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DISTRIBUTED CONSENSUS FOR MULTIAGENT SYSTEMS WITH COMMUNICATION DELAYS AND LIMITED DATA RATE*

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Abstract. This paper considers the average consensus problem for multiagent networks with communication delays and limited data rate. On one hand, communication delays often exist in information acquisition and transmission; on the other hand, only limited state information of agents can be transmitted to their neighbors at each time step due to bandwidth constraints. The average consensus problem becomes much more complicated when both delays and data-rate constraints are to be considered. In this paper, a distributed consensus protocol is proposed based on dynamic encoding and decoding. It is shown that for a connected network, as long as the time delays are bounded, the average consensus can be achieved with a finite communication data rate. In particular, it is shown that merely a one-bit information exchange between each pair of adjacent agents at each time step suffices to guarantee the average consensus.

Key words. multiagent systems, delay systems, quantization, distributed consensus

AMS subject classifications. 68M14, 68Q85, 68W15, 53D50, 68M10, 68P30

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1. Introduction.

Distributed consensus has become a hot research topic in recent years [11], [18], [1], [2], [19]. The problem is widely encountered in real-world applications such as distributed computation, flocking, traffic control, networked control, and flight formation.

The average consensus problem involves designing a distributed protocol or distributed control law such that the states of all the agents converge to the average of their initial states asymptotically or in a finite time. It is noted that, in practice, on one hand, communication delays are unavoidable in information acquisition and transmission, and it has been shown that under the conventional protocols, consensus may not be achieved when there exist communication delays [20]; on the other hand, in digital communication networks, communication channels have only finite capacities; that is, an agent can transmit only finite bits of information to its neighbors at each time step. Hence, both communication delays and data-rate constraints should be taken into consideration when designing a consensus protocol.

Recently, consensus with quantized communication has drawn the attention of the systems and control community. A gossip-algorithm-based approximate average consensus protocol is provided in [13] under the assumption that the states of agents are integer-valued. Under the same assumption, Nedic et al. [17] analyze the quantization effect on average consensus and give an upper bound for the consensus error. In [7], three kinds of update strategies are considered based on both deterministic

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and probabilistic quantizers. The average consensus problem is solved by using logarithmic quantizers with infinite level in [4] and [5]. Under the assumption that the quantization errors are white noises, two coding schemes are provided in [22], and conditions under which the consensus can be achieved are obtained. Meanwhile, by considering a Cayley graph, the average consensus problem is analyzed in [8]. Other works for the consensus problem with additive noises can be found in [10], [15], [12], and [16]. By introducing dynamic encoding and decoding schemes, it is proved in [6] and [14] that the average consensus problem can be solved by using only finite level quantizers. Moreover, Li et al. [14] prove that the number of transmitting bits at each time step can be reduced to one by properly selecting the controller parameters.

For networks with communication delays, much work on consensus protocol design and analysis has been done. An initial study on this problem can be found in [18] using a frequency domain method, and a necessary and sufficient condition on the upper bound of time delays is provided under the assumption that all the delays are equal and time-invariant. Bliman and Ferrari-Trecate [1] provide a systematic way to analyze the continuous-time average consensus problem with transmission delays and give a sufficient condition for achieving average consensus with bounded communication delays. For the discrete-time case, the consensus problem for systems with dynamically changing topologies and time-varying delays is considered in [3] and [20]. Their results show that a multiagent system can reach consensus under some connectivity condition, irrespective of time delays. For multiagent systems with time-varying delays and noises in transmission channels, the consensus problem is studied in stochastic sense in [16]. Both strong consensus and mean square consensus are investigated under fixed and switching topologies.

It can be seen that most of the above works deal with the consensus problem with communication delays and limited data rate separately. Note that time delays and data-rate constraints coexist in real multiagent networks. However, for networks with both communication delays and data-rate constraints, the average consensus problem cannot be solved by the existing methods. Moreover, for the finite-level quantization, quantization errors cannot be regarded as additive white noises, so the method in [16] is not applicable. In this paper, we shall consider the average consensus problem for undirected networks by considering communication delays and data-rate constraints simultaneously.

Based on the previous work [14], we adopt a dynamic encoding and decoding scheme with finite-level quantizers. We design a distributed protocol with error compensation. Note that the time delays are to be considered in the encoder-decoder and the protocol design. If we write the whole system into an augmented system, it can be seen that the system is time-varying and contains multiple delays in the states. The existing results cannot be applied to analyze the convergence of the augmented system. Due to the presence of state delays, the diagonalization approach in [14] cannot be used. The state augmentation approach in [20], which is used to solve consensus with time-varying transmission delays, is not applicable since the whole system becomes nonlinear due to quantization. To overcome this difficulty, an auxiliary system is introduced to represent the system in two coupled parts—a persistent part which converges to a delay-free version of the original system and a delay part which contains all the effects of the delays. Motivated by [16], we can get a stability condition which is given in terms of the network topology, the upper bound of the time delay, the number of quantization levels, and the scaling function to be designed for coding. We show that if the network is connected and the time delays are bounded, then for any given finite-level uniform quantizer, the consensus gain and the scaling function can be chosen properly such that the average consensus can be achieved asymptotically. In particular, if the control parameters are properly chosen, the average consensus can be guaranteed with merely a one-bit information exchange between agents at each time step.

The remainder of the paper is organized as follows: In section 2 we formulate the consensus problem under investigation. The protocol design and consensus analysis are given in section 3. Some examples are given in section 4. Conclusions are drawn in section 5.

Some remarks on notation are given as follows. Denote by $\chi_\Omega(t)$ the indicator function, i.e., $\chi_\Omega(t) = 0$ if $t \notin \Omega$ and $\chi_\Omega(t) = 1$ if $t \in \Omega$. For a given matrix A, we denote its (i,j)th element as $A^{i,j}$. We denote $A \geq 0$ if A is a nonnegative matrix, i.e., all the elements of A are nonnegative. For given matrices A and B, we denote $A \geq B$ if $A - B$ is a nonnegative matrix. The transpose of a matrix A is denoted by A'. The ceiling and floor functions are, respectively, denoted by $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$. Denote by $\delta_{i,j}$ the Kronecker delta function, i.e., $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. We denote by $\mathbf{1}_n$ the n dimensional column vector with every element equal to 1. We denote the Euclidian norm and infinity norm by $\| \cdot \|_2$ and $\| \cdot \|_\infty$, respectively. The transition matrix of $M(k)$ is defined as

$$\Pi_{i,j}^M = \begin{cases} M(i)M(i-1)\cdots M(j), & i \geq j, \\ I, & i < j. \end{cases}$$

2. Problem statement. In this paper, we consider N agents with the following dynamics:

$$x_i(k+1) = x_i(k) + u_i(k), \quad i = 1, 2, \ldots, N. \tag{1}$$

The communication topology among agents is modeled as a graph, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with the vertex set $\mathcal{V}$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. We denote by $a_{i,j} \geq 0$ the weighting on the edge (j,i). If $(j,i) \notin \mathcal{E}$, then $a_{i,j} = 0$; otherwise, $a_{i,j} > 0$. The set of neighbors of vertex i is defined by $\mathcal{N}_i = \{j \mid j \in \mathcal{V},\ (j,i) \in \mathcal{E}\}$. If $a_{i,j} = a_{j,i}$ for any pair of vertices, the associated communication graph is called an undirected graph. For an undirected graph, we call $\deg_i = \sum_{j \in \mathcal{V}} a_{i,j}$ the degree of i and $D^* = \max_i \deg_i$ the degree of $\mathcal{G}$.

The Laplacian matrix L of the graph $\mathcal{G}$ is defined as that for any $i, j \in \mathcal{V}$ and $i \neq j$, $L^{i,j} = -a_{i,j}$ and $L^{i,i} = \deg_i$. There is a path from vertex i to vertex j if there exists a sequence $l_1, \ldots, l_p \in \mathcal{V}$ satisfying $(i, l_1), (l_1, l_2), \ldots, (l_{p-1}, l_p) \in \mathcal{E}(\mathcal{G})$, where $l_p = j$ and $i, l_1, \ldots, l_p$ are distinct vertices. Given a graph $\mathcal{G}$, it contains a spanning tree if there exists at least one vertex i such that for any other vertex j, there is a path from i to j. If an undirected graph contains a spanning tree, it is connected. Some properties of the Laplacian matrix are recalled below.

LEMMA 2.1. *For an undirected graph $\mathcal{G}$ with $\mathcal{V} = \{1, 2, \ldots, N\}$, the corresponding Laplacian matrix L has the following properties:*

1. $\lambda_1(L) = 0$*;*
2. $\lambda_2(L) > 0$ *if and only if the graph is connected;*
3. $\|L\|_2 = \lambda_N(L)$ *and* $\|L\|_\infty = 2D^*$*,*

where $\lambda_i(L)$ is the ith smallest eigenvalue of L. Moreover, for Laplacian matrices L_1 and L_2 satisfying $L_1 \leq L_2$, we have $\|L_1\|_2 \leq \|L_2\|_2$ and $\|L_1\|_\infty \leq \|L_2\|_\infty$.

Proof. The proof follows directly from the properties of the Laplacian matrix [9] and the details are omitted. ∎

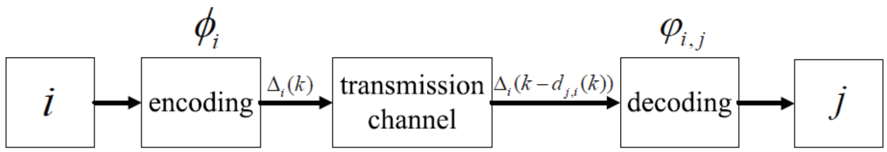

FIG. 1. *The transmission channel from i to j.*

In this paper, we assume that transmission channels are noiseless digital channels without packet dropouts. The time delay of the transmission channel from i to j is denoted by $d_{i,j}$, where $d_{i,j}$ is a nonnegative integer. The communication between each pair of neighbors is as follows: at each time instant k, the agent i encodes its state $x_i(k)$ by an encoder ϕ_i into a symbolic data $\Delta_i(k)$, which will be sent to agent j through the channel. Due to the time delay in the transmission channel from i to j, the symbolic data received by the neighbor node j at time instant k is $\Delta_i(k-d_{j,i}(k))$. Then agent j uses a decoder $\varphi_{i,j}$ to get an estimate of agent i's state. The whole communication process from agent i to j is shown in Figure 1.

We say that $\{u_1(t), u_2(t), \dots, u_N(t), t = 0, 1, \dots\}$ is a distributed protocol over digital network $\mathcal{G}$ if $u_i(t)$ depends only on the states of the ith agent, the state of encoder ϕ_i, and the outputs of decoders $\varphi_{j,i}$, $j \in N_i$. In this paper, we will consider how to design proper encoders, decoders, and a distributed protocol $\{u_1(t), u_2(t), \dots, u_N(t), t = 0, 1, \dots\}$ such that

$$\lim_{k\to\infty} x_i(k) = \frac{1}{N}\sum_{i=1}^{N} x_i(0), \quad i = 1, 2, \dots, N.$$

We introduce the following assumptions to communication channels and topology.

Assumption 1. The communication graph $\mathcal{G}$ is a connected undirected graph.

Assumption 2. The time delays of all channels are bounded and symmetric, i.e., $d_{i,j}(k) = d_{j,i}(k)$, and $\max_{(i,j)\in\mathcal{E}} d_{i,j}(k) \le d$, where d is a positive integer.

Assumption 3. Each agent has a memory of its past d states, and data transmitted through a channel is with a time stamp.

Assumption 4. There is a known constant C_x such that $\max_i |x_i(0)| \le C_x$.

Remark 2.1. We assume that the delays are symmetric in each transmission channel in order to guarantee that the average value of the initial states is preserved at each time step. Meanwhile, we need an upper bound C_x of the initial states to design the protocol. Note that C_x can be estimated in a distributed way.

3. Protocol design and consensus analysis.

3.1. Protocol design. It is noted that the exact consensus cannot be reached by using static quantizers [13]. Therefore, we need to design a dynamic encoder and decoder with finite-level quantization to process the data to be transmitted. The encoder for agent i is given as

$$\begin{cases} \hat{x}_i(k) = g(k-1)\Delta_i(k) + \hat{x}_i(k-1), \quad \hat{x}_i(0) = 0, \\ \Delta_i(k) = q\left[\frac{x_i(k)-\hat{x}_i(k-1)}{g(k-1)}\right], \end{cases} \tag{2}$$

where $\hat{x}_i(k)$ is the encoder state, $g(k) > 0$ is the scaling function to be designed, and $q[\cdot]$ is a quantizer. Figure 2 shows the diagram of the encoder (2). In fact, $\hat{x}_i$

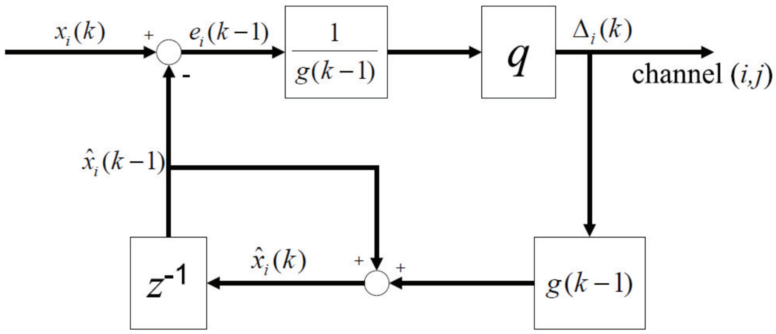

FIG. 2. *The encoder* ϕ_i.

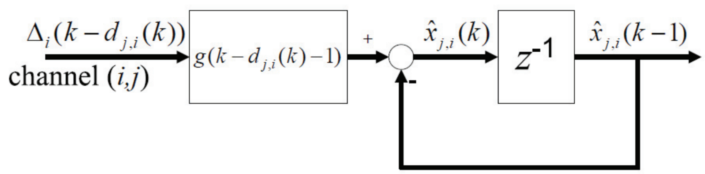

FIG. 3. *The decoder* $\varphi_{i,j}$.

can be viewed as the estimation of the state, and the inverse scaling function $\frac{1}{g(k)}$ is to zoom in the estimation error $e_i(k) \triangleq x_i(k+1) - \hat{x}_i(k)$. Then we quantize the zoomed innovation and send the quantized data $\Delta_i(k)$ to the transmission channel. As shown above, the quantized data received by the neighbor node j at time instant k is $\Delta_i(k - d_{j,i}(k))$. We propose a decoder for agent j associated with the transmission channel from i to j of the form

$$\hat{x}_{j,i}(k) = \hat{x}_{j,i}(k-1)+g(k-d_{j,i}(k)-1)\Delta_i(k-d_{j,i}(k))\chi_{[0,\infty)}(k-d_{j,i}(k)), \hat{x}_{j,i}(0) = 0. \tag{3}$$

The diagram of the decoder is given in Figure 3. It is straightforward that $\hat{x}_{j,i}(k) = \hat{x}_i(k - d_{j,i}(k))$.

The scaling function $g(\cdot)$ should satisfy that $\lim_{k\to\infty} g(k) = 0$. In this paper, we choose exponential function $g_0\gamma^k$ with $0 < \gamma < 1$ as the scaling function $g(k)$. We consider a finite-level uniform quantizer which is defined as

$$q[x] = \begin{cases} K, & x \geq K - \frac{1}{2}, \\ l, & l - \frac{1}{2} \leq x < l + \frac{1}{2}, \\ & l = 1, 2, \ldots, K-1, \\ 0, & -\frac{1}{2} < x < \frac{1}{2}, \\ -q[-x], & x \leq -\frac{1}{2}. \end{cases} \tag{4}$$

From (4) we can see that the number of quantization levels is $2K+1$. No information will be sent when the output of quantizer is 0, in which case the communication channel from agent i to j, $j \in \mathcal{N}_i$, is required to be capable of transmitting $\lceil \log_2(2K) \rceil$ bits.

We propose a distributed protocol as

$$u_i(k) = h \sum_{j\in\mathcal{N}_i} a_{i,j}\chi_{[0,\infty)}(k - d_{i,j}(k)) \left[\hat{x}_{i,j}(k) - \hat{x}_i(k - d_{i,j}(k))\right], \tag{5}$$

where h is the consensus gain function to be designed.

3.2. Consensus analysis. Define the sub-Laplacian matrix $L_l(k)$ as

$$L_l^{i,j}(k) = \begin{cases} -a_{i,j}\delta_{d_{i,j}(k),l}\chi_{[l,\infty)}(k), & i \neq j, \\ \sum\limits_{j\in\mathcal{N}_i} a_{i,j}\delta_{d_{i,j}(k),l}\chi_{[l,\infty)}(k), & i = j, \end{cases}$$

where $L_l(k)$ is the Laplacian matrix of the subgraph [1] corresponding to time delay l. It is noted that $L_l(k) = L_l'(k)$ thanks to the symmetry of delays. According to the definition of $L_l(k)$, we have

$$L_l(k)\mathbf{1}_N = 0,\ \mathbf{1}_N' L_l(k) = 0, \quad l = 0, 1, \ldots, d. \tag{6}$$

Denote

$$X(k) = \mathrm{col}\left\{x_1(k), x_2(k), \ldots, x_N(k)\right\},\ \hat{X}(k) = \mathrm{col}\left\{\hat{x}_1(k), \hat{x}_2(k), \ldots, \hat{x}_N(k)\right\},$$
$$\tilde{X}(k) = X(k) - \hat{X}(k),\ \xi(k) = X(k) - J_N X(k),$$

where $J_N = \mathbf{1}_N\mathbf{1}_N'/N$. It is noted that $\xi(k)$ stands for the disagreement, i.e., the deviations of the states from the average value at time instant k. The closed-loop system can be written in the compact form

$$\begin{cases} X(k+1) = X(k) - h\sum\limits_{i=0}^{d} L_i(k)\hat{X}(k-i), \\ \hat{X}(k+1) = \hat{X}(k) + g(k)q\left[\frac{X(k+1)-\hat{X}(k)}{g(k)}\right], \end{cases} \tag{7}$$

where the quantizer $q[\cdot]$ is carried out elementwise. According to (6) and (7) we get

$$\frac{1}{N}\sum_{i=1}^{N} x_i(k+1) = \frac{1}{N}\mathbf{1}_N' X(k+1) = \frac{1}{N}\mathbf{1}_N' X(k) = \frac{1}{N}\sum_{i=1}^{N} x_i(k), \tag{8}$$

which means that the state average is preserved at each step. Thus, the average consensus problem is solved if and only if

$$\lim_{k\to\infty} \xi(k) = 0. \tag{9}$$

In view of (6) and (7), we can easily see that

$$\begin{aligned} e(k) &\triangleq X(k+1) - \hat{X}(k) \\ &= \tilde{X}(k) + h\sum_{i=0}^{d} L_i(k)\left[\tilde{X}(k-i) - X(k-i)\right] \\ &= \tilde{X}(k) + h\sum_{i=0}^{d} L_i(k)\left[\tilde{X}(k-i) - \xi(k-i)\right]. \end{aligned} \tag{10}$$

Note that $(I - J_N)L_i(k) = L_i(k) = L_i(k)(I - J_N)$, which together with (7) and (10) yields

$$\xi(k+1) = \xi(k) - h\sum_{i=0}^{d} L_i(k)\xi(k-i) + h\sum_{i=0}^{d} L_i(k)\tilde{X}(k-i), \tag{11a}$$

$$\tilde{X}(k+1) = e(k) - g(k)q\left[\frac{e(k)}{g(k)}\right]. \tag{11b}$$

It is noted that there are multiple delays in system (11a) and the matrices $L_i(k)$, $i = 0, \ldots, d$, are time-varying, which poses challenges for consensus analysis. We shall provide a way to choose the consensus gain h, the parameters g_0 and γ of the scaling function, and quantization level $2K+1$ such that (9) holds. By looking at (11a)–(11b), it is easy to see that (9) holds when $\lim_{k\to\infty} \tilde{X}(k) = 0$ and the system (11a) is stable.

Denote $w(k) = \xi(k)\frac{1}{g(k)}$, $z(k) = \tilde{X}(k)\frac{1}{g(k)}$. According to the definition of $g(k)$, it follows from (11a)–(11b) that

$$\begin{cases} w(k+1) = \gamma^{-1}w(k) - h\sum_{i=0}^{d} L_i(k)\gamma^{-i-1}w(k-i) \\ \qquad\qquad + h\sum_{i=0}^{d} L_i(k)\gamma^{-i-1}z(k-i), & \text{(12a)} \\ z(k+1) = \gamma^{-1}\left\{\bar{e}(k) - q\left[\bar{e}(k)\right]\right\}, & \text{(12b)} \end{cases}$$

where

$$\bar{e}(k) = e(k)\frac{1}{g(k)} = z(k) + h\sum_{i=0}^{d} L_i(k)\gamma^{-i}z(k-i) - h\sum_{i=0}^{d} L_i(k)\gamma^{-i}w(k-i). \tag{13}$$

Note that $\gamma z(k+1)$ is equal to the quantization error of $\bar{e}(k)$. If $\bar{e}(k)$ never makes the quantizers saturate, then we have $\|z(k)\|_\infty \le \frac{1}{2\gamma}$. If the delay system (12a) is further stable, $\sup_{k\ge 0}\|w(k)\|_\infty$ must be bounded, which leads to (9) when $g(k)$ satisfies that $\lim_{k\to\infty} g(k) = 0$. Actually, if any of the quantizers is saturated, the problem becomes complicated since the boundedness of $z(k)$ as well as $w(k)$ cannot be guaranteed. Then, we note that the average consensus is solved if the following problem can be solved.

PROBLEM. *Find the consensus gain function h, scaling function parameters g_0, γ, and quantization level $2K+1$ such that*

1. *system* (12a) *is stable;*
2. *$\bar{e}(k)$ never makes the quantizers saturate.*

In the rest of the paper, we shall consider the above problem instead.

Remark 3.1. The two systems (12a) and (12b) play a key role in the average consensus problem. However, the two systems are coupled and (12b) is a nonlinear system. Moreover, it is very challenging to analyze the stability of system (12a) since it is a time-varying system with multiple state delays. If we apply the augmentation approach to system (12a), we have the augmented system of the form

$$W(k+1) = \begin{bmatrix} \frac{I-hL_0(k)}{\gamma} & \frac{-hL_1(k)}{\gamma^2} & \cdots & \frac{-hL_d(k)}{\gamma^{d+1}} \\ I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix} W(k)$$
$$+ \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} h\sum_{i=0}^{d} L_i(k)\gamma^{-i-1}z(k-i),$$

where $W(k) = col\{w(k), w(k-1), \dots, w(k-d)\}$. However, it is not clear how to choose h such that the augmented system is stable. In this paper we introduce an auxiliary system. Based on the analysis of the auxiliary system, we can show how to select the control gain h such that system (12a) is stable.

Consider the auxiliary system

$$\begin{cases} w_a(k+1) = A(k)w_a(k) + H(k),\ k \geq d, & \text{(14a)} \\ H(k) = \sum_{i=1}^{d} B_i(k)H(k-i) + h\sum_{i=0}^{d} L_i(k)\gamma^{-i-1}z(k-i), & \text{(14b)} \end{cases}$$

with initial conditions $w_a(d) = w(d)$, $H(0) = \cdots = H(d-1) = 0$, where $A(k)$ and $B_i(k)$, $i = 1, 2, \dots, d$, satisfy

$$A(k) + B_1(k) = \gamma^{-1}\left(I - hL_0(k)\right), \tag{15}$$

$$B_1(k)A(k-1) - B_2(k) = \gamma^{-2}hL_1(k), \tag{16}$$

$$\vdots$$

$$B_{d-1}(k)A(k-d+1) - B_d(k) = \gamma^{-d}hL_{d-1}(k), \tag{17}$$

$$B_d(k)A(k-d) = \gamma^{-d-1}hL_d(k). \tag{18}$$

By simple calculation, it can be shown that $w(k) = w_a(k)$ for all $k \geq d$. Note that given the initial conditions $A(i)$, $i = 0, 1, \dots, d-1$, the matrices $A(k)$ and $B_i(k)$, $i = 1, 2, \dots, d$, can be calculated iteratively by (15)–(18). Define

$$\rho_A(t,\theta) = t^2 \frac{(1-\theta)^{-d} - 1 - d\theta}{\theta\left(\theta + 1 + d\theta - (1-\theta)^{-d}\right)}, \tag{19}$$

$$\rho_B(t,\theta,l) = \sqrt{N}\left(t^2 + \theta\rho_A(t)\right)\frac{(1-\theta)^{-d} - 1 - d\theta}{\theta^2(1-\theta)^{ld}}, \tag{20}$$

where

$$\theta \in \{a:\ a + 1 + da > (1-a)^{-d},\ a \in (0,1)\}. \tag{21}$$

It is obvious that the set in (21) is not empty for any given positive integer d. For the auxiliary system (14b), we have the following lemma, whose proof can be found in Appendix A.

Lemma 3.1. *Consider system* (14a)–(14b). *Denote*

$$\Delta_A(k) = \gamma A(k) - I + hL(k),\ \Delta_{B_j}(k) = B_j(k) - \gamma^{-j}h\sum_{i=j}^{d} L_i(k),\ j = 1, \dots, d, \tag{22}$$

where $L(k) = \sum_{i=0}^{d} L_i(k)$. *For any given constant* θ *satisfying* (21), *constant* $\bar\lambda$ *satisfying* $\bar\lambda \geq \sup_k \lambda(L(k))$, *if*

$$\|\Delta_A(k)\|_2 < \rho_A(\bar\lambda,\theta)h^2,\ k = 0, 1, \dots, d-1, \tag{23}$$

and $0 < h < h_1(\bar\lambda,\theta)$, *where*

$$h_1(\bar\lambda,\theta) = \frac{2\theta}{\sqrt{\bar\lambda^2 + 4\rho_A(\bar\lambda,\theta)\theta} + \bar\lambda}, \tag{24}$$

then for all $k \geq d$*,* $A(k)$ *is invertible and*

$$\|\Delta_A(k)\|_2 < \rho_A(\bar{\lambda},\theta)h^2. \tag{25}$$

Furthermore, given any $l > 0$*, if* $(1-\theta)^l < \gamma < 1$*, then*

$$\|\Delta_{B_j}(k)\|_\infty < \rho_B(\bar{\lambda},\theta,l)h^2, \quad j = 1,\dots,d. \tag{26}$$

In what follows, we shall consider the problem in two cases separately, including constant transmission delays and time-varying transmission delays.

Case 1. *Constant delays.* In this case, we shall consider that all the communication delays are time invariant. It is obvious that

$$\sum_{l=0}^{d} L_l(k) = L \quad \forall\ k \geq d.$$

Denote λ_i the ith smallest eigenvalue of L. Before giving the main result, the following lemma is needed, whose proof is given in Appendix B.

LEMMA 3.2. *Consider system* (12a)–(12b)*. Define*

$$M_1(h,\gamma,g_0) = \frac{1}{2\gamma} + \left(\frac{1}{2\gamma} + f_1(h,\gamma,d)\right) 2hD^* \frac{\gamma^{-d}-1}{\gamma^{-1}-1}, \tag{27}$$

where f_1 *is given by*

$$f_1(h,\gamma,k) = p_1^k \frac{2\sqrt{N}C_x}{g_0} + \frac{h\sqrt{N}\lambda_N\left(\gamma^{-d}-1\right)\left(p_1^k-1\right)}{2\gamma\left(1-\gamma\right)\left(p_1-1\right)},\ p_1 = \gamma^{-1} + h\lambda_N\frac{\gamma^{-d}-1}{1-\gamma}. \tag{28}$$

If $g_0 \geq 2\gamma C_x$*, and* $K \geq \lfloor M_1(h,\gamma,g_0) + \frac{1}{2}\rfloor$*, then*

$$\|w(k)\|_2 \leq f_1(h,\gamma,k), \ \ \|z(k)\|_\infty \leq \frac{1}{2\gamma} \quad \forall\ k = 0,1,\dots,d. \tag{29}$$

Define

$$\bar{h}_\theta = \min\left\{h_1(\lambda_N,\theta), \frac{\lambda_2}{\rho_A(\lambda_N,\theta)}, \left(1 + 2D^*(1-\theta)^{-d} + \frac{\rho_B(\lambda_N,\theta,1)}{4}\right)^{-d}\right\}, \tag{30}$$

where h_1 is defined in (24), and

$$\bar{\gamma}_h = 1 - \lambda_2 h + \rho_A(\lambda_N,\theta)h^2, \tag{31}$$

$$K_1(h,\gamma,g_0) = \frac{1}{2\gamma} + \left(\frac{1}{2\gamma} + \bar{w}(h,\gamma)\right) 2hD^* \frac{\gamma^{-d-1}-1}{\gamma^{-1}-1}, \tag{32}$$

with $\bar{w}$ defined as

$$\bar{w}(h,\gamma) = \max\left\{f_1(h,\gamma,d), \frac{h^{\frac{1}{d}}D^*\sqrt{N}(\gamma^{-d-1}-1)}{(1-\gamma)(1-\beta)(\gamma-\bar{\gamma}_h)}\right\}, \tag{33}$$

$$\beta = \max\left\{h^{\frac{1}{d}}, h^{\frac{1}{d}}\frac{2D^*(1-\theta)^{-d} + \rho_B(\lambda_N,\theta,1)/4}{1-h^{\frac{1}{d}}}\right\}. \tag{34}$$

Now we are in the position to give the main result.

THEOREM 3.1. *Consider system* (1) *with the Assumptions* 1–4 *satisfied. Apply the protocol* (5) *with the encoder* (2) *and the decoder* (3). *For any given constant* θ *satisfying* (21), *if the control parameters* h, γ, g_0 *satisfy*

$$h \in (0, \bar{h}_\theta), \ \gamma \in (\bar{\gamma}_h, 1), \ g_0 \geq 2\gamma C_x, \tag{35}$$

and the number of the quantization level satisfies

$$K \geq \left\lfloor K_1(h, \gamma, g_0) + \frac{1}{2} \right\rfloor, \tag{36}$$

we have the following results:

I. *The average consensus problem is solved, and*

$$\lim_{h \to 0^+, \gamma \to 1^-} K_1(h, \gamma, g_0) = \frac{1}{2}; \tag{37}$$

i.e., the lower bound of the number of transmitting bits is 1.

II. *The consensus error* $\xi(k)$ *satisfies*

$$\limsup_{k \to \infty} \frac{\|\xi(k)\|_2}{\gamma^k} \leq \bar{w} g_0, \tag{38}$$

and $r_{\text{asym}} \leq \gamma$, *where*

$$r_{\text{asym}} = \sup_{\xi(0) \neq 0} \lim_{k \to \infty} \|X(k) - J_N X(0)\|_2^{1/k} \tag{39}$$

is the convergence rate of the average consensus defined in [21].

Proof. We shall consider the stability of system (14a)–(14b) with the initial condition of $A(k)$ satisfying

$$\|\Delta_A(k)\|_2 < \rho_A(\lambda_N, \theta) h^2, \quad k = 0, 1, \ldots, d-1,$$

where λ_N is the largest eigenvalue of the Laplacian matrix and θ satisfies (21). Then, according to (30), (35), and Lemma 3.1, we have $\|\Delta_A(k)\|_2 < \rho_A(\lambda_N, \theta) h^2$ for all $k \geq 0$. Note that $(I - J_N)\xi(k) = (I - J_N)^2 X(k) = (I - J_N) X(k) = \xi(k)$. According to the definition of $w(k)$, it is clear that $(I - J_N) w(k) = w(k)$. Since $w_a(k) = w(k)$ for all $k \geq d$, system (14a) can be written as

$$w_a(k+1) = \bar{A}(k) w_a(k) + H(k), \tag{40}$$

where $\bar{A}(k) = A(I - J_N) \triangleq \gamma^{-1}(I - hL - J_N + \Delta_{\bar{A}}(k))$ with $\Delta_{\bar{A}}(k) = \Delta_A(k)(I - J_N)$. Then we have

$$\|\Delta_{\bar{A}}(k)\|_2 \leq \|\Delta_A(k)\|_2 \|I - J_N\|_2 = \|\Delta_A(k)\|_2 < \rho_A(\lambda_N, \theta) h^2.$$

Note that the eigenvalues of $I - hL - J_N$ are 0, $1 - \lambda_2, \ldots, 1 - \lambda_N$. Since the communication graph is connected, from Lemma 2.1, we know that $\lambda_2 > 0$. According to (31) and (35), we have

$$\left\|\bar{A}(k)\right\|_2 \leq \gamma^{-1} \left(\|I - hL - J_N\|_2 + \|\Delta_A(k)\|_2\right) \leq \frac{\bar{\gamma}_h}{\gamma} < 1. \tag{41}$$

Inequality (41) implies the stability of system (40) and equivalently that of (14a).

Next, we shall analyze the stability of (14b). We rewrite system (14b) in the augmented form

$$\bar{H}(k) = B(k)\bar{H}(k-1) + \Gamma h \sum_{i=0}^{d} L_i(k)\gamma^{-i-1} z(k-i), \tag{42}$$

where

$$B(k) = \begin{bmatrix} B_1(k) & B_2(k) & B_3(k) & \cdots & B_d(k) \\ I & & & & \\ & I & & & \\ & & \ddots & & \\ & & & I & \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}', \ \bar{H}(k) = \begin{bmatrix} H'(k) & \cdots & H'(k-d+1) \end{bmatrix}'.$$

By introducing the linear transformation

$$T = \begin{cases} \operatorname{diag}\{I, h^{1/d}I, h^{2/d}I, \ldots, h^{d-1/d}I\}, & d \neq 1, \\ I, & d = 1, \end{cases}$$

we have

$$\bar{B}(k) \triangleq TB(k)T^{-1} = \begin{bmatrix} B_1(k) & \frac{B_2(k)}{h^{1/d}} & \frac{B_3(k)}{h^{2/d}} & \cdots & \frac{B_d(k)}{h^{(d-1)/d}} \\ h^{1/d}I & & & & 0 \\ & h^{1/d}I & & & 0 \\ & & \ddots & & \vdots \\ & & & h^{1/d}I & 0 \end{bmatrix}. \tag{43}$$

By (30), it follows that $\bar{\gamma}_h > 1 - \lambda_2 h - \rho_A(\lambda_N, \theta)h^2 > 1 - \theta$. According to Lemma 3.1 and (35), we know that $\|\Delta_{B_j}(k)\|_\infty < \rho_B(\lambda_N, \theta, 1)h^2$. Moreover, by considering Lemmas 2.1 and 3.1, it follows that

$$\|B_j(k)\|_\infty = \left\| \gamma^{-j} h \sum_{i=j}^{d} L_i(k) + \Delta_{B_j}(k) \right\|_\infty \leq 2h\gamma^{-j}D^* + \rho_B(\lambda_N, \theta, 1)h^2,$$

where D^* is the degree of the graph. Then we calculate the norm of $\bar{B}$ based on (43) as follows:

$$\begin{aligned} \left\|\bar{B}(k)\right\|_\infty &\leq \max\left\{ h^{\frac{1}{d}}, \sum_{i=1}^{d} h^{\frac{1-i}{d}} \|B_i(k)\|_\infty \right\} \\ &\leq \max\left\{ h^{\frac{1}{d}}, \sum_{i=1}^{d} h^{\frac{1-i}{d}} \left(2h\gamma^{-i}D^* + \rho_B(\lambda_N, \theta, 1)h^2\right) \right\} \\ &\leq \max\left\{ h^{\frac{1}{d}}, 2h^{\frac{d+1}{d}} D^* \sum_{i=1}^{d} \left(h^{\frac{1}{d}}\gamma\right)^{-i} + \rho_B(\lambda_N, \theta, 1)h^2 \sum_{i=1}^{d} h^{\frac{1-i}{d}} \right\} \\ &< \max\left\{ h^{\frac{1}{d}}, \frac{2h^{\frac{1}{d}}D^*\gamma^{-d}}{1-\gamma h^{\frac{1}{d}}} + \rho_B(\lambda_N, \theta, 1)h^{\frac{d+1}{d}} \frac{1-h}{1-h^{\frac{1}{d}}} \right\} \\ &< \max\left\{ h^{\frac{1}{d}}, h^{\frac{1}{d}} \frac{2D^*(1-\theta)^{-d} + \rho_B(\lambda_N, \theta, 1)/4}{1-h^{\frac{1}{d}}} \right\} = \beta. \end{aligned} \tag{44}$$

The last inequality is due to that $1-\gamma h^{\frac{1}{d}} > 1-h^{\frac{1}{d}}$ and $h(1-h) \leq \frac{1}{4}$. From the above inequality we can see that $\beta < 1$ if and only if $h < \left(1+2D^*(1-\theta)^{-d}+\frac{\rho_B}{4}\right)^{-d}$, which is satisfied according to (30) and (35). On the other hand, $\beta < 1$ implies the stability of system (42). In other words, (35) implies the stability of system (14a)–(14b).

Last, we shall prove that when $K_1(h,\gamma,g_0)$ is large enough the quantizer will never be saturated for any K satisfying (36). This can be done by induction. It is easy to check that $K_1(h,\gamma,g_0) > M_1(h,\gamma,g_0)$, where M_1 is as defined in (27), which means that $\|\bar{e}(k)\|_\infty \leq K_1(h,\gamma,g_0)$ for $k=0,1,\dots,d$. Suppose that

$$\|\bar{e}(k)\|_\infty \leq K_1(h,\gamma,g_0) \quad \forall k=0,1,\dots,s,$$

where $s \geq d$. According to (12b) we see that $\|z(k)\|_\infty \leq \frac{1}{2\gamma}$ for all $k=0,1,\dots,s+1$, $s \geq d$. Considering (42) with initial condition $\bar{H}(0)=0$, we have

$$\left\|\Pi^B_{i,j}\right\|_\infty \leq \|T\|_\infty \left\|T^{-1}\right\|_\infty \left\|\Pi^{\bar{B}}_{i,j}\right\|_\infty \leq h^{\frac{1-d}{d}}\beta^{i-j+1},$$

and

$$\begin{aligned}\left\|\bar{H}(s)\right\|_\infty &= \left\|\sum_{j=1}^{s}\Pi^B_{s,j+1}\Gamma h\sum_{i=0}^{d}L_i(j)\gamma^{-i-1}z(j-i)\right\|_\infty\\ &\leq h^{\frac{1}{d}}\sum_{j=1}^{s}\beta^{s-j}\|\Gamma\|_\infty\sum_{i=0}^{d}\gamma^{-i-1}\left\|L_i(j)\right\|_\infty\left\|z(j-i)\right\|_\infty\\ &\leq h^{\frac{1}{d}}\sum_{j=0}^{s-1}\beta^{j}\left(\sum_{i=0}^{d}\gamma^{-i-1}\|L\|_\infty\frac{1}{2\gamma}\right)\\ &< h^{\frac{1}{d}}D^*\frac{\gamma^{-d-1}-1}{\gamma(1-\gamma)(1-\beta)}.\end{aligned} \tag{45}$$

Based on (14a), (45), and the fact that $\|v\|_\infty \leq \|v\|_2 \leq \sqrt{n}\|v\|_\infty$ for any $n\times 1$ vector v, we have

$$\begin{aligned}\|w_a(s+1)\|_2 &= \left\|\Pi^{\bar{A}}_{s,d}w_a(d)+\sum_{i=d}^{s}\Pi^{\bar{A}}_{s,i+1}H(i)\right\|_2\\ &\leq \left(\frac{\bar{\gamma}_h}{\gamma}\right)^{s-d+1}\|w_a(d)\|_2+\sum_{i=d}^{s}\left(\frac{\bar{\gamma}_h}{\gamma}\right)^{s-i}\sqrt{N}\|H(i)\|_\infty\\ &\leq \left(\frac{\bar{\gamma}_h}{\gamma}\right)^{s-d+1}\|w_a(d)\|_2+\left[1-\left(\frac{\bar{\gamma}_h}{\gamma}\right)^{s-d+1}\right]\frac{h^{\frac{1}{d}}D^*\sqrt{N}\left(\gamma^{-d-1}-1\right)}{(1-\gamma)(1-\beta)(\gamma-\bar{\gamma}_h)}\\ &\leq \bar{w},\end{aligned} \tag{46}$$

where $\bar{w}$ is defined in (33). The last inequality in (46) is obtained from Lemma 3.2 and $w_a(d)=w(d)$. Since $w_a(k)=w(k)$ for all $k \geq d$, we have $\|w(k)\|_2 \leq \bar{w}$ for all $k=0,1,\dots,s+1$. Then, according to the definition of $\bar{e}(k)$ in (13), it follows from

(46) that

$$\begin{aligned}\|\bar{e}(s+1)\|_\infty &= \left\| z(s+1) + h\sum_{i=0}^{d} L_i(s+1)\gamma^{-i} z(s+1-i) \right. \\ &\quad \left. -h\sum_{i=0}^{d} L_i(s+1)\gamma^{-i} w(s+1-i) \right\|_\infty \\ &\le \frac{1 + h\sum_{i=0}^{d} \|L_i(s+1)\|_\infty \gamma^{-i}}{2\gamma} + h\bar{w}\sum_{i=0}^{d} \|L_i(s+1)\|_\infty \gamma^{-i} \\ &\le \frac{1}{2\gamma} + \left(\frac{1}{2\gamma} + \bar{w}\right) 2hD^* \frac{\gamma^{-d-1}-1}{\gamma^{-1}-1} \\ &= K_1(h,\gamma,g_0). \qquad (47)\end{aligned}$$

By induction, we know that $\|\bar{e}(k)\|_\infty \le K_1(h,\gamma,g_0)$ and $\|w(k)\|_2 \le \bar{w}$ for all $k \ge 0$; that is to say, the quantizer will never be saturated if (36) is satisfied. By the definition of $w(k)$ and $0 < \gamma < 1$, (9) is obtained.

From (28), (31), and (44), we have

$$\lim_{h\to 0^+,\gamma\to 1^-} |f(h,\gamma,d)| < +\infty, \quad \lim_{\gamma\to 1^-} \frac{\gamma^{-d-1}-1}{\gamma^{-1}-1} = d+1,$$
$$\lim_{h\to 0^+,\gamma\to 1^-} \frac{h}{\gamma - \bar{\gamma}_h} = \frac{1}{\lambda_2}, \quad \lim_{h\to 0^+} \beta = 0.$$

It follows that

$$\lim_{h\to 0^+,\gamma\to 1^-} K_1(h,\gamma,g_0) = \frac{1}{2},$$

which is (37). So far, we have proved part I.

It is noted that

$$\|\xi(k)\|_2 = \|g(k)w(k)\|_2 \le \bar{w} g_0 \gamma^k,$$

which leads to (38). According to (8) we arrive at

$$\begin{aligned}\lim_{k\to\infty} \|X(k) - J_N X(0)\|_2^{1/k} &= \lim_{k\to\infty} \exp\left\{\frac{1}{k}\ln\|\xi(k)\|_2\right\} \\ &\le \exp\left\{\lim_{k\to\infty} \frac{\ln\|\xi(k)\|_2}{k}\right\} \\ &\le \exp\left\{\lim_{k\to\infty} \frac{\ln\left(\bar{w} g_0 \gamma^k\right)}{k}\right\} \\ &= \gamma.\end{aligned}$$

By this and the definition of r_{asym} we get (39). ∎

Remark 3.2. We have provided the ranges of the consensus gain h and scaling parameter γ such that $w(k)$ and $z(k)$ are uniformly bounded, and consequently, the consensus error $\xi(k)$ converges to 0 at the exponential rate $O(\gamma^k)$. We have worked out a lower bound of the quantization level K, which is a continuous function of h,

γ, and g_0. From (36) we can see that the bit rate can reach 1 when $K_1 < 1$. On the other hand, K_1 can approach $\frac{1}{2}$ when $h \to 0^+$ and $\gamma \to 1^-$. Hence there is a trade-off between the consensus rate and the communication data rate. A lower communication data rate requires a larger γ, which leads to a lower consensus convergence rate $O(\gamma^k)$. On the other hand, a smaller γ can guarantee a faster convergence, but, according to (32), (33), and (36), a higher data rate is required for each channel. This result is coincident with the delay-free case [14].

Remark 3.3. When we design the parameters h, γ, g_0 and calculate the lower bound of quantization level, some knowledge is required, such as the positive lower bound of λ_2, the upper bound of D^*.[1] In fact, this can be estimated for some special graphs. For example, for expander networks, if the communication graph is a d-regular c-expander, we have $\lambda_2 \geq d - \sqrt{d^2 - c^2}$, $D^* = d$.

The above result for the constant time delay case provides a new way to solve such a complicated consensus problem. Based on this result, in the next subsection, we shall consider a more general case in which the communication delays are time-varying.

Case 2. *Time-varying delays.* In this section, we shall consider that the delays in the transmission channels are time-varying. Note that the summation $\sum_{l=0}^{d} L_l(k) \triangleq L(k)$ is no longer a constant matrix due to the time-varying delays. It is much more complicated to analyze the stability of the system than the one with constant delays. Nevertheless, $L(k)$ has the following properties:

$$L \leq \sum_{l=0}^{d} L(k+l), \; L(k) \leq dL. \tag{48}$$

In fact, due to the fact that the time delays are bounded by d, a packet in each channel will be received within d steps, which implies the first inequality in (48). On the other hand, the receiver can receive at most d packets from each channel at one time, which yields the second inequality in (48). Similar to Lemma 3.2, we have the following lemma. The proof is similar to that of Lemma 3.2 by considering (48) and thus is omitted.

LEMMA 3.3. *Consider system* (12a)–(12b). *Define*

$$M_2(h,\gamma,g_0) = \frac{1}{2\gamma} + \left(\frac{1}{2\gamma} + f_2(h,\gamma,d)\right) 2hdD^* \frac{\gamma^{-d}-1}{\gamma^{-1}-1}, \tag{49}$$

where f_2 is given by

$$f_2(h,\gamma,k) = p_2^k \frac{2\sqrt{N}C_x}{g_0} + \frac{hd\sqrt{N}\lambda_N\left(\gamma^{-d}-1\right)\left(p_2^k-1\right)}{2\gamma\left(1-\gamma\right)\left(p_2-1\right)}, \; p_2 = \gamma^{-1} + h\lambda_N d\frac{\gamma^{-d}-1}{1-\gamma}.$$

If $g_0 \geq 2\gamma C_x$ and $K \geq \lfloor M_2(h,\gamma,g_0) + \frac{1}{2} \rfloor$, then the following inequalities hold for all $k = 0, 1, \ldots, d$:

$$\|w(k)\|_2 \leq f_2(h,\gamma,k), \; \|z(k)\|_\infty \leq \frac{1}{2\gamma}. \tag{50}$$

[1] The upper bound of λ_N can be estimated by $\lambda_N \leq 2D^*$.

Define

$$\bar{h}_{\theta,1} = \min\left\{h_1(\lambda_N d,\theta), \frac{\lambda_2}{\bar{\rho}_A(\theta)}, \left(1 + 2dD^*(1-\theta)^{-1} + \frac{\rho_B(\lambda_N d,\theta,(d+1)^{-1})}{4}\right)^{-d}\right\}, \tag{51}$$

with

$$\bar{\rho}_A(\theta) \triangleq (d+1)\rho_A(\lambda_N d,\theta) + \frac{(1+\theta)^{d+1} - 1 - (d+1)\theta}{\theta^2} \times \left(\frac{\sqrt{d^2\lambda_N^2 + 4\alpha\rho_A(\lambda_N d,\theta)} + \lambda_N d}{2}\right)^2, \tag{52}$$

and

$$\bar{\gamma}_{h,1} = 1 - \lambda_2 h + \bar{\rho}_A(\theta)h^2, \tag{53}$$

$$K_2(h,\gamma,g_0) = \frac{1}{2\gamma} + \left(\frac{1}{2\gamma} + \bar{w}_1(h,\gamma)\right) 2hdD^* \frac{\gamma^{-d-1} - 1}{\gamma^{-1} - 1}, \tag{54}$$

with

$$\bar{w}_1(h,\gamma) \triangleq \max\left\{ \left(\frac{1+\rho_A(\lambda_N d,\theta)h^2}{\gamma}\right)^d f_2(h,\gamma,d), \frac{\left(\frac{1+\rho_A(\lambda_N d,\theta)h^2}{\gamma}\right)^{d+1} - 1}{\left(\frac{1+\rho_A(\lambda_N d,\theta)h^2}{\gamma}\right) - 1} \frac{h^{\frac{1}{d}} dD^*\sqrt{N}\left(\gamma^{-d-1} - 1\right)}{(1-\gamma)(1-\beta_1)\left(\gamma - \gamma^{-d}\bar{\gamma}_{h,1}\right)}\right\}, \tag{55}$$

$$\beta_1 = \max\left\{h^{\frac{1}{d}}, h^{\frac{1}{d}} \frac{2dD^*(1-\theta)^{-1} + \rho_B(\lambda_N d,\theta,(d+1)^{-1})/4}{1 - h^{\frac{1}{d}}}\right\}. \tag{56}$$

Then we have the following result.

THEOREM 3.2. *Consider system* (1) *with time-varying transmission delays. Under Assumptions* 1–4*, we apply the protocol* (5) *with encoder* (2) *and decoder* (3)*. For any given* $\theta > 0$ *satisfying* (21)*, if* h*,* γ*,* g_0 *satisfy*

$$h \in (0, \bar{h}_{\theta,1}),\ \ \gamma \in (\sqrt[d+1]{\bar{\gamma}_{h,1}}, 1),\ \ g_0 \geq 2\gamma C_x, \tag{57}$$

and the number of the quantization level satisfies

$$K \geq \left\lfloor K_2(h,\gamma,g_0) + \frac{1}{2} \right\rfloor, \tag{58}$$

we have the following result:

I. *The average consensus problem is solved. In particular,* $K_2(h,\gamma,g_0)$ *satisfies*

$$\lim_{h\to 0^+,\gamma\to 1^-} K_2(h,\gamma,g_0) = \frac{1}{2}; \tag{59}$$

i.e., the lower bound of the number of transmitting bits is 1.

II. *We have $r_{\text{asym}} \leq \gamma$, and the consensus error $\xi(k)$ satisfies*

$$\limsup_{k\to\infty} \frac{\|\xi(k)\|_2}{\gamma^k} \leq \bar{w}g_0. \tag{60}$$

Proof. The proof is partly similar to that of Theorem 3.1, so only the different part will be pointed out. We consider system (14a)–(14b) with the initial condition of $A(k)$ satisfying

$$\|\Delta_A(k)\|_2 < \rho_A(\lambda_N d, \theta)h^2 \quad \forall k = 0, 1, \ldots, d-1,$$

where θ satisfies (21). By considering (48), we can see that $\sup_{k\geq 0} \lambda(L(k)) \leq \lambda_N d$. Then, according to Lemma 3.1 and (57), we have that $\|\Delta_A(k)\|_2 < \rho_A(\lambda_N d, \theta)h^2$ for all $k \geq 0$. System (14a) can be rewritten as

$$w_a(k+1) = \bar{A}(k)w_a(k) + H(k), \tag{61}$$

where

$$\bar{A}(k) = A(k)(I - J_N) = \gamma^{-1}(I - hL(k) - J_N + \Delta_{\bar{A}}(k)), \tag{62}$$

with $\Delta_{\bar{A}}(k) \triangleq \Delta_A(k)(I-J_N)$. It is clear that $\Delta_{\bar{A}}(k)$ satisfies $\|\Delta_{\bar{A}}(k)\|_2 < \rho_A(\lambda_N d, \theta)h^2$. According to (57), by simple calculation, it is shown that

$$\lambda_N d + \rho_A(\lambda_N d, \theta)h < \frac{\sqrt{d^2\lambda_N^2 + 4\bar{\rho}_A\theta} + \lambda_N d}{2}. \tag{63}$$

By defining $g(k) = \max_{j=0,\ldots,d} \|hL(k+j) - \Delta_{\bar{A}}(k+j)\|_2$, it is easy to check that

$$g(k) \leq \lambda_N dh + \rho_A(\lambda_N d, \theta)h^2 < \theta, \tag{64}$$

and therefore

$$\frac{(1+g(k))^{d+1} - 1 - (d+1)g(k)}{g^2(k)} < \frac{(1+\theta)^{d+1} - 1 - (d+1)\theta}{\theta^2}. \tag{65}$$

Then we have

$$\begin{aligned}
\left\|\Pi_{k+d,k}^{\gamma\bar{A}}\right\|_2 &= \gamma^{d+1}\left\|\Pi_{k+d,k}^{\bar{A}}\right\|_2 \\
&\leq \|(I - J_N - hL(k+d) + \Delta_{\bar{A}}(k+d)) \cdots (I - J_N - hL(k) + \Delta_{\bar{A}}(k))\|_2 \\
&\leq \left\|I - J_N - h\sum_{i=0}^{d} L(k+i)\right\|_2 + (d+1)\rho_A(\lambda_N d, \theta)h^2 \\
&\quad + \left[\, (1+g(k))^{d+1} - 1 - (d+1)g(k)\right] \\
&< \left\|I - J_N - h\sum_{i=0}^{d} L(k+i)\right\|_2 + \left[(d+1)\rho_A(\lambda_N d, \theta)\right. \\
&\quad \left. + \frac{(1+\theta)^{d+1} - 1 - (d+1)\theta}{\theta^2}\left(\frac{\sqrt{d^2\lambda_N^2 + 4\rho_A(\lambda_N d, \theta)\theta} + \lambda_N d}{2}\right)^2\right]h^2 \\
&\leq \bar{\gamma}_{h,1},
\end{aligned} \tag{66}$$

where $\bar{\gamma}_{h,1}$ is as defined in (53). The second to last inequality is due to (63) and (65). From (66) we can see that $\sqrt[d+1]{\bar{\gamma}_{h,1}} < \gamma < 1$ implies $\|\Pi^{\bar{A}}_{k+d,k}\|_2 < 1$, which guarantees the stability of system (61). By considering (64), we know that $\theta > \lambda_N dh + \rho_A(\lambda_N d, \theta)h^2$, which leads to that $\bar{\gamma}_{h,1} > 1-\theta$. According to Lemma 3.1, we have

$$\|\Delta_{B_j}(k)\|_\infty < \rho_B(\lambda_N d, \theta, (d+1)^{-1})h^2,$$

where $\Delta_{B_j}(k)$ is as defined in (22). Again we introduce the augmented system (42) of (14b). Then by applying the same manipulation as in (44), for matrix $\bar{B}(k)$ defined in (43), we have $\left\|\bar{B}(k)\right\|_\infty < \beta_1$, where β_1 is as defined in (56). Note that $\beta_1 < 1$ if and only if

$$h < \left(1 + 2dD^*(1-\theta)^{-1} + \frac{\rho_B(\lambda_N d, \theta, (d+1)^{-1})}{4}\right)^{-d}. \tag{67}$$

Inequality (57) guarantees (67) and therefore the stability of system (42). Briefly stated, h and γ satisfying (57) make sure that the system (14a)–(14b) is stable. Since $w_a(k) = w(k)$ for all $k \geq d$, we can conclude that system (12a) is stable, which means we have solved part 1 of our Problem.

Next, we need to determine K_2 such that the quantizer will not be saturated for any K satisfying (58). It is easy to check that $K_2(h,\gamma,g_0) > M_2(h,\gamma,g_0)$, where M_2 is as defined in (49), which means that $\|\bar{e}(k)\|_\infty \leq K_2(h,\gamma,g_0)$ for $k = 0,1,\dots,d$. Suppose that

$$\|\bar{e}(k)\|_\infty \leq K_2(h,\gamma,g_0) \quad \forall\ k = 0,1,\dots,s, \tag{68}$$

where $s \geq d$. According to (12b) we see that $\|z(k)\|_\infty \leq \frac{1}{2\gamma}$ for all $k = 0,1,\dots,s+1$, $s \geq d$. Considering (42) with initial condition $\bar{H}(0) = 0$, along the same lines of (45), we have

$$\left\|\bar{H}(s)\right\|_\infty < h^{\frac{1}{d}} dD^* \frac{\gamma^{-d-1} - 1}{\gamma(1-\gamma)(1-\beta_1)}. \tag{69}$$

From (62) we get

$$\left\|\bar{A}(k)\right\|_2 \leq \frac{\|I - hL(k) + \Delta_{\bar{A}}(k)\|_2 + \rho_A(\lambda_N d,\theta)h^2}{\gamma} \leq \frac{1+\rho_A(\lambda_N d,\theta)h^2}{\gamma}, \tag{70}$$

which together with (66) results in that, for any $k \geq 0$, $m \geq 0$,

$$\begin{aligned}\left\|\Pi^{\bar{A}}_{k+m,k+1}\right\|_2 &\leq \left(\frac{1+\rho_A(\lambda_N d,\theta)h^2}{\gamma}\right)^{m-\phi_{m,d}} \left(\frac{\bar{\gamma}_{h,1}}{\gamma^{d+1}}\right)^{\phi_{m,d}} \\ &\leq \left(\frac{1+\rho_A(\lambda_N d,\theta)h^2}{\gamma}\right)^{d} \left(\frac{\bar{\gamma}_{h,1}}{\gamma^{d+1}}\right)^{\phi_{m,d}}\end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^{m}\left\|\Pi^{\bar{A}}_{k+i,k+1}\right\|_2 &\leq \frac{1-\left(\gamma^{-d-1}\bar{\gamma}_{h,1}\right)^{\phi_{m,d}}}{1-\gamma^{-d-1}\bar{\gamma}_{h,1}} \sum_{j=0}^{d}\left(\frac{1+\rho_A(\lambda_N d,\theta)h^2}{\gamma}\right)^j \\ &\leq \frac{1-\left(\gamma^{-d-1}\bar{\gamma}_{h,1}\right)^{\phi_{m,d}}}{1-\gamma^{-d-1}\bar{\gamma}_{h,1}} \frac{\left(\frac{1+\rho_A(\lambda_N d,\theta)h^2}{\gamma}\right)^{d+1} - 1}{\left(\frac{1+\rho_A(\lambda_N d,\theta)h^2}{\gamma}\right) - 1},\end{aligned} \tag{71}$$

where $\phi_{m,d} \triangleq \lfloor \frac{m}{d+1} \rfloor$. Then we have

$$
\begin{aligned}
\|w_a(s+1)\|_2 &= \left\| \Pi_{s,d}^{\bar{A}} w_a(d) + \sum_{i=d}^{s} \Pi_{s,i+1}^{\bar{A}} H(i) \right\|_2 \\
&\le \left(\frac{\bar{\gamma}_{h,1}}{\gamma^{d+1}} \right)^{\phi_{s-d+1,d}} \left(\frac{1+\rho_A(\lambda_N d,\theta)h^2}{\gamma} \right)^{d} \|w_a(d)\|_2 \\
&\quad + \frac{1 - \left(\frac{\bar{\gamma}_{h,1}}{\gamma^{d+1}} \right)^{\phi_{s-d+1,d}}}{1 - \frac{\bar{\gamma}_{h,1}}{\gamma^{d+1}}} \frac{\left(\frac{1+\rho_A(\lambda_N d,\theta)h^2}{\gamma} \right)^{d+1} - 1}{\left(\frac{1+\rho_A(\lambda_N d,\theta)h^2}{\gamma} \right) - 1} \frac{h^{\frac{1}{d}} d D^* \sqrt{N} \left(\frac{1}{\gamma^{d+1}} - 1 \right)}{\gamma(1-\gamma)(1-\beta_1)} \\
&\le \bar{w}, \qquad (72)
\end{aligned}
$$

where $\bar{w}$ is defined in (55). The first inequality in (72) follows from (69) and (71), and the second inequality is due to Lemma 3.3 and the definition of $\bar{w}$. Along the same lines of (47), there holds that

$$\|\bar{e}(s+1)\|_\infty \le K_2(h, \gamma, g_0).$$

By induction, we know that $\|\bar{e}(k)\|_\infty \le K_2(h, \gamma, g_0)$ for all $k \ge 0$, which means that the quantizer will never be saturated if (58) is satisfied. By the definition of $w(k)$ and $0 < \gamma < 1$, (9) is obtained. The proof of (59) and (60) is analogous to that of (37) and (38) which is omitted here. ▯

Remark 3.4. Note that the result is more conservative for the time-varying delay case than that for the constant delay case. For the same consensus gain h and scaling parameter γ, we need a higher transmission bit rate for the time-varying delay case. Nevertheless, K_2 can also approach $\frac{1}{2}$ when $h \to 0^+$ and $\gamma \to 1^-$, which implies the attainability of a one-bit information exchange while guaranteeing the average consensus.

4. Numerical example.

We consider a network with five agents and 0–1 weights over the edges. The edges of the graph are initially randomly generated with probability 0.5. By calculation, we have $\lambda_2 = 3$, $\lambda_N = 5$, and $D^* = 4$. The initial conditions of the agents are

$$x_1(0) = 0.8449,\ x_2(0) = 0.7531,\ x_3(0) = 0.7030,\ x_4(0) = 0.2466,\ x_5(0) = 0.0399.$$

We assume that all the transmitted data suffers from one step delay. We choose $h = 0.0005$, $\gamma = 0.9993$, and $g_0 = 2$. According to (32) we have $K_1 = 0.8539$. From Theorem 3.1 we know that a one-bit quantizer can be used. The state trajectories are shown in Figure 4. From the figure we see that the average value of the initial states, 0.5175, is achieved asymptotically.

Following Remark 3.2, there is a trade-off in choosing the parameter h, γ, and the data rate. From the above example we see that one-bit rate communication requires that γ be close enough to 1. In fact, the bit number is a conservative estimate, and in practice, fewer bits may be enough when a smaller γ is chosen. If we take $h = 0.01$ and $\gamma = 0.99$, from (32) we get $K_1 = 19.31$. We still use the one-bit quantizer; the state trajectories are shown in Figure 5. We see that the average consensus is achieved asymptotically as well.

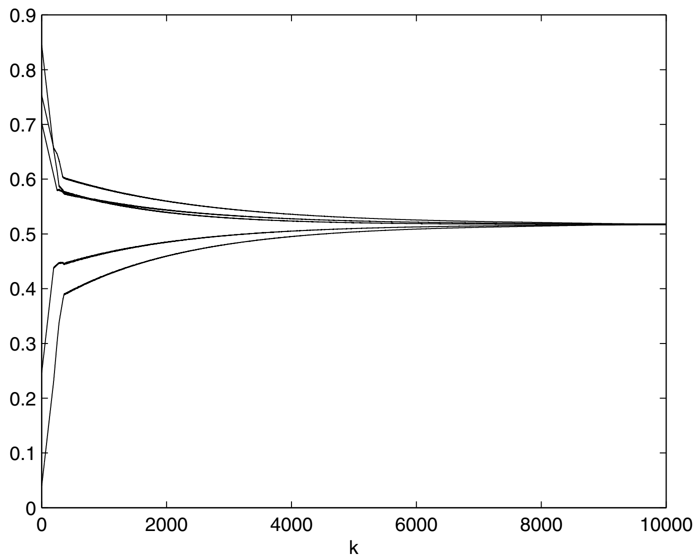

FIG. 4. *State trajectories of five agents with* $h = 0.0005$ *and* $\gamma = 0.9993$.

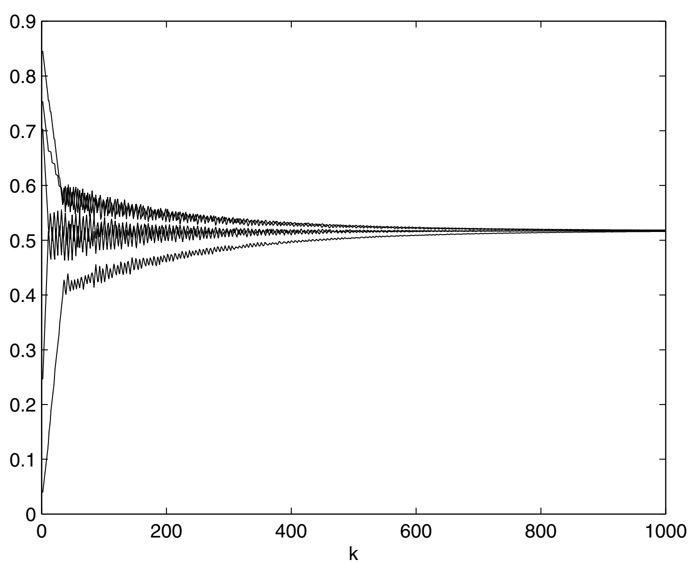

FIG. 5. *State trajectories of five agents with* $h = 0.01$ *and* $\gamma = 0.99$.

5. Conclusions. We have considered the average consensus problem for first order discrete time multiagent systems with communication delays and limited data rate. The dynamical encoding and decoding with finite-level quantization has been adopted. It has been shown that when the network is connected, a distributed protocol can be designed such that the average consensus problem is solvable under a finite-level quantizer. In particular, no matter how large the delays are, we are always able to design a distributed protocol to ensure asymptotic average consensus with an exponential convergence rate using merely a one-bit information exchange between each pair of adjacent agents at each time step.

The encoder and decoder are node-based, which makes the results restrictive. To extend the result to dynamically switching communication graphs, we need to design a channel-based coding strategy, i.e., to design different encoders for different channels. Further, it is possible to consider random packet dropouts and/or random communication graphs when we adopt the channel-based coding strategy. It is worth mentioning that the receipt acknowledgment of each channel is required when the channel-based coding strategy is used.

Appendix A. Proof of Lemma 3.1. Before proving Lemma 3.1, we introduce the following lemma.

LEMMA A.1. *Given a series of $n \times n$ matrices $F_1, F_2, \dots, F_l$ satisfying, for all $i = 1, \dots, l$, $\|F_i\| < 1$, we have the following result:*

$$\begin{aligned} &\left\|(I-F_1)^{-1}(I-F_2)^{-1}\cdots(I-F_l)^{-1}-I\right\| \\ &\leq (1-\|F_1\|)^{-1}(1-\|F_2\|)^{-1}\cdots(1-\|F_l\|)^{-1}-1, \end{aligned} \tag{73}$$

where $\|\cdot\|$ can be any induced norm.

Proof. $\|F_i\| < 1$ guarantees that $I - F_i$ is invertible. In fact, if $I - F_i$ is singular, we can find a vector v satisfying $(I - F_i)\,v = 0$. Thus

$$\|v\|_2 = \|F_i v\|_2 \leq \|F_i\|\,\|v\|_2 < \|v\|_2\,,$$

which is a contradiction. Therefore, such a v does not exist. Then for any $i = 1, 2, \dots, l$, we have

$$(I - F_i)^{-1} = \sum_{k=0}^{\infty} F_i^k,$$

and therefore

$$\begin{aligned} \left\|(I-F_1)^{-1}\cdots(I-F_l)^{-1}-I\right\| &= \left\|\left(\sum_{k=0}^{\infty}F_1^k\right)\left(\sum_{k=0}^{\infty}F_2^k\right)\cdots\left(\sum_{k=0}^{\infty}F_l^k\right)-I\right\| \\ &= \left\|\sum_{1\leq i\leq l}F_i + \sum_{1\leq i\leq j\leq l}F_iF_j + \cdots\right\| \\ &\leq \sum_{1\leq i\leq l}\|F_i\| + \sum_{1\leq i\leq j\leq l}\|F_i\|\,\|F_j\| + \cdots \\ &= (1-\|F_1\|)^{-1}(1-\|F_2\|)^{-1}\cdots(1-\|F_l\|)^{-1}-1, \end{aligned}$$

which is (73). ∎

Proof of Lemma 3.1. It is straightforward that for any given θ satisfying (21), if $h < h_1$, then we have

$$\bar{\lambda}h + \rho_A(\bar{\lambda}, \theta)h^2 \leq \theta. \tag{74}$$

From (23) and (74) we see that $A(k)$ is invertible for $k = 0, 1, \dots, d-1$. Next, we shall show that (23) holds for all $k \geq 0$. Assume that $\|\Delta_A(k)\|_2 < \rho_A(\bar{\lambda}, \theta)h^2$ for

$k = 0, \dots, r-1$, where $r \geq d$. According to (15)–(18), for all $d \leq k \leq r$, we arrive at the following relationships:

$$
\begin{aligned}
I - \gamma A(k) &= \sum_{i=0}^{k} hL_i(k)\gamma^{-i}\left(\Pi_{k-1,k-i}^{A}\right)^{-1} \\
&= \sum_{i=1}^{k} hL_i(k)\big[\,(I-(hL-\Delta_A(k-i)))^{-1}\cdots(I-(hL-\Delta_A(k-1)))^{-1}\,\big] \\
&\quad + hL_0(k) \\
&= hL - \Delta_A(k). \qquad (75)
\end{aligned}
$$

Denote $g(k) = \max_{j=1,\dots,d} \|hL - \Delta_A(k-j)\|_2$. It is clear that for all $k \leq r$, we have $g(k) \leq \theta$. This is due to the fact that $g(k) \leq \bar{\lambda}h + \rho_A(\bar{\lambda},\theta)h^2$. According to (75), we have

$$
\begin{aligned}
\|\Delta_A(r)\|_2 &= \left\| \sum_{i=1}^{d} hL_i(r)\left[\left(\Pi_{r-1,r-i}^{I-hL+\Delta_A}\right)^{-1} - I\right]\right\|_2 \\
&\leq h\,\|L(k)\|_2 \sum_{i=1}^{d}\left[(1-g(r))^{-i} - 1\right] \\
&\leq \bar{\lambda}h\frac{(1-g(r))^{-d} - 1 - dg(r)}{g(r)} \\
&\leq \bar{\lambda}h\left(\bar{\lambda}h + \rho_A(\bar{\lambda},\theta)h^2\right)\frac{(1-g(r))^{-d} - 1 - dg(r)}{g^2(r)} \\
&< h^2\left(\bar{\lambda}^2 + \theta\rho_A(\bar{\lambda},\theta)\right)\frac{(1-\theta)^{-d} - 1 - d\theta}{\theta^2}. \qquad (76)
\end{aligned}
$$

The first inequality comes from Lemmas 2.1 and A.1 and the definition of $g(k)$. The third inequality is due to the fact that $g(r) \leq \bar{\lambda}h + \rho_A(\bar{\lambda},\theta)h^2$. The last inequality is obtained from the inequality $\bar{\lambda}h < \bar{\lambda}h + \rho_A h^2 \leq \theta$. According to the definition of ρ_A and inequality (76) we see that $\|\Delta_A(r)\|_2 < \rho_A(\bar{\lambda},\theta)h^2$. It is clear that $A(r)$ is also invertible. By induction, $\|\Delta_A(k)\|_2 < \rho_A(\bar{\lambda},\theta)h^2$ and $A(k)$ is invertible for all $k \geq 0$.

Next, we shall prove (26). According to (15)–(18) we have, for all $j = 1, 2, \dots, d$,

$$
B_j(k) = \sum_{i=j}^{d} hL_i(k)\gamma^{-i-1}\left(\Pi_{k-j,k-i}^{A}\right)^{-1}. \qquad (77)
$$

Since $(1-\theta)^l < \gamma < 1$, we know that for all $j < d$, $\gamma^{-j} < \gamma^{-d} < (1-\theta)^{-ld}$. Then, by comparing (22) and (77), we have

$$\begin{aligned}\left\|\Delta_{B_j}(k)\right\|_2 &= \left\|\gamma^{-j}\sum_{i=j}^{d} hL_i(k)\left(\gamma^{j-i-1}\left(\Pi^A_{k-j,k-i}\right)^{-1} - I\right)\right\|_2\\ &\leq h\gamma^{-j}\left\|L(k)\right\|_2\sum_{i=j}^{d}\left[(1-g(k))^{-(i-j+1)} - 1\right]\\ &< \bar{\lambda}h\gamma^{-d}\frac{(1-g(k))^{-d} - 1 - dg(k)}{g(k)}\\ &\leq \bar{\lambda}h\gamma^{-d}\left(\bar{\lambda}h + \rho_A(\bar{\lambda},\theta)h^2\right)\frac{(1-g(k))^{-d} - 1 - dg(k)}{g^2(k)}\\ &< h^2\left(\bar{\lambda}^2 + \theta\rho_A(\bar{\lambda},\theta)\right)\frac{(1-\theta)^{-d} - 1 - d\theta}{\theta^2(1-\theta)^{ld}}.\end{aligned}$$

It is known that for any given $n \times n$ matrix B, $\|B\|_\infty \leq \sqrt{n}\,\|B\|_2$; then we have

$$\left\|\Delta_{B_j}(k)\right\|_\infty \leq \sqrt{N}\left\|\Delta_{B_j}(k)\right\|_2 < \rho_B(\bar{\lambda},\theta,l)h^2.$$

Appendix B. Proof of Lemma 3.2.

Proof. Set $k = 0$; then we have

$$\|w(0)\|_2 \leq \sqrt{N}\,\|w(0)\|_\infty = \frac{\sqrt{N}\,\|\xi(0)\|_\infty}{g_0} \leq \frac{2\sqrt{N}C_x}{g_0} = f_1(h,\gamma,0), \tag{78}$$

which is the first inequality of (29). The last inequality of (78) is due to the fact that $\|\xi(0)\|_\infty \leq 2C_x$. According to the definition of $z(k)$, we have

$$\|z(0)\|_\infty = \left\|\frac{\tilde{X}(0)}{g_0}\right\|_\infty = \left\|\frac{X(0)}{g_0}\right\|_\infty \leq \frac{C_x}{g_0} \leq \frac{1}{2\gamma},$$

which is the second inequality of (29).

Next, we assume that (29) is satisfied when $k = 0, 1, \ldots, r$ with $r < d$. It is clear that $f_1(h,\gamma,k)$ is an increasing function due to the fact that $p_1 > 1$. According to the definition of $\bar{e}(k)$ we have that

$$\begin{aligned}\|\bar{e}(r)\|_\infty &\leq \left\|z(r) + h\sum_{i=0}^{r} L_i(r)\gamma^{-i}z(r-i) - h\sum_{i=0}^{r} L_i(r)\gamma^{-i}w(r-i)\right\|_\infty\\ &\leq \left(1 + h\sum_{i=0}^{r}\|L_i(r)\|_\infty\gamma^{-i}\right)\frac{1}{2\gamma} + h\sum_{i=0}^{r}\|L_i(r)\|_\infty\gamma^{-i}f(r)\\ &\leq \left(1 + h\|L\|_\infty\frac{\gamma^{-r-1}-1}{\gamma^{-1}-1}\right)\frac{1}{2\gamma} + h\|L\|_\infty f_1(h,\gamma,r)\frac{\gamma^{-r-1}-1}{\gamma^{-1}-1}\\ &\leq M_1(h,\gamma,g_0).\end{aligned}$$

Because $K \geq \lfloor M_1(h,\gamma,g_0) + \frac{1}{2}\rfloor$, we know that $\bar{e}(r)$ does not make the quantizer saturate, which, according to (12b), leads to the fact that $\|z(r+1)\|_\infty \leq \frac{1}{2\gamma}$.

From (12a), we have

$$\begin{aligned}\|w(r+1)\|_2 &\leq \gamma^{-1}\left\|w(r)-h\sum_{i=0}^{r}L_i(r)\gamma^{-i}w(r-i)\right\|_2+\gamma^{-1}\left\|h\sum_{i=0}^{r}L_i(r)\gamma^{-i}z(r-i)\right\|_2\\ &\leq \gamma^{-1}\left(1+h\sum_{i=0}^{r}\|L_i(r)\|_2\gamma^{-i}\right)f_1(h,\gamma,r)+\gamma^{-1}\frac{\sqrt{N}}{2\gamma}h\sum_{i=0}^{r}\|L_i(r)\|_2\gamma^{-i}\\ &\leq p_1f_1(h,\gamma,r)+\frac{\sqrt{N}}{2\gamma}h\|L\|_2\frac{\gamma^{-r-1}-1}{1-\gamma}\\ &= f_1(h,\gamma,r+1),\end{aligned}$$

which means that (29) holds for $k=r+1$. By induction, the proof is completed. ∎

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