Consensus Control of General Linear Multiagent Systems With Antagonistic Interactions and Communication Noises

Jiangping Hu, *Senior Member, IEEE*, Yanzhi Wu, Tao Li, *Senior Member, IEEE*, and Bijoy K. Ghosh, *Fellow, IEEE*

***Abstract*—This technical note considers a consensus problem of high-order multiagent systems with antagonistic interactions and communication noises. The interaction network associated with the multiagent system is modeled by a signed graph (called coopetition network) and the agent dynamics is described by a general linear system. A novel stochastic-approximation based control strategy is designed for each agent by using the relative state information from its neighbors. Additionally, convergence of a consensus error system is analyzed by the stochastic stability theory. Finally, the effectiveness of our results is demonstrated by simulation.**

***Index Terms*—Communication noise, coopetition network, linear multiagent systems, mean square bipartite consensus, stochastic-approximation gain.**

I. Introduction

Recently, distributed consensus problems of multiagent systems have attracted much attention in the control community due to their widespread applications in systems such as unmanned air vehicles, satellite formation flying, and distributed reconfigurable sensor networks, etc. In most cases, communication between agents is degraded due to thermal, fading channel, and quantization noises during encoding and decoding. The effect of these noises should, therefore, be carefully considered while investigating realistic problems with multiagent systems.

In this technical note, we consider a multiagent system with a general linear system as follows:

$$\dot{x}_i = Ax_i + Bu_i, \quad i = 1, \dots, N \tag{1}$$

where $x_i = \mathrm{col}(x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n$ represents the state of agent i, and $u_i \in \mathbb{R}^m$ is the control input. It is assumed that (A, B) is a stabilizable pair. For agent $i \in \mathcal{V}$, a relative state measurement from its neighbor j is defined by

$$z_{ij}^{[1]} = |\mathcal{A}_{ij}|\,(x_i - \mathrm{sign}(\mathcal{A}_{ij})(x_j + \rho_{ij} n_{ij}))$$

where $\{n_{ij}(t) \in \mathbb{R}^n, i, j = 1, 2, \dots, N\}$ are independent standard white noises, and $\{\rho_{ij} \in \mathbb{R}^{n\times n}, i, j = 1, 2, \dots, N\}$ representing the noisy intensities are positive definite diagonal matrices. It is assumed that $\rho_{ij} = \rho_{ji}$.

Our objective is to design a distributed control for the system (1) by using the relative state measurements to guarantee the mean square bipartite consensus, i.e.,

$$\lim_{t\to\infty} E\left\{\left\|x_i(t) - \frac{1}{N}\sum_{j=1}^{N} s_i s_j x_j(t)\right\|^2\right\} = 0 \tag{2}$$

where $E\{\cdot\}$ denotes the mathematical expectation, and s_i equals 1 or -1. It is noted that (2) is equivalent to the following two equations:

$$\lim_{t\to\infty} E\left\{\left\|x_i(t) - \frac{1}{N}\sum_{j=1}^{N} s_j x_j(t)\right\|^2\right\} = 0 \ \text{ for } s_i = 1$$

$$\lim_{t\to\infty} E\left\{\left\|x_i(t) + \frac{1}{N}\sum_{j=1}^{N} s_j x_j(t)\right\|^2\right\} = 0 \ \text{ for } s_i = -1.$$

To the best of our knowledge, very few results have been obtained for consensus control of high-order multiagent systems with communication noises due to great challenges in the controller design and the convergence analysis.

The main contributions of this note are listed as follows.

1) This note proposes a distributed control strategy that is not dependent on the structure of the state matrix of the agent dynamics. Moreover, the strategy does not need the absolute state information of the agents own and only uses the relative state information in the neighborhood.
2) A novel stochastic-approximation type protocol is developed to attenuate communication noises. Through the protocol, the primary term in the closed-loop system is noise free, which is beneficial for convergence analysis.
3) This note investigates a bipartite consensus control of cooperative-competitive multiagent systems with communication noises, which is analyzed from a new perspective.

The remainder of this note is organized as follows. Some helpful preliminary results are presented in Section II. A mean square bipartite consensus problem is solved in Section III with a dynamic output-feedback control strategy. Simulation results are presented in Section IV, and concluding remarks are given in Section V.

Notations: Throughout this note, the following notations are used. $I_n \in \mathbb{R}^{n\times n}$ represents the identity matrix. $\mathbf{1}$ denotes a column vector with elements all being 1. $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues, respectively. $P > 0$ $(P \geq 0)$ denotes a

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J. Hu and Y. Wu are with the School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu 611731, China (e-mail: hujp@uestc.edu.cn; 291846071@qq.com).

T. Li is with the Key Laboratory of Pure Mathematics and Mathematical Practice, School of Mathematical Sciences, East China Normal University, Shanghai 200241, China (e-mail: tli@math.ecnu.edu.cn).

B. K. Ghosh is with the School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu 611731, China, and the Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409-1042, USA (e-mail: bijoy.ghosh@ttu.edu).

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positive definite (positive semidefinite) matrix. χ_s denotes the indicator function. $\mathrm{col}(\cdot)$ denotes a concatenation. $\mathrm{diag}(\cdot)$ represents a diagonal matrix. $\mathrm{sign}(\cdot)$ denotes a sign function.

II. Preliminary Results

A. Motivations

Distributed control of multiagent systems with communication noises was firstly concerned in [1]–[3], where the agent dynamics is first order. A stochastic delayed consensus problem was considered in [4]. Furthermore, necessary and sufficient conditions were provided in [5] for an average consensus problem of continuous-time second-order multiagent systems with communication noises and fixed topology. Although some distributed control strategies have been developed for first-order or second-order multiagent systems with communication noises in the past few years, the results are scarce in the case when the agent dynamics is high order. A mean square consensus problem was studied in [6] for continuous-time high-order multiagent systems with communication noises and fixed topology. Furthermore, a consensus problem was investigated for discrete-time single-input multiagent systems in [7] and for multi-input multi-output continuous-time multiagent systems in [8]. However, the control strategies proposed in [6]–[8] need the absolute state information of the agents own, which motivates us to further consider a distributed control design with only relative state information in the neighborhood.

Another motivation is to consider high-order multiagent systems with cooperative-competitive interactions. It is well known that cooperation and competition coexist in many complex systems where two teams compete, for example, activators inhibitors in biological systems, markets and social networks with two competing cartels, and competing robots. As a specific social dynamics, opinion formation was investigated for trust-mistrust networks through DeGroot-type or Laplacian-type dynamics in [9] and [10]. Generally, consensus, polarization, and fragmentation can emerge on coopetition networks even though they have connected topologies [11]. Particularly, polarization or bipartite consensus, where all agents reach a final state with identical magnitude but with opposite sign, was investigated extensively in the control community [12], [13], for coopetition networks with first-order agent dynamics. An interval bipartite consensus problem was solved in [14] for first-order multiagent systems. Some adaptive control strategies were proposed for bipartite consensus of reaction-diffusion neural networks in [15] and high-order multiagent systems in [16] and [17]. Cooperative and bipartite output synchronizations were investigated by using the absolute state information in [18] for heterogenous linear multiagent systems. However, till now, there is no result investing high-order coopetitive multiagent systems with communication noises.

B. Algebraic Graph Theory

It is convenient to use a signed graph to model the coopetition network. An undirected signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is introduced, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of vertices representing agents, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of edges, and $\mathcal{A} = [\mathcal{A}_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix. When the interaction weight $\mathcal{A}_{ij} \neq 0$, it means that there is a connection between agents i and j. Specifically, if $\mathcal{A}_{ij} > 0$, the interaction between agents i and j is cooperative, and competitive if $\mathcal{A}_{ij} < 0$. It is assumed that the signed graph $\mathcal{G}$ is simple, i.e., $\mathcal{A}_{ii} = 0$ for $i \in \mathcal{V}$. The undirected signed graph $\mathcal{G}$ is structurally balanced, if the set of vertices $\mathcal{V}$ can be divided into two disjoint subsets $\mathcal{V}_1 = \{1, 2, \ldots, N_0\}$ and $\mathcal{V}_2 = \{N_0 + 1, \ldots, N\}$, which satisfy $\mathcal{V} = \mathcal{V}_1 \bigcup \mathcal{V}_2, \mathcal{V}_1 \bigcap \mathcal{V}_2 = \emptyset$, $\mathcal{A}_{ij} \geq 0, \forall i, j \in \mathcal{V}_p$ $(p \in 1, 2)$, $\mathcal{A}_{ij} \leq 0, \forall i \in \mathcal{V}_p, j \in \mathcal{V}_q, p \neq q$ $(p, q \in \{1, 2\})$.

For a structurally balanced graph, a gauge transformation is defined by $S = \mathrm{diag}(s_1, s_2, \ldots, s_N) \in \mathbb{R}^{N\times N}$, where $s_i = 1$ when vertex $i \in \mathcal{V}_1$ and $s_i = -1$ when vertex $i \in \mathcal{V}_2$. Obviously, the gauge transformation is an orthogonal matrix. Additionally, the adjacency matrix $\mathcal{A}$ is similar to a nonnegative matrix through the gauge transformation.

A signed Laplacian matrix is defined by

$$L = C_r - \mathcal{A}$$

where C_r is a diagonal matrix with the diagonal elements being $c_{r,ii} = \sum_{j=1}^{N} |\mathcal{A}_{ij}|$ for $i = 1, 2, \ldots, N$. The signed Laplacian matrix has an important spectral property as follows.

Lemma 1 (see [11] and [12]): Let L be the signed Laplacian matrix of the undirected signed graph $\mathcal{G}$. If $\mathcal{G}$ is structurally balanced and connected, then the following statements are true.

1) L has a simple zero eigenvalue with $s = \mathrm{col}(s_1, s_2, \ldots, s_N) \in \mathbb{R}^N$ as its zero eigenvector, and all the nonzero eigenvalues are strictly positive.
2) The smallest nonzero eigenvalue $\lambda_2(L)$ of the matrix L satisfies

$$\lambda_2(L) = \min_{x^T \mathbf{s}=0, x\neq 0} \frac{x^T L x}{x^T x}.$$

III. Mean Square Average Bipartite Consensus

In this note, a dynamic output-feedback controller is designed for agent i as

$$\begin{cases} u_i = (a(t) + c)K \sum_{j\in N_i} \beta_{ij} \rho_{ij}^{-1} z_{ij}^{[1]} - cK \sum_{j\in N_i} \beta_{ij} \varrho_{ij}^{-1} z_{ij}^{[2]} \\ \dot{\theta}_i = (A + BK_1)\theta_i - r_1 a(t) \sum_{j\in N_i} z_{ij}^{[2]} - r_2 a(t) \sum_{j\in N_i} z_{ij}^{[1]} \end{cases} \tag{3}$$

where θ_i is the internal state of the controller, $z_{ij}^{[2]} = |\mathcal{A}_{ij}|(\theta_i - \mathrm{sign}(\mathcal{A}_{ij})(\theta_j + \varrho_{ij} n_{ij}))$ is the relative internal state, $\beta_{ij} = \beta_{ji} \in \mathbb{R}^{n\times n}$ is a positive definite diagonal matrix satisfying $\beta_{ij}\rho_{ij}^{-1} = \beta_\rho I_n$, and β_ρ is a positive constant. c, r_1, and r_2 are positive constants. $K \in \mathbb{R}^{m\times n}$ and $K_1 \in \mathbb{R}^{m\times n}$ are feedback gain matrices. The time-varying gain $a(t) > 0$ is a uniformly continuous function.

Remark 1: The controller (3) uses only the relative information of agent i from its neighbors, which is different from the existing control strategies in [6]–[8]. Additionally, the appropriate gains $a(t)$ and β_{ij} can be selected such that the term, which plays an essential role for consensus, in the controller (3) is $cK \sum_{j\in N_i} \beta_{ij}\rho_{ij}^{-1} |\mathcal{A}_{ij}|(x_i - \mathrm{sign}(\mathcal{A}_{ij})x_j) - cK \sum_{j\in N_i} \beta_{ij}\varrho_{ij}^{-1} |\mathcal{A}_{ij}|(\theta_i - \mathrm{sign}(\mathcal{A}_{ij})\theta_j)$.

Remark 2: If the time-varying gain $a(t)$ satisfies $\int_0^\infty a^2(s)ds < \infty$, from Barbalat's Lemma, it is not difficult to show that $a(t)$ is bounded in $[0, +\infty)$, i.e., $|a(t)| < \bar{a}$ for a positive constant $\bar{a}$.

Some assumptions are listed as follows.

(A_1) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is structurally balanced.
(A_2) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is connected.
(A_3) $\int_0^\infty a(s)ds = \infty$.
(A_4) $\int_0^\infty a^2(s)ds < \infty$.

Applying the controller (3) to the multiagent system (1) results in the following closed-loop system:

$$\begin{cases} \dot{x} = (I_N \otimes A)x + c\beta_\rho(L \otimes (BK))x - c\left[(I_N \otimes (BK))\Pi\right]\theta \\ \qquad + a(t)\beta_\rho(L \otimes (BK))x - a(t)\left[(I_N \otimes (BK))D\right]\eta \\ \dot{\theta} = (I_N \otimes (A + BK_1))\theta - r_1 a(t)(L \otimes I_n)\theta + r_1 a(t) D_1 \eta \\ \qquad - r_2 a(t)(L \otimes I_n)x + r_2 a(t) D_2 \eta \end{cases} \tag{4}$$

where $x = \mathrm{col}(x_1, x_2, \ldots, x_N) \in \mathbb{R}^{Nn}$, $\theta = \mathrm{col}(\theta_1, \theta_2, \ldots, \theta_N) \in \mathbb{R}^{Nn}$, $D = \mathrm{diag}(d_1, d_2, \ldots, d_N) \in \mathbb{R}^{Nn\times N^2 n}$, $d_i = (\mathcal{A}_{i1}\beta_{i1}, \mathcal{A}_{i2}\beta_{i2},$

$\ldots, \mathcal{A}_{iN}\beta_{iN}) \in \mathbb{R}^{n\times Nn}$, $\Pi = (\Pi_{ij})_{N\times N} \in \mathbb{R}^{Nn\times Nn}$, $\Pi_{ii} = \sum_{k=1}^{N} |\mathcal{A}_{ik}|\beta_{ik}\varrho_{ik}^{-1}$, $\Pi_{ij} = -\mathcal{A}_{ij}\beta_{ij}\varrho_{ij}^{-1}$ for $i \neq j$, $D_1 = \mathrm{diag}(\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_N) \in \mathbb{R}^{Nn\times N^2 n}$, $\hat{d}_i = (\mathcal{A}_{i1}\varrho_{i1}, \mathcal{A}_{i2}\varrho_{i2}, \ldots, \mathcal{A}_{iN}\varrho_{iN}) \in \mathbb{R}^{n\times Nn}$, $D_2 = \mathrm{diag}(\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_N) \in \mathbb{R}^{Nn\times N^2 n}$, $\bar{d}_i = (\mathcal{A}_{i1}\rho_{i1}, \mathcal{A}_{i2}\rho_{i2}, \ldots, \mathcal{A}_{iN}\rho_{iN}) \in \mathbb{R}^{n\times Nn}$, $\eta = \mathrm{col}(n_1, n_2, \ldots, n_N) \in \mathbb{R}^{N^2 n}$, and $n_i = \mathrm{col}(n_{i1}, n_{i2}, \ldots, n_{iN}) \in \mathbb{R}^{Nn}$.

Define $\bar{s} = s \otimes I_n \in \mathbb{R}^{Nn\times n}$. Then, we have the following lemma for the matrix Π.

Lemma 2: If $\mathcal{G}$ is structurally balanced and connected, Π has a simple zero eigenvalue satisfying $\Pi\bar{s} = 0$, and all the nonzero eigenvalues are strictly positive.

Proof: From the definition, one has

$$\Pi = \begin{pmatrix} \sum_{k=1}^{N} |\mathcal{A}_{1k}|\beta_{1k}\varrho_{1k}^{-1} & -\mathcal{A}_{12}\beta_{12}\varrho_{12}^{-1} & \cdots & -\mathcal{A}_{1N}\beta_{1N}\varrho_{1N}^{-1} \\ -\mathcal{A}_{21}\beta_{21}\varrho_{21}^{-1} & \sum_{k=1}^{N} |\mathcal{A}_{2k}|\beta_{2k}\varrho_{2k}^{-1} & \cdots & -\mathcal{A}_{2N}\beta_{2N}\varrho_{2N}^{-1} \\ \vdots & \vdots & \cdots & \vdots \\ -\mathcal{A}_{N1}\beta_{N1}\varrho_{N1}^{-1} & -\mathcal{A}_{N2}\beta_{N2}\varrho_{N2}^{-1} & \cdots & \sum_{k=1}^{N} |\mathcal{A}_{Nk}|\beta_{Nk}\varrho_{Nk}^{-1} \end{pmatrix}.$$

It is not difficult to see that $\Pi\bar{s} = 0$. Additionally, Π is similar to a Laplacian matrix associated with an unsigned undirected graph through an orthogonal transformation $S \otimes I_n$. Thus, the conclusion follows with a similar proof of Lemma 1. ∎

An average bipartite consensus error is defined for agent i by $\xi_{1i} = x_i - \frac{1}{N}\sum_{j=1}^{N} s_i s_j x_j \in \mathbb{R}^n$, or, in a compact form, $\xi_1 = \mathrm{col}(\xi_{11}, \xi_{12}, \ldots, \xi_{1N}) = ((I_N - \frac{1}{N}ss^T) \otimes I_n)x$. Let $M = I_N - \frac{1}{N}ss^T$ and $M_\xi = \mathrm{diag}(M \otimes I_n, M \otimes I_n)$. It is noted that the matrix M satisfies $ML = LM = L$. Since $(s^T \otimes I_n)\Pi = \Pi(s \otimes I_n) = 0$, one has

$$(M \otimes I_n)(I_N \otimes (BK))\Pi = (I_N \otimes (BK))\Pi(M \otimes I_n). \quad (5)$$

Let $x_\theta = \mathrm{col}(x, \theta) \in \mathbb{R}^{2Nn}$. A variable of change is given by $\xi = \mathrm{col}(\xi_1, \xi_2) = M_\xi x_\theta$. From (5), the system (4) is transformed to the following Itô stochastic differential equation:

$$d\xi = A_c\xi dt + a(t)B_c\xi dt + a(t)D_c dW(t) \quad (6)$$

where

$$A_c = \begin{pmatrix} I_N \otimes A + c\beta_\rho L \otimes (BK) & -c(I_N \otimes (BK))\Pi \\ 0 & I_N \otimes (A + BK_1) \end{pmatrix}$$

$$B_c = \begin{pmatrix} \beta_\rho L \otimes (BK) & 0 \\ -r_2 L \otimes I_n & -r_1 L \otimes I_n \end{pmatrix}$$

and $D_c = \mathrm{col}(-(M \otimes BK)D, r_1(M \otimes I_n)D_1 + r_2(M \otimes I_n)D_2)$. $W(t) = \mathrm{col}(w_1, w_2, \ldots, w_N)$ is an $N^2 n$ dimensional standard Brownian motion, $w_i = \mathrm{col}(w_{i1}, w_{i2}, \ldots, w_{iN})$, and $\int_0^t n_{ij} ds = w_{ij}$.

Now a main result is given to provide a sufficient condition for the mean square average bipartite consensus.

Theorem 1: Consider the multiagent system (1). Suppose that Assumptions $(A_1) - (A_4)$ hold. Let $P > 0$ and $P_1 > 0$ be the solutions of the algebraic Riccati equation (ARE) $f(\mathcal{P}) = 0$ with $\kappa = k$ and $\kappa = k_1$, respectively, where

$$f(\mathcal{P}) = A^T\mathcal{P} + \mathcal{P}A - \mathcal{P}BB^T\mathcal{P} + \kappa I_n = 0. \quad (7)$$

Then, the mean square average bipartite consensus can be achieved for all the agents under the distributed controller (3) with the feedback gain matrices $K = -B^T P$ and $K_1 = -\frac{1}{2}B^T P_1$.

Proof: Consider a Lyapunov function candidate

$$V = \xi^T\bar{P}\xi \quad (8)$$

where $\bar{P} = \begin{pmatrix} L\otimes P & 0 \\ 0 & L\otimes P_1 \end{pmatrix}$. According to Lemma 1, one has

$$\begin{aligned} \xi^T\bar{P}\xi &= \xi_1^T L \otimes P\xi_1 + \xi_2^T L \otimes P_1\xi_2 \\ &\geq \lambda_2(L)\lambda_{\min}(P)\xi_1^T\xi_1 + \lambda_2(L)\lambda_{\min}(P_1)\xi_2^T\xi_2 \\ &\geq c_2\xi^T\xi > 0 \quad \forall\xi \neq 0 \end{aligned} \quad (9)$$

where $c_2 = \min(\lambda_2(L)\lambda_{\min}(P), \lambda_2(L)\lambda_{\min}(P_1))$. Therefore, the Lyapunov function candidate V satisfies

$$c_2\xi^T\xi \leq V \leq \lambda_{\max}(\bar{P})\xi^T\xi. \quad (10)$$

The derivative of V along the trajectory determined by (6) is given by

$$\begin{aligned} dV = {} & \xi^T\left[\bar{P}A_c + A_c^T\bar{P} + a(t)(\bar{P}B_c + B_c^T\bar{P})\right]\xi dt \\ & + a^2(t)tr(D_c^T\bar{P}D_c)dt + 2a(t)\xi^T\bar{P}D_c dW(t). \end{aligned} \quad (11)$$

When the gain matrices are selected, respectively, as $K = -B^T P$ and $K_1 = -\frac{1}{2}B^T P_1$, one has

$$\begin{aligned} & \xi^T\left[\bar{P}A_c + A_c^T\bar{P} + a(t)\left(\bar{P}B_c + B_c^T\bar{P}\right)\right]\xi \\ & \leq \xi_1^T\left[L \otimes (PA + A^T P) - 2c\beta_\rho\left(L^2 \otimes (PBB^T P)\right)\right]\xi_1 \\ & \quad + 2\xi_1^T\left[c\left(L \otimes (PBB^T P)\right)\Pi - r_2 a(t)L^2 \otimes P_1\right]\xi_2 \\ & \quad + \xi_2^T\left[L \otimes \left(A^T P_1 + P_1 A - P_1 BB^T P_1\right)\right]\xi_2. \end{aligned} \quad (12)$$

For the signed Laplacian matrix L, there exists a unitary matrix $U = (\frac{s}{\sqrt{N}}\ U_1)$ such that $U^T LU = \Lambda = \mathrm{diag}\big(\lambda_1(L), \lambda_2(L), \ldots, \lambda_N(L)\big)$, where $\lambda_i(L)$ are the eigenvalues of L. Define the variable of changes by $\hat{\xi}_1 = \mathrm{col}(\hat{\xi}_{11}, \hat{\xi}_{12}, \ldots, \hat{\xi}_{1N}) = (U^T \otimes I_n)\xi_1$ and $\hat{\xi}_2 = \mathrm{col}(\hat{\xi}_{21}, \hat{\xi}_{22}, \ldots, \hat{\xi}_{2N}) = (U^T \otimes I_n)\xi_2$. By properly choosing positive constants c, β_ρ such that $c\beta_\rho\lambda_2(L) \geq \frac{1}{2}$ and according to the AREs (7), the inequality (12) can be transformed to

$$\begin{aligned} & \xi^T(\bar{P}A_c + A_c^T\bar{P})\xi + a(t)\xi^T(\bar{P}B_c + B_c^T\bar{P})\xi \\ & \leq \hat{\xi}_1^T\left[\Lambda \otimes \left(PA + A^T P\right) - 2c\beta_\rho\left(\Lambda^2 \otimes (PBB^T P)\right)\right]\hat{\xi}_1 \\ & \quad + 2\xi_1^T\left[c\left(L \otimes (PBB^T P)\right)\Pi - r_2 a(t)L^2 \otimes P_1\right]\xi_2 \\ & \quad + \hat{\xi}_2^T\left[\Lambda \otimes (A^T P_1 + P_1 A - P_1 BB^T P_1)\right]\hat{\xi}_2 \\ & \leq -k\lambda_2(L)\xi_1^T\xi_1 - k_1\lambda_2(L)\xi_2^T\xi_2 \\ & \quad + 2\xi_1^T\left[c\left(L \otimes (PBB^T P)\right)\Pi - r_2 a(t)\left(L^2 \otimes P_1\right)\right]\xi_2 \end{aligned}$$

or

$$\xi^T\left(\bar{P}A_c + A_c^T\bar{P}\right)\xi + a(t)\xi^T\left(\bar{P}B_c + B_c^T\bar{P}\right)\xi \leq -\xi^T Q\xi \quad (13)$$

where $Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$, $Q_{11} = k\lambda_2(L)I_{Nn}$, $Q_{12} = -c(L \otimes (PBB^T P))\Pi + r_2 a(t)(L^2 \otimes P_1)$, $Q_{21} = -c\Pi^T(L \otimes (PBB^T P)) + r_2 a(t)(L^2 \otimes P_1)$, and $Q_{22} = k_1\lambda_2(L)I_{Nn}$. Furthermore, let $\bar{Q}_1 = r_2(L^2 \otimes P_1)$ and $\bar{Q}_2 = c(L \otimes (PBB^T P))\Pi$, then one has $Q_{12} = Q_{21}^T = a(t)\bar{Q}_1 - \bar{Q}_2$. It is not difficult to obtain

$$\begin{aligned} Q_{12}^T Q_{12} &\leq a(t)(\bar{Q}_1 - \bar{Q}_2)^T(\bar{Q}_1 - \bar{Q}_2) + a^2(t)\bar{Q}_1^T\bar{Q}_1 + \bar{Q}_2^T\bar{Q}_2 \\ &\leq \bar{a}(\bar{Q}_1 - \bar{Q}_2)^T(\bar{Q}_1 - \bar{Q}_2) + \bar{a}^2\bar{Q}_1^T\bar{Q}_1 + \bar{Q}_2^T\bar{Q}_2. \end{aligned}$$

By using Remark 2, Lemma 2, Schur complement lemma, and selecting $c, r_2, a(t), k, k_1$ properly such that the following conditions hold:

$$\begin{cases} k\lambda_2(L) > 0 \\ k_1\lambda_2(L)I_{Nn} - \frac{1}{k\lambda_2(L)}\left(\bar{a}(\bar{Q}_1 - \bar{Q}_2)^T(\bar{Q}_1 - \bar{Q}_2)\right. \\ \qquad \left. + \bar{a}^2\bar{Q}_1^T\bar{Q}_1 + \bar{Q}_2^T\bar{Q}_2\right) > 0 \end{cases}$$

it follows that the symmetry matrix Q is positive definite.

From (10), (11), and (13), one has

$$\begin{aligned} dV &= \xi^T\left(\bar{P}A_c + A_c^T\bar{P}\right)\xi dt + a(t)\xi^T\left(\bar{P}B_c + B_c^T\bar{P}\right)\xi dt \\ &\quad + a^2(t)tr\left(D_c^T\bar{P}D_c\right)dt + 2a(t)\xi^T\bar{P}D_c dW \\ &\le -\xi^T Q\xi + a^2(t)tr\left(D_c^T\bar{P}D_c\right)dt + 2a(t)\xi^T\bar{P}D_c dW \\ &\le -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(\bar{P})\bar{a}}a(t)Vdt + a^2(t)tr\left(D_c^T\bar{P}D_c\right)dt \\ &\quad + 2a(t)\xi^T\bar{P}D_c dW(t). \end{aligned} \tag{14}$$

Next, we will show

$$E\left[\int_{t_0}^{t} a(s)\xi^T\bar{P}D_c dW(t)\right] = 0 \tag{15}$$

for $t \ge t_0 \ge 0$. Define a stopping time $\tau_{\bar{m}}^{t_0} \triangleq \inf\{t \in [t_0, T] : V(t) \ge \bar{m}\}$, where $t_0 \ge 0, T \ge t_0, \bar{m} > 0$. Then, from (14), one has

$$\begin{aligned} &E[V(t\wedge\tau_{\bar{m}}^{t_0})\chi_{t\le\tau_{\bar{m}}^{t_0}}] - E[V(t_0)] \\ &\le -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(\bar{P})\bar{a}}\int_{t_0}^{t} a(s)V(s)ds + tr(D_c^T\bar{P}D_c)\int_{t_0}^{t} a^2(s)ds \\ &\le tr(D_c^T\bar{P}D_c)\int_{t_0}^{t} a^2(s)ds \end{aligned}$$

which means that $E[V(t\wedge\tau_{\bar{m}}^{t_0})\chi_{t\le\tau_{\bar{m}}^{t_0}}] \le M_1 < \infty$, where M_1 is a positive constant and is dependent on t_0 and T. It is noted that $\lim_{M_1\to\infty} t\wedge\tau_{\bar{m}}^{t_0} = t$, then one has $\sup_{t_0\le t\le T} E[V(t)] \le M_1$ by Fatou lemma. Thus, $\forall t \in [t_0, T]$, one has

$$E\left[\int_{t_0}^{t} a^2(s)V(s)ds\right] \le \sup_{t_0\le t\le T} E[V(t)]\int_{t_0}^{T} a^2(s)ds < \infty.$$

Since T can be arbitrary, thus, one has

$$E\left[\int_{t_0}^{t} a^2(s)V(s)ds\right] < \infty \quad \forall t \ge t_0.$$

Furthermore, one has

$$E\left[\int_{t_0}^{t} a^2(s)\|\xi^T\bar{P}D_c\|^2 ds\right] \le \|\bar{P}\|\|D_c\|^2 E\left[\int_{t_0}^{t} a^2(s)V(s)ds\right]$$

which leads to (15).

From the analysis presented previously, for any $t \ge 0$ and $h > 0$, one has

$$\begin{aligned} E[V(t+h)] - E[V(t)] &\le -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(\bar{P})\bar{a}}\int_{t}^{t+h} a(s)V(s)ds \\ &\quad + tr(D_c^T\bar{P}D_c)\int_{t}^{t+h} a^2(s)ds \\ &\triangleq V_1(t, t+h). \end{aligned}$$

Thus, one has

$$\begin{aligned} \limsup_{h\to 0^+}\frac{E[V(t+h)] - E[V(t)]}{h} &\le \limsup_{h\to 0^+}\frac{V_1(t,t+h)}{h} \\ &= -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(\bar{P})\bar{a}}a(t)E[V(t)] \\ &\quad + tr(D_c^T\bar{P}D_c)a^2(t). \end{aligned}$$

By using the comparison lemma, one has

$$\begin{aligned} E[V(t)] &\le tr(D_c^T\bar{P}D_c)\int_0^t a^2(s)e^{-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(\bar{P})\bar{a}}\int_s^t a(\tau)d\tau}ds \\ &\quad + E[V(0)]e^{-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(\bar{P})\bar{a}}\int_0^t a(s)ds}. \end{aligned} \tag{16}$$

From Assumption (A_4), that is, $\int_0^\infty a^2(s)ds < \infty$, one has for any given $\epsilon > 0$, there exists $T^* > 0$ such that $\int_{T^*}^\infty a^2(s)ds < \epsilon$. For the right-side term in (16), one has

$$\begin{aligned} 0 &\le tr(D_c^T\bar{P}D_c)\int_0^t a^2(s)e^{-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(\bar{P})\bar{a}}\int_s^t a(\tau)d\tau}ds \\ &= tr(D_c^T\bar{P}D_c)\int_0^{T^*} a^2(s)e^{-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(\bar{P})\bar{a}}\int_s^t a(\tau)d\tau}ds \\ &\quad + tr(D_c^T\bar{P}D_c)\int_{T^*}^t a^2(s)e^{-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(\bar{P})\bar{a}}\int_s^t a(\tau)d\tau}ds \\ &\le tr(D_c^T\bar{P}D_c)e^{-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(\bar{P})\bar{a}}\int_{T^*}^t a(\tau)d\tau}\int_0^\infty a^2(s)ds \\ &\quad + tr(D_c^T\bar{P}D_c)\int_{T^*}^\infty a^2(s)ds \\ &\le o(1) + tr(D_c^T\bar{P}D_c)\epsilon \end{aligned}$$

as $t \to \infty$. By the arbitrariness of ϵ, one has

$$\lim_{t\to\infty} tr(D_c^T\bar{P}D_c)\int_0^t a^2(s)e^{-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(\bar{P})\bar{a}}\int_s^t a(\tau)d\tau}ds = 0.$$

Thus, according to (16), one has $\lim_{t\to\infty} E[V(t)] = 0$, furthermore, $\lim_{t\to\infty} E\|\xi(t)\|^2 = 0$, which implies $\lim_{t\to\infty} E\|\xi_1(t)\|^2 = 0$. Therefore, the mean square average bipartite consensus is achieved for the multiagent system. ∎

Remark 3: In the controller (3), we assume that the stochastic-approximation gain $a(t)$ is uniformly continuous in the interval $[0, \infty)$, which plays a key role in the proof of Theorem 1. Additionally, when the interaction weights a_{ij} are constrained to be nonnegative numbers, the controller (3) can guarantee an average consensus for cooperative multiagent systems, which is a great extension of the existing results.

Remark 4: Structural balance is an important assumption on the coopetition network $\mathcal{G}$, which is a sufficient condition to ensure the bipartite consensus of cooperative-competitive multiagent systems. In fact, the equilibrium set of the multiagent systems is determined by $\{\mathbf{s}\otimes c(t) | c(t) \in \mathbb{R}^n\}$, where

$$c(t) = \frac{1}{N}\left(s^T \otimes e^{At}\right)\begin{pmatrix} x_1(0) \\ \vdots \\ x_N(0) \end{pmatrix} \in \mathbb{R}^n.$$

Thus, it is noted that the linearly independent group depends on the initial states $x(0) = \mathrm{col}(x_1(0), \ldots, x_N(0))$ and the state matrix A.

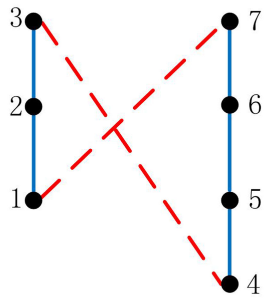
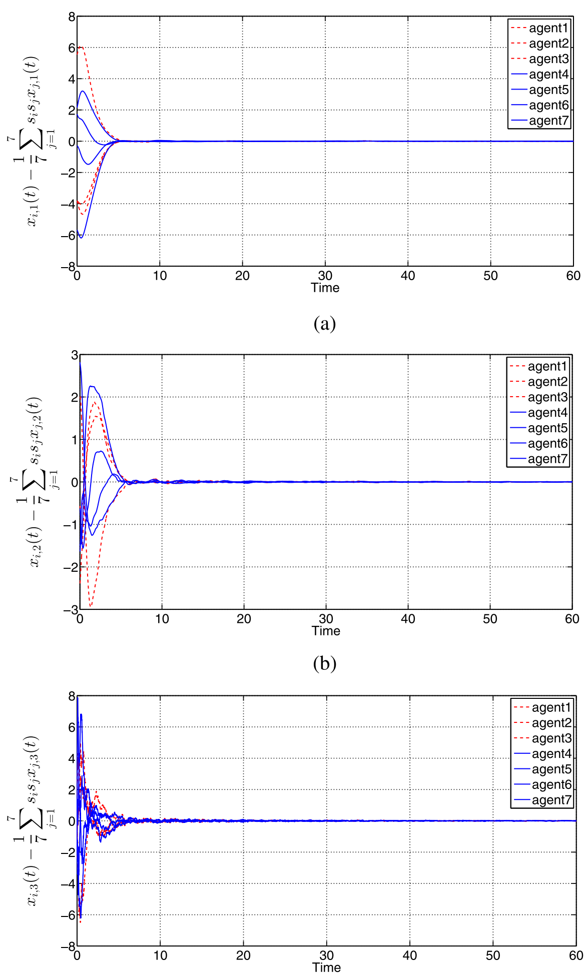


Fig. 1. Coopetition network $\mathcal{G}$.



(a)

(b)

(c)

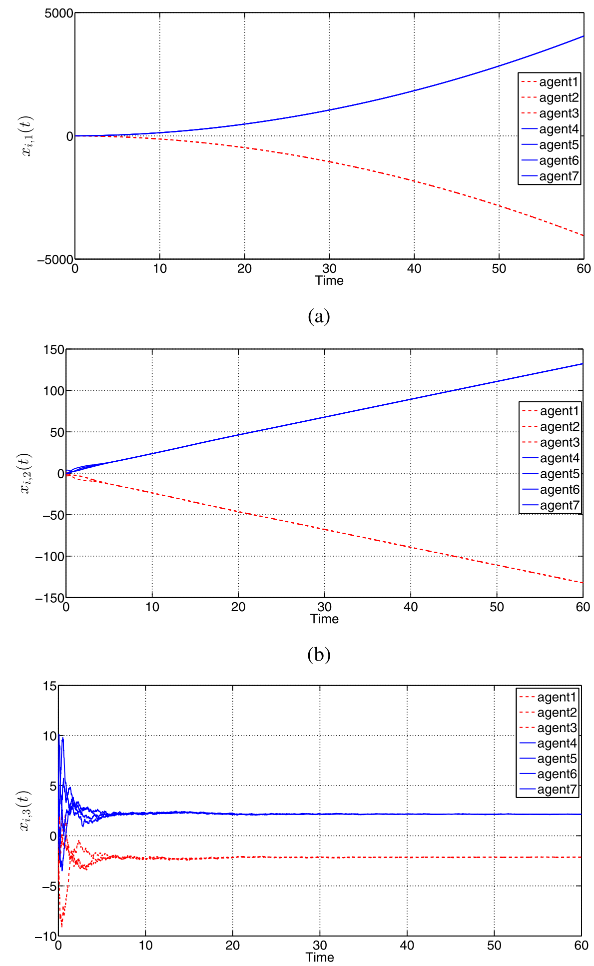
Fig. 2. Evolution of the consensus errors $\xi_{1i}(t) \in \mathbb{R}^3$ $(i = 1, \ldots, 7)$.

IV. Simulation

A simulation example is presented in this section. Suppose that the agent dynamics (1) is described by a third-order linear system and seven agents interact each other on the coopetition network, as illustrated in Fig. 1. The blue solid edges and the red dash edges are used to describe cooperative and competitive relationships, respectively. From the definition of the structural balance, $\mathcal{G}$ is structurally balanced and can be divided into two competitive subnetworks $\mathcal{V}_1$ and $\mathcal{V}_2$, where $\mathcal{V}_1 = \{1, 2, 3\}$ and $\mathcal{V}_2 = \{4, 5, 6, 7\}$. The gauge transformation matrix $S = \mathrm{diag}(1, 1, 1, -1, -1, -1, -1)$. The eigenvalues of the Laplacian matrix L are given by $\lambda_1(L) = 0, \lambda_2(L) = 0.753, \lambda_3(L) = 0.753, \lambda_4(L) = 2.445, \lambda_5(L) = 2.445, \lambda_6(L) = 3.8019$, and $\lambda_7(L) = 3.8019$. The zero eigenvector of the Laplacian matrix L is given by $\mathbf{s} = \mathrm{col}(1, 1, 1, -1, -1, -1, -1)$. Take $c = 4$, $r_1 = 1$, $r_2 = 2$, $k = 10$, and $k_1 = 20$. The two positive definite positive matrices are, respectively, given by

$$P = \begin{pmatrix} 20.0277 & 15.0554 & 3.1623 \\ 15.0554 & 26.9903 & 6.3333 \\ 3.1623 & 6.3333 & 4.7609 \end{pmatrix} \quad \text{and}$$

$$P_1 = \begin{pmatrix} 38.6284 & 27.3039 & 4.4721 \\ 27.3039 & 48.2632 & 8.6376 \\ 4.4721 & 8.6376 & 6.1053 \end{pmatrix}.$$



(a)

(b)

(c)

Fig. 3. Evolution of the agent states $x_i(t) \in \mathbb{R}^3$ $(i = 1, \ldots, 7)$.

The feedback gain matrices are selected as $K = (-3.1623\ \ -6.3333\ \ -4.7609)$ and $K_1 = (-2.2361\ -4.3188\ -3.0527)$. The time-varying gain is designed as $a(t) = \frac{1}{t+20}$. The intensity matrices of the communication noises $\rho_{ij} \in \mathbb{R}^{3\times 3}$ and $\varrho_{ij} \in \mathbb{R}^{3\times 2}$ for $i, j = 1, 2, \ldots, 7$ are arbitrarily positive definite diagonal matrices such that $\beta_\rho = \frac{1}{4}$. Under the proposed controller (3), the average consensus errors all approach to zero in mean square, as shown in Fig. 2. Simultaneously, the seven agents reach an average bipartite consensus, as illustrated in Fig. 3, that is, $E\|x_i(t) - \frac{1}{7}\sum_{j=1}^{7} s_j x_j(t)\|^2 \to 0$ for $i =$

$1, 2, 3$, while $E\|x_i(t) + \frac{1}{7}\sum_{j=1}^{7} s_j x_j(t)\|^2 \to 0$ for $i = 4, 5, 6, 7$, respectively, as time t goes to infinity.

V. Conclusion

In this note, an average bipartite consensus problem has been investigated for general linear multiagent systems with antagonistic interactions and communication noises. A distributed dynamic output-feedback controller together with a stochastic-approximation gain has been designed for each agent to achieve bipartite consensus in mean square. A simulation example has been presented to validate the proposed control strategy. Some future topics are to design optimal adaptive control strategies for linear multiagent systems with unknown time-varying disturbances, communication noises, and optimal control parameters.

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