Course: Commutative Algebra

Homework 8 (due to next Friday, 4/20/2012)

All rings are assumed commutative with identity.

1. Let R be a ring, D is a multiplicatively closed subset of R containing 1. Show that
   (a) $D^{-1}(IJ) = D^{-1}(I)D^{-1}(J)$, where I and J are ideals of R;
   (b) $D^{-1}(M \otimes_R N) \cong D^{-1}M \otimes_{D^{-1}R} D^{-1}N.$, where M and N are R-modules.

2. Suppose D is a multiplicatively closed subset of the polynomial ring $\mathbb{C}[x]$ in one variable containing 1 and $I_a = (x-a)$ for $a \in \mathbb{C}$. Show that
$$D^{-1}(\cap_{a\in\mathbb{C}} I_a) \neq \cap_{a\in\mathbb{C}}(D^{-1}I_a).$$
   Thus localization does not commute with infinite intersections.

3. Define the support of an R-module M by
$$\mathrm{Supp}M = \{p \in \mathrm{Spec}R | M_p \neq 0\}.$$
   Show that if M is a finitely generated and P is a prime of R, then $P \in \mathrm{Supp}M$ if and only if P contains the annihilator of M.

4. Suppose D is a multiplicatively closed subset of ring R containing 1. Let $N' \subseteq D^{-1}M$ be an $D^{-1}R$-submodule, and let $\pi : M \to D^{-1}M$ be the usual map and $N = \pi^{-1}(N')$. Show that $N' = D^{-1}N$.