Course: Commutative Algebra

Homework 11 (due to next Friday, 5/11/2012)

R is a commutative ring with identity.

1. Let $R = \mathbb{Z}_{(2)}$ be the localization of $\mathbb{Z}$ at the prime ideal (2). Prove that $\mathrm{Jac}R = (2)$ is the ideal generated by 2. If $M = \mathbb{Q}$, prove that $M/2M$ is a finitely generated R-module but that M is not finitely generated over R.

2. If R is a Noetherian ring, prove that the Zariski topology on $\mathrm{Spec}R$ is discrete (i.e., every subset is Zariski open and Zariski closed) if and only if R is Artinian.

3. Suppose I is the ideal $(x_1, x_2^2, x_3^3, \cdots)$ in the polynomial ring $k[x_1, x_2, \cdots]$ where k is a field and let R be the quotient ring $k[x_1, x_2, \cdots]/I$. Prove that the image of the ideal $(x_1, x_2, \cdots)$ in R is the unique prime ideal in R but is not finitely generated. Deduce that R is a local ring of Krull dimension 0 but is not Artinian.

4. * If R is a local ring then every finitely generated projective R-module is a free R-module.