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Brief paper

Continuous-time and sampled-data-based average consensus with logarithmic quantizers ☆

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ABSTRACT

This paper considers the average consensus problem for multi-agent systems with continuous-time first-order dynamics. Logarithmic quantization is considered in the communication channels, and continuous-time and sampled-data-based protocols are proposed. For the continuous-time protocol, we give an explicit upper bound of the consensus error in terms of the initial states, the quantization density and the parameters of the network graph. It is shown that in contrast with the case with uniform quantization, the consensus error in the logarithmic quantization case is always uniformly bounded, independent of the quantization density, and the β-asymptotic average consensus is ensured under the proposed protocol, i.e. the asymptotic consensus error converges to zero as the sector bound β of the logarithmic quantizer approaches zero. For the sampled-data-based protocol, we give sufficient conditions on the sampling interval to ensure the β-asymptotic average consensus. Numerical examples are given to demonstrate the effectiveness of the protocols.

1. Introduction

Distributed consensus has become a hot research topic in recent years (Cao, Morse, & Anderson, 2008; Jadbabaie, Jie, & Morse, 2003; Liu, Xie, & Lewis, 2011; Olfati-Saber & Murray, 2004). The problem is widely encountered in the real world, for example, in distributed computation, flocking, traffic control, networked control and formation flight.

The average consensus problem means to design a networked interaction protocol such that the states of all the agents converge to the average of their initial states asymptotically or in finite time. The difference between the states of the agents and the averaged initial value is called the consensus error. When transmission channels of the network are precise analog channels, the consensus error exponentially converges to zero under the protocol given in Cao et al. (2008), Olfati-Saber and Murray (2004), etc. However, when we consider that each agent is able to communicate with its neighbors by digital communications, the problem becomes complicated due to quantization which is nonlinear in nature. For integer-valued consensus problems, an average consensus protocol is designed in Kashyap, Basar, and Srikant (2007) based on the gossip algorithm. In Nedic, Olshevsky, Ozdaglar, and Tsitsiklis (2009), the quantization effect on the average consensus is analyzed and an upper bound for the consensus errors is given. When the states of agents are real-valued, distributed consensus of discrete-time agent dynamics based on quantized communication has attracted much attention in the past few years. In early works, the quantization error or quantization uncertainties are always modeled as white noises. Three kinds of update strategies including totally quantized, partially quantized and compensating strategies are considered in Carli, Fagnani, Frasca, and Zampieri (2010) based on both deterministic quantization and probabilistic quantization. Based on the assumption that the quantization errors are white noises, two coding schemes are provided by Yildiz and Scaglione (2008) and conditions under which consensus is achieved are obtained. Other works for the consensus problem with additive noises in channels can be found in Huang and Manton (2009), Kar and Moura (2009), Li and Zhang (2009a) and Liu, Xie, and Zhang (2011).

Besides the above work, in a stochastic analog communication framework, much effort has been made to model the real communication environment of digital networks. For the case with

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discrete-time dynamics, readers are referred to Carli, Bullo, and Zampieri (2010), Frasca, Carli, Fagnani, and Zampieri (2009), Li, Fu, and Xie (2011), Li and Xie (2011) and Liu, Li, and Xie (2011). In Frasca et al. (2009), the average-consensus problem is considered based on static uniform quantizers, and an upper bound for the consensus errors is derived. The case with dynamic logarithmic quantizers is considered in Carli, Bullo et al. (2010). The asymptotic exact average-consensus under a limited data rate is addressed in Li et al. (2011), Li and Xie (2011) and Liu, Li et al. (2011). It is proved in Li et al. (2011) that one bit information exchange per transmission between each pair of adjacent agents suffices for exponentially fast exact average-consensus. The result of Li et al. (2011) is then generalized to the case with communication delays in Liu, Li et al. (2011) and with link failures in Li and Xie (2011).

In practical systems, it is more natural to model the dynamics of agents in a continuous-time setting and easier to implement static quantizers, so quantized consensus with continuous-time agent dynamics and static quantizers is of particular interest to the control communities in recent years. The case for quantized communication with uniform quantizers is dealt with in Ceragioli, De Persis, and Frasca (2011) and Frasca (2012). In Frasca (2012), the case with general static quantizers has also been studied. It is proved that there is a finite time, after which all the states of agents will enter into a region bounded by the quantization interval. Note that this result cannot ensure that the consensus error is uniformly bounded when logarithmic quantizer is adopted.

In this paper, we propose continuous-time and sampled-data-based average consensus protocols based on static logarithmic quantizers. For the continuous-time protocol, we give an explicit upper bound of the consensus error in terms of the initial states, the quantization density and the parameters of the network graph. It is shown that different from the case with uniform quantizers, the consensus error is always uniformly bounded no matter how coarse the quantization density is, and the β-asymptotic average consensus is guaranteed, i.e. the consensus error will converge to zero as the sector bound β of the quantizer approaches zero. For the sampled-data based protocol, we give sufficient conditions on the sampling interval to ensure the β-asymptotic average consensus. Existing works on sampled-data-based consensus protocol with precise communication channels can be found in Gao, Wang, Xie, and Wu (2009), Ren and Cao (2008), Xie, Liu, Wang, and Jia (2009), Yu, Zheng, Chen, Ren, and Cao (2011), etc., and with additive random noises can be found in Li and Zhang (2009b). The preliminary version of the paper was presented at the 31st Chinese Control Conference (Liu, Li, Xie, Fu, & Zhang, 2012).

Some remarks on notations are given as follows: we denote $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ the set of real numbers, real n-dimensional column vectors and real $m \times n$ matrices, respectively. For a vector or matrix A, its transpose is denoted by A'. We denote by $\|\cdot\|_1$ and $\|\cdot\|_2$ the 1-norm and Euclidean norm, respectively. We use $\mathbf{1}$ to denote a column vector with every element of 1. $\mathrm{diag}\{a_1, \ldots, a_n\}$ is a diagonal matrix with a_i, $i = 1, \ldots, n$ the diagonal entries. Given a square matrix M with all the eigenvalues being real, we denote by $\lambda_i(M)$ and $\lambda_{\max}(M)$ the ith smallest eigenvalue and the largest eigenvalue of M, respectively. The floor function is denoted by $\lfloor\cdot\rfloor$. $\Psi_{i,j}^M$ is defined as $\Psi_{i,j}^M = M(i-1)\cdots M(j)$, when $i > j$, and $\Psi_{i,j}^M = I$ otherwise.

2. Preliminaries

In this section, we shall introduce some notation of graph, and review some basics of logarithmic quantization and discontinuous ordinary differential equations which are essential to the later development.

2.1. Notation of graph

A digraph is denoted by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}}, A_{\mathcal{G}}\}$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of nodes with i representing the ith agent; $\mathcal{E}_{\mathcal{G}}$ is the set of edges which are represented by a pair of node indices (i, j). We consider that $(i, j) \in \mathcal{E}$ if and only if node i can send its information to node j. In this case, we call node i the parent node and node j the child node. The set of neighbors of the ith agent is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$. We assume that $(i, i) \notin \mathcal{E}$. The matrix $A_{\mathcal{G}} = [a_{i,j}] \in \mathbb{R}^{N\times N}$ is called the weighted adjacency matrix associated with $\mathcal{G}$. If $j \in \mathcal{N}_i$, it has $a_{i,j} > 0$; otherwise $a_{i,j} = 0$. A digraph $\mathcal{G}$ is balanced if the in-degree $\deg_{\text{in}}(i) \triangleq \sum_{j\in\mathcal{V}\setminus\{i\}} a_{i,j}$ and the out-degree $\deg_{\text{out}}(i) \triangleq \sum_{j\in\mathcal{V}\setminus\{i\}} a_{j,i}$ are equal for all $i \in \mathcal{V}$. We denote by $d(\mathcal{G}) = \max_i \deg_{\text{in}}(i)$ the degree of graph $\mathcal{G}$.

2.2. Concepts in logarithmic quantization

A quantizer $q(\cdot) : \mathbb{R} \to \Gamma$ is a mapping from $\mathbb{R}$ to the set Γ of quantized levels, where Γ is finite or denumerable. The quantizer $q(\cdot)$ is called logarithmic if it has the form

$$\Gamma = \{\pm w_{(i)} : w_{(i)} = \rho^i w_{(0)}, i = 0, \pm 1, \pm 2, \ldots\} \cup \{0\},$$
$$0 < \rho < 1, \quad w_{(0)} > 0.$$

The associated quantizer $q(\cdot)$ is defined as follows:

$$q(x) = \begin{cases} w_{(i)}, & \text{if } \frac{1}{1+\beta} w_{(i)} < x \le \frac{1}{1-\beta} w_{(i)} \\ 0, & \text{if } x = 0 \\ -q(-x), & \text{if } x < 0 \end{cases} \tag{1}$$

where $\beta = \frac{1-\rho}{1+\rho} \in (0, 1)$ is called sector bound (Fu & Xie, 2005). The quantization density for quantizer (1) is defined as $\frac{-2}{\ln\rho}$. It is noted that the smaller the β, the more the number of quantization levels in any given subset of $\mathbb{R}$. From (1) we can see that the quantization error satisfies the following sector bound condition:

$$q(x) - x = \Delta x, \quad \exists \Delta \in [-\beta, \beta), \ \forall x \in \mathbb{R}, \tag{2}$$

i.e. $q(x)$ must lie in the sector bound interval $[(1-\beta)x, (1+\beta)x)$ around x.

2.3. Discontinuous differential equations

Consider the following ordinary differential equation:

$$\dot{x}(t) = F(x(t)), \qquad x(0) = x_0 \in \mathbb{R}^d, \tag{3}$$

where $F(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is not necessarily continuous. Classical solutions of (3) might not exist due to the discontinuity of F. The most basic question is the notion of a solution. In this paper we understand the solutions of (3) in terms of the following differential inclusions:

$$\dot{x}(t) \in \mathcal{K}[F](x) \triangleq \bigcap_{\varepsilon>0} \mathrm{co}\{F(B(x, \varepsilon))\}, \tag{4}$$

where $B(x, \varepsilon)$ is the open ball centered at x with radius ε, $\mathrm{co}\{\cdot\}$ denotes the convex closure. A Krasovskii solution of (3) on $[0, t_1] \in \mathbb{R}$ is an absolutely continuous map $x : [0, t_1] \to \mathbb{R}^d$ that satisfies (4) for almost all $t \in [0, t_1]$. A solution is complete if $t_1 \to \infty$. The Krasovskii solution is considered throughout this paper since the set of Krasovskii solutions includes Filippov and Carathéodory solutions and results about Krasovskii solutions also hold for Filippov and Carathéodory solutions (Ceragioli et al., 2011).

3. Problem statement

In this section, we shall formulate the consensus problem to be studied for multi-agent systems. The agent i is assumed to have the following dynamics:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \ldots, N, \tag{5}$$

where $x_i(t) \in \mathbb{R}$ is the state information of agent i, $u_i(t) \in \mathbb{R}$ is the control input. The communication graph is denoted by $\mathcal{G} =$

$\{\mathcal{V}, \mathcal{E}, A\}$ with $\mathcal{V} = \{1, 2, \dots, N\}$ and the corresponding Laplacian matrix is L. Assume that agent i can receive its neighbors' quantized state information

$$y_{j,i}(t) = q(x_j(t)), \quad j \in \mathcal{N}_i,\ i \in \mathcal{V}, \tag{6}$$

where $q(\cdot)$ is defined as in (1). According to (2), $y_{j,i}(t)$ can be written as

$$y_{j,i}(t) = (1 + \Delta_j(t))x_j(t), \quad \Delta_j(t) \in [-\beta, \beta). \tag{7}$$

A control strategy $u_i(t)$ is called a distributed protocol if $\forall i \in \mathcal{V}$, $u_i(t)$ is a function of $\{y_{j,i}(s), 0 \le s \le t, j \in \mathcal{N}_i \cup \{i\}\}$, where $y_{i,i}(t) \triangleq q(x_i(t))$. Average consensus means to design a distributed protocol over $\mathcal{G}$ such that for any initial conditions $\{x_i(0), i \in \mathcal{V}\}$, the closed-loop system satisfies $\lim_{t\to\infty} x_i(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(0) \triangleq \bar{x}(0), \forall i \in \mathcal{V}$. Since quantization is involved in the communication, the exact average consensus cannot be easily achieved. As such, we will introduce the concept of β-asymptotic average consensus under logarithmic quantization. Denote

$$X(t) = [x_1(t), \dots, x_N(t)]', \qquad \delta(t) = X(t) - JX(0),$$

$$J = \frac{1}{N}\mathbf{1}\mathbf{1}', \qquad \Delta(t) = \mathrm{diag}\{\Delta_1(t), \dots, \Delta_N(t)\}. \tag{8}$$

Then, we have the following definition.

Definition. Under logarithmic quantization, the distributed protocol is admissible if for any given $\beta \in (0, 1)$, the closed-loop system satisfies

$$\sup_{t\ge 0} \|X(t)\|_2 < \infty, \quad \forall X(0) \in \mathbb{R}^N.$$

Moreover, a β-asymptotic average consensus is achieved under the admissible protocol if

$$\lim_{\beta\to 0} \limsup_{t\to\infty} \|\delta(t)\|_2 = 0, \quad \forall X(0) \in \mathbb{R}^N. \tag{9}$$

In the definition we call it β-asymptotic average consensus when (9) is satisfied because the upper bound of the disagreement δ has asymptotic property with respect to β.

4. Continuous-time protocol

In this section, we consider a continuous-time distributed protocol, which is proposed as follows:

$$u_i(t) = \sum_{j\in\mathcal{N}_i} \left[y_{j,i}(t) - y_{i,i}(t)\right], \quad i \in \mathcal{V}. \tag{10}$$

The following assumptions are introduced.

A1. $\mathcal{G}$ contains a spanning tree.
A2. $\mathcal{G}$ is a balanced digraph.

The above assumptions are very common for the average consensus problem. Only when the graph contains a spanning tree, all the agents are connected. Assumption A2 guarantees that the average value of the initial states is preserved. Define $\hat{L} = (L + L')/2$. Substituting the protocol (10) into the system (5)–(6) leads to

$$\dot{X}(t) = -L(I + \Delta(t))X(t), \tag{11}$$

where $\Delta(t)$ is defined as in (8). Note that the term on the right hand side is discontinuous. According to Cortes (2006), it can be shown that there exist complete Filippov and hence complete Krasovskii solutions to the following inclusion:

$$\dot{X}(t) \in \mathcal{K}[-L(I + \Delta(t))X(t)] = -LX(t) - L\mathcal{K}[\Delta(t)X(t)],$$

which are absolutely continuous functions of time. Note that there may exist more than one solutions to (11).

We first introduce the following lemmas.

Lemma 1 (Lin & Jia, 2008). *Under Assumptions* A1–A2, *there exists a set of vectors $\{\phi_1, \phi_2, \dots, \phi_{N-1}\}$ as a standard orthogonal basis of the column space of L such that*

(1) *the matrix $\Phi = \left[\frac{\mathbf{1}}{\sqrt{N}}, \phi\right]$ is a standard orthogonal matrix, where $\phi = [\phi_1, \phi_2, \dots, \phi_{N-1}]$; and*
(2) *$\Phi' L \Phi = \mathrm{diag}(0, \tilde{L})$, where $\tilde{L} \in \mathbb{R}^{(N-1)\times(N-1)}$ is with all its eigenvalues having positive real parts.*

Lemma 2. *Under Assumptions* A1–A2, *the Laplacian matrix of the graph satisfies*

$$\lambda_i(\hat{L}) \le \|L\|_2, \quad i = 1, 2, \dots, N. \tag{12}$$

Proof. According to Olfati-Saber and Murray (2004), we know that $\hat{L}$ is positive semi-definite. From Lemma 1 we have

$$\Phi'\hat{L}\Phi = \frac{\Phi' L\Phi + \Phi' L'\Phi}{2} = \mathrm{diag}\left(0, \frac{\tilde{L} + \tilde{L}'}{2}\right),$$

Then we know that $\frac{\tilde{L}+\tilde{L}'}{2}$ is positive definite and

$$\lambda_1(\hat{L}) = 0, \qquad \lambda_i(\hat{L}) = \lambda_i(\Phi'\hat{L}\Phi) = \lambda_{i-1}\left(\frac{\tilde{L} + \tilde{L}'}{2}\right) > 0,$$

$\forall i = 2, \dots, N$.

Letting $\|L\|_2 = a$ and $\lambda_{\max}(\hat{L}) = b$, we know that $\left(I - \frac{\tilde{L}}{a}\right)'\left(I - \frac{\tilde{L}}{a}\right) \ge 0$, which together with $\|L\|_2^2 = \lambda_{\max}(\Phi' L' \Phi \Phi' L \Phi) = \lambda_{\max}(\tilde{L}'\tilde{L})$ leads to

$$\lambda_{\max}\left(I + \frac{\tilde{L}'\tilde{L}}{a^2}\right) = 2 \ge \lambda_{\max}\left(\frac{\tilde{L} + \tilde{L}'}{a}\right) \ge \frac{2b}{a}.$$

So (12) is obtained. □

Now, we are in the position to show our main results.

Theroem 3. *Under Assumptions* A1–A2 *and protocol* (10), *for any given $\beta \in (0, 1)$, the Krasovskii solutions $X(t)$ to* (11) *satisfy*

$$\sup_{t\ge 0} \|X(t)\|_2 \le \|X(0)\|_1, \tag{13}$$

and

$$\limsup_{t\to\infty} \|\delta(t)\|_2 \le \|X(0)\|_1 \sqrt{\frac{\beta\|L\|_2}{\lambda_2(\hat{L})}}, \tag{14}$$

i.e. the protocol is admissible for the system (5)–(6). *In particular, if $\beta \in \left(0, \frac{\lambda_2(\hat{L})}{\|L\|_2}\right)$, then*

$$\limsup_{t\to\infty} \|\delta(t)\|_2 \le \sqrt{\frac{\sqrt{N}\beta\|L\|_2\|X(0)\|_1|\bar{x}(0)|}{\lambda_2(\hat{L}) - \beta\|L\|_2}}, \tag{15}$$

i.e. β-asymptotic average consensus can be achieved.

Remark 4. From Theorem 3, we can see that the 2-norm of the consensus error is upper bounded by $\|X(0)\|_1\sqrt{\frac{\|L\|_2}{\lambda_2(\hat{L})}}$. This means that no matter how loose the logarithmic quantization is, the consensus error is always uniformly bounded, independent of the parameters of the logarithmic quantization. Especially, if the network is undirected, this bound is given by $\|X(0)\|_1\sqrt{\frac{\lambda_{max}(L)}{\lambda_2(L)}}$.

Before proving Theorem 3, the following lemma in Guo (1990, p. 5) is introduced.

Lemma 5. *If the time-varying matrix $\Theta(t)$ is locally integrable (with respect to t), then the solution of the differential equation*

$$\dot{x}(t) = \Theta(t)x(t), \qquad x(0) = x_0,$$

satisfies $\|x(t)\|_1 \leq \|x_0\|_1 \exp\left\{\int_0^t \mu\big(\Theta(\tau)\big)d\tau\right\}$, *where for any matrix* $M = [m_{i,j}] \in \mathbb{R}^{m\times m}$, $\mu(\cdot)$ *is defined as*

$$\mu(M) = \max_j \left\{ m_{j,j} + \sum_{i=1, i\neq j}^{m} |m_{i,j}| \right\}.$$

Now we shall provide the proof of Theorem 3.

Proof. Since $\Delta(t)$ is uniformly bounded, the matrix $L(I+\Delta(t))$ in (11) is locally integrable. By Assumption A2, we know that $\sum_{i=1,i\neq j}^N a_{j,i} = \sum_{i=1,i\neq j}^N a_{i,j}$. Then according to the definition of L, we can see that for any given $\tau \geq 0$,

$$\mu\big(-L(I+\Delta(\tau))\big) = \max_j \Big\{\big[|1+\Delta_j(\tau)| - (1+\Delta_j(\tau))\big]\deg_{\text{in}}(i)\Big\},$$

which together with $\Delta_j(\tau) \in [-\beta, \beta)$ gives

$$\mu\big(-L(I+\Delta(\tau))\big) \equiv 0, \quad \forall\, \tau \geq 0. \tag{16}$$

According to (11), (16) and Lemma 5, we have

$$\|X(t)\|_2 \leq \|X(t)\|_1 \leq \|X(0)\|_1 \exp\left\{\int_0^t \mu\big(-L(I+\Delta(\tau))\big)d\tau\right\} = \|X(0)\|_1,$$

which implies (13). Based on $\|\delta(t)\|_2 = \|(I-J)X(t)\|_2 \leq \|X(t)\|_2$, we have that $\sup_{t\geq 0}\|\delta(t)\|_2 \leq \|X(0)\|_1$.

Since $\mathcal{G}$ is balanced, it is noted that $JL = 0$ and $JX(t) \equiv JX(0)$. Therefore,

$$\dot{\delta}(t) = -L(I+\Delta(t))X(t) \in -L\delta(t) - L\mathcal{K}[\Delta(t)X(t)]. \tag{17}$$

Define $V(t) = \|\delta(t)\|_2^2$ and v any vector satisfying $v \in \mathcal{K}[\Delta(t)X(t)]$. According to Olfati-Saber and Murray (2004), it has

$$\delta'(t)\hat{L}\delta(t) \geq \lambda_2(\hat{L})\|\delta(t)\|_2^2.$$

Since $\|v\|_2 \leq \beta\|X(t)\|_2 \leq \beta\|X(0)\|_1$, (17) leads to

$$\begin{aligned}(\nabla V)'\dot{\delta}(t) &= \delta'(t)\dot{\delta}(t) + \dot{\delta}'(t)\delta(t)\\ &= -2\delta'(t)\hat{L}\delta(t) - 2\delta'(t)Lv\\ &\leq -2\lambda_2(\hat{L})V(t) + 2\beta\|L\|_2\|X(0)\|_1^2.\end{aligned} \tag{18}$$

The last inequality is due to $\|\delta(t)\|_2 \leq \|X(0)\|_1$. By Assumptions A1–A2, we have $\lambda_2(\hat{L}) > 0$. According to the generalized LaSalle invariance principle (Cortes, 2006), (18) implies that

$$\limsup_{t\to\infty} V(t) \leq \frac{\beta\|L\|_2\|X(0)\|_1^2}{\lambda_2(\hat{L})}.$$

(14) follows by considering the definition of V.

On the other hand, from (17) and $X(t) = \delta(t) + JX(0)$ we have

$$\dot{\delta}(t) = -L(I+\Delta(t))\delta(t) - L\Delta(t)JX(0). \tag{19}$$

According to Lemma 2, $0 < \frac{\lambda_2(\hat{L})}{\|L\|_2} < 1$. For any $\beta \in \left(0, \frac{\lambda_2(\hat{L})}{\|L\|_2}\right)$, there exists $\eta \in (0,1)$ such that $\beta = \frac{\lambda_2(\hat{L})(1-\eta)}{\|L\|_2}$. Then we have

$$\begin{aligned}\dot{V}(t) &= \delta'(t)\dot{\delta}(t) + \dot{\delta}'(t)\delta(t)\\ &= -2\delta'(t)(\hat{L} + L\Delta(t))\delta(t) - 2\delta'(t)L\Delta(t)JX(0)\\ &\leq -2\big(\lambda_2(\hat{L}) - \beta\|L\|_2\big)V(t)\\ &\quad + 2\|\delta(t)\|_2\|L\|_2\|\Delta\|_2\|JX(0)\|_2\\ &\leq -2\eta\lambda_2(\hat{L})V(t) + 2\sqrt{N}\beta\|L\|_2\|X(0)\|_1|\bar{x}(0)|,\end{aligned}$$

which implies that

$$\limsup_{t\to\infty} V(t) \leq \frac{\sqrt{N}\beta\|L\|_2\|X(0)\|_1|\bar{x}(0)|}{\eta\lambda_2(\hat{L})}.$$

According to the definition of η, (15) is obtained. $\square$

Remark 6. In Frasca (2012), a generalized quantization is studied. It is proved that the consensus error is bounded by the quantization interval. Since the logarithmic quantization interval is not uniformly bounded, the result in Frasca (2012) cannot guarantee that the consensus error is uniformly bounded for the logarithmic quantization case. In this paper, Theorem 3 shows that the consensus error is uniformly bounded and the upper bound is only related to the initial states, the communication graph and quantization density. On the other hand, the convergence rate in this paper is exponential, which is faster than the one for the uniform quantizer case in Ceragioli et al. (2011). Furthermore, it also shows the asymptotic property with respect to quantization density. As β goes to zero, the exact average consensus will be achieved.

Remark 7. Comparing with uniform quantization (Ceragioli et al., 2011; Frasca, 2012), logarithmic quantization can make the consensus error exponentially converge into a small bounded interval. In particular, if the quantization density is large, the consensus error is also controlled by the average value of initial conditions, which means that the asymptotic consensus error is in a sector bound interval around the average value. It can be shown that for the case of logarithmic quantization, the dependency of the consensus error bound on the initial states is inherent. In fact, for system (11), it can be shown that

$$\begin{aligned}\lim_{t\to\infty} X(t) &= \frac{X(0)}{N}\begin{bmatrix}1+\varepsilon_1 & \cdots & 1+\varepsilon_1\\ \vdots & \ddots & \vdots\\ 1+\varepsilon_N & \cdots & 1+\varepsilon_N\end{bmatrix}\\ &= \begin{bmatrix}1+\varepsilon_1\\ \vdots\\ 1+\varepsilon_N\end{bmatrix}\bar{x}(0),\end{aligned}$$

where $\sum_{i=1}^N \varepsilon_i = 0$, $|\varepsilon_i| = O\big(\sqrt{\beta}\big)$.

5. Sampled-data based protocol

In this section, we consider a sampled-data setting where the measurements are made at discrete sampling times and the control inputs are based on zero-order hold. Assuming that the sampling interval is h, we propose the distributed protocol as $\forall t \in [kh, (k+1)h)$,

$$u_i(t) = \sum_{j\in\mathcal{N}_i}\big[q(x_j(kh)) - q(x_i(kh))\big]. \tag{20}$$

The discretized model with zero-order hold can be written as

$$x_i(k+1) = x_i(k) + hu_i(k), \quad k = 0, 1, \ldots \tag{21}$$

where we omit the sampling time interval to simplify the notation. Then we provide our results for the sampled-data-based protocol.

Theroem 8. *Under Assumptions* A1–A2, *if the sampling interval h satisfies* $0 < h < \frac{1}{(1+\beta)d(\mathcal{G})}$, *then the protocol* (20) *is admissible to the system* (5)–(6), *and for any given* $\beta \in (0, 1)$, *the closed-loop system satisfies*

$$\sup_{t\geq 0}\|X(t)\|_2 \leq \|X(0)\|_1, \tag{22}$$

and

$$\limsup_{t\to\infty} \|\delta(t)\|_2 \le h\beta \min\{c_1(h)\|X_0\|_1, c_2(h,\beta)|\bar{x}(0)|\}, \tag{23}$$

where $c_1(h)$ *and* $c_2(h,\beta)$ *are bounded for any* $(h,\beta) \in \left(0, \frac{1}{(1+\beta)d(\mathcal{G})}\right) \times (0,1)$ *and* $\lim_{\beta\to 0} c_2 < \infty$, *i.e.* β*-asymptotic average consensus can be achieved.*

Proof. Substituting (20) into the discrete-time system (21) leads to

$$X(k+1) = (I - hL - hL\Delta(k))X(k). \tag{24}$$

Since $\beta < 1$ and $\mathcal{G}$ is balanced, $(I+\Delta(k))L'$ is a valid Laplacian matrix. Thanks to $h < \frac{1}{(1+\beta)d(\mathcal{G})}$, $I - h(I+\Delta(k))L'$ is a stochastic matrix, which implies that $\|I - hL - hL\Delta(k)\|_1 = \|I - hL' - h\Delta(k)L'\|_\infty = 1$. Taking 1-norm on both sides of (24) yields

$$\begin{aligned}\|X(k+1)\|_1 &\le \|I - hL - hL\Delta(k)\|_1 \|X(k)\|_1 \\ &= \|X(k)\|_1 \le \cdots \le \|X(0)\|_1.\end{aligned} \tag{25}$$

This together with $\|X(k)\|_2 \le \|X(k)\|_1$ leads to (22).

Substituting $X(k) = \delta(k) + JX(0)$ into (24) and noting the fact that $J\mathbf{1} = 0$, we have

$$\delta(k+1) = (I - hL)\delta(k) - hL\Delta(k)X(k). \tag{26}$$

Next, we introduce the linear transformation $\tilde{\delta}(k) = \Phi'\delta(k)$, where Φ is defined as in Lemma 1. According to Lemma 1 and (26), we can see that $\tilde{\delta}(k) = [0, \tilde{\delta}_2'(k)]'$, where $\tilde{\delta}_2(k)$ satisfies

$$\tilde{\delta}_2(k+1) = (I - h\tilde{L})\tilde{\delta}_2(k) - h\phi' L\Delta(k)X(k). \tag{27}$$

Since $\mathcal{G}$ contains a spanning tree, according to the definition of $\tilde{L}$, $I - h\tilde{L}$ is a Schur matrix. Therefore, there exists a positive definite matrix P such that $(I - h\tilde{L})'P(I - h\tilde{L}) - P = -I$. Define a Lyapunov function $V(k) = \tilde{\delta}_2'(k)P\tilde{\delta}_2(k)$. Then we have

$$\begin{aligned}V(k+1) - V(k) &= \tilde{\delta}_2'(k+1)P\tilde{\delta}_2(k+1) - \tilde{\delta}_2'(k)P\tilde{\delta}_2(k) \\ &= -\|\tilde{\delta}_2(k)\|_2^2 - 2h\tilde{\delta}_2'(k)(I - h\tilde{L})'P\phi' L\Delta(k) \\ &\quad \times X(k) + h^2 X'(k)\Delta(k)L'\phi P\phi' L\Delta(k)X(k) \\ &\le -\|\tilde{\delta}_2(k)\|_2^2 + h^2\beta^2\|X(k)\|_2^2\|L\|_2^2\|P\|_2 \\ &\quad + 2h\beta\|\tilde{\delta}_2(k)\|_2\|I - h\tilde{L}\|_2\|P\|_2\|L\|_2\|X(k)\|_2,\end{aligned}$$

which guarantees that

$$\begin{aligned}\limsup_{k\to\infty} \|\tilde{\delta}_2(k)\|_2 &\le h\beta\|I - h\tilde{L}x\|_2\|P\|_2\|L\|_2\|X(k)\|_2 \\ &\quad + h\beta\|L\|_2\|X(k)\|_2\sqrt{\|P\|_2 + \|I - h\tilde{L}\|_2^2\|P\|_2^2}.\end{aligned} \tag{28}$$

It is clear that $\|I - h\tilde{L}\|_2$, $\|P\|_2$ and $\|L\|_2$ are all bounded. According to (17), we have

$$\dot{\delta}(t) = -L\delta(t_s) - L\Delta(t_s)X(t_s), \tag{29}$$

where $t_s = \left\lfloor \frac{t}{h} \right\rfloor h$ is the last sampling time instant before t. Since the right hand side of (29) is constant within $[t_s, t_s + h)$, we then have

$$\delta(t) = [I - (t - t_s)L]\delta(t_s) - (t - t_s)L\Delta(t_s)X(t_s). \tag{30}$$

It is known that $\|\delta(t)\|_2 = \|\tilde{\delta}_2(t)\|_2$, which together with (30) yields

$$\|\delta(t)\|_2 \le \left(1 + h\|L\|_2\right)\|\tilde{\delta}_2(t_s)\|_2 + \beta h\|L\|_2\|X(t_s)\|_2.$$

Since $\|X(k)\|_2 \le \|X(0)\|_1$, we have

$$\begin{aligned}\limsup_{t\to\infty} \|\delta(t)\|_2 &\le h\beta\|L\|_2\|X(0)\|_1\Big[1 + \left(1 + h\|L\|_2\right)\|I - h\tilde{L}\|_2 \\ &\quad \times \|P\|_2 + \left(1 + h\|L\|_2\right)\sqrt{\|P\|_2 + \|I - h\tilde{L}\|_2^2\|P\|_2^2}\Big].\end{aligned} \tag{31}$$

Letting

$$\begin{aligned}c_1 &= \|L\|_2\big[1 + \left(1 + h\|L\|_2\right)\|I - h\tilde{L}\|_2\|P\|_2 \\ &\quad \times \left(1 + h\|L\|_2\right)\sqrt{\|P\|_2 + \|I - h\tilde{L}\|_2^2\|P\|_2^2}\big],\end{aligned} \tag{32}$$

we can obtain the first expression on the right side of (23).

On the other hand, system (26) can be further written as

$$\delta(k+1) = (I - hL - hL\Delta(k))\delta(k) - hL\Delta(k)JX(0). \tag{33}$$

It has been shown that $(I - hL - hL\Delta(k))'$ is a stochastic matrix. Note that $(I - hL - hL\Delta(k))'$ has the same type (Wolfowitz, 1963) with $I - hL$ which is a primitive matrix. Meanwhile, thanks to $h < \frac{1}{(1+\beta)d(\mathcal{G})}$, the positive numbers in $(I - hL - hL\Delta(k))'$ are uniformly bounded away from 0. Then, according to Seneta (2006) and Wolfowitz (1963), there exists a matrix $R(j)$ with rows of the same vector $r(j)$ such that $(\Psi_{i,j}^{I-hL-hL\Delta})'$ converges to $R(j)$ exponentially as $i \to \infty$, i.e. for all $i \ge j$,

$$\left\|(\Psi_{i,j}^{I-hL-hL\Delta})' - R(j)\right\|_2 \le p(h,\beta)[\gamma(h,\beta)]^{i-j}, \tag{34}$$

where $p(h,\beta) \in (0,\infty)$ and $\gamma(h,\beta) \in (0,1)$ for all $h \in \left(0, \frac{1}{(1+\beta)d(\mathcal{G})}\right)$ and $\beta \in (0,1)$. Furthermore, it has $\lim_{\beta\to 0}\gamma(h,\beta) < 1$.

From (33), we can write the system of $\tilde{\delta}(k)$ as follows:

$$\begin{bmatrix} 0 \\ \tilde{\delta}_2(k+1)\end{bmatrix} = A(k)\begin{bmatrix} 0 \\ \tilde{\delta}_2(k)\end{bmatrix} - \begin{bmatrix} 0 \\ h\phi' L\Delta(k)JX(0)\end{bmatrix}, \tag{35}$$

where

$$\begin{aligned}A(k) &= \Phi'(I - hL - hL\Delta(k))\Phi \\ &= \begin{bmatrix} 1 & 0 \\ -\phi' hL(I+\Delta(k))\frac{\mathbf{1}}{\sqrt{N}} & I - \phi' hL(I+\Delta(k))\phi\end{bmatrix}.\end{aligned}$$

It follows that for all $i \ge j$,

$$\begin{aligned}\left\|(\Psi_{i,j}^{A})' - \begin{bmatrix}\sqrt{N}r(j)\Phi \\ 0\end{bmatrix}\right\|_2 &= \left\|\Phi'\left((\Psi_{i,j}^{I-hL-hL\Delta})' - R(j)\right)\Phi\right\|_2 \\ &\le p(h,\beta)[\gamma(h,\beta)]^{i-j}.\end{aligned}$$

According to the definition of A, we have for all $i \ge j$,

$$\left\|\Psi_{i,j}^{I-\phi' hL(I+\Delta)\phi}\right\|_2 \le p(h,\beta)[\gamma(h,\beta)]^{i-j}. \tag{36}$$

Now we consider the following subsystem:

$$\tilde{\delta}_2(k+1) = (I - \phi' hL(I+\Delta)\phi)\tilde{\delta}_2(k) - h\phi' L\Delta(k)JX(0), \tag{37}$$

which comes directly from (35). Then we have

$$\begin{aligned}\tilde{\delta}_2(k) &= \Psi_{k,0}^{I-\phi' hL(I+\Delta)\phi}\tilde{\delta}_2(0) \\ &\quad - h\sum_{i=0}^{k-1}\Psi_{k,i+1}^{I-\phi' hL(I+\Delta)\phi}\phi' L\Delta(k)JX(0),\end{aligned} \tag{38}$$

which together with (36) yields that

$$\begin{aligned}\|\tilde{\delta}_2(k)\|_2 &\le p(h,\beta)[\gamma(h,\beta)]^k\|\tilde{\delta}_2(0)\|_2 + h\beta\sqrt{N}|\bar{x}(0)| \\ &\quad\cdot\|L\|_2\sum_{i=0}^{k-1} p(h,\beta)[\gamma(h,\beta)]^i \\ &= p(h,\beta)[\gamma(h,\beta)]^k\|\tilde{\delta}_2(0)\|_2 + h\beta\sqrt{N}|\bar{x}(0)| \\ &\quad\cdot\|L\|_2\frac{p(h,\beta)(1-[\gamma(h,\beta)]^k)}{1-\gamma(h,\beta)}.\end{aligned}$$

Note that $\forall k \ge 0$, $\|\delta(k)\| = \|\tilde{\delta}_2(k)\|$, then we have

$$\lim_{k\to\infty}\|\delta(k)\| \le h\beta\sqrt{N}|\bar{x}(0)|\,\|L\|\frac{p(h,\beta)}{1-\gamma(h,\beta)}. \quad (39)$$

On the other hand, according to (5) and (21), we know that

$$\begin{aligned}X(t) &= X(t_s) + (t-t_s)u(t_s) \\ &= X(t_s) - (t-t_s)L(I+\Delta(t_s))X(t_s),\end{aligned} \quad (40)$$

where $t_s = \lfloor \frac{t}{h} \rfloor h$ is the last sampling time instant before t. It is clear that $t - t_s < h$. Substituting $X(t) = \delta(t) + JX(0)$ into (40) leads to that

$$\delta(t) = \delta(t_s) - (t-t_s)L(I+\Delta(t_s))\delta(t_s) - (t-t_s)L\Delta(t_s)JX(0).$$

Then we have

$$\begin{aligned}\limsup_{t\to\infty}\|\delta(t)\|_2 &\le (1+h(1+\beta)\|L\|_2)\lim_{k\to\infty}\|\delta(k)\|_2 \\ &\quad + h\beta\|L\|_2\sqrt{N}|\bar{x}(0)| \\ &\le (1+h(1+\beta)\|L\|_2)\,h\beta\sqrt{N}|\bar{x}(0)|\,\|L\|_2 \\ &\quad\cdot\frac{p(h,\beta)}{1-\gamma(h,\beta)} + h\beta\|L\|_2\sqrt{N}|\bar{x}(0)|.\end{aligned} \quad (41)$$

Let $c_2 = \sqrt{N}\|L\|_2\left[1 + \frac{p(h,\beta)}{1-\gamma(h,\beta)}(1+h(1+\beta)\|L\|_2)\right]$, which together with (31) results in (23). □

Remark 9. Theorem 8 shows that when the sampling interval h is less than $\frac{1}{(1+\beta)d(\mathcal{G})}$, sampled-data based protocol (20) is admissible and β-asymptotic average consensus is guaranteed. The upper bound of consensus error is proportional to β and the average value $\bar{x}(0)$. In particular, if $\bar{x}(0) = 0$, we can guarantee the exact consensus.

6. Numerical example

In this section, we shall give some examples to demonstrate the proposed protocols. We consider system (5) with 3 agents connecting end to end. The communication graph is shown in Fig. 1. The corresponding adjacency matrix A and Laplacian matrix L are given below:

$$A = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}, \qquad L = \begin{bmatrix}1 & 0 & -1\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}.$$

It is clear that the graph $\mathcal{G}$ is balanced and contains a spanning tree. The initial values of the 3 agents are $x_1(0) = 1.42$, $x_2(0) = 1.83$ and $x_3(0) = 1.25$ with the average $\frac{1}{3}\left[x_1(0)+x_2(0)+x_3(0)\right] = 1.5$. The logarithmic quantizer (1) with $w_{(0)} = 1$ is applied. We first apply the continuous-time average consensus protocol (10). According to Theorem 3, we know that β could be freely chosen from $(0, 1)$ and the smaller the β is, the closer the states are to the average. We then select a different β and compare the consensus errors.

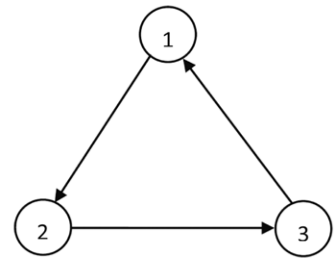

Fig. 1. Communication graph $\mathcal{G}$.

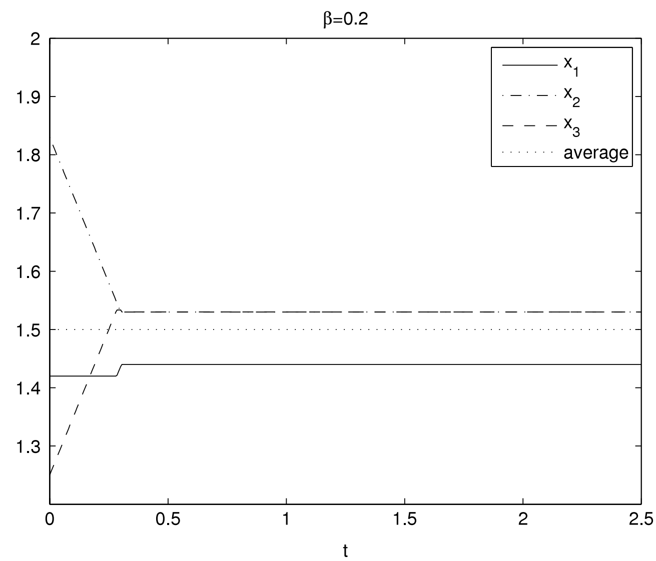

Fig. 2. State trajectories corresponding to the continuous-time protocol with $\beta = 0.2$.

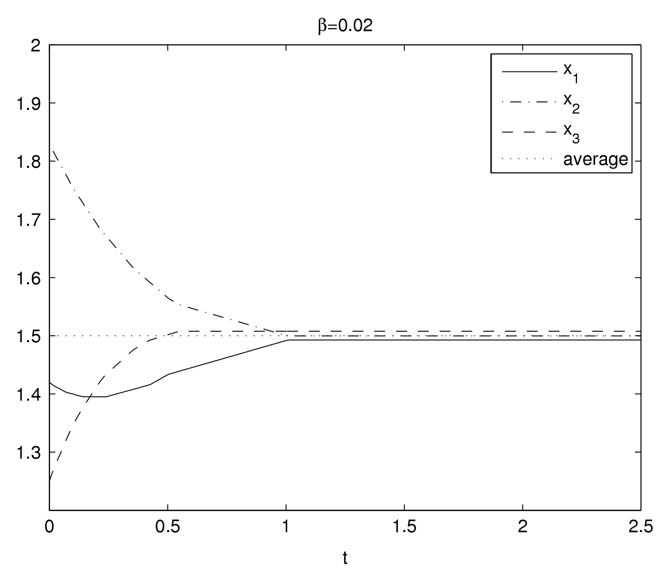

Fig. 3. State trajectories corresponding to the continuous-time protocol with $\beta = 0.02$.

Figs. 2 and 3 show the trajectories of the states with $\beta = 0.2$ and $\beta = 0.02$, respectively.

Next, the consensus protocol (20) is applied to system (5). We choose $h = 0.5 = \frac{1}{2d(\mathcal{G})} < \frac{1}{(1+\beta)d(\mathcal{G})}$. The state trajectories are shown in Figs. 4 and 5. In Fig. 4, $\beta = 0.2$, the states are uniformly bounded and the consensus errors are relatively large compared with the ones in Fig. 5, in which $\beta = 0.02$. Figs. 4 and 5 verify that when $\beta \to 0$, the average consensus errors also goes to 0.

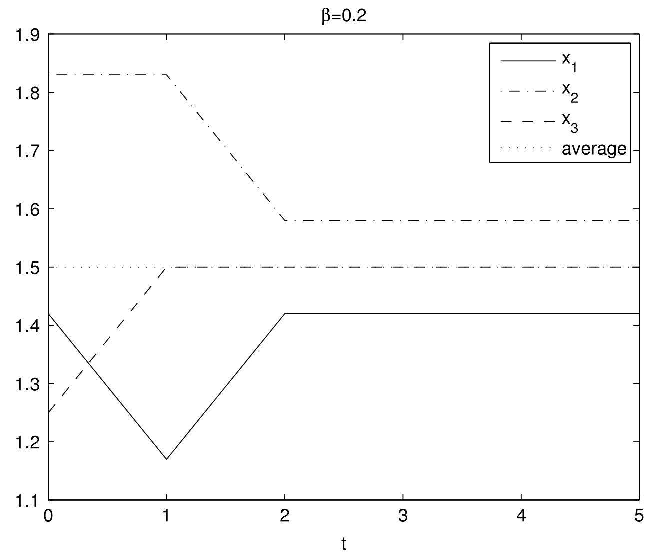

Fig. 4. State trajectories with $\beta = 0.2$.

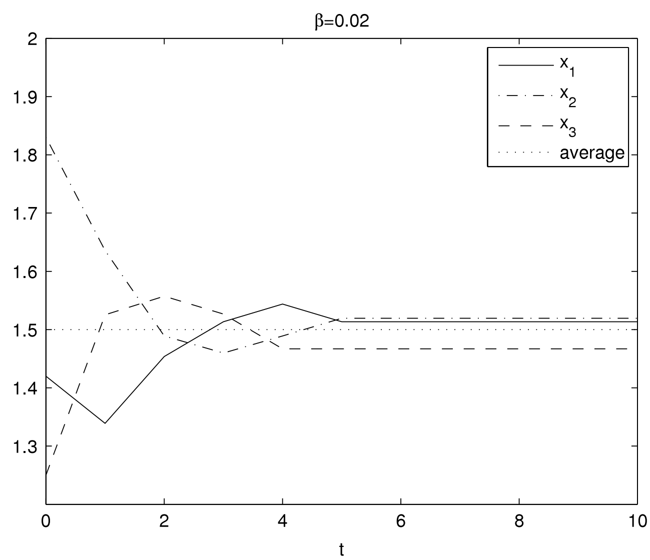

Fig. 5. State trajectories with $\beta = 0.02$.

Finally, we shall change the initial condition by $x_2(0) = -2.67$ in order that the average value is zero. By applying logarithmic quantizer with $\beta = 0.2$, we use the two kinds of protocols (10) and (20). We still take $h = 0.5$ for the sampled-data based protocol. According to Theorems 3 and 8, exact consensus can be achieved asymptotically. Figs. 6 and 7 show the state trajectories under the two protocols, respectively.

7. Conclusion

In this paper we have considered the average consensus problem for multi-agent systems with logarithmic quantization in communication channels. The agents are homogeneous and with first order continuous dynamics. Two protocols have been proposed based on continuous and sampled measurements, respectively. It has been proved that when the sampling rate is high enough and the quantization density is high enough, all the states of the agents are uniformly bounded and the average consensus error will converge to zero as the sector bound β approaches zero. Some numerical examples have been provided to demonstrate the results.

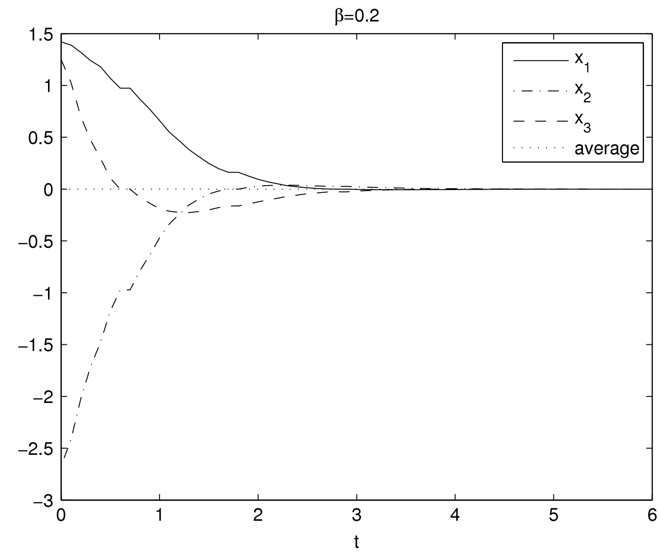

Fig. 6. Continuous-time protocol with zero average.

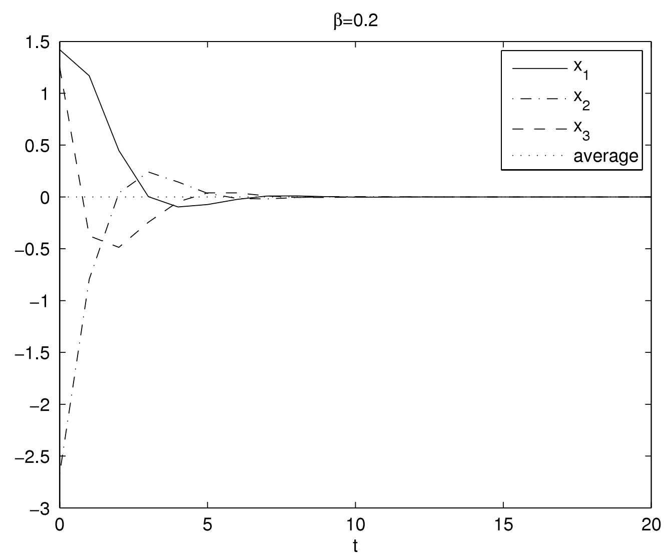

Fig. 7. Sampled-data based protocol with zero average.

References

Cao, M., Morse, A. S., & Anderson, B. D. O. (2008). Reaching a consensus in a dynamically changing environment: a graphical approach. *SIAM Journal on Control and Optimization*, *47*(2), 575–600.

Carli, R., Bullo, F., & Zampieri, S. (2010). Quantized average consensus via dynamic coding/decoding schemes. *International Journal of Robust and Nonlinear Control*, *20*(2), 156–175.

Carli, R., Fagnani, F., Frasca, P., & Zampieri, S. (2010). Gossip consensus algorithms via quantized communication. *Automatica*, *46*(1), 70–80.

Ceragioli, F., De Persis, C., & Frasca, P. (2011). Discontinuities and hysteresis in quantized average consensus. *Automatica*, *47*(9), 1916–1928.

Cortes, J. (2006). Finite-time convergent gradient flows with applications to network consensus. *Automatica*, *42*(11), 1993–2000.

Frasca, P. (2012). Continuous-time quantized consensus: convergence of Krasovskii solutions. *Systems & Control Letters*, *61*(2), 273–278.

Frasca, P., Carli, R., Fagnani, F., & Zampieri, S. (2009). Average consensus on networks with quantized communication. *International Journal of Robust and Nonlinear Control*, *19*(16), 1787–1816.

Fu, M., & Xie, L. (2005). The sector bound approach to quantized feedback control. *IEEE Transactions on Automatic Control*, *50*(11), 1698–1711.

Gao, Y., Wang, L., Xie, G., & Wu, B. (2009). Consensus of multi-agent systems based on sampled-data control. *International Journal of Control*, *82*(12), 2193–2205.

Guo, L. (1990). *Time-varying stochastic systems: stability, estimation and control*. Jilin, China: Jilin Science and Technology Press.

Huang, M., & Manton, J. H. (2009). Coordination and consensus of networked agents with noisy measurements: stochastic algorithms and asymptotic behavior. *SIAM Journal on Control and Optimization*, *48*(1), 134–161.

Jadbabaie, A., Jie, L., & Morse, A. S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, *48*(6), 988–1001.
Kar, S., & Moura, J. M. F. (2009). Distributed consensus algorithms in sensor networks with imperfect communication: Link failures and channel noise. *IEEE Transactions on Signal Processing*, *57*(1), 355–369.
Kashyap, A., Basar, T., & Srikant, R. (2007). Quantized consensus. *Automatica*, *43*(7), 1192–1203.
Li, T., Fu, M., Xie, L., & Zhang, J.-F. (2011). Distributed consensus with limited communication data rate. *IEEE Transactions on Automatic Control*, *56*(2), 279–292.
Li, T., & Xie, L. (2011). Distributed consensus over digital networks with limited bandwidth and time-varying topologies. *Automatica*, *47*(9), 2006–2015.
Li, T., & Zhang, J.-F. (2009a). Mean square average-consensus under measurement noises and fixed topologies: necessary and sufficient conditions. *Automatica*, *45*(8), 1929–1936.
Li, T., & Zhang, J. F. (2009b). Sampled-data based average consensus with measurement noises: Convergence ananlysis and uncertainty principle. *Science in China, Series F: Information Sciences*, *52*(11), 2089–2103.
Lin, P., & Jia, Y. (2008). Average consensus in networks of multi-agents with both switching topology and coupling time-delay. *Physica A. Statistical Mechanics and its Applications*, *387*(1), 303–313.
Liu, S., Li, T., & Xie, L. (2011). Distributed consensus for multiagent systems with communication delays and limited data rate. *SIAM Journal on Control and Optimization*, *49*(6), 2239–2262.
Liu, S., Li, T., Xie, L., Fu, M., & Zhang, J. (2012). Sampled-data based average consensus with logarithmic quantizer. In *Proceedings of the 31st Chinese control conference* (pp. 6141–6146). Hefei.
Liu, S., Xie, L., & Lewis, F. L. (2011). Synchronization of multi-agent systems with delayed control input information from neighbors. *Automatica*, *47*(10), 2152–2164.
Liu, S., Xie, L., & Zhang, H. (2011). Distributed consensus for multi-agent systems with delays and noises in transmission channels. *Automatica*, *47*(5), 920–934.
Nedic, A., Olshevsky, A., Ozdaglar, A., & Tsitsiklis, J. N. (2009). On distributed averaging algorithms and quantization effects. *IEEE Transactions on Automatic Control*, *54*(11), 2506–2517.
Olfati-Saber, R., & Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, *49*(9), 1520–1533.
Ren, W., & Cao, Y. (2008). Convergence of sampled-data consensus algorithms for double-integrator dynamics. In *Proceedings of the 47th IEEE conference on decision and control* (pp. 3965–3970).
Seneta, E. (2006). *Non-negative matrices and Markov chains*. Springer.
Wolfowitz, J. (1963). Products of indecomposable, aperiodic, stochastic matrices. *Proceedings of the American Mathematical Society*, *14*(5), 733–737.
Xie, G., Liu, H., Wang, L., & Jia, Y. (2009). Consensus in networked multi-agent systems via sampled control: fixed topology case. In *Proceedings of the American control conference* (pp. 3902–3907).
Yildiz, M. E., & Scaglione, A. (2008). Coding with side information for rate-constrained consensus. *IEEE Transactions on Signal Processing*, *56*(8), 3753–3764.
Yu, W., Zheng, W. X., Chen, G., Ren, W., & Cao, J. (2011). Second-order consensus in multi-agent dynamical systems with sampled position data. *Automatica*, *47*(7), 1496–1503.

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