Course: Commutative Algebra

Homework 4 (due to next Friday, 3/16/2012)

All rings are assumed commutative with identity.

1. Show that $V = \mathcal{Z}(x^2 - y^2 z)$ is the smallest algebraic set in $\mathbb{R}^3$ containing the points $S = \{(st, s, t^2) | s, t \in \mathbb{R}\}$. Show that S is not Zariski closed in V. Do the same over $\mathbb{C}$, but show that in this case $S = V$ is closed.
2. Prove that if Q_1 and Q_2 are both P-primary then so is $Q_1 \bigcap Q_2$.
3. Prove that if Q_1 and Q_2 are both M-primary, where M is a maximal ideal, then so are $Q_1 + Q_2$ and $Q_1 Q_2$.
4. Suppose $\varphi : R \to S$ is a surjective ring homomorphism. Prove that an ideal Q in R containing the kernel of φ is primary if and only if $\varphi(Q)$ is primary in S.
5. Suppose $\varphi : R \to S$ is a ring homomorphism.
    - Suppose I is an ideal of R containing $ker\varphi$ with minimal primary decomposition $I = Q_1 \bigcap \cdots \bigcap Q_m$ with $\text{rad}Q_i = P_i$. If φ is a surjective homomorphism prove that $\varphi(I) = \varphi(Q_1) \bigcap \cdots \bigcap \varphi(Q_m)$, where $\text{rad}\varphi(Q_i) = \varphi(P_i)$, is a minimal primary decomposition of $\varphi(I)$.
    - Suppose I is an ideal of S with minimal primary decomposition $I = Q_1 \bigcap \cdots \bigcap Q_m$ with $\text{rad}Q_i = P_i$. Prove that $\varphi^{-1}(I) = \varphi^{-1}(Q_1) \bigcap \cdots \bigcap \varphi^{-1}(Q_m)$, where $\text{rad}\varphi^{-1}(Q_i) = \varphi^{-1}(P_i)$, is a primary decomposition of $\varphi^{-1}(I)$, and is minimal if φ is surjective.

P.S. For the example (6) which was talked in the class today, the condition on the ring R should be P. I. D not U. F. D. Then the following proof is correct. Thank Jiawei Hu for pointing out the mistake.

If R is U. F. D we also can show that any $< a >$-primary ideal Q can be written as $< a >^n$ for a irreducible and $n = 1, 2, \cdots$:

Proof : If $xy \in < a >^n$ then we can write $xy = ra^n$ for some $r \in R$. If $x \notin < a >^n$ then $y = as$ for some $s \in R$ by noticing that $xy = ra^n$. So $y^n \in < a >^n$, i.e., $< a >^n$ is primary. Since $< a >$ is the minimal prime ideal containing $< a >^n$, rad$< a >^n = < a >$.

Conversely, suppose Q is a $< a >$-primary ideal, and let n be the largest integer with $Q \subseteq < a >^n$. If $q \in Q, q \notin < a >^{n+1}$ then $q = ra^n$ for some $r \in R$ and $r \notin < a >$. If $a^n \notin Q$, since $q = ra^n \in Q$, $r \in \text{rad}Q = < a >$, a contradiction. So $Q = < a >^n$.