华东师范大学硕士研究生期末试卷

2010—2011 学年第一学期

课程：代数曲面

从下面8题中任选5题, 每题20分，多做得分可累加，最高到100分。

All the problems are over complex number $\mathbb{C}$:

1. The following two cases of compact complex surfaces are not projective:

(1) Show that any compact complex surface homeomorphic to $S^1 \times S^3$ is not a projective surface;

(2) Let $\sigma : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at 12 points $p_1, \cdots, p_{12}$ on a smooth cubic plane curve C. Let $\tilde{C} \subset X$ be the strict transform of C. Show that $\tilde{C}$ is an elliptic curve with $\tilde{C}^2 = -3$. We know that $\tilde{C}$ can be contracted via an analytic morphism $f : X \to Y$, where Y is an analytic surface. Show that Y is not projective if the 12 points are in general positions.

2. Let D be an effective $\mathbb{Q}$-divisor on a smooth complex surface X. $\mu : X' \to X$ is a resolution of singularities of D such that total transform of D has simple normal crossing support. Define an ideal sheaf $\mathcal{I}(D) := \mu_* \mathcal{O}_{X'}(K_{X'/X} - [\mu^* D])$ which is independent of the resolution. Let $X = \mathbb{C}^2$ with coordinates x and y, and let $D \subset \mathbb{C}^2$ be the cuspidal curve defined by $y^2 = x^3$. Show that $\mathcal{I}(\frac{5}{6}D) = (x, y)$ and $\mathcal{I}(cD) = \mathcal{O}_X$ for $0 < c < \frac{5}{6}$.

3. Let $a_1, \cdots a_n$ be integers. Define the *scroll* $\mathbb{F}(a_1, \cdots, a_n)$ as the quotient of $(\mathbb{C}^2 \backslash 0) \times (\mathbb{C}^n \backslash 0)$ by an action of $\mathbb{C}^* \times \mathbb{C}^*$. Write t_1, t_2 for coordinates on $\mathbb{C}^2$ and $x_1, \cdots, x_n$ on $\mathbb{C}^n$, and $(\lambda, \nu) \in \mathbb{C}^* \times \mathbb{C}^*$. The action is given as follows:

$$(\lambda, 1) : (t_1, t_2; x_1, \cdots, x_n) \mapsto (\lambda t_1, \lambda t_2; \lambda^{-a_1} x_1, \cdots, \lambda^{-a_n} x_n);$$

$$(1, \mu) : (t_1, t_2; x_1, \cdots, x_n) \mapsto (t_1, t_2; \mu x_1, \cdots, \mu x_n).$$

(1) Show that $\mathbb{F}(a) \cong \mathbb{P}^1$ for any $a \in \mathbb{Z}$;

(2) Show that $\mathbb{F}(0, 0) \cong \mathbb{P}^1 \times \mathbb{P}^1$;

(3)* Show that $\mathbb{F}(a_1, a_2) \cong \mathbb{F}(a_1 + b, a_2 + b)$, for any $a_1, a_2, b \in \mathbb{Z}$.

4. Let X be an irreducible, smooth and projective surface with an ample divisor H. Show that the set

$$Q := \{Z \in NS(X) : Z^2 > 0\}$$

has two connected components

$$Q^+ := \{Z \in Q : Z \cdot H > 0\} \quad \text{and} \quad Q^- := \{Z \in Q : Z \cdot H < 0\}.$$

If X is an abelian surface, show that $\overline{NE}(X) = \bar{Q}^+$; and if $\dim NS(X) \geq 3$, then $\overline{NE}(X)$ is a 'round' cone, i.e. every point on the boundary of $\overline{NE}(X)$ is extremal.

5. Let X be a smooth projective surface in $\mathbb{P}^n$. If for a smooth hyperplane section C such that $\mathcal{O}_C(K_C) = \mathcal{O}_C(C)$ and $H^0(X, \mathcal{O}_X(kC)) \to H^0(C, \mathcal{O}_C(kK_C))$ is surjective for all $k \geq 0$, show that X is a K_3 surface.

6. Let A be a line and B be a smooth conic in $\mathbb{P}^2$ defined by $z = 0$ and $xy - z^2 = 0$ respectively. Choose another line L in $\mathbb{P}^2$ defined by $x - y = 0$ which meets B at two distinct points p, q, and which also meets A at r. Consider a cubic pencil (linear system of dimension 1) in $\mathbb{P}^2$ generated by $A+B$ and $3L$, i.e. $\lambda(A+B)+\mu(3L)$, for $[\lambda : \mu] \in \mathbb{P}^1$. First blow up at three points p, q, r and get three exceptional curves E_1, E_2, E_3. Then blow up again three times at the intersection points of the strict transforms of A, B with E_1, E_2, E_3 and get three new exceptional curves E_1', E_2', E_3'. Finally, blow up again three times at the intersection points of the strict transforms of A and B with the three exceptional curves E_1', E_2', E_3' and get again three new exceptional curves E_1'', E_2'', E_3''. Show that $\mathbb{P}^2$ becomes Y which is an elliptic fibration with four singular fibers over $\mathbb{P}^1$ after those blowing-ups and find the $[\lambda : \mu]$ for the corresponding singular fibers.

7. If X is a K_3 surface and D is a divisor on X, show that

(1) $D^2 \geq -2$ implies $H^0(D) \neq 0$ or $H^0(-D) \neq 0$;

(2) $D^2 \geq 0$ implies $h^0(D) \geq 2$ or $h^0(-D) \geq 2$ or $D \equiv 0$;

(3) if D is reduced and irreducible curve with $D^2 < 0$, then $D \cong \mathbb{P}^1$ and $D^2 = -2$ (we call D is a -2-curve);

(4) if D is effective with $h^0(D) = 1$, then $D'^2 \leq -2$ for every divisor D' with $0 < D' \leq D$, and in particular D is a sum of -2-curves with $D^2 \leq -2$.

8. * Let X be a K_3 surface and D any effective divisor on X. Show that

(1) we can subtract of an effective sum of -2-curves $F = \sum n_i \Gamma_i$, to get $M = D - F$ such that M is effective and nef (possibly zero), $M^2 \geq D^2$ and $H^0(X, \mathcal{O}_X(M)) = H^0(X, \mathcal{O}_X(D))$;

(2) if $D > 0$ is nef and $D^2 = 0$ then $D \equiv aE$, where E is a reduced and irreducible curve and a an integer;

(3) if D is nef and big then either $|D|$ has no fixed part or $D = aE + \Gamma$, where E is a reduced and irreducible curve and Γ an irreducible -2-curve such that $E \cdot \Gamma = 1$.