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Consensus of multi-agent systems with general linear dynamics via dynamic output feedback control

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Abstract: This study addresses consensus problems of multi-agent systems (MASs) using dynamic output feedback control under both fixed and switching topologies. We aim to exploit the information structure for the consensusability of MASs. Necessary and sufficient conditions are presented in terms of detectability and stabilisability of the agents, graph topology, and some matrix inequality constraints. These conditions explicitly reveal how consensusability is affected by the intrinsic dynamics of the agents, the communication topology and available information. In addition, this paper provides several constructive procedures for protocol design to achieve consensus, and establishes the so-called separation principle, which simplifies the design procedure greatly.

1 Introduction

The interplay between control and communications has been attracting increasing attentions in recent years. It becomes well known that communication constraints and information flow can significantly affect the performance of networked control systems, especially for networked multi-agent systems (MASs). The consensus problem is one of the fundamental problems for networked MASs [1–5]. Consensus protocols for continuous-time systems are presented using a static state feedback control in [1, 2, 4, 6–9], while for the discrete-time counterpart, necessary and sufficient conditions for consensusability are given in [10, 11].

However, when some states are unmeasurable, the output feedback control needs to be considered. Under mild assumptions, a sufficient condition for MASs to reach consensus via static output feedback control (SOFC) is presented in [7]. In [12], a sufficient condition for single-input-single-output (SISO) MASs is presented. Note that the applicability of the SOFC-based protocols is inherently limited given their existence conditions and design difficulty, for example, stabilisability and detectability generally cannot guarantee the existence of an SOFC and the control design usually involves bilinear matrix inequalities (BMIs). Hence, dynamic output feedback control (DOFC)-based protocols have been studied in the literature [13–17]. In [13], a necessary and sufficient condition using simultaneous stabilisation for a local controller to stabilise certain formation dynamics is given. In [18], an extended result on SISO system for double-integrator dynamics [1] is provided. An observer-based controller is proposed in [14], where a separation principle-like condition is presented. In [12, 15, 19, 20], the DOFCs are also observer-based. This paper, different from the above literature, will explore a more general form of DOFC and its prosperities.

In addition, all those DOFC-based protocols [12, 15, 19, 20] are designed only for fixed topologies. Few works are concerned with switching topologies [1, 6, 21, 22] even for static feedback control. In [6], a framework for fixed and switching topologies is proposed for agents with simple dynamics. In [1], the MASs with double-integrator dynamics are investigated. In [23], a design strategy for double-integral systems is introduced to handle strongly connected, directed and unbalanced graphys under a switching network configuration, although the properties of balanced graphs are still used. In [24], the notion of globally reachable node under dynamically changing interaction topology is studied, whereas [17] is concerned with a leader-follower consensus under switching topology. In [8], a protocol is proposed under weak connection and balance assumption for directed graph, which is a strong assumption. However, to the best of our knowledge, little work has been conducted on consensus via DOFC under switching topologies and the influence of information levels [16]. In [21], an equivalence is established between robust stabilisation of some uncertain systems and the consensus of output feedback. And then a controller is designed by tuning a time-scale parameter for consensus. However, the design is under some strict assumptions, for example, the underlining MAS is square system.

In this paper, we derive further necessary and sufficient conditions on consensus using generalised DOFCs under various information structures. With full local information, we demonstrate that consensusability is equivalent to stabilisability via a single dynamic output feedback controller, given the connectivity of the network and homogeneity of agents. With partial local information, however, the situation becomes complicated. Specifically, for some particular information structures, we prove that the consensusability is simply equivalent to the stabilisability and detectability of agents under a digraph containing a spanning tree. Meanwhile, the separation principle is valid under some special structures. Besides, the benefits of additional local information are also discussed. In addition, switching (time-varying) topologies are considered for both digraph and undirected graph. We show that under the assumption of consistent weak connectivity together with dwell time or balanced digraph, the results for static topologies are still applicable. When the strong assumption is invalid, our results link consensusability to a joint spanning tree and the system structure.

This paper is organised as follows. Section 2 starts with problem statements, followed by some preliminaries. Section 3 considers the generalised DOFC-based protocols for digraph. Section 4 addresses the consensus problem under switching topology for both digraph and undirected graph. Section 5 uses an example of linearised satellite systems for illustration. The last section concludes this paper and provides directions for future research.

The notations used in this paper are standard. $\mathcal{R}$ and $\mathcal{C}$ denote the real and complex spaces, respectively. A^{T} (A^*) is the transpose (conjugate) of A. $A > 0$ ($A \geq 0$) means that A is positive definite (semidefinite). $\emptyset$ is the empty set. $\jmath$ is the pure imaginary number such that $\jmath^2 = -1$. $\mathbf{1}$ is a vector with one as its entries. $\otimes$ is the symbol for the Kronecker product. The communication topology among agents is represented by a digraph $\mathcal{G}$, which consists of a node set $\mathcal{V} = \{1, 2, \ldots, n\}$ and an edge set $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$. Denote the adjacency matrix of $\mathcal{G}$ as $\mathcal{A}_g = [a_{ij}] \in \mathcal{R}^{n\times n}$ and the in-degree matrix as $\bar{\mathcal{D}}$, where $a_{ij} = 1$ if there is a link $i \leftarrow j$ and $a_{ij} = 0$ otherwise. Then the Laplacian matrix of a digraph $\mathcal{G}$ with n vertices $\mathcal{L}_g \in \mathcal{R}^{n\times n}$ can be represented as $\mathcal{L}_g = \bar{\mathcal{D}} - \mathcal{A}_g$. Let λ_i be any eigenvalue of $\mathcal{L}_g$. For agent i, denote its neighbour set as $\mathcal{N}(i)$. A digraph $\mathcal{G}$ contains a spanning tree if there exists a root (node) such that there exists a directed path from this node to every other node [14]. In addition, $\mathcal{L}_g$ of a digraph $\mathcal{G}$ has no eigenvalues with negative real part and at least one zero eigenvalue. $\mathcal{L}_g$ has only one zero eigenvalue if and only if $\mathcal{G}$ contains a spanning tree [6, 7]. Without loss of generality, assume that $i = 2, \ldots, n$, $i = 2, \ldots, n$ are the eigenvalues of $\mathcal{L}_g$ and $0 \leq Re(\lambda_2) \leq \cdots \leq Re(\lambda_n)$.

2 Problem formulation and preliminaries

2.1 Problem statement

Consider the following MAS

$$\dot{x}_i = Ax_i + Bu_i \tag{1}$$

$$y_i = Cx_i, \quad i = 1, 2, \ldots, n \tag{2}$$

where $x_i \in \mathcal{R}^{n_x}$, $u_i \in \mathcal{R}^r$, $y_i \in \mathcal{R}^m$ are the state, control input and output of the agent i, respectively. Note that the agents are homogeneous. Define α as the first column of $\mathcal{A}_g$ by deleting the first entry a_{11} and $\mathcal{L}$ as a principal minor by deleting the first column and first row of $\mathcal{L}_g$. Thus, $\mathcal{L}_g$ can be expressed as

$$\mathcal{L}_g = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} & -\alpha^{\mathrm{T}} \\ -\mathcal{L}\cdot\mathbf{1} & \mathcal{L} \end{pmatrix} \tag{3}$$

For system (1)–(2), if there exists a $u(t) \in \mathcal{U}$, where $\mathcal{U}$ is a set of admissible control inputs, such that

$$\lim_{t\to\infty} \|x_j(t) - x_i(t)\| = 0, \quad i, j = 1, \ldots, n$$

then the system is deemed to asymptotically reach a consensus.

The main objective of this paper is to examine consensusability using the following distributed DOFC-based consensus protocol for the agent i, $i \in \mathcal{V}$

$$\dot{\xi}_i = M\xi_i + Ny_i + V\sum_{j=1}^{n} a_{ij}(\xi_j - \xi_i) + H\sum_{j=1}^{n} a_{ij}(y_j - y_i) \tag{4}$$

$$u_i = S\xi_i + Ry_i + L\sum_{j=1}^{n} a_{ij}(\xi_j - \xi_i) + K\sum_{j=1}^{n} a_{ij}(y_j - y_i) \tag{5}$$

where M, N, L, K, S, R, V, H are gain matrices with appropriate dimensions and ξ_i is the internal state. When the pairs (M, S), (N, R), (V, L) and (H, K) are non-zero, the protocol of agent i uses full local information (y_i and ξ_i) from itself and its neighbours (including y_j and ξ_j, $j \in \mathcal{N}(i)$). However, it is not always true that all local information are available to an agent. For instance, assume that y_i represents the location information of agent i obtained by GPS. When the GPS is not available, y_i becomes unknown. In this case, we may only have the relative information $y_i - y_j$ (e.g. the distance between the two agents) obtained by some on-board sensors such as laser range finders. Thus N and R are zero matrices. We will present some basic properties of the DOFC-based consensus protocols using full local information in Section 3.1 and discuss partial local information cases in Section 3.2.

2.2 Preliminaries

Lemma 1: Suppose that the digraph $\mathcal{G}$ contains a spanning tree. (A, B) is stabilisable if and only if $\mathbb{K}_c = \{F{:}A + \lambda_i BF \text{ is Hurwitz}, i = 2, \ldots, n\} \neq \emptyset$. (A, C) is detectable if and only if $\mathbb{K}_o = \{L{:}A + \lambda_i LC \text{ is Hurwitz}, i = 2, \ldots, n\} \neq \emptyset$.

Proof: The proof procedure follows from a constructive method by applying Finsler's lemma [25]. Details can be found in [16].

Remark 1: With the aid of Lemmas 1 and 16 in [16], we can derive the consensusability condition for static state/output feedback protocols. In fact, for static state feedback protocol [4], consensusability is equivalent to the stabilisability of (A, B), if $\mathcal{G}$ contains a spanning tree. Under the precondition of Lemma 16 in [16], the consensusability of the output feedback protocols [4] is equivalent to the existence of a certain set $\mathbb{K}$ defined in [16].

Lemma 2: For any undirected graph $\mathcal{G}$ with the corresponding Laplacian matrix $\mathcal{L}_g$, the matrix $\mathcal{L} + \mathbf{1}\cdot\alpha^{\mathrm{T}}$ is similar to a diagonal matrix.

Proof: Based on (3), we can see that $\underline{S}^{-1}\mathcal{L}_g\underline{S} = \begin{pmatrix} 0 & -\alpha^{\mathrm{T}} \\ 0 & \mathcal{L} + \mathbf{1}\cdot\alpha^{\mathrm{T}} \end{pmatrix} \doteq \check{\mathcal{L}}_g$, where $\underline{S} = \begin{pmatrix} 1 & 0 \\ \mathbf{1} & I_{n-1} \end{pmatrix}$. Since $\mathcal{L}_g$ is symmetric as $\mathcal{G}$ is undirected, there exists a matrix T such that $T^{-1}\mathcal{L}_g T$ is a diagonal matrix defined as Λ. Thus $\check{\mathcal{L}}_g$ is also similar to Λ, that is, for any eigenvalue λ_i of $\check{\mathcal{L}}_g$

$$\operatorname{rank}(\lambda_i I_n - \check{\mathcal{L}}_g) = n - r(\lambda_i) \tag{6}$$

where $r(\lambda_i)$ is the degree of the algebraic multiplicity of λ_i. Note that because of the special structure of $\check{\mathcal{L}}_g$, any eigenvalue of $\mathcal{L} + \mathbf{1}\cdot\alpha^{\mathrm{T}}$ is also $\check{\mathcal{L}}_g$'s. Now we consider two cases depending on whether $\mathcal{G}$ contains at least one spanning tree.

If $\mathcal{G}$ has at least one spanning tree, choose an eigenvalue $\lambda_j \neq 0$ of $\mathcal{L} + \mathbf{1}\cdot\alpha^{\mathrm{T}}$. If $\mathcal{L} + \mathbf{1}\cdot\alpha^{\mathrm{T}}$ is not diagonisable, $\operatorname{rank}(\lambda_j I_{n-1} - (\mathcal{L} + \mathbf{1}\cdot\alpha^{\mathrm{T}})) > n - 1 - r(\lambda_j)$. Meanwhile, $\operatorname{rank}(\lambda_j I_n - \check{\mathcal{L}}_g) = 1 + \operatorname{rank}(\lambda_j I_{n-1} - (\mathcal{L} + \mathbf{1}\cdot\alpha^{\mathrm{T}}))$. Thus we obtain a contradiction to (6). Hence, $\mathcal{L} + \mathbf{1}\cdot\alpha^{\mathrm{T}}$ is diagonisable.

Otherwise, we can always divide $\mathcal{G}$ into several connected components (subgraphs). To illustrate, suppose there are two components $\mathcal{G}_1$ and $\mathcal{G}_2$ whose Laplacian matrices are $\mathcal{L}_{g_1} \in \mathcal{R}^{n_1\times n_1}$ and $\mathcal{L}_{g_2} \in \mathcal{R}^{n_2\times n_2}$, respectively. Then $\mathcal{L}_g = \begin{pmatrix} \mathcal{L}_{g_1} & 0 \\ 0 & \mathcal{L}_{g_2} \end{pmatrix}$. Define $\bar{S} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$, where $S_i = \begin{pmatrix} 1 & 0 \\ \mathbf{1} & I_{n_i-1} \end{pmatrix}$, $i = 1,2$. Then $\bar{S}^{-1}\mathcal{L}_g\bar{S} = \begin{pmatrix} \check{\mathcal{L}}_{g_1} & 0 \\ 0 & \check{\mathcal{L}}_{g_2} \end{pmatrix}$, where $\check{\mathcal{L}}_{g_i}$ has an analogous structure to $\check{\mathcal{L}}_g$. Since there always exists an invertible matrix X such that $\underline{S} = \bar{S}X$, we can apply the aforementioned result to obtain the conclusion. □

3 Dynamic output feedback consensus and separation principle

3.1 Consensus protocols with full local information

This subsection presents a result when full local information is available.

Theorem 3: Suppose that digraph $\mathcal{G}$ contains a spanning tree. There exists a protocol (4)–(5) such that consensus is asymptotically reached for all initial states if and only if $\mathbb{K}_d \neq \emptyset$, where

$$\mathbb{K}_d = \{\bar{\mathcal{K}}: \bar{\mathcal{A}} + \bar{\mathcal{B}}\bar{\mathcal{K}}\bar{\mathcal{C}}_i \text{is Hurwitz }, i = 2, \ldots, n\} \tag{7}$$

Furthermore, for a given $\bar{K} \in \mathbb{K}_d$, we have

$$v(t) \to (\mathbf{1}r^{\mathrm{T}} \otimes e^{\mathcal{A}})v(0), \quad t \to \infty \tag{8}$$

where

$$\bar{\mathcal{A}} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{\mathcal{B}} = \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}, \quad \bar{\mathcal{K}}_1 = \begin{pmatrix} R & S \\ N & M \end{pmatrix} \tag{9}$$

$$\bar{\mathcal{K}}_2 = \begin{pmatrix} K & L \\ H & V \end{pmatrix}, \quad \hat{\mathcal{C}} = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \tag{10}$$

$$\bar{\mathcal{K}} = [\bar{\mathcal{K}}_1, \bar{\mathcal{K}}_2], \quad \bar{\mathcal{C}} = \begin{pmatrix} \hat{\mathcal{C}} \\ \hat{\mathcal{C}} \end{pmatrix}, \quad \bar{\mathcal{C}}_i = \begin{pmatrix} \hat{\mathcal{C}} \\ \lambda_i\hat{\mathcal{C}} \end{pmatrix} \tag{11}$$

$v = [v_1^{\mathrm{T}}, v_2^{\mathrm{T}}, \ldots v_n^{\mathrm{T}}]^{\mathrm{T}}$ and $v_i = [x_i^{\mathrm{T}}\xi_i^{\mathrm{T}}]^{\mathrm{T}}$, $r^{\mathrm{T}} = [r_1, \ldots, r_n] \in \mathcal{R}^n$ is the left eigenvector of $\mathcal{L}_g$ associated with the eigenvalue 0, satisfying $r^{\mathrm{T}}\mathbf{1} = 1$.

Proof: Owing to the homogeneity of agents, without loss of generality, we can choose the difference between vector indices i and 1, that is

$$\delta_i = x_1 - x_i$$

$$\zeta_i = \xi_1 - \xi_i$$

Then

$$\dot{\delta}_i = (A + BRC)\delta_i + BKC\left(\sum_{j=1}^{n}(a_{ij} - a_{1j})\delta_j - \sum_{j=1}^{n}a_{ij}\delta_i\right) + BS\zeta_i + BL\left(\sum_{j=1}^{n}(a_{ij} - a_{1j})\zeta_j - \sum_{j=1}^{n}a_{ij}\zeta_i\right)$$

$$\dot{\zeta}_i = M\zeta_i + NC\delta_i + HC\left(\sum_{j=1}^{n}(a_{ij} - a_{1j})\delta_j - \sum_{j=1}^{n}a_{ij}\delta_i\right) + V\left(\sum_{j=1}^{n}(a_{ij} - a_{1j})\zeta_j - \sum_{j=1}^{n}a_{ij}\zeta_i\right)$$

that is

$$\dot{\varsigma} = [I \otimes \mathcal{A} - (\mathcal{L} + \mathbf{1}\cdot\alpha^{\mathrm{T}}) \otimes \mathcal{B}]\varsigma \tag{12}$$

where

$$\varsigma_i = \begin{pmatrix} \delta_i \\ \zeta_i \end{pmatrix}, \quad \varsigma = \begin{pmatrix} \varsigma_2 \\ \vdots \\ \varsigma_n \end{pmatrix}$$

$$\mathcal{A} = \begin{pmatrix} A + BRC & BS \\ NC & M \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} BKC & BL \\ HC & V \end{pmatrix}$$

Noting the structure of Laplacian matrix $\mathcal{L}_g$ defined in (3), we can easily deduce the following conditions:

1. The remaining $n-1$ eigenvalues of $\mathcal{L}_g$ (except 0) are determined by $\mathcal{L} + \mathbf{1}\cdot\alpha^{\mathrm{T}}$;
2. There exist a Jordan matrix J and an invertible matrix T such that $T^{-1}(\mathcal{L} + \mathbf{1}\cdot\alpha^{\mathrm{T}})T = J$.

Thus $(T \otimes I)^{-1}[I \otimes \mathcal{A} - (\mathcal{L} + \mathbf{1}\cdot\alpha^{\mathrm{T}}) \otimes \mathcal{B}](T \otimes I) = I \otimes \mathcal{A} - J \otimes \mathcal{B}$. This implies that the eigenvalues of $[I \otimes \mathcal{A} - (\mathcal{L} + \mathbf{1}\cdot\alpha^{\mathrm{T}}) \otimes \mathcal{B}]$ are governed by $\bar{\mathcal{A}} + \bar{\mathcal{B}}\bar{\mathcal{K}}\bar{\mathcal{C}}_i$. Hence, the consensus is reached if and only if $\bar{\mathcal{A}} + \bar{\mathcal{B}}\bar{\mathcal{K}}\bar{\mathcal{C}}_i$ is Hurwitz for a certain $\bar{\mathcal{K}}$, $i = 2, \ldots, n$.

To prove the second part, we define

$$\bar{T} = [\mathbf{1}\sharp] \in \mathcal{R}^{n\times n}, \quad \bar{T}^{-1} = \begin{pmatrix} r^{\mathrm{T}} \\ \sharp \end{pmatrix},$$

$$\bar{T}^{-1}\mathcal{L}_g\bar{T} = \bar{\mathcal{J}} = \begin{pmatrix} 0 & 0 \\ 0 & \bar{J}_d \end{pmatrix}$$

where $\sharp$ denotes entries we are not interested in and $\bar{J}_d$ is a Jordan matrix with non-zero eigenvalues of $\mathcal{L}_g$ as its

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diagonal entries. It is easy to observe that $\dot{\upsilon} = (I \otimes \mathcal{A} + \mathcal{L}_g \otimes \mathcal{B})\upsilon$, which gives the solution

$$\begin{aligned} \upsilon(t) &= e^{(I\otimes\mathcal{A}+\mathcal{L}_g\otimes\mathcal{B})t}\upsilon(0) \\ &= (\bar{T} \otimes I)e^{(I\otimes\mathcal{A}+\bar{\mathcal{J}}\otimes\mathcal{B})t}(\bar{T}^{-1} \otimes I)\upsilon(0) \\ &= (\bar{T} \otimes I)\begin{pmatrix} e^{\mathcal{A}t} & 0 \\ 0 & e^{(I\otimes\mathcal{A}+\bar{\mathcal{J}}_d\otimes\mathcal{B})t} \end{pmatrix}(\bar{T}^{-1} \otimes I)\upsilon(0) \end{aligned} \tag{13}$$

By Theorem 3, $(I \otimes \mathcal{A} + \bar{\mathcal{J}} \otimes \mathcal{B})$ is Hurwitz. Hence, (8) holds. Thus, we complete the proof. □

Remark 2: Based on Theorem 3, we can see that the convergent rate is determined by $\bar{\mathcal{K}}$. However, the convergence (consensus) value is not influenced by $\bar{\mathcal{K}}_2$ from (8). Note that here $\bar{\mathcal{K}}$ is a variable matrix without structure constraint, thus we can easily derive that $\mathbb{K}_d \neq \emptyset$ if and only if $\hat{\mathbb{K}}_{d1} = \{\hat{\mathcal{K}}_1 : \bar{\mathcal{A}} + \bar{\mathcal{B}}\hat{\mathcal{K}}_1\hat{\mathcal{C}} \text{ is Hurwitz}\} \neq \emptyset$.

3.2 Partial local information and separation principle

Consider the similarity transformation $\Theta_i = \Pi\Psi_i\Pi^{-1}$, where $\Psi_i = \bar{\mathcal{A}} + \bar{\mathcal{B}}\bar{\mathcal{K}}\bar{\mathcal{C}}_i$ and $\Pi = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$. Θ_i can be simplified as

$$\begin{pmatrix} A + BS_1 + \lambda_i BL_1 & \lambda_i BL + BS \\ 0 & A - NC + BRC + \lambda_i(-HC + BKC) \end{pmatrix}$$

by defining

$$\begin{aligned} S = S_1 - RC, L = L_1 - KC \\ -A + NC + M - BS_1 = 0, HC - BL_1 + V = 0 \end{aligned} \tag{14}$$

where S_1, L_1, N_1 and H_1 are auxiliary matrices. Now by rearranging the controller gains and applying Lemma 1, we have the following result [16].

Corollary 4: Assume that the controller gain $\bar{\mathcal{K}}$ satisfies (14) and one of the following additional constraints

$$N = N_1 + BR, \quad H = BK, \quad S_1 = 0 \tag{15}$$

$$H = H_1 + BK, \quad N = BR, \quad S_1 = 0 \tag{16}$$

$$H = H_1 + BK, \quad N = BR, \quad L_1 = 0 \tag{17}$$

Then there exists a protocol (4)–(5) such that consensus is asymptotically reached for all initial states if and only if (1) (A, B) is stabilisable; (2) (A, C) is detectable; and (3) if A is not Hurwitz, $\mathcal{G}$ contains a spanning tree.

Remark 3: The results stated in Corollary 4 together with Lemma 1 are interesting. They lead to the so-called separation principle. For example, under the constraints (14) and (15), we can design the gains L_1 and N_1 separately. L_1 can be designed using the procedure introduced in the proof of Lemma 1. For N_1, simply choose one such that $A - N_1C$ is stable. R and K can be arbitrarily assigned. And then L, N, S, H, M and V can be obtained accordingly. Another interesting case is that $N = N_1 + BR$, $H = BK$, $L_1 = 0$, $K \neq 0$ together with (14), which is not related to λ_i.

Now we consider some special cases of Corollary 4. First, we consider the case that N and R are zero, that is, y_i is unknown for agent i, which will reduce the DOFC protocol (4)–(5) to observer-based protocols satisfying the condition of Corollary 4

$$\text{Protocol 1:} \begin{cases} \dot{\xi}_i = A\xi_i + BL\sum_{j=1}^{n} a_{ij}(\xi_j - \xi_i) + H\sum_{j=1}^{n} a_{ij}\chi_{ji} \\ u_i = L\sum_{j_1}^{n} a_{ij}(\xi_j - \xi_i) \end{cases} \tag{18}$$

$$\text{Protocol 2:} \begin{cases} \dot{\xi}_i = A\xi_i + BS\xi_i + H\sum_{j=1}^{n} a_{ij}\chi_{ji} \\ u_i = S\xi_i \end{cases} \tag{19}$$

where $\chi_{ji} = y_j - C\xi_j - y_i + C\xi_i$. Note that (19) is suggested in [12, 14] and (18) is proposed for discrete-time cases in [10].

Second, we consider the case that besides $N = 0$, $R = 0$, the gains V and L are also zero matrices, that is, both y_i and $\xi_j - \xi_i$ are unknown. Note that this can reduce the communication load. If (18) or (19) is used, then the consensusability is achieved only if A is stable. When A is not stable, a stronger condition may be enforced. For example, if further $S = 0$ (or $H = 0$), then $\mathbb{K} = \{L : A + \lambda_i BLC \text{ is Hurwitz }, i = 2, \ldots, n\} \neq \emptyset$ is the necessary and sufficient condition for $\mathbb{K}_d \neq \emptyset$ (in this case, M must be stable).

Third, let us look at some other interesting cases. For example, in the case that $y_j - y_i$ is unknown, if $K = 0$, $R = 0$, $H = 0$, $S = 0$, $M = A - NC$, $V = BL$, then the separation principle holds. For the case of $V = 0$, $L = 0$, [13] provides some similar results. In addition, if one of the following conditions holds: (1) $R = 0$, $N = 0$, $H = 0$, $S = 0$, $M = A$, $V - BKC - BL = 0$; (2) $R = 0$, $N = 0$, $H = 0$, $L = 0$, $M - A - BS = 0$, $V = BKC$; (3) $\bar{\mathcal{K}}_1$ is a zero matrix, the problem is similar to the one with static output feedback protocol and can be partially solved using Lemma 16 in [16].

Remark 4: Now a natural question is what is the benefit of design with more information. First, we can see that the existence condition for consensusability and formationability with less information is generally stronger than that with more information. For example, the existence condition of the aforementioned protocols with $N = 0$, $R = 0$, $L = 0$ and $V = 0$ is stronger than that of (18) and (19). Second, the extra variables in the formulation may bring more freedom in pole/eigenstructure assignment, thus the consensus convergence rate can be faster. Third, the consensus value may be adjusted. For example, using the protocols (19) such that $\mathbb{K}_d \neq 0$, $x_i \to (r^{\mathrm{T}} \otimes e^{A})X(0)$ when $t \to \infty$, where $X(0) = [x_1(0)^{\mathrm{T}}, \ldots, x_n(0)^{\mathrm{T}}]^{\mathrm{T}}$. If we add a $R \neq 0$ to the protocols (19) in (4)–(5), then $x_i \to (r^{\mathrm{T}} \otimes e^{A+BRC})X(0)$ when $t \to \infty$.

Remark 5: Formation control can be viewed as a special case of consensus with additional formation vectors $\mathcal{H} = (h_1^{\mathrm{T}}, \ldots, h_n^{\mathrm{T}})^{\mathrm{T}}$ [16], that is, if there exists a $u(t) \in \mathcal{U}$, such that $\lim_{t\to\infty} \|x_j(t) - x_i(t) - (h_j - h_i)\| = 0$, $i, j = 1, \ldots, n$, then the system asymptotically reaches a formation $\mathcal{H}$. When the information exchange of internal state is allowed, we design the formation protocol as follows

$$\dot{\xi}_i = M(\xi_i - h_i) + V\sum_{j=1}^{n} a_{ij}\bar{\zeta}_{ji} + N(y_i - Ch_i) + H\sum_{j=1}^{n} a_{ij}\chi_{ji} \tag{20}$$

$$u_i = S(\xi_i - h_i) + L\sum_{j=1}^{n} a_{ij}\bar{\zeta}_{ji} + R(y_i - Ch_i) + K\sum_{j=1}^{n} a_{ij}\chi_{ji} \tag{21}$$

where $\bar{\zeta}_{ji} = \xi_j - \xi_i - h_j + h_i$ and χ_{ji} are defined in (18). By introducing the new vectors $\bar{\delta}_i = x_1 - x_i - h_1 + h_i$ and $\bar{\zeta}_{1i} = \xi_1 - \xi_i - h_1 + h_i$, we can obtain the following result:

Corollary 5: Consider the system (1)–(2) and assume that $\mathcal{G}$ contains a spanning tree. Then, there exists a protocol (20)–(21) such that the prescribed formation is asymptotically reached for all initial states if and only if $\mathbb{K}_d \neq \emptyset$ and $A(h_i - h_j) = 0$, $i, j = 1, \ldots, n$.

Note that besides the choice of the controller gains, the position of h_i in the structure also affects the formationability. In fact, if we slightly change (21) to

$$u_i = S\xi_i + R(y_i - Ch_i) + L\sum_{j=1}^{n} a_{ij}\bar{\zeta}_{ji} + K\sum_{j=1}^{n} a_{ij}\chi_{ji}$$

then the formationability is equivalent to $\mathbb{K}_d \neq \emptyset$ and $(A + BS)(h_i - h_j) = 0$, $i, j = 1, \ldots, n$. Obviously, the matrix S introduces some flexibility for the formation control design.

4 Switching topologies

Consider an infinite sequence of non-empty, bounded and contiguous time intervals $[t_k, t_{k+1})$, $k = 0, 1, \ldots$, with $t_0 = 0$, $t_{k+1} - t_k \leq T$ for some constant $T > 0$. Assume that for each $[t_k, t_{k+1})$, there is a sequence of non-overlapping subintervals $[t_k^0, t_{k+1}^1), \ldots, [t_k^j, t_{k+1}^{j+1}), \ldots, [t_k^{m_k-1}, t_{k+1}^{m_k})$, $t_k = t_k^0$, $t_{k+1} = t_{k+1}^{m_{k+1}}$ satisfying $t_{k+1}^{j+1} - t_{k+1}^j \geq \tau$, $0 \leq j \leq m_k - 1$ for some integer $m_k > 0$ and a given constant $\tau > 0$ such that $\mathcal{G}(t)$ is not changed for $t \in [t_k^j, t_{k+1}^{j+1})$ and changed at time t_{k+1}^{j+1}. Let $\mathbb{S}$ denote an index set for all possible digraphs defined on the vertices $\{1, 2, \ldots, n\}$ and $\mathcal{S} \subset \mathbb{S}$ be the index set of actual digraphs of switching topologies of system dynamics. There is no doubt that $\mathcal{S}$ is a finite set if $n < \infty$.

It is well-known that a switching system is asymptotically stable if all individual subsystems are asymptotically stable and the switching is sufficiently slow, that is, τ is large enough, so as to allow the transient effects to dissipate after each switch [26]. If we assume that the dwell time τ is larger than the lower-bound/critical dwell time as stated in [26], then there exists a protocol (18) or (19) such that a consensus is asymptotically reached for all initial states if (A, B) is stabilisable, (A, C) is detectable, and the finite switching topology $\mathcal{G}(t)$ always contains a spanning tree. Motivated by [8], we can also derive a similar conclusion for the weakly connected and balanced digraph $\mathcal{G}(t)$ without constraint on the dwell time.

Using the consensusability result in Corollary 4, we can easily derive a protocol for sufficiently slow-switching connected topologies, as long as the switching topologies are known *a priori*, or at least the minimum non-zero eigenvalue of all the Laplacian matrices is known *a priori*. In a sense, this knowledge on system topologies is global. If such knowledge is unknown, there are two options. The first one is based on a greedy-like algorithm. Given that there are only a finite number of connected topologies, we can always find the minimum non-zero eigenvalue for each topology, although it may be conservative since not all topologies are used. The second option is to use an adaptive procedure to find proper eigenvalue-related coupling weights [27]. However, the spanning tree condition and slow switching condition are conservative, which motivates us to relax the requirement by using the concept of the so-called joint spanning tree in the following.

Consider the graph $\mathcal{G}(t)$ for $t \in [t_k, t_{k+1})$ as having a joint spanning tree if the union of the graphs $\mathcal{G}(t)$ for $t \in [t_k, t_{k+1})$ has a spanning tree. For a $\sigma(t) \in \mathcal{S}$, $t \in [t_k, t_{k+1})$, denote the associated graph as $\mathcal{G}_{\sigma(t)}$, whose components are $S^1_{\sigma(t)}, \ldots, S^{\bar{n}_{\sigma(t)}}_{\sigma(t)}$ arranged in the order of the numbers of vertices (from largest to smallest). Without loss of generality, assume that there exist some components $S^1_{\sigma(t)}, \ldots, S^{n_{\sigma(t)}}_{\sigma(t)}$, $n_{\sigma(t)} \leq \bar{n}_{\sigma(t)}$, whose $\mathcal{L}^j + \mathbf{1}\alpha^{j^{\mathrm{T}}}$, $j = 1, \ldots, n_{\sigma(t)}$, have eigenvalues of positive real part, where $\mathcal{L}^j$ and α^j are defined similarly to $\mathcal{L}$ and α. It means that these components are not isolated vertices. Define $\ell(\sigma(t)) = \left\{v : v \in \bigcup_{j=1}^{n_{\sigma(t)}} \mathbb{V}(S^j_{\sigma(t)})\right\}$, where $\mathbb{V}(S^j_{\sigma(t)})$ is the index set of all vertices of the corresponding components $S^j_{\sigma(t)}$ and $|\mathbb{V}(S^j_{\sigma(t)})| \geq 2$. We can easily derive that if $\mathcal{G}(t)$ contains a joint spanning tree for each interval $[t_k, t_{k+1})$, then $\bigcup_{t\in[t_k,t_{k+1})} \ell(\sigma(t)) = \{1, \ldots, n\}$.

Theorem 6: Assume that $\mathcal{G}(t)$ is undirected and A has no eigenvalue of positive real part. There exists a protocol (18) (or (19)) such that consensus is asymptotically reached for all initial states if (A, B) is stabilisable, (A, C) is detectable, and the switching topology $\mathcal{G}(t)$ contains a joint spanning tree for each interval $[t_k, t_{k+1})$.

Proof: We only consider the case (19). A constructive method similar to Lemma 1 is applied. Let $\lambda_i^\sigma = \alpha_i^\sigma + \jmath\beta_i^\sigma$ be the eigenvalues of all Laplacian matrices $\mathcal{L}_{g(\sigma)}$ of system dynamics, $\sigma(t) \in \mathcal{S}$. Arrange all the non-zero α_i^σ for all $\sigma(t) \in \mathcal{S}$ into a set $\mathbb{A}$. Index all λ_i^σ as the set $\mathbb{L}$.

It is easy to find that $\Psi_l = \bar{\mathcal{A}} + \bar{\mathcal{B}}\bar{\mathcal{K}}\bar{\mathcal{C}}_l$, $l \in \mathbb{L}$, is similar to $\Theta_l = \begin{pmatrix} A + BS & BS \\ 0 & A - \lambda_l HC \end{pmatrix}$. Construct the H using the method in Lemma 1, $H = \frac{\bar{\tau}}{2}P^{-1}C^{\mathrm{T}}$, where P is a solution to $A^{\mathrm{T}}P + PA - \tau C^{\mathrm{T}}C < 0$, $A^{\mathrm{T}}P + PA \leq 0$, $P > 0$, $\tau > 0$, $\bar{\tau} = \frac{\tau}{\alpha}$ and $\alpha = \min \mathbb{A}$. Construct $S = -B^{\mathrm{T}}\bar{Q}^{-1}$, where $\bar{Q}$ is the solution of $A\bar{Q} + \bar{Q}A^{\mathrm{T}} - BB^{\mathrm{T}} < 0$, $\bar{Q} > 0$. There are two cases based on whether $\lambda_l = 0$.

When $\lambda_l \neq 0$, for certain $W_l > 0$, $(A - \lambda_l HC)^{\mathrm{T}}P + P(A - \lambda_l HC) = -W_l < 0$. Now the task is to find a common Lyapunov matrix $\mathcal{P} = \begin{pmatrix} Q & 0 \\ 0 & \phi P \end{pmatrix} > 0$ for all Θ_l such that the Lyapunov inequality holds, that is, $\mathcal{P}\Theta_l + \Theta_l^{\mathrm{T}}\mathcal{P} < 0$, where $Q = \bar{Q}^{-1}$. That is possible since for $\mathcal{W}_l = W_c - \frac{1}{\phi}QBSW_l^{-1}S^{\mathrm{T}}B^{\mathrm{T}}Q$ where $W_c = \phi((A + BS)^{\mathrm{T}}Q + Q(A + BS))$, by adjusting ϕ, that is,

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making ϕ large enough, $\mathcal{W}_l < 0$ always holds for $\forall l \in \mathbb{L}$ as $\mathbb{L}$ is a finite set. Thus, $\mathcal{P}\Theta_l + \Theta_l^{\mathrm{T}}\mathcal{P} < 0$ holds for $\forall l \in \mathbb{L}$. Define $\bar{\mathcal{P}} = \Pi^{\mathrm{T}}\mathcal{P}\Pi$. Then

$$\bar{\mathcal{P}}\Psi_l + \Psi_l^{\mathrm{T}}\bar{\mathcal{P}} = \Pi^{\mathrm{T}}(\mathcal{P}\Theta_l + \Theta_l^{\mathrm{T}}\mathcal{P})\Pi < 0$$

For $\forall l \in \mathbb{L}$, there exists a $\varphi > 0$ such that $\bar{\mathcal{P}}\Psi_l + \Psi_l^{\mathrm{T}}\bar{\mathcal{P}} + \varphi I < 0$. When $\lambda_l = 0$, we have $\bar{\mathcal{P}}\Psi_l + \Psi_l^{\mathrm{T}}\bar{\mathcal{P}} \le 0$.

Owing to the fact stated in Lemma 2, there exist invertible matrices T_l such that $(T_l \otimes I)^{-1}[I \otimes \mathcal{A} - (\mathcal{L}^l + \mathbf{1}\cdot\alpha^{l^{\mathrm{T}}})\otimes \mathcal{B}](T_l \otimes I) = I \otimes \Theta_l$. $T_l^{\mathrm{T}}(\mathcal{L}^l + \mathbf{1}\cdot\alpha^{l^{\mathrm{T}}})T_l = \Lambda_l = \mathrm{diag}\{\lambda_l^{(1)}, \ldots, \lambda_l^{(l_n)}\}$, where $\lambda_l^{(j)}$ are corresponding eigenvalues. Thus, we define $\bar{\varsigma} = (T_l \otimes I)\varsigma$. Since T_l is invertible, instead of proving the stability of ς, we can prove the stability of $\bar{\varsigma}$.

In the following, we use the Lyapunov stability theory [28] with a similar technique in [17] to prove the consensusability. Define the Lyapunov function for the switching system $\dot{\bar{\varsigma}}$ similarly to (12)

$$V(t) = \bar{\varsigma}^{\mathrm{T}}(I \otimes \bar{\mathcal{P}})\bar{\varsigma}$$

Clearly, $V(t) > 0$ for $\bar{\varsigma} \neq 0$.

$$\begin{aligned}\dot{V}(t) &= \bar{\varsigma}^{\mathrm{T}}((I \otimes \mathcal{A} - \Lambda_l \otimes \mathcal{B})(I \otimes \bar{\mathcal{P}}) \\ &\quad + (I \otimes \bar{\mathcal{P}})(I \otimes \mathcal{A} - \Lambda_l \otimes \mathcal{B}))\bar{\varsigma} \\ &= \bar{\varsigma}^{\mathrm{T}}(I \otimes (\bar{\mathcal{P}}\Psi_l + \Psi_l^{\mathrm{T}}\bar{\mathcal{P}}))\bar{\varsigma} < -\varphi \sum_{j \in \ell(l)} \bar{\varsigma}_j^{\mathrm{T}}\bar{\varsigma}_j\end{aligned}$$

Consider the infinite sequences $V(t_i)$, $i = 0, 1, \ldots$, based on Cauchy's convergence criteria. There exists a positive number $\wp(\epsilon)$ for any $\epsilon > 0$ such that $\forall k \ge \wp(\epsilon)$, $|V(t_{k+1}) - V(t_k)| < \epsilon$ or $|\int_{t_k}^{t_{k+1}} \dot{V}(t)dt| < \epsilon$. that is, in the interval $[t_k, t_{k+1})$

$$\int_{t_k^0}^{t_k^1} \dot{V}(t)\mathrm{d}t + \cdots + \int_{t_k^{m_k-1}}^{t_k^{m_k}} \dot{V}(t)\mathrm{d}t > -\epsilon$$

For each sub-integral, $\int_{t_k^i}^{t_k^{i+1}} \dot{V}(t)\mathrm{d}t < -\int_{t_k^i}^{t_k^{i+1}} \varphi \sum_{j \in \ell(t_k^i)} \bar{\varsigma}_j^{\mathrm{T}}\bar{\varsigma}_j\,\mathrm{d}t < -\int_{t_k^i}^{t_k^i+\tau} \varphi \sum_{j \in \ell(t_k^i)} \bar{\varsigma}_j^{\mathrm{T}}\bar{\varsigma}_j\,\mathrm{d}t$. Thus $\epsilon > \varphi \int_{t_k^0}^{t_k^0+\tau} \sum_{j \in \ell(t_k^0)} \bar{\varsigma}_j^{\mathrm{T}}\bar{\varsigma}_j\,\mathrm{d}t + \cdots + \int_{t_k^{m_k-1}}^{t_k^{m_k-1}+\tau} \sum_{j \in \ell(t_k^{m_k-1})} \bar{\varsigma}_j^{\mathrm{T}}\bar{\varsigma}_j\,\mathrm{d}t$. Since m_k is finite, for $k > \wp(\epsilon)$, $\epsilon > \varphi \int_{t_k^i}^{t_k^i+\tau} \sum_{j \in \ell(t_k^i)} \bar{\varsigma}_j^{\mathrm{T}}\bar{\varsigma}_j\,\mathrm{d}t$, $i = 0, 1, \ldots, m_k - 1$, that is

$$\lim_{t \to \infty} \int_{t_k^i}^{t_k^i+\tau} \sum_{j \in \ell(t_k^i)} \bar{\varsigma}_j^{\mathrm{T}}\bar{\varsigma}_j\,\mathrm{d}s = 0, \quad i = 0, 1, \ldots, m_k - 1$$

which implies that $\lim_{t\to\infty} \sum_{i=0}^{m_k-1} \int_{t_k^i}^{t_k^i+\tau} \sum_{j \in \ell(t_k^i)} \bar{\varsigma}_j^{\mathrm{T}}\bar{\varsigma}_j\,\mathrm{d}s = 0$.

Since $\bigcup_{t \in [t_k, t_{k+1})} \ell(\sigma(t)) = \{1, \ldots, n\}$ owing to the assumption of joint spanning tree, we can derive that $\lim_{t\to\infty} \int_{t_k^i}^{t_k^i+\tau} \sum_{j=1}^n \alpha_j \bar{\varsigma}_j^{\mathrm{T}}\bar{\varsigma}_j\,\mathrm{d}s = 0$, where $\alpha_j > 0$ are some positive integers. The fact $\dot{V}(t) < 0$ implies that $\bar{\varsigma}$ is bounded and so is $\dot{\bar{\varsigma}}$. Thus $\sum_{j=1}^n \alpha_j \bar{\varsigma}_j^{\mathrm{T}}\bar{\varsigma}_j$ is uniformly continuous. Applying Barbalat's lemma [28], we can see that $\lim_{t\to\infty} \sum_{j=1}^n \alpha_j \bar{\varsigma}_j^{\mathrm{T}}\bar{\varsigma}_j\,\mathrm{d}s = 0$. Hence, $\lim_{t\to\infty} \bar{\varsigma}_j = 0$ can be derived. So is ς_j. □

Remark 6: The results stated in Sections 3 and 4 can be extended to the leader–follower case. Suppose that the leader's dynamics satisfy

$$\dot{x}_0 = Ax_0 \tag{22}$$

$$y_0 = Cx_0 \tag{23}$$

and the other agents are followers with dynamics (1)–(2). Define a diagonal matrix $\mathcal{D} \in \mathcal{R}^{n\times n}$ with its diagonal entries $d_i = 1$ or, 0, that is, if the link $(0, i)$ exists, then $d_i = 1$, otherwise $d_i = 0$. Denote the new graph as $\bar{\mathcal{G}}$ which contains the leader agent 0 and following agents $1, \ldots, n$. Denote the Laplacian matrix of $\bar{\mathcal{G}}$ as $\mathcal{L}_g$. Consider the DOFC-based consensus protocol using the relative information as follows

$$\begin{aligned}\dot{\xi}_i = M\xi_i - Vd_i\xi_i + V\sum\nolimits_{j=1}^n a_{ij}(\xi_j - \xi_i) \\ + H\sum\nolimits_{j=1}^n a_{ij}(y_j - y_i) + Hd_i(y_0 - y_i)\end{aligned} \tag{24}$$

$$\begin{aligned}u = S\xi_i - Ld_i\xi_i + L\sum\nolimits_{j=1}^n a_{ij}(\xi_j - \xi_i) \\ + K\sum\nolimits_{j=1}^n a_{ij}(y_j - y_i) + Kd_i(y_0 - y_i)\end{aligned} \tag{25}$$

Some results for fixed topologies have been presented in [16]. For switching topologies, we have the following result.

Theorem 7: Assume that $\bar{\mathcal{G}}$ is undirected and A has no eigenvalue of positive real-part. There exists a protocol (24)–(25) under the equation constraint ($K = 0$, $S = 0$, $M = A$, $V = BL - HC$) or ($K = 0$, $L = 0$, $M = A - BS$, $V = HC$) such that leader–follower consensus is asymptotically reached for all initial states if (A, B) is stabilisable, (A, C) is detectable, and the switching topology $\bar{\mathcal{G}}(t)$ contains a joint spanning tree rooted at 0 for each interval $[t_k, t_{k+1})$.

5 Simulation example

Consider the following linearised dynamics of satellite systems [14].

$$\ddot{\bar{x}}_i - 2w_0\dot{\bar{y}}_i = u_{x_i}$$

$$\ddot{\bar{y}}_i + 2w_0\dot{\bar{x}}_i - 3w_0^2\bar{y}_i = u_{y_i}$$

$$\ddot{z}_i + w_0^2 z_i = u_{z_i}$$

where $r_i = [\bar{x}_i, \bar{y}_i, \bar{z}_i]^{\mathrm{T}}$ is the position component of the ith satellite in the rotating coordinate, $u_i = [u_{x_i}, u_{y_i}, u_{z_i}]^{\mathrm{T}}$ is control input and w_0 denotes the angular rate of the virtual satellite. $i = 1, \ldots, 4$. The communication topology $\mathcal{G}_1$ is shown in Fig. 1. We assume that only the r_i is known, while their speed is unknown. Then the above equations can be formulated into a 6-state ($x_i = [r_i^{\mathrm{T}}\dot{r}_i^{\mathrm{T}}]^{\mathrm{T}}$) and 3-input ($u_i$) system (1), that is

$$A = \begin{bmatrix} 0_3 & I_3 \\ A_1 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0_3 \\ I_3 \end{bmatrix}, \quad C = \begin{bmatrix} I_3 & 0_3 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3w_0^2 & 0 \\ 0 & 0 & -w_0^2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 2w_0 & 0 \\ -2w_0 & 0^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

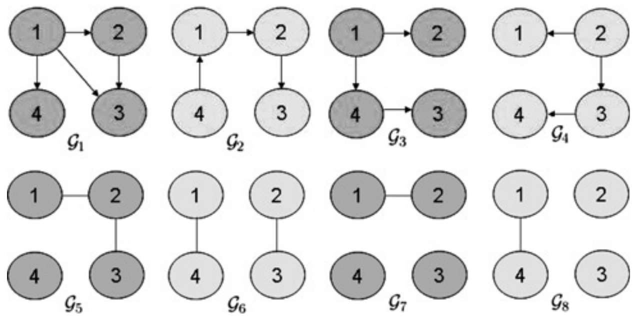

Fig. 1 *Communication topologies*

where I_3 is the 3×3 identity matrix and 0_3 is the 3×3 zero matrix. We intend to design a formation with $h_2 = (-100, 100, 0)$, $h_2 = (-100, 100, 0)$, $h_4 = (-100, 10, 173.21)$, $h_4 = (-100, 10, 173.21)$. Assume that $w_0 = 0.001$.

For the fixed topology case, we simply design a protocol (18). Through a simple LMI formulation as stated in Lemma 15 of [16] and Remark 3, we can obtain the following solution for (18):

$$L = \begin{bmatrix} -1.131 & 0.0005 & 0 & -0.5687 & 0.0001 & 0 \\ -0.0005 & -1.131 & 0 & -0.0001 & -0.5687 & 0 \\ 0 & 0 & -1.131 & 0 & 0 & -0.5687 \end{bmatrix}$$

$$H = \begin{bmatrix} 0.5688 & 0.0001 & 0 \\ -0.0001 & 0.5688 & 0 \\ 0 & 0 & 0.5687 \\ 1.131 & 0.0005 & 0 \\ -0.0005 & 1.131 & 0 \\ 0 & 0 & 1.131 \end{bmatrix}$$

Fig. 2 shows the formation trajectories using the controller (18), where △ denotes the initial positions with agent number and ○ the final positions after 50 s. We can see that the desired formation is achieved. Fig. 3 shows the leader–follower formation control, where △ denotes the initial positions of the followers and * the initial position of the leader.

Now let us look at the switching topology case. Note that A has no eigenvalue with positive real part. First, we consider the situation that all graphs contain the spanning tree. For example, suppose that the communication topologies uniformly switch among $\mathcal{G}_k$, $k = 1,2,3,4$, where $\mathcal{G}_k$ is defined in Fig. 1. We simulate the system behaviours for different switching intervals with $\tau = 0.5$ s and $\tau = 0.05$ s, respectively. Fig. 4 shows the corresponding formation control trajectories for $\tau = 0.5$ s and Fig. 5 for $\tau = 0.05$ s. Second, we consider the joint spanning tree cases. The corresponding switching topologies are $\mathcal{G}_k$, $k = 5,6,7,8$, as shown in Fig. 1. Note that all graphs contain no spanning tree. We also assume that the topology will change at time intervals of 0.5 and 0.05 s, respectively. Figs. 6 and 7 show the formation trajectories under such switching topologies.

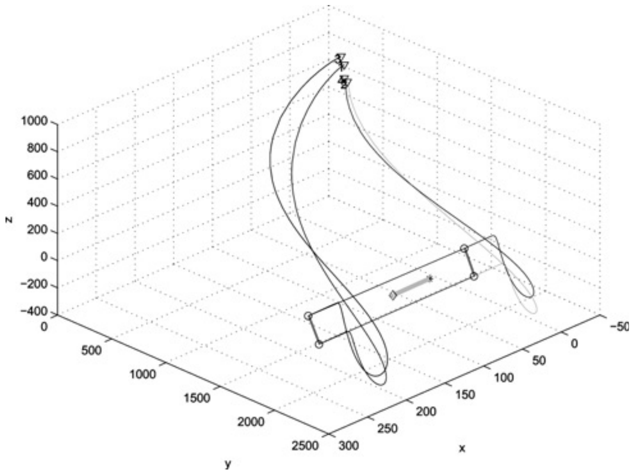

Fig. 3 *Leader–follower formulation using controller (18)*

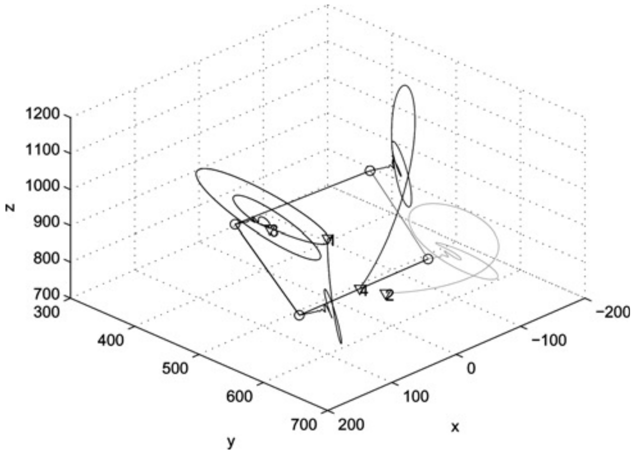

Fig. 4 *Formulation under switching digraphs: $\tau = 0.5$ s*

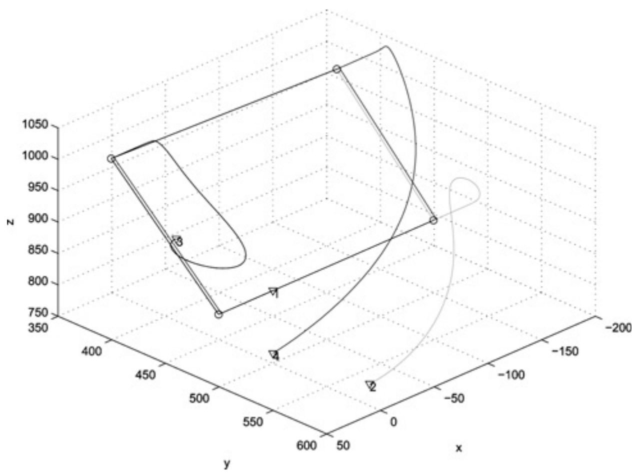

Fig. 2 *Leaderless formulation using controller (18)*

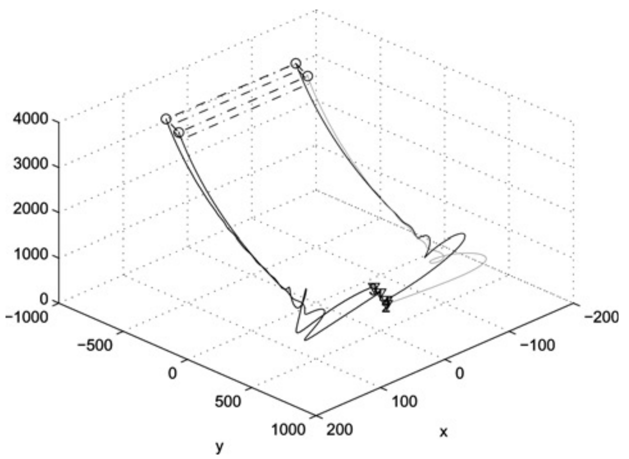

Fig. 5 *Formulation under switching digraphs: $\tau = 0.05$ s*

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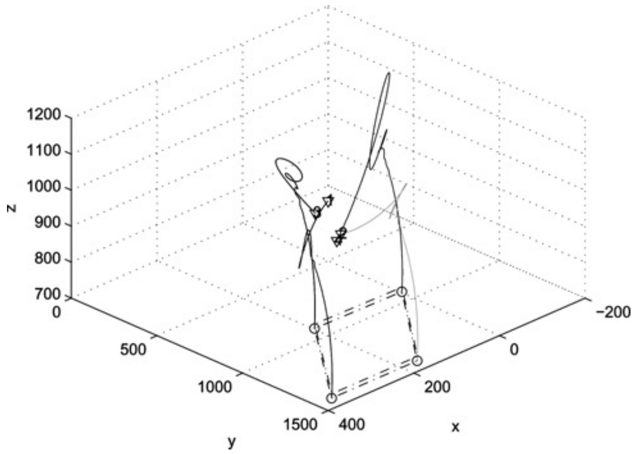

Fig. 6 *Formulation under non-connected switching graphs: τ = 0.5 s*

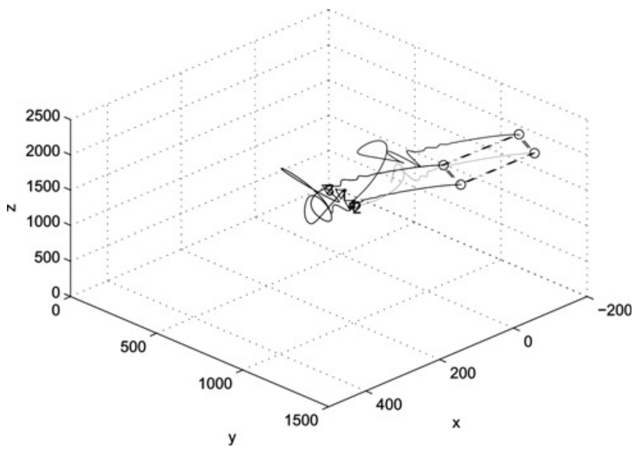

Fig. 7 *Formulation under non-connected switching graphs: τ = 0.05 s*

6 Conclusion

This paper presented some necessary and sufficient conditions for consensusability via dynamic output feedback. Most of these conditions were proved by constructive methods. The proposed protocols can not only be used for static communication topologies but also for switching communication topologies as long as the minimum non-zero eigenvalue of the Laplacian matrices of these topologies are known *a priori*. Our future work will further investigate the properties of these protocols such as the consensus under switching digraph. We will also address the corresponding problems for the cases with constrained input and partial consensus situations.

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