (25) we can get (21). □

Remark 11: The uniformity in Theorem 4.1 refers to that there exists $A\in\mathcal{F}, P(A)=1$, and a nonnegative finite random variable C independent of N,i, such that for any $\omega\in A$, $\limsup_{T\to\infty}(1/T)\int_0^T\|y_i^0\|^2+\|u_i^0\|^2dt\leq C(\omega)$. So, the closed-loop stability is retained when N becomes arbitrarily large. This kind of stability can be viewed as a generalization to LPSMAS for the almost sure stability in the sense of time average of the single-agent stochastic systems ([27]).

V. Laws of Large Numbers of the Closed-Loop System

In this section, the performance of the IPM will be analyzed as the approximation of the PSA. The analysis in Section IV shows that the performance of the IPM has no influence on the closed-loop stability. However, as it is well-known that the optimality and robustness of the adaptive control based upon certainty equivalence principle often depend on whether or not the parameters estimate is convergent to their true values. Likewise, the analysis in Section VI will show that the asymptotic optimality of the decentralized control law based on the Nash certainty equivalence principle also depends on whether or not the PSA converges to the IPM as N increases to infinity.

The main results in this section will be summed up as Theorems 5.1–5.2, which are actually two types of measures for evaluating the accuracy of the IPM as the approximation of the PSA.

For the auxiliary system (14) and the closed-loop system, we have the following lemmas, whose proofs are put into Appendix C.

Lemma 5.1: For the system (2), if Assumptions A1), A3)–A5) hold, then the unique solution $Ey_\theta(t)$ of the auxiliary system (14) is uniformly bounded and equicontinuous on $\mathcal{M}$, namely, $\sup_{\theta\in\mathcal{M}}\|Ey_\theta\|_\infty<\infty$, and for any $\epsilon>0$, there exists $\delta>0$, such that for any $\theta,\theta'\in\mathcal{M}, t\in[0,\infty)$, $\|Ey_\theta(t)-Ey_{\theta'}(t)\|<\epsilon$, provided $\|\theta-\theta'\|<\delta$.

Lemma 5.2: For the system (2), if Assumptions A1), A3)–A5) hold, then under the control law (19), the closed-loop solution (22) satisfies: for almost all $\omega\in\Omega$ and any given $T\geq 0$, $\{\|\bar{y}_N^0(t)-E\bar{y}_N^0(t)\|^2, N\geq 1\}$ is uniformly Lebesgue integrable on $[0,T]$. Here $\bar{y}_N^0(t)=(1/N)\sum_{i=1}^N y_i^0(t)$.

Lemma 5.3: For the system (2), if Assumptions A1), A3)–A5) hold, then under the control law (19), the closed-loop solution (22) satisfies

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\left\|\bar{y}_N^0(t)-E\bar{y}_N^0(t)\right\|^2dt=0\ a.s.$$

where $\bar{y}_N^0(t)$ is defined in Lemma 5.2.

Theorem 5.1: (Law of large numbers in the sense of $\mathcal{L}_2$-norm) For the system (2), if Assumptions A1)–A5) hold, then under the control law (19), the closed-loop solution (22) satisfies

$$\lim_{N\to\infty}\int_0^T\left\|\bar{y}_N^0(t)-y^*(t)\right\|^2dt=0,\quad\forall T\geq 0,\ a.s. \tag{26}$$

where $\bar{y}_N^0(t)$ is defined in Lemma 5.2.

Proof: By (22) and (15) we have

$$Ey_i^0(t)=Ey_{\theta_i}(t)=Ey_\theta(t)|_{\theta=\theta_i}. \tag{27}$$

This together with (22) gives

$$y_i^0(t) - Ey_i^0(t) = e^{G_i t}(y_i(0) - y_0) + \int_0^t e^{G_i(t-\tau)} DdW_i(\tau). \tag{28}$$

From Assumption A4) and Remark 8 (ii) it follows that $E\|y_i^0(t) - Ey_i^0(t)\|^2 = E\|e^{G_i t}(y_i(0) - y_0)\|^2 + E\|\int_0^t e^{G_i(t-\tau)} DdW_i(\tau)\|^2 \le \kappa^2 e^{-2\rho t} E\|y_1(0) - y_0\|^2 + \|D\|^2\kappa^2/2\rho$, which together with Theorem 2.8 in [27] implies

$$\lim_{N\to\infty} \left\|\bar{y}_N^0(t) - E\bar{y}_N^0(t)\right\| \to 0 \ a.s., \quad \forall t \ge 0. \tag{29}$$

Notice that $\forall T \ge 0$

$$\begin{aligned}\int_0^T &\left\|\bar{y}_N^0(t) - y^*(t)\right\|^2 dt \\ &\le 2\int_0^T \left\|\bar{y}_N^0(t) - E\bar{y}_N^0(t)\right\|^2 dt \\ &\quad + 2T \sup_{t\ge 0} \left\|E\bar{y}_N^0(t) - y^*(t)\right\|^2. \end{aligned} \tag{30}$$

Then, by Lemma 5.2 and (29) we have

$$\lim_{N\to\infty} \int_0^T \left\|\bar{y}_N^0(t) - E\bar{y}_N^0(t)\right\|^2 dt = 0, \quad \forall T \ge 0, \ a.s. \tag{31}$$

By (4), (14), and (27), we have

$$E\bar{y}_N^0(t) - y^*(t) = \int_{\mathbb{R}^{N_\theta}} Ey_\theta(t) dF_N(\theta) - \int_{\mathbb{R}^{N_\theta}} Ey_\theta(t) dF(\theta). \tag{32}$$

By Lemma 5.1, $Ey_\theta(t)$ is uniformly bounded and equicontinuous. Hence, by (32), Assumption A2) and Corollary 1.1.5 in [35], we have

$$\lim_{N\to\infty} \sup_{t\ge 0} \left\|E\bar{y}_N^0(t) - y^*(t)\right\| = 0. \tag{33}$$

Therefore, by (30) and (31) we get (26). □

Theorem 5.2: (Law of large numbers in the sense of time average) For the system (2), if Assumptions A1)–A5) hold, then under the control law (19), the closed-loop solution (22) satisfies

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left\|\bar{y}_N^0(t) - y^*(t)\right\|^2 dt = 0 \ a.s. \tag{34}$$

Proof: Similar to (30), we have

$$\begin{aligned}\limsup_{T\to\infty} &\frac{1}{T}\int_0^T \|\bar{y}_N^0(t) - y^*(t)\|^2 dt \\ &\le 2\limsup_{T\to\infty} \frac{1}{T}\int_0^T \|\bar{y}_N^0(t) - E\bar{y}_N^0(t)\|^2 dt \\ &\quad + 2\sup_{t\ge 0} \|E\bar{y}_N^0(t) - y^*(t)\|^2. \end{aligned}$$

Hence, from Lemma 5.3 and (33) we get (34). □

Remark 12: From Theorem 5.1, it can be seen that the closed-loop system satisfies: for almost all $\omega \in \Omega$, $\{\bar{y}_N^0(t), N \ge 1\} \subseteq \mathcal{L}_2([0,T])$, $\bar{y}_N^0 \xrightarrow[N\to\infty]{\mathcal{L}_2} y^*$, $\forall T \ge 0$. Theorem 5.2 presents another type of measure for evaluating the approaching of the PSA to the IPM, which emphasizes for the stationary case. Since we select the time-averaged stationary cost function (3) for agents, Theorem 5.2 will play a key role in the optimality analysis of the closed-loop system in Section VI.

We can construct a subspace of $\mathcal{C}_n$

$$\mathcal{C}_n^{Pb} \triangleq \left\{x \in \mathcal{C}_n \mid \limsup_{T\to\infty} \frac{1}{T}\int_0^T \|x(t)\|^2 dt < \infty\right\}$$

which has a clear physical meaning: a family of all n dimensional finite average power continuous functions. It is obvious that $\mathcal{C}_n^b \subseteq \mathcal{C}_n^{Pb}$. We can define an equivalence relationship on $\mathcal{C}_n^{Pb}$ denoted by $\sim$: for any $x, y \in \mathcal{C}_n^{Pb}$, $x \sim y$ if and only if $\limsup_{T\to\infty}(1/T)\int_0^T \|x(t) - y(t)\|^2 dt = 0$. Denote the equivalent class of x by $[x]$, and define on the quotient space $\mathcal{C}_n^{Pb}/\sim$: for any $[x] \in \mathcal{C}_n^{Pb}$, $\|[x]\|_{Pb} \triangleq (\limsup_{T\to\infty}(1/T)\int_0^T \|x(t)\|^2 dt)^{1/2}$. It can be shown that $\|\cdot\|_{Pb}$ is a norm on $\mathcal{C}_n^{Pb}/\sim$, and $(\mathcal{C}_n^{Pb}/\sim, \|\cdot\|_{Pb})$ is a normed space. Theorem 4.1 and Theorem 5.2 tell us that for almost all $\omega \in \Omega$, $\{[\bar{y}_N^0], N \ge 1\} \subseteq \mathcal{C}_n^{Pb}$, and $\lim_{N\to\infty} \|[\bar{y}_N^0] - [y^*]\|_{Pb} = 0$.

From the point view of the decentralized control law design, the IPM can be regarded as the approximation of the PSA. On the other hand, it characterizes the macroscopic behavior, of which $(1/N)\sum_{j=1}^N y_j$ is the approximation. In the LPSMAS of biology and economics, there may be some intrinsic game mechanism leading to a deterministic patten on the layer of the macro scope. Laws of large numbers may play a role in bridging between the microscopic uncertainty and the macroscopic determinacy.

VI. Optimality Analysis

Below we will investigate the optimality of the decentralized control law $\{\mathbf{U}^N = \{u_i^0, 1 \le i \le N\}, N \ge 1\}$ with respect to the corresponding sequence of cost groups $\{\mathbf{J}^N = \{J_i^N, 1 \le i \le N\}, N \ge 1\}$. Particularly, in Theorem 6.1 we will show that for any $N \ge 1$, the control group $\{u_i^0, 1 \le i \le N\}$ is suboptimal in the sense of Nash-equilibrium, namely

$$J_i^N(u_i^0, u_{-i}^0) \le \inf_{u_i \in \mathcal{U}_{g,i}} J_i^N\left(u_i, u_{-i}^0\right) + \epsilon_1^N \ a.s.$$

The fact that the decentralized control law is not optimal is essentially due to the information restriction of control design, that is, the IPM is used to construct the control law instead of the PSA. However, owing to the laws of large numbers of the closed-loop system, the PSA $(1)/(N)\sum_{i=1}^N y_i$ converges to the IPM y^* almost surely. Therefore, the optimality loss incurred by information restriction will tend to zero when the number of agents $N \to \infty$. In fact, we can prove that $\epsilon^{N-1} \xrightarrow[N\to\infty]{a.s.} 0$. Hence, the decentralized control law is almost surely optimal.

Let

$$J_i^N\left(u_i^0,u_{-i}^0\right) = \limsup_{T\to\infty}\frac{1}{T}\int_0^T\left(\left\|y_i^0-\gamma\bar{y}_N^0\right\|_Q^2+\left\|u_i^0\right\|_R^2\right)dt \quad (35)$$

$$J_i\left(u_i^0,\gamma y^*\right) = \limsup_{T\to\infty}\frac{1}{T}\int_0^T\left(\left\|y_i^0-\gamma y^*\|\right\|_Q^2+\left\|u_i^0\right\|_R^2\right)dt \quad (36)$$

$$J_i^N\left(u_i,u_{-i}^0\right) = \limsup_{T\to\infty}\frac{1}{T}\int_0^T\left(\left\|y_i|_{u_i}-\gamma\frac{1}{N}\sum_{j=1}^N y_j|_{u_i,u_{-i}^0}\right\|_Q^2+\|u_i\|_R^2\right)dt \quad (37)$$

where $y_i|_{u_i}$ refers to the closed-loop solution corresponding to some control $u_i\in U_{g,i}$ for the system (2); and $y_j|_{u_i,u_{-i}^0}=y_j^0$ if $j\neq i, y_j|_{u_i,u_{-i}^0}=y_i|_{u_i}$ otherwise. Then we have Lemmas 6.1–6.1, whose proofs are put into Appendix D.

Lemma 6.1: For the system (2) and the cost function (3), if Assumptions A1), A3)–A5) hold, then there exists a nonnegative finite random variable C_1, such that

$$\sup_{N\geq 1}\max_{1\leq i\leq N} J_i^N(u_i^0,u_{-i}^0)\leq C_1\ a.s. \quad (38)$$

Lemma 6.2: For the system (2) and the cost function (3), if Assumptions A1), A3)–A5) hold, then there exists a nonnegative finite random variable C_2, such that

$$\sup_{N\geq 1}\max_{1\leq i\leq N}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\left\|\frac{\gamma}{N}\sum_{j=1,j\neq i}^N y_j^0(t)\right\|_Q^2 dt\leq C_2\ a.s.$$

Lemma 6.3: For the system (2) and the cost function (3), if Assumptions A1), A3)–A5) hold, then we have

$$J_i^N\left(u_i^0,u_{-i}^0\right)\leq J_i\left(u_i^0,\gamma y^*\right)+O\left(\epsilon_1^N\right)\ a.s. \quad (39)$$

Lemma 6.4: For the system (2) and the cost function (3), if Assumptions A1), A3)–A5) hold, then we have

$$J_i\left(u_i^0,\gamma y^*\right)\leq\inf_{u_i\in\mathcal{U}_{g,i}}J_i^N\left(u_i,u_{-i}^0\right)+O\left(\epsilon_1^N\right)+O(N^{-1}) \quad (40)$$

where

$$\epsilon_1^N=\left(\limsup_{T\to\infty}\frac{1}{T}\int_0^T\left\|y^*(t)-\bar{y}_N^0(t)\right\|^2dt\right)^{1/2}. \quad (41)$$

Theorem 6.1: For the system (2) and the cost function (3), if Assumptions A1)–A5) hold, then the sequence of control groups $\{\mathbf{U}^N=\{u_i^0,1\leq i\leq N\},N\geq 1\}$ is an almost sure asymptotic Nash-equilibrium with respect to the corresponding cost groups $\{\mathbf{J}^N=\{J_i^N,1\leq i\leq N\},N\geq 1\}$.

Proof: From Theorem 5.2 and (41) it follows that $\epsilon_1^N\xrightarrow[N\to\infty]{a.s.}0$. By Lemmas 6.3 and 6.4 we have

$$J_i^N\left(u_i^0,u_{-i}^0\right)\leq\inf_{u_i\in\mathcal{U}_{g,i}}J_i^N\left(u_i,u_{-i}^0\right)+O\left(\epsilon_1^N\right)+O(N^{-1})\ a.s.$$

which together with Definition 2.4 leads to the conclusion. □

By now, we have considered the case with linearly coupled cost function given by (3). In fact, we may consider a more general case with the coupled cost function:

$$\tilde{J}_i^N\left(u_i,u_{-i}^N\right)=\limsup_{T\to\infty}\frac{1}{T}\int_0^T\left\{\left\|y_i-\phi\left(\frac{1}{N}\sum_{j=1}^N y_j\right)\right\|_Q^2+\|u_i\|_R^2\right\}dt \quad (42)$$

where $\phi(\cdot)$ is continuous and for a constant $\gamma_1>0$, satisfies

$$\|\phi(x)-\phi(y)\|_Q^2\leq\gamma_1\|x-y\|_Q^2. \quad (43)$$

It can be seen that if 1) $\phi(\cdot)$ is Lipschitz continuous and Q is positive definite, or 2) $\phi(x)=\gamma x, x\in\mathbb{R}^n$, then (43) holds. So, (3) is a special case of (42) with $\phi(x)=\gamma x$.

Based on the Nash certainty equivalence principle, the design procedure of the decentralized control law for the system (2) and the cost function (42) is the same as that for (2) and (3). The decentralized control law is given by

$$\tilde{u}_i^0(t)=-R^{-1}B_i^TP_iy_i(t)+R^{-1}B_i^T\tilde{\xi}_i(t)$$

where $\tilde{\xi}_i(t)=\int_t^\infty e^{-G_i^T(t-\tau)}(Q\phi(y^*(\tau))-P_if(\tau))d\tau$. Here y^* is the unique solution of (17) with the operator $\mathcal{T}$ redefined as: the $\gamma x(s)$ is replaced by $\phi(x)$ in (16). Similar to Theorem 6.1, we have the following result.

Theorem 6.2: For the system (2) and the cost function (42), if Assumptions A1)–A5) hold, then the sequence of control groups $\{\mathbf{U}^N=\{\tilde{u}_i^0,1\leq i\leq N\},N\geq 1\}$ is an almost sure asymptotic Nash-equilibrium with respect to the corresponding cost groups $\{\mathbf{J}^N=\{\tilde{J}_i^N,1\leq i\leq N\},N\geq 1\}$.

Remark 13: The cost-coupled MAS considered in this paper can be found in many engineering problems, such as the power control of wireless communication networks ([36]), the production output adjustment ([26]) and the distributed optimal consensus ([37]). Obviously, it would be more applicable and meaningful if one could extend the result of this paper to the dynamics-coupled case. However, in that case, the change of the control strategy of a single agent may result in the change of the states of all the other agents, which brings essential difficulty to the analysis of the asymptotic optimality of the control law synthesized based on the Nash certainty equivalence principle. Thus, the dynamics-coupled case is a hard topic and needs further investigation.

VII. Numerical Example

Consider a social foraging model ([8]). Different from the deterministic noise in [8], agents are driven by Gaussian white noises. The dynamical equation of agent i in $\mathbf{S}^N$ is

$$dv_i=\left(av_i+bu_i+\frac{\partial J_p(x)}{\partial x}\right)dt+\sigma dw_i$$

where v_i is its velocity, $a=-k/m, k\geq 0$ is the damping factor, $m>0$ is its mass, $b=1/m,\sigma>0, J_P(x)$ is the

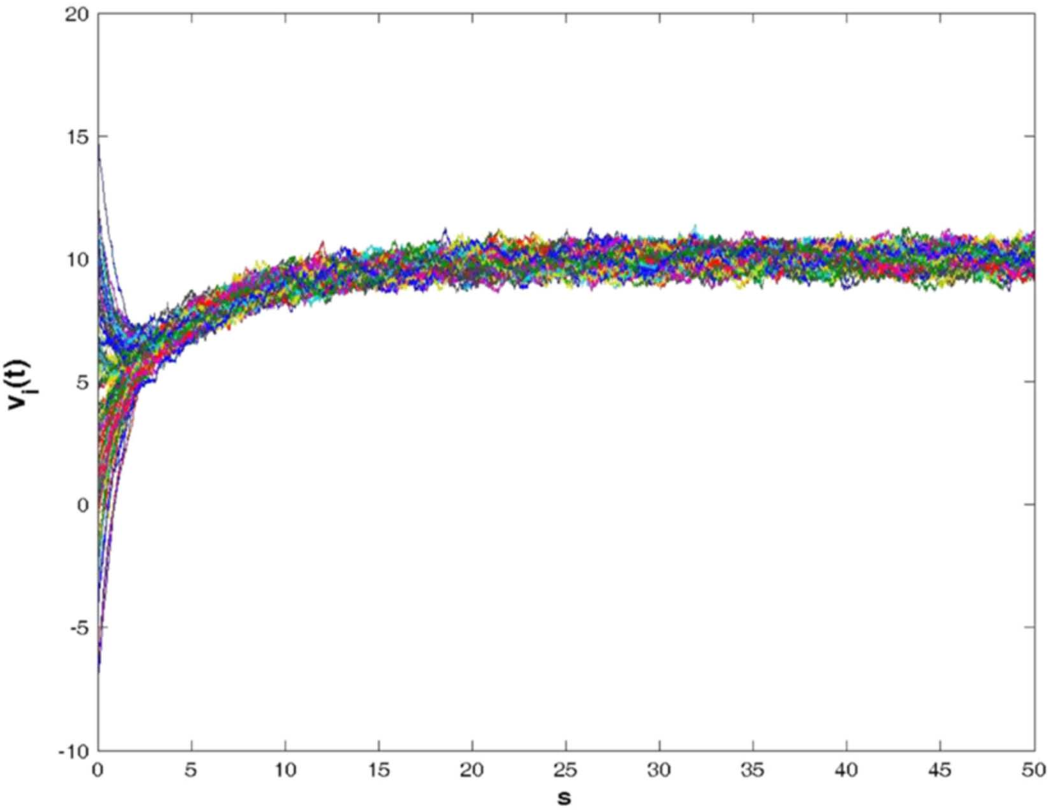

Fig. 1. Trajectories of agents' states when $N = 100$.

nutrient profile description function, $J_p(x) = \bar{R}x + r_p, \bar{R} \neq 0 \cdot \{v_i(0), i \geq 1\}$ are i.i.d r.v.s. with normal distribution $N(v_0, \sigma_v^2); \{w_i(t), \mathcal{F}_t^i, i \geq 1\}$ is a sequence of independent standard Brownian motions.

The cost function of agent i is described by

$$J_i^N(u_i, u_{-i}) = \limsup_{T\to\infty} \frac{1}{T} \times \int_0^T \left(\left(v_i - \frac{\gamma}{N}\sum_{j=1}^N v_j \right)^2 + r u_i^2 \right) dt$$

where $\gamma > 0, r > 0$.

Let $\theta \equiv (a, b), P(\theta) = (ar + \sqrt{a^2r^2 + b^2r})/(b^2), G(\theta) = a - (b^2P(\theta))/(r) = -\sqrt{a^2 + b^2/r}$. Then, according to Theorem 3.2, to ensure Assumption A5), it needs only to require that $(b^2\gamma)/((ra^2 + b^2)) < 1$. It can be easily seen that when $\gamma = 1, k > 0$ or $\gamma < 1$, the above inequality holds. In this case, similar to the method suggested by [38], from the auxiliary system (14) one can get $\xi_\theta(t) = (\bar{R}\gamma + G(\theta)\bar{R}P(\theta))/(-G^2(\theta) + b^2\gamma/r) - (\gamma)/[\lambda_0 + G(\theta)](v_0 - v(\infty))e^{\lambda_0 t}$, where $\lambda_0 = -\sqrt{a^2 + b^2(1-\gamma)/r}, v(\infty) = (aR)/(-a^2 + b^2(\gamma - 1)/r)$. Hence, by (19) we have the decentralized control law $u_i^0(t) = -br^{-1}P(\theta)v_i(t) + br^{-1}\xi_\theta(t)$.

Taking $k = 0.1, m = b = r = 1, \bar{R} = 3, \sigma = 0.5, v_0 = 4, \sigma_v^2 = 5, \gamma = 0.98$, when the number of agents $N = 100$, the trajectories of the closed-loop system are shown in Fig. 1.

Letting N increase from 1 to 800, the curves of ϵ_1^N are shown in Fig. 2, where (a) is for ϵ_1^N along with N, (b) is for $\epsilon_1^N\sqrt{N}$ along with N. It can be seen that the rate of ϵ_1^N approaching to zero is approximately $O(N^{-1/2})$.

VIII. Concluding Remarks

The decentralized control problem for LPSMAS with coupled stochastic cost functions is investigated, including the control design and the closed-loop analysis. For the control design, we first present the tracking-like quadratic optimal control, then the approximation of the PSA by state aggregation, at last the decentralized control law based on the Nash-certainty equivalence principle. For the closed-loop analysis, by probability limit theory (such as the strong law of large numbers and weakly convergence of empirical distributions), we answer the following three fundamental questions: 1) under the decentralized control law designed, whether the closed-loop system is stable, and whether the stability is independent of the number of agents N (namely, uniform with respect to N); 2) As $N \to \infty$, whether the PSA converges to the IPM; 3) whether the decentralized control law designed is asymptotically optimal with respect to the coupled stochastic cost functions.

It is worth pointing out that, the cost-coupled LPSMAS considered in this paper is a kind of individual-population interacting MAS. For this kind of systems, the result obtained in this paper indicates that the overall impact on a given agent by the population of all agents is nearly deterministic with probability one, and the individual impact is neglectable as the number of agents increases to infinity. This phenomenon is similar in spirit

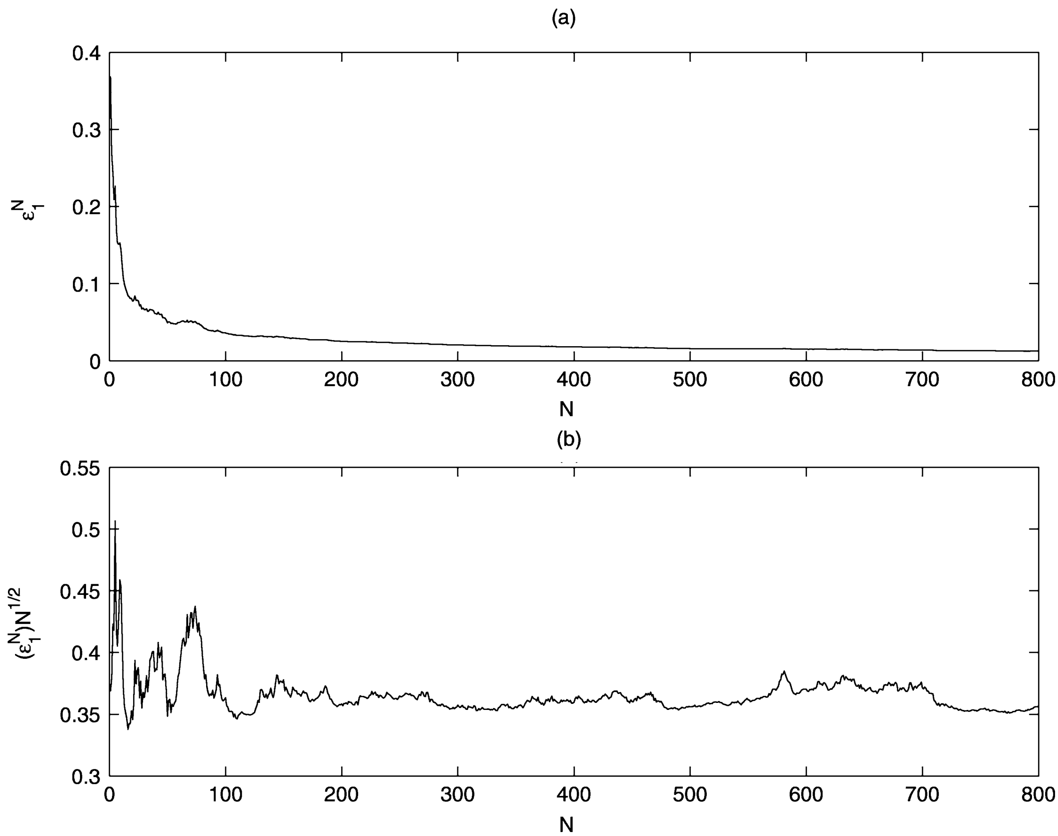

Fig. 2. Curves of ϵ_1^N and $\epsilon_1^N\sqrt{N}$ with respect to N.

to the concept of Wardrop-equilibrium in the context of transportation networks ([6], [39]). For the decentralized control of this kind of MAS, many important issues are still open and need to be investigated, such as, how to design optimal decentralized (adaptive) controls for the cases with parametric uncertainties, unmodelled dynamics ([40], [41]), unknown disturbances or partial measurement information.

APPENDIX A
BASIC LEMMAS IN PROBABILITY THEORY

Lemma A.1: Consider a wide stationary n-dimensional Gaussian process $X(t) = (x_1(t), \ldots, x_n(t))^T$ with zero-mean and continuous in mean square. If its self-covariance matrix $R(s) = E(X(t)X^T(t+s))$ is absolutely integrable, that is, $\int_0^\infty \|R(s)\|ds < \infty$, then $\lim_{T\to\infty} \frac{1}{T}\int_0^T \|X(t)\|^2 dt = E\|X(0)\|^2$ a.s.

Proof: It needs only to show

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|x_i(t)\|^2 dt = E\|x_i(0)\|^2 \; a.s. \tag{A.1}$$

Notice that $x_i(t)$ is zero-mean wide stationary Gaussian and continuous in mean square, the covariance function $r_i(s) = E(x_i(t)x_i(t+s))$ satisfies $\int_0^\infty |r_i(t)|dt = \int_0^\infty \|e_i R(s) e_i^T\|dt \le \int_0^\infty \|R(s)\|ds < \infty$, where e_i denotes the n-dimensional vector with the ith component being 1, and the others being 0. Then, we can conclude that the spectrum function $F_i(\lambda)$ of $x_i(t)$ is absolutely continuous, and hence, (A.1) holds ([42]). □

Lemma A.2: If all the eigenvalues of A have negative real part, then the solution $y(t)$ of following differential equation

$$dy(t) = Ay(t)dt + GdW(t), \quad y(0) = 0 \tag{A.2}$$

satisfies

$$E\|y(t)\|^{2m} \le (2m-1)!!\|G\|^{2m} \times \left(\lambda_1^2/\lambda_2\right)^m \left(1 - e^{-m\lambda_1^{-1}t}\right) \tag{A.3}$$

where $m = 1, 2, \ldots, (2m-1)!!$ means the double factorial of $2m-1, P$ is the unique positive definite solution of $PA + A^TP = -I, \lambda_1 = \lambda_{\max}(P)$, and $\lambda_2 = \lambda_{\min}(P)$.

Proof: By (A.2) and Itô formula we have

$$\begin{aligned} d(y^T(t)Py(t))^m &= -m(y^T(t)Py(t))^{m-1}y^T(t)y(t)dt \\ &\quad + m(y^T(t)Py(t))^{m-1}tr(PGG^T)dt \\ &\quad + 2m(m-1)(y^T(t)Py(t))^{m-2}y^T(t) \\ &\quad \times PGG^TPy(t)dt \\ &\quad + 2m(y^T(t)Py(t))^{m-1}y^T(t)PGdW(t). \end{aligned} \tag{A.4}$$

Let $\tau_N = \inf\{t \ge 0 : \|y(t)\| \ge N\}$. Then, by (A.4), we have

$$\begin{aligned} &E\left(\left(y^T(t\wedge\tau_N)\,Py(t\wedge\tau_N)\right)^m \chi_{\{t\le\tau_N\}}\right) \\ &\quad \le -m\lambda_2^{m-1}\int_0^t E\left(\|y(s\wedge\tau_N)\|^{2m}\chi_{\{s\le\tau_N\}}\right)ds \\ &\qquad + m\lambda_1^m\|G\|^2 \end{aligned}$$

$$\int_0^t E\left(\|y(s\wedge\tau_N)\|^{2(m-1)}\chi_{\{s\leq\tau_N\}}\right)ds + 2m(m-1)\lambda_1^2\|G\|^2 \int_0^t E\left(\|y(s\wedge\tau_N)\|^{2(m-1)}\chi_{\{s\leq\tau_N\}}\right)ds \leq m(2m-1)\lambda_1^m\|G\|^2 \int_0^t E\left(\|y(s\wedge\tau_N)\|^{2(m-1)}\chi_{\{s\leq\tau_N\}}\right)ds.$$

Therefore, $\lambda_2^m E(\|y(t\wedge\tau_N)\|^{2m}\chi_{\{t\leq\tau_N\}}) \leq m(2m-1)\lambda_1^m \|G\|^2\int_0^t E(\|y(s\wedge\tau_N)\|^{2(m-1)}\chi_{\{s\leq\tau_N\}})ds$. Let $S_t^N(m) = E(\|y(t\wedge\tau_N)\|^{2m}\chi_{\{t\leq\tau_N\}}), m \geq 0$. Then the above inequality can be rewritten as $S_t^N(m)\leq m(2m-1)(\lambda_1/\lambda_2)^m \|G\|^2\int_0^t S_s^N(m-1)ds$. Noticing $S_t^N(0)\leq 1$, by induction we have

$$S_t^N(m)\leq (2m-1)!!(\lambda_1/\lambda_2)^{m(m+1)/2}\|G\|^{2m}t^m. \quad \text{(A5)}$$

Since $y(t)$ is continuous in terms of t almost surely, $t\wedge\tau_N \xrightarrow[N\to\infty]{a.s.} t$. Hence, by (A.5) and the Fatou lemma we obtain $E\|y(t)\|^{2m}\leq(2m-1)!!(\lambda_1/\lambda_2)^{m(m+1)/2}\|G\|^{2m}t^m$. Furthermore, $\int_0^T E(\|y(t)\|^{2m}dt<\infty, \forall\, m\geq 1, \forall\, T\geq 0$. This together with (A.4) gives

$$\frac{dE(y^T(t)Py(t))^m}{dt}\leq -m\lambda_1^{-1}E(y^T(t)Py(t))^m +m(2m-1)\lambda_1\|G\|^2E(y^T(t)Py(t))^{m-1}. \quad \text{(A.6)}$$

Set $V_t(m)=E(y^T(t)Py(t))^m$. Then from $V_0(m)=0$, (A.6) and the comparison theorem (Theorem 2.6.4 in [43]), it follows that

$$V_t(m)\leq m(2m-1)\|G\|^2\lambda_1\int_0^t e^{-\lambda_1^{-1}m(t-\tau)}V_\tau(m-1)d\tau. \quad \text{(A.7)}$$

By $V_t(0)\equiv 1$ we have $V_t(1)\leq \|G\|^2\lambda_1^2(1-e^{-m\lambda_1^{-1}t})$. Hence, when $m=1$, we have $V_t(m)\leq(2m-1)!!\|G\|^{2m}\lambda_1^{2m}(1-e^{-m\lambda_1^{-1}t})$. Suppose for some $m\geq 1$, we have already got $V_t(m-1)\leq(2m-3)!!\|G\|^{2m-2}\lambda_1^{2m-2}(1-e^{-(m-1)\lambda_1^{-1}t})$. Then it follows from (A.7) that $V_t(m)\leq(2m-1)!!m\|G\|^{2m}\lambda_1^{2m-1}\int_0^t e^{-m\lambda_1^{-1}(t-\tau)}d\tau = (2m-1)!!\|G\|^{2m}\lambda_1^{2m}(1-e^{-m\lambda_1^{-1}t})$. Therefore, by induction, (A.3) is true for all $m\geq 1$. □

APPENDIX B
PROOF OF THEOREM 3.1

The proofs of (i) and (ii) can be found in [44]. Here we need only to show (iii), (iv), and (v).

(iii): The general solution of (9) can be expressed as $\xi(t)=e^{-G^Tt}\xi(0)+\int_0^t e^{-G^T(t-\tau)}(Pf^b(\tau)-C^TQy^d(\tau))d\tau$. Since all the eigenvalues of G have negative real part, there exists $\kappa_1>0$ and $\rho_1>0$ such that $\|e^{Gt}\|\leq\kappa_1e^{-\rho_1t},\forall t\geq 0$. Let $M_{f^b}=\|f^b\|_\infty$ and $M_{y^d}=\|y^d\|_\infty$. Then, when $\xi(0)=-\int_0^\infty e^{G^T\tau}(Pf^b(\tau)-C^TQy^d(\tau))d\tau\triangleq\xi^*(0)$, one can get the solution (10). Furthermore, by

$$\|\xi^*(t)\|\leq\left(\|P\|M_{f^b}+\|C\|\|Q\|M_{y^d}\right)\kappa_1/\rho_1,\ \forall t\geq 0, \quad \text{(B.1)}$$

we have $\sup_{t\geq 0}\|\xi^*(t)\|<\infty$, i.e., $\xi^*(t)\in\mathcal{C}_n^b$. Notice that for any initial value $\xi(0)=\xi^*(0)+\Delta\xi(0),\Delta\xi(0)\neq 0$, the corresponding solution is $\xi(t)=e^{-G^Tt}\Delta\xi(0)+\xi^*(t)$. Since $\xi^*(t)\in\mathcal{C}_n^b$ and all the eigenvalues of $-G^T$ have positive real part, $\xi(t)$ is unbounded. Thus, the solution of (10) is unique in $\mathcal{C}_n^b$, i.e., (iii) holds.

(iv), (v): From (7) and Itô formula it follows that

$$\begin{aligned}\int_0^T(x(t)^TC^TQCx(t))dt &= x^T(0)Px(0)-x^T(T)Px(T)\\ &+\int_0^T(x^T(t)PBR^{-1}B^TPx(t)\\ &+2u^T(t)B^TPx(t)+2f^T(t)Px(t))dt\\ &+2\int_0^T x^T(t)PDdW(t)+TtrPDD^T.\end{aligned}$$

By (7), (9), and (10) we have

$$\begin{aligned}\int_0^T 2\left(f^{b^T}(t)Px(t)-y^{d^T}(t)QCx(t)\right)dt &= 2(\xi^{*^T}(T)x(T)-\xi^T(0)x(0))\\ &-2\int_0^T\{\xi^T(t)BR^{-1}B^TPx(t)\\ &+\xi^T(t)(Bu(t)+f^b(t))\}dt\\ &-2\int_0^T\xi^{*^T}(t)DdW(t).\end{aligned} \quad \text{(B.2)}$$

Let $J(T,u)=\int_0^T(\|y(t)-y^d(t)\|_Q^2+\|u(t)\|_R^2)dt$. Then, by (B.2) we have

$$\begin{aligned}J(T,u) &= x^T(0)Px(0)-x^T(T)Px(T)\\ &+2(\xi^{*^T}(T)x(T)-\xi^{*^T}(0)x(0))+TtrPDD^T\\ &+\int_0^T\left\{\|u(t)+R^{-1}B^TPx-R^{-1}B^T\xi^*(t)\|_R^2\right\}dt\\ &+\int_0^T\left(\|y^d(t)\|_Q^2-\|B^T\xi^*(t)\|_{R^{-1}}^2-2\xi^{*^T}(t)f^b(t)\right)dt\\ &+2\int_0^T(x^T(t)P-\xi^{*^T}(t))DdW(t).\end{aligned} \quad \text{(B.3)}$$

This together with (iii) implies that for all $u\in\mathcal{U}$,

$$\begin{aligned}J(u) &= \limsup_{T\to\infty}\frac{1}{T}J(T,u)\\ &\geq \liminf_{T\to\infty}\frac{1}{T}\{x^T(0)Px(0)-x^T(T)Px(T)\\ &+2(\xi^{*^T}(T)x(T)-\xi^{*^T}(0)x(0))\}+trPDD^T\\ &+\limsup_{T\to\infty}\frac{1}{T}\int_0^T\left(\|y^d(t)\|_Q^2-\|B^T\xi^*(t)\|_{R^{-1}}^2\right.\\ &\left.-2\xi^{*^T}(t)f^b(t)\right)dt\\ &+\liminf_{T\to\infty}\frac{2}{T}\int_0^T(x^T(t)P-\xi^{*^T}(t))DdW(t).\end{aligned} \quad \text{(B.4)}$$

Recalling that $u \in \mathcal{U}$ and Lemma 12.3 of [27], we have

$$\int_0^T (x^T(t)P - \xi^{*^T}(t))DdW(t) = O(T^{1/2+\epsilon}),\ \forall\ \epsilon > 0 \tag{B.5}$$

which together with (B.4) gives

$$\begin{aligned} J(u) \geq \limsup_{T\to\infty} \frac{1}{T}\int_0^T & \left(\|y^d(t)\|_Q^2 - \|B^T\xi^*(t)\|_{R^{-1}}^2\right. \\ & -\|B^T\xi^*(t)\|_{R^{-1}}^2 \\ & \left.- 2\xi^{*^T}(t)f^b(t)\right)dt + trPDD^T \triangleq J^*\ a.s. \end{aligned}$$

Take

$$\tilde{u}(t) = -R^{-1}B^TPx(t) + R^{-1}B^T\xi^*(t). \tag{B.6}$$

Then, the closed-loop system is

$$dx(t) = (Gx(t) + BR^{-1}B^T\xi^*(t) + f^b(t))dt + DdW(t). \tag{B.7}$$

Thus, from Lemma 12.4 of [27] and (iii)

$$\int_0^T \|x(t)\|^2dt = O(T)\ a.s. \tag{B.8}$$

Let $z(t) = \int_0^T e^{G(t-\tau)}DdW(\tau)$, and $Q = \int_0^\infty e^{G\tau}DD^T e^{G^T\tau}d\tau$. Then, we have $GQ + QG^T + DD^T = O$, and by Lemma 12.3 of [27], $\lim_{t\to\infty}(z(t)z^T(t))/t = GQ + QG^T + DD^T$. Thus, $\lim_{t\to\infty}(z^T(t)z(t))/t = 0$ a.s. Furthermore, from (B.7) and (iii) it follows that $\|x(T)\|^2 = o(T)$ a.s. Thus, by (B.6) and (B.8) we have $\tilde{u} \in \mathcal{U}$; by (B.1), (B.4), and (B.5), $J(\tilde{u}) = J^*$. Hence, $\tilde{u}$ is the optimal control u^*, and the corresponding cost value is J^*, i.e., (11) and (12) both hold. □

APPENDIX C
PROOFS OF LEMMAS 5.1, 5.2, AND 5.3

Proof of Lemmas 5.1: Let

$$\begin{aligned} I_{3_\theta}(t) &= e^{G(\theta)t}y_0 + \int_0^t e^{G(\theta)(t-\tau)}f(\tau)d\tau \\ I_{4_\theta}(t) &= \int_0^t e^{G(\theta)(t-\tau)}L(\theta) \\ &\quad\times\left(\int_\tau^\infty e^{-G^T(\theta)(\tau-s)}Q\gamma y^*(s)ds\right)d\tau \\ I_{5_\theta}(t) &= \int_0^t e^{G(\theta)(t-\tau)}L(\theta) \\ &\quad\times\left(\int_\tau^\infty e^{-G^T(\theta)(\tau-s)}P(\theta)f(s)ds\right)d\tau. \end{aligned}$$

Then (15) can be rewritten as $Ey_\theta(t) = I_{3_\theta}(t)+I_{4_\theta}(t)-I_{5_\theta}(t)$. Hence, to prove the lemma, we need only to show that $I_{3_\theta}(t), I_{4_\theta}(t)$ and $I_{5_\theta}(t)$ are uniformly bounded and equicontinuous.

From Remark 8 (ii) we know that for any $t \geq 0, \theta \in \mathcal{M}$

$$\begin{aligned} |I_{3_\theta}(t)| &\leq \|y_0\|\kappa + M_f\int_0^t \left\|e^{G(\theta)(t-\tau)}\right\|d\tau \\ &\leq \|y_0\|\kappa + M_f\kappa/\rho \\ |I_{4_\theta}(t)| &\leq M_B^2\|R^{-1}\|\|Q\|\|\gamma\|M_{y^*} \\ &\quad\times\int_0^t \left\|e^{G(\theta)(t-\tau)}\right\|\int_\tau^\infty \left\|e^{-G^T(\theta)(\tau-s)}\right\|dsd\tau \\ &\leq M_B^2\|R^{-1}\|\|Q\|\|\gamma\|M_{y^*}\kappa^2/\rho^2 \\ |I_{5_\theta}(t)| &\leq M_B^2M_P\|R^{-1}\|\|Q\|\|\gamma\|M_f\kappa^2/\rho^2. \end{aligned}$$

Thus, $I_{3_\theta}(t), I_{4_\theta}(t)$, and $I_{5_\theta}(t)$ are uniformly bounded. We now show the equicontinuity of $I_{3_\theta}(t), I_{4_\theta}(t)$, and $I_{5_\theta}(t)$.

By Dyson's expansion ([45])

$$e^{A+B} - e^A = \int_0^1 e^{(1-t)A}Be^{t(A+B)}dt$$

we have

$$e^{G(\theta)t} - e^{G(\theta')t} = \int_0^1 e^{(1-s)G(\theta')t}(G(\theta) - G(\theta'))te^{sG(\theta)t}ds$$

and hence, $\forall\theta, \theta' \in \mathcal{M}, t \geq 0$

$$\begin{aligned} &\left\|e^{G(\theta)t} - e^{G(\theta')t}\right\| \\ &\quad\leq \|G(\theta) - G(\theta')\|\int_0^1 \kappa^2te^{-\rho(1-s)t}e^{-\rho st}ds \\ &\quad\leq \|G(\theta) - G(\theta')\|\kappa^2te^{-\rho t}. \end{aligned} \tag{C.1}$$

This together with

$$\begin{aligned} &\left\|\int_0^t e^{G(\theta)(t-\tau)}f(\tau)d\tau - \int_0^t e^{G(\theta')(t-\tau)}f(\tau)d\tau\right\| \\ &\quad\leq M_f\int_0^t \|e^{G(\theta)\tau} - e^{G(\theta')\tau}\|d\tau \\ &\quad\leq \frac{\kappa^2M_f}{\rho^2}\|G(\theta) - G(\theta')\| \end{aligned}$$

implies that there exists $\lambda_3 > 0$ independent of t, θ, and θ' such that $\|I_{3_{\theta}(t)} - I_{3_{\theta'}(t)}\| \leq \lambda_3\|G(\theta) - G(\theta')\|$. Thus, by the uniform continuity of $G(\theta)$ on $\mathcal{M}$ we see that $I_{3_\theta}(t)$ is equicontinuous.

Denote

$$\begin{aligned} &\triangle_1(\theta, \theta', t) \\ &\quad= \left\|\int_0^t e^{G(\theta)(t-\tau)}L(\theta)\right. \\ &\qquad\times\left(\int_\tau^\infty e^{-G^T(\theta)(\tau-s)}Q\gamma y^*(s)ds\right)d\tau \\ &\qquad-\int_0^t e^{G(\theta')(t-\tau)}L(\theta') \\ &\qquad\left.\times\left(\int_\tau^\infty e^{-G^T(\theta)(\tau-s)}Q\gamma y^*(s)ds\right)d\tau\right\| \\ &\triangle_2(\theta, \theta', t) \\ &\quad= \left\|\int_0^t e^{G(\theta')(t-\tau)}L(\theta')\right. \\ &\qquad\times\left(\int_\tau^\infty e^{-G^T(\theta)(\tau-s)}Q\gamma y^*(s)ds\right)d\tau \end{aligned}$$

$$- \int_0^t e^{G(\theta')(t-\tau)} L(\theta') \times \left(\int_\tau^\infty e^{-G^T(\theta')(\tau-s)} Q\gamma y^*(s) ds \right) d\tau \Bigg\|.$$

Then $\|I_{4_\theta(t)} - I_{4_{\theta'}(t)}\| \le \triangle_1(\theta,\theta',t) + \triangle_2(\theta,\theta',t)$. Notice that from (C.1)

$$\begin{aligned}
&\triangle_1(\theta,\theta',t)\\
&\le \|Q\|\|\gamma\| M_{y^*} \int_0^t \|e^{G(\theta)(t-\tau)} L(\theta) - e^{G(\theta')(t-\tau)} L(\theta')\| \times \left(\int_0^\infty \left\| e^{G^T(\theta)s} \right\| ds \right) d\tau\\
&\le \|Q\|\|\gamma\| M_{y^*} \kappa/\rho \left(M_B^2 \|R^{-1}\| \|G(\theta) - G(\theta')\| \frac{\kappa^2}{\rho^2} + \|L(\theta) - L(\theta')\| \kappa/\rho \right)\\
&\triangle_2(\theta,\theta',t)\\
&\le \|Q\|\|\gamma\| M_{y^*} \left(\int_0^\infty \|e^{G^T(\theta)s} - e^{G^T(\theta')s}\| ds \right) \times \int_0^\infty \|e^{G(\theta')\tau} L(\theta')\| d\tau\\
&\le \|Q\|\|\gamma\| M_{y^*} M_B^2 \|R^{-1}\| 2\kappa^3/\rho^3 (\|G(\theta) - G(\theta')\|).
\end{aligned}$$

Then, there exists $\lambda_4 > 0$ independent of t, θ, and θ' such that $\|I_{4_\theta(t)} - I_{4_{\theta'}(t)}\| \le \lambda_4(\|G(\theta) - G(\theta')\| + \|L(\theta) - L(\theta')\|)$. This together with the uniform continuity of $G(\theta)$ and $L(\theta)$ on $\mathcal{M}$ implies the equicontinuity of $I_{4_\theta}(t)$.

Similarly, we can show that there exists $\lambda_5 > 0$ independent of t, θ, and θ' such that $\|I_{5_\theta(t)} - I_{5_{\theta'}(t)}\| \le \lambda_5(\|P(\theta) - P(\theta')\| + \|G(\theta) - G(\theta')\| + \|L(\theta) - L(\theta')\|)$. From this and the uniform continuity of $G(\theta), L(\theta)$, and $P(\theta)$ on $\mathcal{M}$ the equicontinuity of $I_{5_\theta}(t)$ follows. Thus, the lemma is true. □

Proof of Lemma 5.2: Let $z_i(t) = \int_0^t e^{G_i(t-\tau)} D dW_i(\tau)$. Then, in order to prove the lemma, by (28) we need only to show

$$P\left\{\omega : \left\{ \|\frac{1}{N}\sum_{i=1}^N e^{G_i t}(y_i(0) - Ey_i(0))\|^2, N \ge 1 \right\} \text{ is uniformly Lebesgue integrable on } [0,T], \forall T \ge 0 \right\} = 1 \quad \text{(C.2)}$$

$$P\left\{\omega : \left\{ \left\| \frac{1}{N}\sum_{i=1}^N z_i(t) \right\|^2, N \ge 1 \text{ is uniformly Lebesgue integrable on} [0,T],\ \forall T \ge 0 \right\}\right\} = 1. \quad \text{(C.3)}$$

From Assumption A4) and Theorem 2.8 in [27] it follows that $\lim_{N\to\infty}(1/N)\sum_{i=1}^N (\|y_i(0) - y_0\|^2 - E\|y_i(0) - y_0\|^2) = 0$ a.s. By using Assumption A4) again we have $\sup_{N\ge1}(1/N)\sum_{i=1}^N E\|y_i(0) - y_0\|^2 < \infty$, and hence, $P\{\sup_{N\ge1}(1/N)\sum_{i=1}^N \|y_i(0) - Ey_i(0)\|^2 < \infty\} = 1$. From Remark 8 (ii) we see that $\int_0^T \sup_{\theta\in\mathcal{M}} \|e^{G(\theta)t}\|^2 dt < \infty, \forall T \ge 0$. Thus, by Jensen inequality ([32]) we have $P\{\omega : \int_0^T \sup_{N\ge1} \|(1/N)\sum_{i=1}^N e^{G_i t}(y_i^0(0) - y_0)\|^2 dt < \infty, \forall T \ge 0\} = 1$. This together with the properties of uniformly integrability [see ([32], pp.94)] ensures (C.2).

We now prove (C.3). Let $z_\theta(t) = \int_0^t e^{G(\theta)(t-\tau)} D dW(\tau)$. From Remark 8 (iii) and Lemma A.2 it follows that $\forall T \ge 0$

$$\sup_{\theta\in\mathcal{M}} \int_0^T E\|z_\theta(t)\|^4 < \infty,\ \sup_{\theta\in\mathcal{M}} \int_0^T E\|z_\theta(t)\|^8 < \infty. \quad \text{(C.4)}$$

Hence, by Cauchy inequality we have

$$\begin{aligned}
E\left(\int_0^T \|z_i(t)\|^4 dt \right)^2 &\le T \int_0^T E\|z_i(t)\|^8 dt\\
&\le T \sup_{\theta\in\mathcal{M}} \int_0^T E\left\| \int_0^t e^{G(\theta)(t-\tau)} D dW(\tau) \right\|^8 dt\\
&= T \sup_{\theta\in\mathcal{M}} \int_0^T E\|z_\theta(t)\|^8 dt < \infty,\ \forall T \ge 0\\
&1 \le i \le N.
\end{aligned}$$

This together with Theorem 2.8 in [27] implies

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \int_0^T \left(\|z_i(t)\|^4 - E\|z_i(t)\|^4 \right) dt = 0\ a.s. \quad \text{(C.5)}$$

By (C.4) we get $\sup_{N\ge1}(1/N)\sum_{i=1}^N E\int_0^T \|z_i(t)\|^4 dt < \infty$. Therefore, by (C.5) and Jensen inequality, we know that for any given positive integer $K \ge 1$, there is $A_K \in \mathcal{F}, P(A_K) = 1$, such that

$$\sup_{N\ge1} \int_0^K \left\| \frac{1}{N}\sum_{i=1}^N z_i(t) \right\|^4 dt < \infty,\ \forall \omega \in A_K. \quad \text{(C.6)}$$

Take $A = \bigcap_{K=1}^\infty A_K$. Then for any $\omega \in A$ and $T \ge 0$, it follows from $\omega \in A_{\lfloor T\rfloor+1}$ and (C.6) that $\sup_{N\ge1}\int_0^T \|(1/N)\sum_{i=1}^N z_i(t)\|^4 dt \le \sup_{N\ge1}\int_0^{\lfloor T\rfloor+1} \|(1/N)\sum_{i=1}^N z_i(t)\|^4 dt < \infty$. Thus, $A \subseteq \{\omega : \sup_{N\ge1}\int_0^T \|(1/N)\sum_{i=1}^N z_i(t)\|^4 dt < \infty, \forall T \ge 0\}$. Here $\lfloor T\rfloor$ denotes the largest integer less than or equal to T. Noticing $P(A) = 1$, by the property of uniform integrability [see ([32], pp. 102)], one can get (C.3) immediately. Thus, the lemma is true. □

Proof of Lemma 5.3.: Let $z_i^{(1)}(t) = \int_{-\infty}^0 e^{G_i(t-\tau)} D dW_i(\tau)$ and $z_i^{(2)}(t) = \int_{-\infty}^t e^{G_i(t-\tau)} D W_i(\tau)$. Then, (28) can be rewritten as

$$y_i^0(t) - Ey_i^0(t) = e^{G_i t}(y_i(0) - Ey_i(0)) - z_i^{(1)}(t) + z_i^{(2)}(t).$$

So, to show the lemma, we need only to show

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left\| \frac{1}{N}\sum_{i=1}^N z_i^{(1)}(t) \right\|^2 dt = 0\ a.s. \quad \text{(C.7)}$$

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left\| \frac{1}{N}\sum_{i=1}^N z_i^{(2)}(t) \right\|^2 dt = 0\ a.s. \quad \text{(C.8)}$$

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left\| \frac{1}{N}\sum_{i=1}^N (e^{G_i(t)}(y_i(0) - Ey_i(0))) \right\|^2 dt = 0\ a.s. \quad \text{(C.9)}$$

Denote $\Sigma_\theta(t) = e^{-G(\theta)\tau}D, \phi_\theta(t) = \int_{-\infty}^{t}\Sigma(\tau)dW(\tau)\triangleq(\phi_\theta^1(t),\ldots,\phi_\theta^n(t))^T$. Then by Itô formula and Remark 8 (ii) one can get

$$\sup_{\theta\in\mathcal{M}} E\|\phi_\theta(0)\|^4 < \infty. \tag{C.10}$$

Hence, by Theorem 2.8 in [27] there exists $B\in\mathcal{F}, P(B)=1$, such that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\left(\left\|\int_{-\infty}^{0}e^{-G_i\tau}DdW_i(\tau)\right\|^2 - E\left\|\int_{-\infty}^{0}e^{-G_i\tau}DdW_i(\tau)\right\|^2\right)=0,\ \forall\omega\in B. \tag{C.11}$$

From (C.10) it follows that $\sup_{N\ge1}(1/N)\sum_{i=1}^N E\|\int_{-\infty}^0 e^{-G_i\tau}\ DdW_i(\tau)\|^2 \le \sup_{\theta\in\mathcal{M}}E\|\phi_\theta(0)\|^2 \triangleq M_\phi < \infty$. By (C.11) we know that for any $\omega\in B$, there is $N_1(\omega)>0$ such that for all $N\ge N_1(\omega)$

$$\frac{1}{N}\sum_{i=1}^{N}\left\|\int_{-\infty}^{0}e^{-G_i\tau}DdW_i(\tau)\right\|^2 \le M_\phi+1. \tag{C.12}$$

For any $\epsilon>0$, by Remark 8 (ii) there is $T_1>0$ such that

$$\|e^{G_it}\|^2 \le \sup_{\theta\in M}\|e^{G(\theta)t}\|^2 \le \frac{\epsilon}{M_\phi+1},\quad \forall i\ge1,\ \forall t\ge T_1.$$

This together with (C.12) implies that $\limsup_{T\to\infty}(1/T)\int_{T_1}^{T}\|(1/N)\sum_{i=1}^N z_i^{(1)}(t)\|^2dt \le \limsup_{T\to\infty}\frac{1}{T}\int_{T_1}^{T}[\epsilon/(M_\phi+1)](1/N)\sum_{i=1}^N\|\int_{-\infty}^0 e^{-G_i\tau}DdW_i(\tau)\|^2dt\le\epsilon, \forall N\ge N_1(\omega)$.

Therefore, by $\limsup_{T\to\infty}(1/T)\int_0^{T_1}\|(1/N)\sum_{i=1}^N z_i^{(1)}(t)\|^2dt = 0, P(B)=1$ and the arbitrariness of ϵ we have (C.7).

Recall that $(1/N)\sum_{i=1}^N z_i^{(2)}(t)$ is a wide stationary Gaussian process with zero-mean and continuous in mean square. Then, by some direct calculation we know that $\int_0^\infty\|R(s)\|ds<\infty$, where

$$R(s)=E\left(\frac{1}{N}\sum_{i=1}^{N}z_i^{(2)}(t)\frac{1}{N}\sum_{i=1}^{N}z_i^{(2)^T}(t+s)\right).$$

Thus, it follows from Lemma A.1 that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\|\frac{1}{N}\sum_{i=1}^N z_i^{(2)}(t)\right\|^2dt = \frac{1}{N^2}\sum_{i=1}^N E\left\|z_i^{(2)}(0)\right\|^2 \le \frac{\|D\|^2\kappa^2}{2\rho N} = O\left(\frac{1}{N}\right)\ a.s.$$

Hence, (C.8) is true.

From Assumption A4) and Theorem 2.8 in [27] we have $\lim_{N\to\infty}(1/N)\sum_{i=1}^N(\|y_i^0-Ey_i(0)\|^2 - E\|y_i^0-Ey_i(0)\|^2)=0$, a.s. Similar to the proof of (C.7) one can get (C.9). Thus, the lemma is true. □

Appendix D
Proofs of Lemmas 6.1–6.4

Proofs of Lemmas 6.1–6.2: The proof of Lemma 6.2 is very similar to that of Lemma 6.1. So, here we only show Lemma 6.1.

Let $C_0 = \sup_{N\ge1}\max_{1\le i\le N}\limsup_{T\to\infty}(1/T)\int_0^T(\|y_i^0(t)\|^2+\|u_i^0(t)\|)^2dt$. Then, by (35) we have

$$\begin{aligned}J_i^N\left(u_i^0,u_{-i}^0\right) &\le \limsup_{T\to\infty}\frac{1}{T}\int_0^T\left(\|Q\|\|y_i^0 - \gamma\bar{y}_N^0\|^2+\|R\|\|u_i^0\|^2\right)dt\\ &\le (2\|Q\|+\|R\|)\limsup_{T\to\infty}\frac{1}{T}\times\int_0^T\left(\|y_i^0\|^2+\|u_i^0\|^2\right)dt\\ &\quad + 2\|\gamma\|\|Q\|\limsup_{T\to\infty}\frac{1}{T}\int_0^T\|\bar{y}_N^0\|^2dt\\ &\le (2\|Q\|+\|R\|+2\|\gamma\|\|Q\|)C_0\triangleq C_1.\end{aligned}$$

From Theorem 4.1 we know that C_1 is independent of N, i. Thus, (38) holds, that is, Lemma 6.1 holds. □

Proof of Lemma 6.3: From (35) and Theorem 4.1 we have

$$\begin{aligned}J_i^N\left(u_i^0,u_{-i}^0\right) &\le J_i\left(u_i^0,\gamma y^*\right)+\limsup_{T\to\infty}\frac{1}{T}\times\int_0^T\left(\gamma y^*-\gamma\bar{y}_N^0\right)^TQ\left(\gamma y^*-\gamma\bar{y}_N^0\right)dt\\ &\quad+\limsup_{T\to\infty}\frac{1}{T}\int_0^T 2\left(\gamma y^*-\gamma\bar{y}_N^0\right)^TQ\left(y_i^0-\gamma y^*\right)dt\\ &\le J_i\left(u_i^0,\gamma y^*\right)+\|Q\|\|\gamma\|^2\left(\epsilon_1^N\right)^2\\ &\quad+\limsup_{T\to\infty}\frac{2\|Q\|\|\gamma\|}{T}\int_0^T\|y^*-\bar{y}_N^0\|\|y_i^0-\gamma y^*\|dt\\ &\le J_i\left(u_i^0,\gamma y^*\right)+o\left(\epsilon_1^N\right)\\ &\quad+2\|Q\|\|\gamma\|\epsilon_1^N\left(\limsup_{T\to\infty}\frac{1}{T}\int_0^T\|y_i^0-\gamma y^*\|^2dt\right)^{1/2}\\ &= J_i(u_i^0,\gamma y^*)+O(\epsilon_1^N)\ a.s.\end{aligned}$$

Thus, (39) holds, that is, Lemma 6.3 holds. □

Proof of Lemma 6.4: By the notation of Lemma 6.1 and Lemma 6.2, let $\overline{y}_{-i}^0=(1/N)\sum_{j=1,j\ne i}^N y_j^0(t)$ and

$$\Omega^0=\left\{\omega:\sup_{N\ge1}\max_{1\le i\le N}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\|\gamma\overline{y}_{-i}^0\|_Q^2dt \le C_2, \sup_{N\ge1}\max_{1\le i\le N}J_i^N(u_i^0,u_{-i}^0)\le C_1\right\}.$$

Then, by Lemma 6.1–6.2 we have $P(\Omega^0)=1$. For any $\omega\in\Omega^0$, since $u_i^0\in\mathcal{U}_{l,i}\subseteq\mathcal{U}_{g,i}, \inf_{u_i\in\mathcal{U}_{g,i}}J_i^N(u_i,u_{-i}^0)\le$

$J_i^N(u_i^0, u_{-i}^0)$. Thus, we need only to consider such $u_i \in \mathcal{U}_{g,i}$ that satisfies the following condition:

$$J_i^N(u_i, u_{-i}^0) \le J_i^N(u_i^0, u_{-i}^0). \quad \text{(D.1)}$$

From Lemma 6.1 and (37), (D.1) it follows that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \|(1-\gamma N^{-1})y_i - \gamma \overline{y}_{-i}^0\|_Q^2 dt = J_i^N(u_i, u_{-i}^0) \le C_1. \quad \text{(D.2)}$$

Noticing $\|(1-\gamma N^{-1})y_i|_{u_i}\|_Q^2 \le 2\|(1-\gamma N^{-1})y_i|_{u_i} - \gamma\overline{y}_{-i}^0\|_Q^2 + 2\|\gamma\overline{y}_{-i}^0\|_Q^2$, by (D.2) and Lemma 6.2 we get

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\int_0^T \left\|(1-\gamma N^{-1})y_i|_{u_i}\right\|_Q^2 dt\\ &\le 2\limsup_{T\to\infty}\frac{1}{T}\int_0^T \left\|(1-\gamma N^{-1})y_i|_{u_i} - \gamma\overline{y}_{-i}^0\right\|_Q^2 dt\\ &\quad + 2\limsup_{T\to\infty}\frac{1}{T}\int_0^T \left\|\gamma\overline{y}_{-i}^0\right\|_Q^2 dt = 2(C_1+C_2). \quad \text{(D.3)}\end{aligned}$$

Noticing $\limsup_{T\to\infty}(1/T)\int_0^T \|(1-\gamma N^{-1})y_i|_{u_i}\|_Q^2 dt \ge (1-2\gamma N^{-1})\limsup_{T\to\infty}\frac{1}{T}\int_0^T \|y_i|_{u_i}\|_Q^2 dt$, letting $N\to\infty$, by (D.3) we have

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T \|y_i|_{u_i}\|_Q^2 dt \le C_3 \triangleq 2(C_1+C_2)/(1-2\gamma N^{-1}). \quad \text{(D.4)}$$

From (36), (37) it is easy to see that

$$\begin{aligned}J_i^N(u_i,u_{-i}^0) &= \limsup_{T\to\infty}\frac{1}{T}\int_0^T (\|y_i|_{u_i} - \gamma y^*\\ &\quad + \gamma y^* - \gamma\overline{y}_{-i}^0\|_Q^2 + \|u_i\|_R^2)dt\\ &\ge \liminf_{T\to\infty}\frac{2\gamma}{T}\int_0^T \left(y^* - \overline{y}_N^0\right.\\ &\quad \left.+\overline{y}_N^0 - \overline{y}_{-i}^0\right)^T Q(y_i|_{u_i} - \gamma y^*)dt\\ &\quad + \limsup_{T\to\infty}\frac{1}{T}\int_0^T (\|y_i|_{u_i} - \gamma y^*\|_Q^2 + \|u_i\|_R^2)dt\\ &\ge J_i\left(u_i^0, \gamma y^*\right) + I_1^N + I_2^N \quad \text{(D.5)}\end{aligned}$$

where

$$\begin{aligned}I_1^N &= \liminf_{T\to\infty}\frac{2\gamma}{T}\int_0^T \left(y^* - \overline{y}_N^0\right)^T Q(y_i|_{u_i} - \gamma y^*)dt\\ I_2^N &= \liminf_{T\to\infty}\frac{2\gamma}{T}\int_0^T 1(1/N)(y_i^0 - y_i|_{u_i})^T Q(y_i|_{u_i} - \gamma y^*)dt.\end{aligned}$$

By the Schwarz inequality, (41) and (D.4) we have

$$\begin{aligned}|I_1^N| &\le 2\gamma\sqrt{\|Q\|}\epsilon_1^N \left(\limsup_{T\to\infty}\frac{2}{T}\right.\\ &\quad \left.\times\int_0^T \left(\|y_i|_{u_i}\|_Q^2 + \|\gamma y^*\|_Q^2\right)dt\right)^{1/2}\\ &\le 2\|\gamma\|\sqrt{2\|Q\|}\epsilon_1^N \left(C_3\right.\\ &\quad \left.+\|\gamma\|^2\|Q\|M_{y_*}^2\right)^{1/2} = O(\epsilon_1^N) \quad \text{(D.6)}\end{aligned}$$

$$\begin{aligned}|I_2^N| &\le \frac{2}{N}\left(\limsup_{T\to\infty}\frac{1}{T}\int_0^T \|\gamma y_i^0 - \gamma y_i|_{u_i}\|_Q^2 dt\right)^{1/2}\\ &\quad\times\left(\limsup_{T\to\infty}\frac{1}{T}\int_0^T \|y_i|_{u_i} - \gamma y^*\|_Q^2 dt\right)^{1/2}\\ &\le \frac{4\sqrt{2}}{N}\left(C_3 + \|\gamma\|^2\|Q\|M_{y_*}^2\right)^{1/2}\\ &\quad\times\left(\|Q\|\|\gamma\|^2\limsup_{T\to\infty}\frac{1}{T}\int_0^T \|y_i^0\|^2 dt\right.\\ &\quad\left.+\limsup_{T\to\infty}\frac{1}{T}\int_0^T \|\gamma y_i|_{u_i}\|_Q^2 dt\right)^{1/2}\\ &\le \frac{4\sqrt{2}}{N}(C_3 + \|\gamma\|^2\|Q\|M_{y_*}^2)^{1/2}(\|Q\|\|\gamma\|^2 C_0\\ &\quad + \gamma^2 C_3)^{1/2} = O(1/N). \quad \text{(D.7)}\end{aligned}$$

Thus, from (D.5), (D.6), (D.7) and $P(\Omega^0) = 1$ we get (40). Thus, Lemma 6.4 holds. □

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for their detailed and insightful comments.

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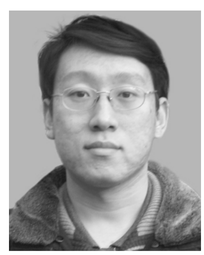

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