Course: Commutative Algebra

Homework 10 (due to next Friday, 5/4/2012)

R is a commutative ring with identity.

1. Suppose U is a Zariski open subset of $\mathrm{Spec}R$. Show that $\mathcal{O}(U)$ is a ring with identity and if U' is an open subset of U then the natural restrict map from $\mathcal{O}(U)$ to $\mathcal{O}(U')$ is a homomorphism of rings.

2. Suppose $\varphi : \mathrm{Spec}\mathbb{Z}[x] \to \mathrm{Spec}\mathbb{Z}$ to be the natural map. Prove that the elements in the fiber over (p) of φ are homeomorphic with the elements in $\mathrm{Spec}(\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{F}_p)$.

3. Prove that $X = \mathrm{Spec}R$ is irreducible if and only if $X_f \cap X_g \neq \emptyset$ for any two nonempty principal open sets X_f and X_g. Deduce that X is irreducible if and only if the nilradical of R is a prime ideal.

4. Suppose D is a multiplicatively closed subset of R. Show that the localization homomorphism $R \to D^{-1}R$ induces a homeomorphism from $\mathrm{Spec}(D^{-1}R)$ to the collection of prime ideals P of R with $P \cap D = \emptyset$.