Course: Commutative Algebra

Homework 2 (due to next Friday)

1. Suppose M is a Noetherian R-module and $\varphi : M \to M$ is an R-module endomorphism of M. Prove that $ker(\varphi^n) \bigcap Image(\varphi^n) = 0$ for $n >> 0$ and if φ is surjective then φ is an isomorphism.

2. Over finite field $\mathbb{F}_2$, let $V = \{(0,0),(1,1)\} \subseteq \mathbb{A}^2$. Find $\mathcal{I}(V)$.

3. Let $V = \mathcal{Z}(xy - z) \subseteq \mathbb{A}^3$. Prove that V is isomorphic to $\mathbb{A}^2$. Is $V = \mathcal{Z}(xy - z^2)$ isomorphic to $\mathbb{A}^2$?

4. Let $V = \mathcal{Z}(xz - y^2, yz - x^3, z^2 - x^2y) \subseteq \mathbb{A}^3$.

   - Prove that the map $\varphi : \mathbb{A}^1 \to V$ defined by $\varphi(t) = (t^3, t^4, t^5)$ is a surjective morphism.

   - Describe the corresponding k-algebra homomorphism $\tilde{\varphi} : A(V) \to A(\mathbb{A}^1)$ explicitly.

   - Prove that φ is not an isomorphism.