Course: Commutative Algebra

Homework 5 (due to next Friday, 3/23/2012)

All rings are assumed commutative with identity. k is a field.

1. Let $I = (x^2, xy, xz, yz) \lhd k[x, y, z]$. Prove that a primary decomposition of I is $I = (x, y) \bigcap (x, z) \bigcap (x, y, z)^2$, determine the isolated and embedded primes of I, and find rad$I$.

2. Suppose that R is a Noetherian ring, I, $J \lhd R$, rad$I \not\supseteq J$ and $I = Q_1 \bigcap \cdots \bigcap Q_m$ is a minimal primary decomposition, with $P_i = \text{rad}Q_i$. Then
$$\bigcup_{n=0}^{\infty}(I : J^n) = \bigcap_{\{i|J\not\subseteq P_i\}} Q_i,$$
where $(I : J^n) := \{r \in R | rJ^n \subseteq I\}$.

3. Show that every associated prime ideal for a radical ideal is isolated (Suppose that a primary decomposition exists).

4. Suppose R is a Noetherian ring. Prove that R is either an integral domain, has nonzero nilpotent elements, or has at least two minimal prime ideals.

5. *Prove that a Noetherian integral domain R is U.F.D if and only if for every $a \in R$ the isolated primes associated to the principle ideal (a) are principle ideals.