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Brief paper

Distributed consensus over digital networks with limited bandwidth and time-varying topologies[☆]

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ABSTRACT

In this paper, we consider discrete-time distributed average-consensus with limited communication data-rate and time-varying communication topologies. We design a distributed encoding-decoding scheme based on quantization of scaled innovations and a control protocol based on a symmetric compensation method. We develop an adaptive scheme to select the numbers of quantization levels according to whether the associated channel is active or not. We prove that if the network is jointly connected, then under the protocol designed, average-consensus can be asymptotically achieved, and the convergence rate is quantified. Especially, if the duration of any link failure in the network is bounded, then the control gain and the scaling function can be selected properly such that 5-level quantizers suffice for asymptotic average-consensus with an exponential convergence rate.

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1. Introduction

In recent years, distributed coordination over multi-agent networks has become one of the most dynamic directions of networked control systems and complexity science. When digital communications are adopted, due to the finite channel capacity, only a finite number of bits of information can be exchanged between neighbors at each time step. The communication between agents is a combined process of encoding, transmission, receiving and decoding. Similar to the traditional single-agent control theory, the research on multi-agent coordination started initially without considering the communication constraint, then with the deepening of the study, various aspects of communication networks were gradually paid attention to by researchers. Recently, quantized consensus or consensus with quantized communications has drawn the attention of more and more researchers (Kashyap, Basar, & Srikant, 2007; Carli, Fagnani, Frasca, & Zampieri, 2007; Frasca, Carli, Fagnani, & Zampieri, 2009; Carli, Bullo, & Zampieri, 2010a; Carli, Fagnani, Frasca, & Zampieri, 2010b; Li, Fu, Xie, & Zhang, 2011).

Most of the existing works on distributed consensus with quantized communications assume time-invariant communication topologies. It is well-known that in multi-agent networks the communication topology is often time-varying due to many reasons such as link failure or the change of environment. The change of network topologies due to link failures is a kind of passive switching. A well distributed consensus algorithm should be robust against this kind of switching. In some other cases, the network topology may be changed on purpose, for example, the network switches among different modes according to high-level commands for performance optimization of the whole system. Due to the limited bandwidth, distributed consensus with quantized communications and time-varying topologies is of significance from both theoretic and engineering points of view. Dimarogonas and Johansson (2008) considered distributed consensus based on quantized relative state information with static infinite-level uniform and logarithmic quantizers and proved that if the communication graph remains a tree for all consecutive time intervals between switching points then consensus can be achieved provided the quantization density is sufficiently high. Nedić, Olshevsky, Ozdaglar, and Tsitsiklis (2009) considered quantized average-consensus with time-varying topology and static infinite-level uniform quantizers. They proved that if the

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communication graph is jointly connected, then approximate average-consensus can be achieved. Carli et al. (2010b) and Lavaei and Murray (2009a,b) considered a random gossip algorithm with quantized communications based on an infinite-level uniform quantizer. Carli et al. (2010b) proved that if the edges selected form a connected graph, then the states of agents converge to an approximate average of the initial values up to the size of the quantization interval. More performance analysis can be found in Lavaei and Murray (2009a,b). Kar and Moura (2009) considered quantized average-consensus with random link failures by static infinite-level and finite-level uniform quantizers. They added a random dither before quantization to make the quantization error a "white" noise and proved that if the sequence of Laplacian matrices is i.i.d and the mean graph is connected all the time, then all agents' states eventually enter into a small neighborhood of the average of the initial states with high probability. Note that most of the above literature concentrates on static infinite-level quantizers and the steady-state error is not zero.

In this paper, we consider discrete-time average-consensus with limited communication bandwidth. The agents have real-valued states and communicate with each other through undirected digital networks. The network topology can be time-varying. There are two fundamental difficulties. One is that the communication channels between agents have only limited capacity, so only finite-level quantizers can be used, which may result in unbounded quantization errors. The other is that the neighbors of a given agent may change with time, which may result in a mismatch between the encoder and decoder designed for the fixed topology case (Li et al., 2011). To overcome these difficulties, we design a distributed encoding-decoding scheme based on quantization of scaled innovations, where a scaling function is used to avoid the saturation of the finite-level quantizers. Here, unlike the fixed topology case, each communication channel has its own encoder and decoder. The number of quantization levels of each quantizer depends on the status of the associated communication channel. We propose a control protocol based on symmetric compensation and design an adaptive scheme to select the numbers of quantization levels. The number of quantization levels of each quantizer is tuned on-line according to whether the associated channel is active or not at the last step. By using this method, we prove that if the network is jointly connected, then under the protocol designed, average-consensus can be asymptotically achieved without steady-state error, which means that average-consensus can be achieved with arbitrary precision as time goes on. We show that the convergence rate is no lower than that of the scaling function. Especially, if the duration of any link failures in the network is bounded, then the control gain and the scaling function can be selected properly such that 5-level quantizers suffice for asymptotic average-consensus with an exponential convergence rate.

The remainder of this paper is organized as follows. In Section 2, we present the model of the network and formulate the problem to be investigated. In Section 3, we propose an encoding-decoding scheme and a control protocol. In Section 4, we consider how to select the control parameters to ensure asymptotic average-consensus and give the convergence rate of the closed-loop system. In Section 5, we give a numerical example to demonstrate the validity of our protocol. In Section 6, some concluding remarks and future research topics are provided.

The following notation will be used throughout this paper: **1** denotes a column vector with all ones. I denotes the identity matrix with an appropriate size. For a given set $\mathcal{S}$, the number of its elements is denoted by $|\mathcal{S}|$. For a given vector or matrix A, its transpose is denoted by A^T, its ∞-norm is denoted by $\|A\|_\infty$, and its Euclidean norm is denoted by $\|A\|$. For a given positive number x, the logarithm of x with base 2 is denoted by $\log_2(x)$; the maximum integer less than or equal to x is denoted by $\lfloor x \rfloor$, and the minimum integer greater than or equal to x is denoted by $\lceil x \rceil$. For two real symmetric matrices A and B, we denote $A \le B$ if $B - A$ is positive semi-definite.

2. Model description and problem formulation

We consider a network of N agents with the dynamics

$$x_i(t+1) = x_i(t) + u_i(t), \quad t = 0, 1, \ldots, \ i = 1, 2, \ldots, N, \tag{1}$$

where $x_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ are, respectively, the state and input of the ith agent. The communications between agents at time t are represented by the undirected graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{A}(t)\}$, $t = 1, 2 \ldots$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of agents, and $\mathcal{A}(t) = [a_{ij}(t)] \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix of $\mathcal{G}(t)$ with element $a_{ij}(t) = 1$ or 0 indicating whether or not there is an active communication channel from j to i.

Since the communication graphs are undirected, $\mathcal{A}(t)$ is a symmetric matrix. The neighborhood of agent i at time t is defined as $N_i(t) = \{j \in \mathcal{V} \mid a_{ij}(t) = 1\}$. The cardinal number of $N_i(t)$ is called the degree of i at time t and is denoted by $d_i(t)$. Denote $\mathcal{N}_i = \bigcap_{t=1}^{\infty}\bigcup_{k=t}^{\infty} N_i(k)$.

The Laplacian matrix of $\mathcal{G}(t)$ is defined as $\mathcal{L}(t) = \mathcal{D}(t) - \mathcal{A}(t)$, where $\mathcal{D}(t) = \mathrm{diag}(d_1(t), \ldots, d_N(t))$. We represent the communication channel from j to i by pair (j, i). A sequence of active communication channels $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ is called a path from agent i_1 to agent i_k. The graph $\mathcal{G}(t)$ is called a connected graph if for any $i, j \in \mathcal{V}$, there is a path from i to j.

For the sequence $\{\mathcal{G}(t), t = 1, 2, \ldots\}$, we have the following assumption.

(A1) $\quad \mathcal{N}_i = \bigcup_{t=1}^{\infty} N_i(t)$.

Remark 1. Observe that $\mathcal{N}_i$ represents the set of agents which are the neighbors of agent i for an infinite number of times. Assumption (A1) means that if agent j is a neighbor of agent i for some time, then it is also a neighbor of i for an infinite number of times. Note that channels which only exist for a finite period will have no essential impact on the protocol design and closed-loop analysis.

In this paper, we aim to design an encoding-decoding scheme using a finite number of bits of data and an interaction protocol to achieve asymptotic average-consensus.

3. Protocol design

We assume that only symbolic data can be exchanged between agents. For each communication channel, the state value of the sender is firstly encoded to symbolic data and then transmitted, after the data is received, a decoder is used by the receiver to get an estimate of the sender's state. For agent i and agent j, where $j \in \mathcal{N}_i$, $i = 1, 2, \ldots, N$, the encoder ϕ_{ij} associated with i for channel (i, j) is designed as

$$\begin{cases} \xi_{ij}(0) = 0, \\ \xi_{ij}(t) = g(t-1)a_{ij}(t)s_{ij}(t) + \xi_{ij}(t-1), \\ s_{ij}(t) = q_t^{ij}\left(\dfrac{x_i(t) - \xi_{ij}(t-1)}{g(t-1)}\right), \quad t = 1, 2, \ldots, \end{cases} \tag{2}$$

where $\xi_{ij}(t)$ is the internal state of ϕ_{ij}, $x_i(t)$ and $s_{ij}(t)$ are the input and the output of ϕ_{ij}, respectively. Here, $q_t^{ij}(\cdot)$ is a quantizer, and $g(t) > 0$ is a scaling function. For agent $l \notin \mathcal{N}_i$, the channel (i, l) will never be active, and it is unnecessary to design an encoder for (i, l). Therefore, there are $|\mathcal{N}_i|$ encoders associated with agent i in total.

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The decoder φ_{ij} associated with agent j for channel (i, j) is given by

$$\begin{cases} \widehat{x}_{ij}(0) = 0, \\ \widehat{x}_{ij}(t) = g(t-1)a_{ij}(t)s_{ij}(t) + \widehat{x}_{ij}(t-1), \quad t = 1, 2, \ldots, \end{cases} \tag{3}$$

where $\widehat{x}_{ij}(t)$ is the output of φ_{ij}. When the channel (i, j) is active, after $s_{ij}(t)$ is received by agent j, the decoder φ_{ij} is activated to obtain the estimate $\widehat{x}_{ij}(t)$ of $x_i(t)$.

Remark 2. To our best knowledge, the idea of using dynamic encoding-decoding schemes for distributed consensus was firstly pursued in Carli et al. (2007). Here, the structures of the encoder (2) and the decoder (3) are similar to those in the previous work (Carli & Bullo, 2009; Carli et al., 2010a; Li et al., 2011), however, due to the time-variation of the communication topologies, there are two main differences.

- For the case with fixed communication topology, both the updating gain and the quantizer of the encoder are fixed (Li et al., 2011), while for the case with time-varying topologies, the parameters of the encoders and decoders, such as the updating gain $a_{ij}(t)$ and the numbers of the quantization levels of $q_t^{ij}(\cdot)$ are adjusted adaptively to the status of the channel (i, j).
- For the case with fixed communication topology, each agent needs only to maintain one encoder (Li et al., 2011), while for the case with time-varying topologies, since the status of the links to the neighbors are changed with time, we need to design $|\mathcal{N}_i|$ encoders for the ith agent.

Remark 3. In the above, to compute the outputs of encoder ϕ_{ij} and decoder φ_{ij} at time t, both agent i and agent j need to know $a_{ij}(t)$, that is, whether the communication channel between them is active. Note that if the switching of the communication topology is predesigned, then $a_{ij}(t)$ is obviously known, and if the status of the communication channel cannot be known *a priori*, such as the case with intermittent link failures or packet dropouts, then we can use a separate feedback communication channel to know $a_{ij}(t)$. The feedback channel is assumed to be reliable and each feedback will cost one bit of information transmission between neighbors.

We adopt finite-level uniform symmetric quantizers, that is, quantizer $q_t^{ij}(\cdot)$ takes the form:

$$q_K(y) = \begin{cases} 0, & -1/2 < y < 1/2, \\ i, & \dfrac{2i-1}{2} \le y < \dfrac{2i+1}{2}, \\ & i = 1, 2, \ldots, K-1, \\ K, & y \ge \dfrac{2K-1}{2}, \\ -q(-y), & y \le -1/2, \end{cases} \tag{4}$$

where K may change with time. Note that for the quantizer q_K, the number of quantization levels is $2K + 1$.

Remark 4. In the fixed topology case, we assume that the output 0 of the quantizer will not be sent and the number of bits of quantizer q_K is $\lceil \log_2(2K) \rceil$ (Li et al., 2011). Here, in order to avoid confusion with the case of inactive channels, the output 0 will be sent to the neighbors explicitly, so the number of bits of quantizer q_K is $\lceil \log_2(2K+1) \rceil$.

We denote the number of quantization levels of $q_t^{ij}(\cdot)$ by $K_{ij}(t)$, which depends on time, and propose a distributed protocol as

$$u_i(t) = h \sum_{j \in \mathcal{N}_i} a_{ij}(t)(\widehat{x}_{ji}(t) - \xi_{ij}(t)),$$
$$t = 0, 1, \ldots, \quad i = 1, 2, \ldots N, \tag{5}$$

where $h > 0$ is the control gain. Denote $e_{ji}(t) = \xi_{ji}(t) - x_j(t)$, $\Delta_{ij}(t) = s_{ij}(t+1) - \frac{x_i(t+1) - \xi_{ij}(t)}{g(t)}$.

From the following theorem, we can see that the protocol (2), (3) and (5) can preserve the average state of the closed-loop system.

Theorem 3.1. *If Assumption (A1) holds, then under the protocol (2), (3) and (5), the closed-loop system satisfies*

$$\frac{1}{N}\sum_{i=1}^{N} x_i(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(0), \quad t = 1, 2, \ldots \tag{6}$$

Proof. From (2), (3), it can be seen that $\xi_{ji}(t) \equiv \widehat{x}_{ji}(t), t = 0, 1 \ldots$. Then by the definition of $e_{ij}(t)$ and (5), we know that

$$\begin{aligned} u_i(t) &= h \sum_{j \in \mathcal{N}_i} a_{ij}(t)(\widehat{x}_{ji}(t) - \xi_{ij}(t)) \\ &= h \sum_{j \in \mathcal{N}_i} a_{ij}(t)[\widehat{x}_{ji}(t) - x_j(t) + x_j(t) - x_i(t) + x_i(t) - \xi_{ij}(t)] \\ &= h \sum_{j \in \mathcal{N}_i} a_{ij}(t)[e_{ji}(t) - e_{ij}(t)] + h \sum_{j \in \mathcal{N}_i} a_{ij}(t)[x_j(t) - x_i(t)]. \end{aligned}$$

By Assumption (A1), we know that $\sum_{j \in \mathcal{N}_i} a_{ij}(t)[x_j(t) - x_i(t)] = \sum_{j=1}^{N} a_{ij}(t)[x_j(t) - x_i(t)]$. Furthermore, noting that $a_{ij}(t) \equiv a_{ji}(t)$, it can be easily obtained that $\sum_{i=1}^{N} u_i(t) = 0$, which together with (1) leads to (6). □

Remark 5. Note that the control input (5) can be divided into $\sum_{j \in \mathcal{N}_i} a_{ij}(t)(\widehat{x}_{ji}(t) - x_i(t))$ and an error compensation term $\sum_{j \in \mathcal{N}_i} a_{ij}(t)(x_i(t) - \xi_{ij}(t))$. Due to the information loss during encoding-decoding, the control input $\sum_{j \in \mathcal{N}_i} a_{ij}(t)(\widehat{x}_{ji}(t) - x_i(t))$ is deviated from $\sum_{j \in \mathcal{N}_i} a_{ij}(t)(x_j(t) - x_i(t))$ by the weighted sum of estimation errors for the neighbor's states $\sum_{j \in \mathcal{N}_i} a_{ij}(t)e_{ji}(t)$. From the proof of Theorem 3.1, it can be seen that this estimation error sum can be compensated by $\sum_{j \in \mathcal{N}_i} a_{ij}(t)(x_i(t) - \xi_{ij}(t))$, which is just the weighted sum of estimation errors for $x_i(t)$ by the neighbors. The key idea is that for each channel (j, i), though the "receiving error" $e_{ji}(t)$ is unknown by agent i, the "sending error" $e_{ij}(t)$ can be obtained by using the state $x_i(t)$ minus the internal state $\xi_{ij}(t)$ of the encoder ϕ_{ij} of agent i itself. Thus, agent i is able to make compensation for the receiving error of agent j based on its sending error. By the symmetry of the whole network, the sum of overall receiving errors can be compensated and the average of the system is preserved.

4. Convergence analysis

We make the following assumptions:

(A2) There is a constant $d^* > 0$, such that $\sup_{t \ge 0} d_i(t) \le d^*$, $i = 1, 2, \ldots, N$.

(A3) $\max_i |x_i(0)| \le C_x$, $\max_{ij} |x_i(0) - x_j(0)| \le C_\delta$, where C_x and C_δ are known nonnegative constants.

(A4) There exist an integer $T > 0$ and a real constant $\rho > 0$, such that

$$\inf_{m \ge 0} \lambda_2 \left(\sum_{k=mT+1}^{(m+1)T} \mathcal{L}(k) \right) \ge \rho(T+1),$$

where $\lambda_2(\mathcal{L})$ denotes the second smallest eigenvalue of Laplacian matrix $\mathcal{L}$.

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Remark 6. It can be seen that Assumption (A4) holds if and only if there exist an integer $\widetilde{T} > 0$ and a constant $\widetilde{\rho} > 0$, such that

$$\inf_{m\geq 0} \lambda_2 \left(\frac{1}{\widetilde{T}} \sum_{k=m\widetilde{T}+1}^{(m+1)\widetilde{T}} \mathcal{L}(k) \right) \geq \widetilde{\rho}.$$

This means that the network flow $\{\mathcal{G}(t), t = 1, 2, \ldots\}$ is jointly connected (Jadbabaie, Lin, & Morse, 2003) over intervals $[m\widetilde{T}+1, (m+1)\widetilde{T}]$, $m = 0, 1, \ldots$, with the average algebraic connectivity being uniformly bounded below from zero. For example, if the network switches among a finite number of graphs which are jointly connected over bounded consecutive intervals (Jadbabaie et al., 2003), then (A4) holds.

It can be verified that if Assumption (A4) holds and $h \in (0, \frac{1}{2d^*})$, then $0 < h\rho \leq \frac{h \sup_{m\geq 0} \| \sum_{k=mT+1}^{(m+1)T} \mathcal{L}(k) \|_2}{T+1} \leq \frac{2hd^* T}{T+1} < \frac{T}{T+1}$. Denote $\mu = \sup_{t\geq 0} \frac{g(t)}{g(t+1)}$ and $\tilde{\mu} = \limsup_{t\to\infty} \frac{g(t)}{g(t+1)}$. From (2), (3) and (5), we can see that our protocol can be characterized by the following parameters: the control gain h, the number of quantization levels $K_{ij}(t)$, $t = 1, 2, \ldots, i = 1, 2, \ldots, N, j \in \mathcal{N}_i$, and the scaling function $g(t)$. We will consider how to select the control parameters to ensure asymptotic average-consensus. In the following theorem, we can see that for any given control gain h and scaling function $g(t)$, the number of quantization levels of each communication channel can be selected properly as follows: If the channel is active at time t, then the number of quantization levels at time $t+1$ can be kept constant, while if the channel is inactive, or there is a link failure, then the number of quantization levels of the quantizer can be increased for this channel, such that the uncertainty due to the link failure is counteracted by the improved information accuracy. The key is to make the quantizers unsaturated by selecting the design parameters.

Theorem 4.1. *Suppose Assumptions (A1)–(A4) hold. If $h \in (0, \frac{1}{2d^*})$, $\mu < \frac{1}{(1-\frac{h\rho}{T+1})^{1/2T}}$, and for any $i \in \mathcal{V}, j \in \mathcal{N}_i$, the numbers of quantization levels $K_{ij}(\cdot)$ satisfy*

$$K_{ij}(1) \geq \frac{C_x}{g(0)} - \frac{1}{2}, \tag{7}$$

$$K_{ij}(2) \geq \begin{cases} \dfrac{2hd^* C_\delta + (2hd^*+1)g(0)}{2g(1)} - \dfrac{1}{2}, & a_{ij}(1) = 1, \\ \dfrac{hd^* C_\delta}{g(1)} + \dfrac{\left(hd^* + K_{ij}(1) + \frac{1}{2}\right) g(0)}{g(1)} \\ -\dfrac{1}{2}, & a_{ij}(1) = 0, \end{cases} \tag{8}$$

$$K_{ij}(t+1) \geq \begin{cases} \kappa_{h,\mu} + \dfrac{\mu(2hd^*+1)}{2} - \dfrac{1}{2}, & a_{ij}(t) = 1, \\ \kappa_{h,\mu} + \dfrac{g(t-1)}{g(t)}(hd^* + K_{ij}(t)) \\ + \dfrac{\frac{g(t-1)}{g(t)} - 1}{2}, & a_{ij}(t) = 0, \end{cases}$$
$$t = 2, 3, \ldots \tag{9}$$

where

$$\kappa_{h,\mu} = \frac{\sqrt{2N} C_\delta hd^* \mu^2 (1 - \frac{h\rho}{T+1})^{1/2T}}{g(0)} + \frac{\sqrt{2N} h^2 \mu^2 (d^*)^2}{1 - \left(1 - \frac{h\rho}{T+1}\right)^{1/2T} \mu}, \tag{10}$$

then under the protocol (2), (3) and (5), the closed-loop system satisfies

$$\limsup_{t\to\infty} \frac{\max_{ij} |x_i(t) - x_j(t)|}{g(t)} \leq \frac{\sqrt{2N} hd^* \tilde{\mu}^2}{1 - \left(1 - \frac{h\rho}{T+1}\right)^{1/2T} \tilde{\mu}}. \tag{11}$$

Furthermore, if $g(t) \to 0$, $t \to \infty$, then

$$\lim_{t\to\infty} x_i(t) = \frac{1}{N} \sum_{j=0}^{N} x_j(0), \quad i = 1, 2, \ldots, N. \tag{12}$$

The proof of Theorem 4.1 needs detailed analysis of the closed-loop consensus error equation. Substituting the protocol (2), (3) and (5) into the system (1), the closed-loop system is given by

$$\begin{aligned} x_i(t+1) &= x_i(t) + h \sum_{j=1}^{N} a_{ij}(t)[x_j(t) - x_i(t)] \\ &\quad + h \sum_{j\in\mathcal{N}_i} a_{ij}(t) e_{ji}(t) - h \sum_{j\in\mathcal{N}_i} a_{ij}(t) e_{ij}(t), \\ &\quad i = 1, 2, \ldots, N. \end{aligned} \tag{13}$$

Denote $a_i(t) = [a_{1i}(t), a_{2i}(t), \ldots, a_{Ni}(t)]^T$, $\Sigma_1(t) = \mathrm{diag}\{a_1^T(t), \ldots, a_N^T(t)\}$, $\alpha_i(t) = [a_{i1}(t), a_{i2}(t), \ldots, a_{iN}(t)]^T$, $\Sigma_2(t) = \mathrm{diag}\{\alpha_1^T(t), \ldots, \alpha_N^T(t)\}$, $X(t) = [x_1(t), \ldots, x_N(t)]^T$, $\hat{\Delta}(t) = [\Delta_{11}(t), \Delta_{21}(t), \ldots, \Delta_{N1}(t), \Delta_{12}(t), \ldots, \Delta_{NN}(t)]^T$, $\bar{\Delta}(t) = [\Delta_{11}(t), \Delta_{12}(t), \ldots, \Delta_{1N}(t), \Delta_{21}(t), \ldots, \Delta_{NN}(t)]^T$, $\delta(t) = X(t) - J_N X(t)$, where $J_N = \frac{1}{N}\mathbf{1}\mathbf{1}^T$. Here, $\Delta_{ij}(t)$ is defined as 0 for $i = 1, 2, \ldots, N$, $j \notin \mathcal{N}_i$, $t = 1, 2, \ldots$. The closed-loop system (13) can be rewritten in the vector form

$$\begin{aligned} X(t+1) &= (I - h\mathcal{L}(t))X(t) \\ &\quad + hg(t-1)[\Sigma_1(t)\hat{\Delta}(t-1) - \Sigma_2(t)\bar{\Delta}(t-1)]. \end{aligned}$$

Then from the above and the definition of $\delta(t)$, noting that $\mathcal{L}(t)J_N = J_N\mathcal{L}(t) = 0$ and $J_N[\Sigma_1(t)\hat{\Delta}(t-1) - \Sigma_2(t)\bar{\Delta}(t-1)] = 0$, we have

$$\begin{aligned} \delta(t+1) &= (I - h\mathcal{L}(t))\delta(t) + hg(t-1)[\Sigma_1(t)\hat{\Delta}(t-1) \\ &\quad - \Sigma_2(t)\bar{\Delta}(t-1)], \end{aligned} \tag{14}$$

which gives

$$\begin{aligned} \delta(t+1) &= h \sum_{i=1}^{t} \left[\prod_{j=i+1}^{t} (I - h\mathcal{L}(j)) \right] g(i-1)[\Sigma_1(i)\hat{\Delta}(i-1) \\ &\quad - \Sigma_2(i)\bar{\Delta}(i-1)] + \left[\prod_{k=1}^{t} (I - h\mathcal{L}(k)) \right] \delta(1), \\ &\quad t = 1, 2, \ldots \end{aligned} \tag{15}$$

Here, the consensus error Eq. (14) has the following characteristics. (i) The state matrix of the homogeneous part is the identity matrix minus a time-varying Laplacian matrix. (ii) $\mathbf{1}^T\delta(0)$ and $\mathbf{1}^T[\Sigma_1(t)\hat{\Delta}(t-1) - \Sigma_2(t)\bar{\Delta}(t-1)]$, $t = 1, 2, \ldots$ are always zero. Since the state matrix is time-varying, the diagonalization method used in the fixed topology case (Li et al., 2011) is not available. However, by combining the above characteristics, the properties of graph Laplacian and the method for analyzing time-varying linear systems (Guo, 1993, 1994), we develop Lemmas 4.1 and 4.2 to get the relationship among the consensus error, the control parameters h and $g(t)$, and parameters of the network flow T and ρ.

Lemma 4.1. *Let $\{\mathcal{H}(t),\ t = 1, 2 \ldots\}$ be a sequence of Laplacian matrices of undirected graphs satisfying $\mathcal{H}(t) \leq I$, $t = 1, 2, \ldots$. For any integer $T > 0$, denote*

$$\lambda_m = \frac{1}{1+T} \lambda_2 \left(\sum_{k=mT+1}^{(m+1)T} \mathcal{H}(k) \right), \quad m = 0, 1, \ldots.$$

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For given $t_0 = 1, 2 \ldots$, suppose $x(t_0)$ is a vector satisfying $\mathbf{1}^T x(t_0) = 0$ and let

$$x(t) = \left[\prod_{k=t_0}^{k=t-1} (I - \mathcal{H}(k))\right] x(t_0), \quad t = t_0 + 1, t_0 + 2, \ldots. \tag{16}$$

Then $\mathbf{1}^T x(t) = 0$, and

$$\|x(t)\|^2 \le \left[\prod_{k=\lceil \frac{t_0-1}{T}\rceil}^{\lfloor \frac{t-1}{T}\rfloor - 1} \left(1 - \frac{\lambda_k}{1+T}\right)\right] \|x(t_0)\|^2. \tag{17}$$

Lemma 4.2. *Suppose*

$$y(t+1) \le (1 - \alpha(t))y(t) + \beta(t), \quad t = 1, 2, \ldots, \tag{18}$$

with $0 < \alpha(t) \le 1$, $\beta(t) \ge 0$, $t = 1, 2 \ldots$. Then

$$y(t+1) \le \left[\prod_{k=1}^{t} (1 - \alpha(t))\right] y(1) + \sup_{1\le i\le t} \frac{\beta(i)}{\alpha(i)}. \tag{19}$$

Furthermore, if $\sum_{t=1}^{\infty} \alpha(t) = \infty$, then

$$\limsup_{t\to\infty} y(t) \le \limsup_{t\to\infty} \frac{\beta(t)}{\alpha(t)}. \tag{20}$$

Proof of Theorem 4.1. From (2), we have

$$\begin{cases} \xi_{ij}(t) = x_i(t) + g(t-1)\Delta_{ij}(t-1), & a_{ij}(t) = 1 \\ \xi_{ij}(t) = \xi_{ij}(t-1), & a_{ij}(t) = 0, \end{cases}$$
$$t = 1, 2, \ldots. \tag{21}$$

By this and the definition of $e_{ij}(t)$, we have $a_{ij}(t)e_{ij}(t) = a_{ij}(t)g(t-1)\Delta_{ij}(t-1)$, $t = 1, 2, \ldots$, which together with (13) gives

$$\begin{aligned} x_i(t+1) = x_i(t) &+ h\sum_{j=1}^{N} a_{ij}(t)[x_j(t) - x_i(t)] \\ &+ hg(t-1)\sum_{j\in\mathcal{N}_i} a_{ji}(t)\Delta_{ji}(t-1) \\ &- hg(t-1)\sum_{j\in\mathcal{N}_i} a_{ij}(t)\Delta_{ij}(t-1). \end{aligned} \tag{22}$$

From the above and (21), we know that if $a_{ij}(t) = 1$, then

$$\left|\frac{x_i(t+1) - \xi_{ij}(t)}{g(t)}\right| \le \frac{hd_i(t)\max_{ij}|x_j(t) - x_i(t)|}{g(t)} + \frac{(2hd_i(t)+1)g(t-1)}{g(t)}\max_{ij}|\Delta_{ij}(t-1)|, \tag{23}$$

and if $a_{ij}(t) = 0$, then

$$\begin{aligned} \left|\frac{x_i(t+1) - \xi_{ij}(t)}{g(t)}\right| \le& \left|\frac{x_i(t) - \xi_{ij}(t-1)}{g(t-1)}\right| \frac{g(t-1)}{g(t)} \\ &+ \frac{hd_i(t)\max_{ij}|x_j(t) - x_i(t)|}{g(t)} \\ &+ \frac{2hd_i(t)g(t-1)}{g(t)}\max_{ij}|\Delta_{ij}(t-1)|. \end{aligned} \tag{24}$$

Noting that $\xi_{ij}(0) = 0, \widehat{x}_{ji}(0) = 0, i = 1, 2, \ldots, N, j \in \mathcal{N}_i$, from (5), we know that $x_i(1) = x_i(0)$, $i = 1, 2, \ldots, N$. Then by Assumption (A3) and (7), it follows that

$$\left|\frac{x_i(1) - \xi_{ij}(0)}{g(0)}\right| = \left|\frac{x_i(0)}{g(0)}\right| \le K_{ij}(1) + \frac{1}{2}, \tag{25}$$

which together with (2) leads to

$$\max_{ij} |\Delta_{ij}(0)| \le \frac{1}{2}. \tag{26}$$

Similarly, by Assumptions (A2)–(A3), (8) and (23)–(26), we have

$$\max_{ij} |\Delta_{ij}(1)| \le \frac{1}{2}. \tag{27}$$

Denote $d^*(t) = \max_{1\le i\le N} d_i(t)$, $t = 1, 2, \ldots$, and

$$\lambda_m = \frac{1}{1+T}\lambda_2\left(\sum_{k=mT+1}^{(m+1)T} \mathcal{L}(k)\right), \quad m = 0, 1, \ldots.$$

From $h \in (0, \frac{1}{2d^*})$, we know that $h\mathcal{L}(t) \le I$, $t = 1, 2 \ldots$. Note that for any N dimensional vector x, $\|x\| \le \sqrt{N}\|x\|_\infty$. Then by (15), Assumption (A4) and Lemma 4.1, we have

$$\begin{aligned} \|\delta(t+1)\| \le& \left[\prod_{k=0}^{\lfloor \frac{t}{T}\rfloor - 1}\left(1 - \frac{h\lambda_k}{T+1}\right)\right]^{1/2} \|\delta(1)\| \\ &+ h\sum_{i=1}^{t}\left[\prod_{k=\lceil \frac{i}{T}\rceil}^{\lfloor \frac{t}{T}\rfloor - 1}\left(1 - \frac{h\lambda_k}{T+1}\right)\right]^{1/2} \\ &\times g(i-1)\|\Sigma_1(i)\hat{\Delta}(i-1) - \Sigma_2(i)\bar{\Delta}(i-1)\| \\ \le& \left(1 - \frac{h\rho}{T+1}\right)^{t/2T} \|\delta(1)\| \\ &+ h\sqrt{N}\sum_{k=1}^{t}\left(1 - \frac{h\rho}{T+1}\right)^{(t-k)/2T} \\ &\times g(k-1)2d^*(k)\max_{ij}|\Delta_{ij}(k-1)|, \end{aligned}$$

which together with $\max_{ij}|x_i(t) - x_j(t)| \le \max_{ij}(|x_i(t) - \frac{1}{N}\sum_{l=1}^{N} x_l(t)| + |x_j(t) - \frac{1}{N}\sum_{l=1}^{N} x_l(t)|) \le \sqrt{2}\|\delta(t)\|$ leads to

$$\begin{aligned} \frac{\max_{ij}|x_i(t) - x_j(t)|}{g(t)} \le& \frac{\sqrt{2}\left(1 - \frac{h\rho}{T+1}\right)^{(t-1)/2T}}{g(t)}\|\delta(1)\| \\ &+ \frac{2\sqrt{2N}h}{g(t)}\sum_{k=1}^{t-1}\left(1 - \frac{h\rho}{T+1}\right)^{(t-k-1)/2T} \\ &\times g(k-1)d^*(k)\max_{ij}|\Delta_{ij}(k-1)|. \end{aligned} \tag{28}$$

Denote

$$\begin{aligned} y(t) =& \sum_{k=1}^{t-1}\left(1 - \frac{h\rho}{T+1}\right)^{(t-k-1)/2T} g(k-1) \\ &\times d^*(k)\max_{ij}|\Delta_{ij}(k-1)|. \end{aligned}$$

Then $y(t)$ is the solution of the equation

$$\begin{aligned} y(t+1) =& \left(1 - \frac{h\rho}{T+1}\right)^{1/2T} y(t) \\ &+ g(t-1)d^*(t)\max_{ij}|\Delta_{ij}(t-1)|, \quad y(1) = 0, \\ &t = 1, 2 \ldots. \end{aligned} \tag{29}$$

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Denote $z(t)=\frac{y(t)}{g(t)}$, $\theta(t)=1-(1-\frac{h\rho}{T+1})^{1/2T}\frac{g(t)}{g(t+1)}$. Then by (29), we get

$$z(t+1)=(1-\theta(t))z(t)+\frac{g(t-1)d^*(t)\max_{ij}|\Delta_{ij}(t-1)|}{g(t+1)},$$
$$z(1)=0,\ t=1,2\ldots. \tag{30}$$

From $\frac{g(t)}{g(t+1)}<\frac{1}{(1-\frac{h\rho}{T+1})^{1/2T}}$, we get $0<\theta(t)\le 1$. Then by (30), Assumption (A2) and Lemma 4.2, we have

$$z(t+1)\le\frac{d^*\mu^2}{1-\left(1-\frac{h\rho}{T+1}\right)^{1/2T}\mu}\sup_{0\le k\le t-1}\max_{ij}|\Delta_{ij}(k)|,$$

which together with (28) gives

$$\frac{\max_{ij}|x_i(t)-x_j(t)|}{g(t)}\le\frac{\sqrt{2}\left(1-\frac{h\rho}{T+1}\right)^{(t-1)/2T}}{g(t)}\|\delta(1)\|$$
$$+\frac{2\sqrt{2N}hd^*\mu^2}{1-\left(1-\frac{h\rho}{T+1}\right)^{1/2T}\mu}\sup_{0\le k\le t-2}\max_{ij}|\Delta_{ij}(k)|. \tag{31}$$

Combining the above equation with (23), (24) and Assumption (A2), we get if $a_{ij}(t)=1$, then

$$\left|\frac{x_i(t+1)-\xi_{ij}(t)}{g(t)}\right|\le\frac{\sqrt{2}hd^*\left(1-\frac{h\rho}{T+1}\right)^{(t-1)/2T}}{g(t)}\|\delta(1)\|$$
$$+\frac{2h^2\mu^2(d^*)^2\sqrt{2N}}{1-\left(1-\frac{h\rho}{T+1}\right)^{1/2T}\mu}\sup_{0\le k\le t-2}\max_{ij}|\Delta_{ij}(k)|$$
$$+(2hd^*+1)\frac{g(t-1)}{g(t)}\max_{ij}|\Delta_{ij}(t-1)|,\quad t=2,3\ldots, \tag{32}$$

and if $a_{ij}(t)=0$, then

$$\left|\frac{x_i(t+1)-\xi_{ij}(t)}{g(t)}\right|\le\left|\frac{x_i(t)-\xi_{ij}(t-1)}{g(t-1)}\right|\frac{g(t-1)}{g(t)}$$
$$+\frac{hd^*\sqrt{2}\left(1-\frac{h\rho}{T+1}\right)^{(t-1)/2T}}{g(t)}\|\delta(1)\|$$
$$+\frac{2h^2\mu^2(d^*)^2\sqrt{2N}}{1-\left(1-\frac{h\rho}{T+1}\right)^{1/2T}\mu}\sup_{0\le k\le t-2}\max_{ij}|\Delta_{ij}(k)|$$
$$+\frac{2hd^*g(t-1)}{g(t)}\max_{ij}|\Delta_{ij}(t-1)|,\quad t=2,3\ldots. \tag{33}$$

Now we prove that if $\max_{ij}|\Delta_{ij}(k)|\le\frac{1}{2}$, for all $k=0,1,\ldots,t-1$, $t=2,3,\ldots$, then $\max_{ij}|\Delta_{ij}(t)|\le\frac{1}{2}$. Suppose $\max_{0\le k\le t-1}\max_{ij}|\Delta_{ij}(k)|\le\frac{1}{2}$. If $a_{ij}(t)=1$, then by (9) and (32), we have

$$\left|\frac{x_i(t+1)-\xi_{ij}(t)}{g(t)}\right|\le\frac{\sqrt{2N}C_\delta hd^*\left(1-\frac{h\rho}{T+1}\right)^{(t-1)/2T}}{g(t)}$$
$$+\frac{\sqrt{2N}h^2\mu^2(d^*)^2}{1-\left(1-\frac{h\rho}{T+1}\right)^{1/2T}\mu}+\frac{(2hd^*+1)g(t-1)}{2g(t)}$$
$$\le K_{ij}(t+1)+\frac{1}{2}, \tag{34}$$

while similarly, if $a_{ij}(t)=0$, then by $|\frac{x_i(t)-\xi_{ij}(t-1)}{g(t-1)}|\le K_{ij}(t)+\frac{1}{2}$, (9) and (33), we have $|\frac{x_i(t+1)-\xi_{ij}(t)}{g(t)}|\le K_{ij}(t+1)+\frac{1}{2}$. This together with (34) leads to $\max_{ij}|\Delta_{ij}(t)|\le\frac{1}{2}$. Then by (26), (27) and induction, we conclude that

$$\max_{t\ge 0}\max_{ij}|\Delta_{ij}(t)|\le\frac{1}{2}. \tag{35}$$

From $\mu<\frac{1}{(1-\frac{h\rho}{T+1})^{1/2T}}$, we have

$$\frac{\left(1-\frac{h\rho}{T+1}\right)^{(t-1)/2T}}{g(t)}=\frac{\left(1-\frac{h\rho}{T+1}\right)^{(t-1)/2T}}{g(0)}\prod_{j=1}^{t}\frac{g(j-1)}{g(j)}$$
$$\le\frac{\left(1-\frac{h\rho}{T+1}\right)^{(t-1)/2T}}{g(0)}(\mu)^t\to 0,\quad t\to\infty. \tag{36}$$

From (30) and Lemma 4.2, we have $\limsup_{t\to\infty}z(t)\le\limsup_{t\to\infty}\frac{g(t-1)d^*(t)\max_{ij}|\Delta_{ij}(t-1)|}{\theta(t)g(t+1)}$, which together with (28), (35) and (36) leads to (11). If $g(t)\to 0$, $t\to\infty$, then by (11) and Theorem 3.1, we have (12). □

Remark 7. From (21), (22) and the definition of $\Delta_{ij}(t)$, it can be seen that the overall closed-loop system comes down to a nonlinear system described by

$$\begin{cases}\varphi(t+1)=F(t)\varphi(t)+G(t)\Delta(t-1),\\ \Delta(t)=S(\varphi(t),\Delta(t-1),t),\\ \delta(t)=(I-J_N)X(t),\end{cases} \tag{37}$$

where $\varphi(t)=[X^T(t),\xi^T(t)]^T$, $\xi(t)$ is the internal state vector with $\xi_{ij}(t)$ as its components, $\Delta(t)$ is the quantization error vector with $\Delta_{ij}(t)$ as its components. $S(\cdot,\cdot,\cdot)$ is a discontinuous nonlinear time-dependent vector function due to the nonlinearity of the finite-level quantization. This kind of system includes a nonlinear interaction between the consensus error and the quantization errors. To analyze the system (37), firstly, by (14), Lemmas 4.1 and 4.2, we get an inequality (31) describing how the consensus error is governed by $\sup_{0\le k\le t-2}\max_{ij}|\Delta_{ij}(k)|$. Secondly we substitute the inequality (31) into (23) and (24) to get the upper bound of the quantization errors. Finally, in return we substitute this upper bound into (14) and then get an estimate for the consensus error. This methodology to deal with the nonlinear interaction between the consensus error and the quantization errors is similar in spirit to that for analyzing adaptive control systems (Guo, 1993; Guo & Chen, 1991), which was used to deal with the nonlinear interaction between the system states and the identification error.

Remark 8. The constants μ and $\tilde{\mu}$ reflect the convergence properties of the scaling function $g(t)$ in some sense. If $\lim_{t\to\infty}g(t)=\epsilon_g>0$, then $\tilde{\mu}=1$. By Theorem 4.1, we can see that (11) gives a steady-state error of the closed-loop system: $\limsup_{t\to\infty}\max_{ij}|x_i(t)-x_j(t)|\le\frac{\epsilon_g\sqrt{2N}hd^*}{1-(1-\frac{h\rho}{T+1})^{1/2T}}$. If $\lim_{t\to\infty}g(t)=0$, then the steady-state error is zero and (11) gives the convergence rate of the closed-loop system: $\max_{ij}|x_i(t)-x_j(t)|=O(g(t))$, $t\to\infty$.

Remark 9. From Theorem 4.1, we know that to achieve asymptotic average-consensus, we can select the scaling function $g(t)$ vanishing as time goes on. For examples, $g(t)=g_0\gamma^t$, $g_0>0$, $\gamma\in(0,1)$. In this case, $\mu=1/\gamma$ and the consensus can be achieved exponentially fast. Another way is to select $g(t)=\frac{g_0}{(t+q)^p}$, $g_0>0$, $q>0$, $p>0$. In this case, the convergence is slower, however, since $\lim_{t\to\infty}\frac{g(t-1)}{g(t)}=1$, noting from (9), the number of quantization levels may be reduced in an asymptotic sense.

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As mentioned in Section 1, link failures and recoveries or packet dropouts in communication networks can be modeled as the switching of network topologies. The link failure of channel (i,j) at time t implies $a_{ji}(t)=0$, otherwise, $a_{ji}(t)=1$. Note that control problems for single-agent networked control systems with finite packet dropouts have been widely studied (Xiao, Xie, & Fu, 2009; Xiong & Lam, 2007; Yu, Wang, Chu, & Xie, 2004). In the following, we can see that if the duration of any link failures in the network is bounded, then a constant number of quantization levels can be adopted. Especially, the control gain and the scaling function can be selected properly such that 5-level quantizers ($K=2$) suffice for asymptotic average-consensus with an exponential convergence rate.

For any $i=1,2,\ldots,N$, $j\in\mathcal{N}_i$, denote the first t such that $a_{ij}(t)=1$ by $t_{ij}(1)$, $t_{ij}(k)=\min\{t: t>t_{ij}(k-1), a_{ij}(t-1)=0, a_{ij}(t)=1\}$, $k=2,3,\ldots$, and $s_{ij}(k)=\min\{t: t>t_{ij}(k), a_{ij}(t-1)=1, a_{ij}(t)=0\}$, $k=1,2,\ldots$

We need the following assumption.

(A5) There is an integer $T_R>0$, such that $\sup_{k\geq 0}|t_{ij}(k+1)-s_{ij}(k)|\leq T_R$, $i=1,2\ldots,N$, $j\in\mathcal{N}_i$. Here, $s_{ij}(0)$ is set to zero.

Remark 10. The above defined $s_{ij}(k)$ denotes the start time for the kth failure of channel (i,j), while $t_{ij}(k)$ denotes the start time for the kth recovery of channel (i,j), and $t_{ij}(k+1)-s_{ij}(k)$ represents the duration of the kth failure of channel (i,j). It can be seen that Assumption (A5) means the duration of all link failures is bounded.

Remark 11. Suppose $\mathcal{G}$ is a connected graph without link failure. Due to the link failures, the network topology $\mathcal{G}(t)$ at time t becomes a subgraph of $\mathcal{G}$. If the duration of the link failures is bounded by a positive integer T_R, then Assumptions (A4) and (A5) both hold with $T=T_R+1$ and $\rho=\lambda_2(\mathcal{L})(T_R+1)/(T_R+2)$, where $\mathcal{L}$ is the Laplacian matrix of $\mathcal{G}$.

Theorem 4.2. *Suppose Assumptions (A1)–(A5) hold. For any integer $K\geq 1$, denote*

$$\Omega_{K,T_R}=\Bigg\{(h,\mu)\mid h\in\left(0,\frac{1}{2d^*}\right),\mu\in\left(1,\frac{1}{(1-\frac{h\rho}{T+1})^{1/2T}}\right),\ \kappa_{h,\mu}+\frac{\mu(2hd^*+1)}{2}\leq K+\frac{1}{2},\ \mu^{T_R}K+(\mu hd^*+\kappa_{h,\mu})\frac{\mu^{T_R}-1}{\mu-1}+\frac{\mu^{T_R}-1}{2}\leq K+1\Bigg\}.$$

Then (i) The set Ω_{K,T_R} is nonempty. (ii) For any h and $g(t)$ such that $(h,\mu)\in\Omega_{K,T_R}$, and

$$g(0)\geq\frac{C_x}{K+\frac{1}{2}},\tag{38}$$

$$g(1)\ \geq\ \max\left\{\frac{2hd^*C_\delta+(2hd^*+1)g(0)}{2K+1},\frac{2hd^*C_\delta+(2hd^*+2K+1)g(0)}{2K+3}\right\},\tag{39}$$

under the protocol (2), (3) and (5) with the number of quantization levels satisfying

$$K_{ij}(1)=K,\tag{40}$$

$$K_{ij}(t+1)=\begin{cases}K, & a_{ij}(t)=1,\\ K+1, & a_{ij}(t)=0,\end{cases}\quad t=1,2,\ldots,\tag{41}$$

the closed-loop system satisfies

$$\lim_{t\to\infty}x_i(t)=\frac{1}{N}\sum_{j=0}^{N}x_j(0),\quad i=1,2,\ldots,N.$$

Proof. For any integer $K\geq 1$, from (10), we have $\lim_{h\to 0}[\kappa_{h,1}+\frac{2hd^*+1}{2}]=\frac{1}{2}$, and $\lim_{h\to 0}[\kappa_{h,1}+hd^*]=0$, which implies there exists $h^*\in(0,\frac{1}{2d^*})$, such that

$$\kappa_{h^*,1}+\frac{2h^*d^*+1}{2}<K+\frac{1}{2},\tag{42}$$

and

$$K+(h^*d^*+\kappa_{h^*,1})T_R<K+1.\tag{43}$$

Note that

$$\lim_{\mu\to 1}\kappa_{h^*,\mu}+\frac{\mu(2h^*d^*+1)}{2}=\kappa_{h^*,1}+\frac{2h^*d^*+1}{2},$$

and

$$\lim_{\mu\to 1}\mu^{T_R}K+(\mu h^*d^*+\kappa_{h^*,\mu})\frac{\mu^{T_R}-1}{\mu-1}+\frac{\mu^{T_R}-1}{2}=K+(h^*d^*+\kappa_{h^*,1})T_R.$$

From (42) and (43), we know that there exists $\mu^*\in(1,\frac{1}{(1-\frac{h^*\rho}{T+1})^{1/2T}})$, such that $(h^*,\mu^*)\in\Omega_{K,T_R}$, that is, (i) holds. For any h and $g(t)$, which satisfy $(h,\mu)\in\Omega_{K,T_R}$, we have $h\in(0,\frac{1}{2d^*})$, $\mu<\frac{1}{(1-\frac{h\rho}{T+1})^{1/2T}}$. By (38) and (40), we get (7). By (39), (40) and (41), we have (8). Then similar to Theorem 4.1, we have (26) and (27). Now we prove that if $\max_{ij}|\Delta_{ij}(k)|\leq\frac{1}{2}$, for all $k=0,1,\ldots,t-1$, $t=2,3,\ldots$, then $\max_{ij}|\Delta_{ij}(t)|\leq\frac{1}{2}$. Suppose $\max_{0\leq k\leq t-1}\max_{ij}|\Delta_{ij}(k)|\leq\frac{1}{2}$. By (40), (41) Assumption (A5) and the definition of Ω_{K,T_R}, similar to (32), we have

$$\begin{aligned}&\left|\frac{x_i(t+1)-\xi_{ij}(t)}{g(t)}\right|\leq\kappa_{h,\mu}+\frac{\mu(2hd^*+1)}{2}\\&\leq K+\frac{1}{2}=K_{ij}(t+1)+\frac{1}{2},\\&t_{ij}(k)\leq t\leq s_{ij}(k)-1,\ k=1,2,\ldots,\end{aligned}\tag{44}$$

and by (44), similar to (33), we have

$$\begin{aligned}&\left|\frac{x_i(t+1)-\xi_{ij}(t)}{g(t)}\right|\leq\left|\frac{x_i(t)-\xi_{ij}(t-1)}{g(t-1)}\right|\mu+\kappa_{h,\mu}+hd^*\mu\\&\leq\mu^{t+1-s_{ij}(k)}\left|\frac{x_i(s_{ij}(k))-\xi_{ij}(s_{ij}(k)-1)}{g(s_{ij}(k)-1)}\right|\\&\quad+(\kappa_{h,\mu}+hd^*\mu)\sum_{i=s_{ij}(k)}^{t}\mu^{t-i}\\&\leq\mu^{T_R}\left(K+\frac{1}{2}\right)+(\kappa_{h,\mu}+hd^*\mu)\frac{\mu^{T_R}-1}{\mu-1}\\&\leq K+1+\frac{1}{2}=K_{ij}(t+1)+\frac{1}{2},\\&s_{ij}(k)\leq t\leq t_{ij}(k+1)-1,\ k=0,1,\ldots.\end{aligned}\tag{45}$$

Combining (44) and (45), we get $\max_{ij}|\Delta_{ij}(t)|\leq\frac{1}{2}$. Then similar to Theorem 4.1, we can conclude that (ii) holds. □

In the above theorem, note that K can be selected to be 1, then from (40) and (41), it can be seen that 5-level quantizers can ensure asymptotic average-consensus by properly selecting the control parameters.

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Remark 12. In Li et al. (2011), we proved that for a connected network, asymptotic average-consensus can always be achieved by using a 3-level (one-bit) quantizer.[1] Here, from Theorem 4.2 and Remark 13, we can see that in a connected network with intermittent link failures or packet dropouts, provided the duration is bounded, 5-level (3-bit) quantizers suffice for asymptotic average-consensus. The additional two levels are to compensate for the information loss due to link failures or packet dropouts.

Remark 13. From Theorem 4.2, we can see that the control parameters h and $g(t)$ ($g(0), g(1)$ and μ) are designed off-line due to the requirement of some global network topology information N, ρ and T. Noting that there are consensus algorithms with static infinite-level quantizers which do not need off-line designed parameters based on global information, such as Kashyap et al. (2007) and Lavaei and Murray (2009a), it is interesting to develop on-line parameter design algorithms for distributed consensus with dynamic finite-level quantizers in the future.

Remark 14. In this paper, we assume the communication network is undirected. For future investigation, it would be interesting to investigate the case with directed topologies. The main difficulties may include the failures of the symmetric compensation method and the technique in Lemma 4.1. One may need to develop new compensation and closed-loop analysis methods.

5. Numerical example

In this section, we use a numerical example to demonstrate the validity of our protocol. We consider a three agent network of switching topologies $\mathcal{G}(t) = \{\mathcal{V} = \{1, 2, 3\}, \mathcal{A}(t) = [a_{ij}(t)]_{3\times 3}\}$, where $a_{12}(3k+1) = a_{21}(3k+1) = 0, a_{23}(3k+1) = a_{32}(3k+1) = 0, a_{31}(3k+1) = a_{13}(3k+1) = 1, a_{12}(3k+2) = a_{21}(3k+2) = 1, a_{23}(3k+2) = a_{32}(3k+2) = 0, a_{31}(3k+2) = a_{13}(3k+2) = 0, a_{12}(3(k+1)) = a_{21}(3(k+1)) = 0, a_{23}(3(k+1)) = a_{32}(3(k+1)) = 1, a_{31}(3(k+1)) = a_{13}(3(k+1)) = 0, k = 0, 1, \ldots$. It can be seen that $\inf_{m\geq 0} \lambda_2(\sum_{k=2m+1}^{2(m+1)} \mathcal{L}(k)) \geq 1$. Here, $d^* = 1, T = 2, \rho = 1/3$. The control gain h is taken as 0.01 and the scaling function $g(t)$ is taken as $5(0.99985)^t$. So $\mu = 1/0.99985$, $\kappa_{0.01,1/0.99985} = 1.9204$. According to Theorem 4.1, the number of quantization levels $K_{ij}(t)$, $i, j = 1, 2, 3$ are selected as follows: $K_{ij}(1) = 2$, $K_{ij}(2) = 2$, if $a_{ij}(1) = 1$, or $K_{ij}(2) = 3$, if $a_{ij}(1) = 0$,

$$K_{ij}(t+1) = \begin{cases} 2, & a_{ij}(t) = 1, \\ \left\lceil 1.9204 + \dfrac{0.01 + K_{ij}(t)}{0.99985} + \dfrac{\frac{1}{0.99985} - 1}{2} \right\rceil, & a_{ij}(t) = 0, \end{cases}$$
$$t = 2, 3, \ldots$$

The evolution of the states and $K_{12}(t)$ are shown in Fig. 1. It can be seen that average-consensus can be asymptotically achieved. Note that the result of Theorem 4.1 is generally conservative, fewer quantization levels can be used and the convergence rate can be improved for this example. We take $h = 0.2$ and $g(t) = 5(0.9)^t$. The evolution of the states and $K_{12}(t)$ are shown in Fig. 2. It can be seen that average-consensus is asymptotically achieved with a higher convergence rate.

[1] In the fixed topology case, a 3-level quantizer costs one bit, since the zero level is not sent. See also Remark 4.

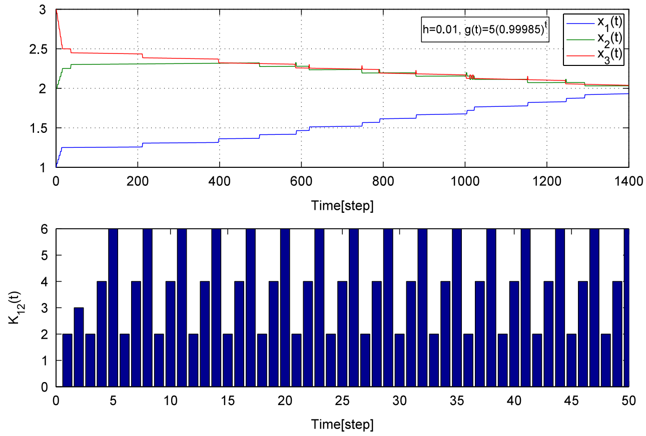

Fig. 1. Curves of states and the number of quantization levels $K_{12}(t)$ with $h = 0.01$ and $g(t) = 5(0.99985)^t$.

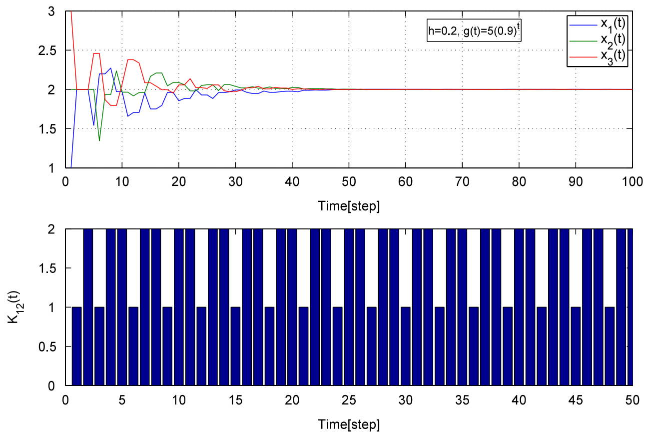

Fig. 2. Curves of states and the number of quantization levels $K_{12}(t)$ with $h = 0.2$ and $g(t) = 5(0.9)^t$.

6. Concluding remarks

In this paper, the discrete-time average-consensus problem with limited communication data-rate and time-varying communication topologies has been considered. A distributed dynamic encoding and decoding scheme has been proposed with finite-level uniform quantizers. The control protocol is designed by a symmetric compensation method and is parameterized by the control gain, the scaling function and the numbers of quantization levels. By selecting the numbers of quantization levels adaptive to the status of the associated communication channels, the closed-loop nonlinear system is converted to a linear time-varying system, and then by the convergence analysis of the linear time-varying system, it is shown that under the protocol designed, average-consensus can be asymptotically achieved provided the network topology is jointly connected. Furthermore, the convergence rate can be designed by selecting a proper scaling function. Especially, if the duration of link failures of all communication channels between agents is bounded, then 5-level quantizers suffice for asymptotic average-consensus with an exponential convergence rate.

It is worthy pointing out that, as a preliminary research, there are still many problems unsolved in this paper. Firstly, we assume all agents are synchronized. In practical networks, the perfect synchronization of local clocks may be difficult. So the asynchronized case is more interesting for future search. Secondly, our analysis is based on a deterministic or pathwise framework. For future research, it will be interesting to investigate how to achieve

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mean square consensus or probabilistic consensus with stochastic switching topology, such as the random gossip algorithm (Boyd, Ghosh, Prabhakar, & Shah, 2006). The case with high-order dynamics or a leader-follower strategy is also interesting.

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Appendix. Proof of lemmas

Proof of Lemma 4.1. Firstly, we prove that

$$\|x((m+1)T+1)\|^2 \le \left(1-\frac{\lambda_m}{1+T}\right)\|x(mT)\|^2, \quad m=0,1,\ldots \tag{46}$$

Let $t_0 = mT+1$, and $t=(m+1)T+1$, $m=0,1,\ldots$. From (16), we know that

$$x(k) = (I-\mathcal{H}(k-1))x(k-1), \quad k=t_0+1, t_0+2,\ldots,t. \tag{47}$$

For any given $j = t_0+1,\ldots,t$, summate the both sides of the above equation from t_0 to j, we have

$$x(j) = x(t_0) - \sum_{i=t_0}^{j-1}\mathcal{H}(i)x(i). \tag{48}$$

Note that $\mathcal{H}(i) \le I$ implies $\mathcal{H}^2(i) \le \mathcal{H}(i)$. Then by Cr inequality and (48), we have

$$\|x(j)-x(t_0)\|^2 \le T\sum_{i=t_0}^{t-1}\|\mathcal{H}(i)x(i)\|^2 \le T\sum_{i=t_0}^{t-1}\|\mathcal{H}^{1/2}(i)x(i)\|^2. \tag{49}$$

By the properties of Laplacian matrices (Godsil & Royle, 2001), we get

$$x^T(t_0)\left(\sum_{k=t_0}^{t-1}\mathcal{H}(k)\right)x(t_0) \ge (1+T)\lambda_m x^T(t_0)x(t_0). \tag{50}$$

Furthermore, note that $x^T(t_0)(\sum_{k=t_0}^{t-1}\mathcal{H}(k))x(t_0) = \sum_{k=t_0}^{t-1}\|\mathcal{H}^{1/2}(k)(x(k)+x(t_0)-x(k))\|^2$, by (49) and $\mathcal{H}(k)\le I$, we have

$$\left[x^T(t_0)\left(\sum_{k=mT+1}^{(m+1)T}\mathcal{H}(k)\right)x(t_0)\right]^{1/2} \le \left(\sum_{k=t_0}^{t-1}\|\mathcal{H}^{1/2}(k)x(k)\|^2\right)^{1/2} + \left(\sum_{k=t_0}^{t-1}\|x(t_0)-x(k)\|^2\right)^{1/2} \le (1+T)\left(\sum_{k=t_0}^{t-1}\|\mathcal{H}^{1/2}(k)x(k)\|^2\right)^{1/2}.$$

This together with (50) gives

$$\lambda_m\|x(t_0)\|^2 \le (1+T)\sum_{k=t_0}^{t-1}\|\mathcal{H}^{1/2}(k)x(k)\|^2. \tag{51}$$

By (47), note that $\mathcal{H}(k-1)-\mathcal{H}^2(k-1)\ge 0$, we have

$$x^T(k)x(k) \le x^T(k-1)x(k-1) - x^T(k-1)\mathcal{H}(k-1)x(k-1), \quad k=t_0+1,\ t_0+2,\ldots,t.$$

Summating the both sides of the above equation from t_0+1 to t and by (51), we get (46).

For general $t_0 = 1,2,\ldots$ and $t=t_0+1, t_0+2,\ldots$, by (46), noting that $\mathbf{1}^T\mathcal{H}(t)=0$, we have

$$\begin{aligned}\|x(t)\|^2 &= \left\|\prod_{k=t_0}^{k=t-1}(I-\mathcal{H}(k))x(t_0)\right\|^2 \\ &\le \left\|\prod_{k=\lfloor\frac{t-1}{T}\rfloor T+1}^{k=t-1}(I-\mathcal{H}(k))\right\|^2\left\|\prod_{k=\lceil\frac{t_0-1}{T}\rceil T+1}^{k=\lfloor\frac{t-1}{T}\rfloor T}(I-\mathcal{H}(k))\right. \\ &\quad\times\left.\left(\prod_{k=t_0}^{k=\lceil\frac{t_0-1}{T}\rceil T}(I-\mathcal{H}(k))\right)x(t_0)\right\|^2. \\ &\le \prod_{k=\lceil\frac{t_0-1}{T}\rceil}^{\lfloor\frac{t-1}{T}\rfloor-1}\left(1-\frac{\lambda_k}{1+T}\right)\|x(t_0)\|^2.\end{aligned} \tag{52}$$

That is, (17) holds. □

Proof of Lemma 4.2. From (18), we have

$$y(t+1) \le \left(\prod_{k=1}^{t}(1-\alpha(k))\right)y(1) + \sup_{1\le i\le t}\frac{\beta(i)}{\alpha(i)}\left[1-\prod_{k=1}^{t}(1-\alpha(k))\right].$$

This together with $0<\alpha(t)\le 1$ and $\beta(t)\ge 0$ leads to (19). Similar to the above inequality, we have

$$y(t+1) \le \left(\prod_{k=n}^{t}(1-\alpha(k))\right)y(n) + \sup_{n\le i\le t}\frac{\beta(i)}{\alpha(i)}\left[1-\prod_{k=n}^{t}(1-\alpha(k))\right], \quad \forall\, t>n\ge 1.$$

From the above and $\sum_{t=1}^{\infty}\alpha(t)=\infty$, we have $\limsup_{t\to\infty}y(t)\le \sup_{i\ge n}\frac{\beta(i)}{\alpha(i)}$, $\forall n \ge 1$. Then by the arbitrariness of n, we have (20). □

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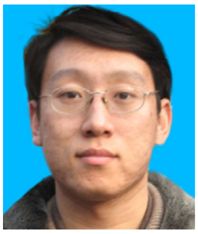

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