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CONTINUOUS AND DISCRETE WAVELET TRANSFORMS*

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Abstract. This paper is an expository survey of results on integral representations and discrete sum expansions of functions in $L^2(\mathbf{R})$ in terms of coherent states. Two types of coherent states are considered: Weyl–Heisenberg coherent states, which arise from translations and modulations of a single function, and affine coherent states, called "wavelets," which arise as translations and dilations of a single function. In each case it is shown how to represent any function in $L^2(\mathbf{R})$ as a sum or integral of these states. Most of the paper is a survey of literature, most notably the work of I. Daubechies, A. Grossmann, and J. Morlet. A few results of the authors are included.

Key words. frame, wavelet, coherent states, integral transform, Gabor transform, wavelet transform, Weyl–Heisenberg group, affine group

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0. Introduction. The representation of a signal by means of its spectrum or Fourier transform is essential to solving many problems both in pure mathematics and in applied science. However, it is in some instances not the most natural or useful way of representing a signal. For example, we often think of music or speech as a signal in which the spectrum evolves over time in a significant way. We imagine that at each instant the ear hears a certain combination of frequencies, and that these frequencies are constantly changing. This time-evolution of the frequencies is not reflected in the Fourier transform, at least not directly. In theory, a signal can be reconstructed from its Fourier transform, but the transform contains information about the frequencies of the signal over all times instead of showing how the frequencies vary with time.

This paper will survey two methods of achieving time-dependent frequency analysis, which we will refer to as the *Gabor transform* and the *wavelet transform*. These transforms are deeply related by the theory of group representations (in §3.5 we summarize the work of H. Feichtinger and K. Gröchenig which demonstrates the relationship). It is therefore not inappropriate to think of the two transforms as different manifestations of a single theory. In fact, the terms "wavelet" and "wavelet transform" have been used in the literature to refer to both types of transforms, with the Gabor transform being called the "Weyl–Heisenberg wavelet transform" and the wavelet transform the "affine wavelet transform." Recently, however, the term *wavelet* has come to be reserved for the affine case, and we adopt this convention here with minor exceptions, namely that we use the term "mother wavelet" when, strictly speaking, it is not a wavelet, and sometimes refer to "affine wavelets" even though that is redundant.

The Gabor transform, named for D. Gabor following his fundamental work in [29], includes and can be illustrated by a technique known as the *short-time Fourier transform*. This transform works by first dividing a signal into short consecutive segments and then computing the Fourier coefficients of each segment. This is a time-frequency localization technique in that it computes the frequencies associated with small portions of the signal. One problem with such a method is that it poorly resolves

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phenomena of duration shorter than the time window. Moreover, shortening the window to increase time resolution can result in unacceptable increases in computational effort, especially if the short-duration phenomenona being investigated do not occur very often.

An equivalent way of describing the short-time Fourier transform is the following. Let f be the signal and g an ideal cutoff function, i.e., the characteristic function of an interval. Chopping up the signal amounts to multiplying f by a translate of g, i.e., by $\overline{g(x-na)}$, where a is the length of the cutoff interval and n is an integer (since g is real, the conjugate is irrelevant here, but will be important later). The Fourier coefficients of this product are then $\int_{-\infty}^{\infty} f(x)\,\overline{g(x-na)}\,e^{-2\pi imx/a}\,dx$, for integers m. In other words, we have computed the inner product of $f(x)$ with $g(x-na)\,e^{2\pi imx/a}$ for $m,n \in \mathbf{Z}$, i.e., with a discrete set of translates and modulates of g.

We now describe the Gabor transform. For simplicity, we will restrict our signals to the class $L^2(\mathbf{R})$, the space of finite energy, one-dimensional signals. This eases the computations, but as we will later mention, neither finite energy nor one dimension are necessary restrictions. Now let $g \in L^2(\mathbf{R})$ be any fixed function, which we call the *mother wavelet*, although, as explained above, this is an abuse of notation. Gabor considered only $g(x) = e^{-rx^2}$, the Gaussian function, but this restriction is not required. In the short-time Fourier transform we considered a discrete set of translates and modulates of g, but let us now consider all possible translates and modulates (we return to the discrete transform below). The *Gabor transform* of g is the operator Ψ_g, which is defined for signals $f \in L^2(\mathbf{R})$ by

$$\Psi_g f(a,b) \;=\; \int_{-\infty}^{\infty} f(x)\,\overline{g(x-a)}\,e^{-2\pi ibx}\,dx,$$

for $a, b \in \mathbf{R}$. If g is concentrated in time at zero and its Fourier transform is concentrated at zero then $\Psi_g f(a,b)$ will give a picture of f at time a and frequency b. This representation is essentially the cross-ambiguity function of f with g. The signal f is completely characterized by the values of $\Psi_g f(a,b)$ and can be recovered via the formula

$$f(x) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_g f(a,b)\,e^{2\pi ibx}\,g(x-a)\,da\,db,$$

when the integral is interpreted in an appropriate way.

We turn now to the wavelet transform, which is formed by taking translations and dilations of a mother wavelet. Specifically, if $g \in L^2(\mathbf{R})$ is the mother wavelet, then the *wavelet transform* of g is the operator Φ_g defined on signals $f \in L^2(\mathbf{R})$ by

$$\Phi_g f(u,v) \;=\; \int_{-\infty}^{\infty} f(x)\,e^{-u/2}\,\overline{g(e^{-u}x-v)}\,dx$$

for $u, v \in \mathbf{R}$. Again, f is characterized by these values and can be recovered by

$$f(x) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_g f(u,v)\,e^{-u/2}\,g(e^{-u}x-v)\,du\,dv,$$

where this integral must be interpreted appropriately. This continuous version of the wavelet transform can be considered a cross-affine-ambiguity function.

The wavelet transform acts as a time and frequency localization operator in the following way. Roughly speaking, if u is a large negative number, and g a function in $L^2(\mathbf{R})$, then $e^{-u/2}g(e^{-u}x)$ is highly concentrated about the point $x = 0$, yet still has the same energy as the original function. As u approaches $-\infty$, $e^{-u/2}g(e^{-u}x)$

becomes more and more concentrated about $x = 0$. Thus the functions $\Phi_g f(u, v)$, thought of as functions of v for each fixed u, display the information in f at various levels of resolution or frequency bands. That is, as u approaches $-\infty$, $\Phi_g f(u, v)$ displays the small-scale, higher-frequency, features of the signal f. As u approaches $+\infty$, the coarser, lower frequency, features are displayed. Moreover, as u approaches $-\infty$, the wavelet transform gives sharper and sharper time resolution.

So far we have discussed only the continuous Gabor and wavelet transforms. The fundamental paper [15] by I. Daubechies, A. Grossmann, and Y. Meyer gave a solid mathematical footing to discrete versions of both transforms (discrete in the sense of using a discrete lattice of translates and modulates or translates and dilates, rather than the entire plane of possibilities). These discrete versions were developed specifically for $L^2(\mathbf{R})$ and were based on the concept of Hilbert space *frames*, an idea originally introduced in 1952 by R. J. Duffin and A. C. Schaeffer in [18] in connection with nonharmonic Fourier series. Daubechies, Grossmann, and Meyer, along with R. Coifman, A. J. E. M. Janssen, S. Mallat, J. Morlet, P. Tchamitchian, and others have extensively developed this theory, especially in the case of the wavelet transform. A major advance was the discovery of smooth mother wavelets whose set of discrete translates and dilates forms an orthonormal basis for $L^2(\mathbf{R})$. This is especially important since it has been shown that smooth mother wavelets with good decay cannot generate orthonormal bases in the Gabor case.

At about the same time, a fundamentally different approach was being taken by M. Frazier and B. Jawerth in [25]. They developed a discrete wavelet transform which allowed functions in a large class of spaces besides just $L^2(\mathbf{R})$ to be analyzed. Later, H. Feichtinger realized that the same could be done for the Gabor case (see [20]), and then, together with K. Gröchenig, unified the Gabor and wavelet transforms into a single theory, showing that a large class of transforms give rise to discrete representations of functions [22]–[24].

In this paper, we survey the literature on the Gabor and wavelet transforms in both the continuous and discrete cases. A few new results of the authors are included, but the tone is intended to be essentially expository. For clarity, we concentrate our study on the space $L^2(\mathbf{R})$ and the techniques evolved from [15], but try to indicate the unification achieved by Feichtinger and Gröchenig.

We summarize in §1 the mathematical notations and definitions used throughout the paper, and provide in §2 some background on frames, which allow us to describe discrete representations of Hilbert spaces such as $L^2(\mathbf{R})$. In particular, if $\{x_n\}$ is a frame then we show how to write any x in the space as $x = \sum c_n x_n$. This representation of x need not be unique, but will have certain properties which make it easy to use. In particular, the scalars c_n are known and computable.

In §3 we discuss the continuous versions of the Gabor and wavelet transforms, and show how both arise as representations of groups on $L^2(\mathbf{R})$. We briefly outline the Feichtinger–Gröchenig theory, showing how any representation will give rise to a discrete transform.

In §§4 and 5 we describe the discrete Gabor and wavelet transforms. In §4, we show how to find a lattice of points $\{(na, mb)\}_{m,n \in \mathbf{Z}}$ so that $\{e^{2\pi imbx} g(x - na)\}_{m,n \in \mathbf{Z}}$ will form a frame for $L^2(\mathbf{R})$, which, following [15], we call a Weyl–Heisenberg frame. This implies that any f in $L^2(\mathbf{R})$ can be written as a discrete sum of the frame elements, i.e.,

$$f(x) = \sum_{m,n} c_{mn} \, e^{2\pi imbx} g(x - na),$$

where the scalars c_{mn} are easily computable. As an aid to analysis of these frames we also discuss the Zak transform, which allows us to prove various results about the interdependence of the mother wavelet and the lattice points. This section contains some new results by the authors.

Finally, in §5, we construct frames of the form $\{a^{-n/2}g(a^{-n}x - mb)\}_{m,n\in\mathbf{Z}}$, called affine frames. We also discuss in this section the Meyer wavelet, a smooth mother wavelet which generates an affine orthonormal basis for $L^2(\mathbf{R})$, and multiresolution analysis, a concept that has been developed to analyze the Meyer and related wavelets, and that is proving to have a large impact on both theoretical mathematics and signal processing applications.

1. Notation and Definitions. For the convenience of the reader we provide in this section a summary of the mathematical notations and definitions used in this paper. A familiarity with Fourier series, Fourier transforms, and Hilbert spaces is helpful; we refer the reader to the general references [30], [40] or any other standard work on real or harmonic analysis.

1.1. Basic symbols. $\mathbf{C}$ will represent the complex numbers. The modulus of a complex number $z \in \mathbf{C}$ is denoted by $|z|$, the complex conjugate by $\overline{z}$. $\mathbf{R}$ is the real number line thought of as the time axis, and $\hat{\mathbf{R}}$ the real line thought of as the frequency axis. The set of integers is $\mathbf{Z}$. The torus group $\mathbf{T}$ is the unit circle in $\mathbf{C}$, i.e., $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$. We identify $\mathbf{T}$ with the interval $[0,1)$ by associating the number $t \in [0,1)$ with the complex number $e^{2\pi it} \in \mathbf{T}$.

Sequences and series with undefined limits are to be taken over $\mathbf{Z}$, and integrals with undefined limits are over $\mathbf{R}$. Unless otherwise indicated, integration is always with respect to Lebesgue measure. The Lebesgue measure of a set $E \subset \mathbf{R}$ is denoted by $|E|$. A property is said to hold almost everywhere, denoted a.e., if the set of points where it fails has Lebesgue measure zero. All functions f are defined on the real line and are complex-valued, unless otherwise indicated.

DEFINITION 1.1.1.

(1) The *support* of a complex-valued function f, denoted $\operatorname{supp}(f)$, is the closure in $\mathbf{R}$ of $\{x \in \mathbf{R} : f(x) \neq 0\}$.

(2) The *essential supremum* of a real-valued f is $\operatorname{ess\,sup}_{x\in\mathbf{R}} f(x) = \inf\{\lambda \in \mathbf{R} : f(x) \leq \lambda \text{ a.e.}\}$. Its *essential infimum* is $\operatorname{ess\,inf}_{x\in\mathbf{R}} f(x) = \sup\{\lambda \in \mathbf{R} : f(x) \geq \lambda \text{ a.e.}\}$.

(3) The *characteristic function* of a set $E \subset \mathbf{R}$ is $\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$

(4) The *Kronecker delta* is $\delta_{xy} = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$

DEFINITION 1.1.2. Given $1 \leq p < \infty$, we define the Lebesgue space $L^p(\mathbf{R}) = \{f : \|f\|_p = \left(\int |f(x)|^p\,dx\right)^{1/p} < \infty\}$. For $p = \infty$ we take $L^\infty(\mathbf{R}) = \{f : \|f\|_\infty = \operatorname{ess\,sup}_{x\in\mathbf{R}} |f(x)| < \infty\}$. It is well known that, for $1 \leq p \leq \infty$, $L^p(\mathbf{R})$ is a Banach space with norm $\|\cdot\|_p$, and that $L^2(\mathbf{R})$ is a Hilbert space with inner product $\langle f, g\rangle = \int f(x)\,\overline{g(x)}\,dx$. The *Cauchy–Schwarz inequality* states that, as in any Hilbert space, $|\langle f, g\rangle| \leq \|f\|_2\,\|g\|_2$.

DEFINITION 1.1.3. Given a Hilbert space H with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, and a sequence $\{x_n\}$ of elements of H.

(1) We say that x_n *converges* to $x \in H$, and write $x_n \to x$, if $\lim_{n\to\infty} \|x - x_n\| = 0$.

(2) We write $\sum x_n = x$, and say that the series $\sum x_n$ *converges* to x, if $s_N \to x$, where $s_N = \sum_{-N}^{N} x_n$. The series converges *unconditionally* if every rearrangement also converges.

(3) The *span* of $\{x_n\}$ in H is the set of all finite linear combinations of the x_n, i.e., $\operatorname{span}\{x_n\} = \left\{\sum_{-N}^{N} c_n x_n : N > 0, c_n \in \mathbf{C}\right\}$.

(4) $\{x_n\}$ is *orthogonal* if $\langle x_m, x_n\rangle = 0$ whenever $m \neq n$.

(5) $\{x_n\}$ is *orthonormal* if it is orthogonal and $\|x_n\| = 1$ for all n.

(6) $\{x_n\}$ is *complete* if $\operatorname{span}\{x_n\}$ is dense in H, or equivalently, if the only element $x \in H$ which is orthogonal to every x_n is $x = 0$.

Given an orthonormal sequence $\{e_n\}$ in a Hilbert space H, it can be shown that the following statements are equivalent:

(1) $\{e_n\}$ is complete.

(2) $\sum |\langle x, e_n\rangle|^2 = \|x\|^2$ for all $x \in H$.

(3) $x = \sum \langle x, e_n\rangle e_n$ for all $x \in H$.

An orthonormal sequence satisfying these equivalent conditions is called an *orthonormal basis*. Statement (2) is referred to as the *Plancherel formula* for orthonormal bases. In statement (3), it follows that the coefficients $\langle x, e_n\rangle$ are unique, i.e., x cannot be written $x = \sum c_n x_n$ in any other way. This is in contrast to the situation in §2, where we will obtain decompositions which are not unique.

1.2. Operators.

DEFINITION 1.2.1. Assume H and K are Hilbert spaces with norms $\|\cdot\|_H, \|\cdot\|_K$ and inner products $\langle\cdot,\cdot\rangle_H, \langle\cdot,\cdot\rangle_K$, respectively, and that $S: H \to K$.

(1) S is *linear* if $S(ax + by) = aSx + bSy$ for all $x, y \in H$ and $a, b \in \mathbf{C}$.

(2) S is *1-1* or *injective* if $Sx \neq Sy$ whenever $x \neq y$.

(3) The *range* of S is $\operatorname{Range}(S) = \left\{Sx : x \in H\right\}$.

(4) S is *onto* or *surjective* if $\operatorname{Range}(S) = K$.

(5) S is *bijective* if it is both injective and surjective.

(6) The *norm* of S is $\|S\| = \sup\left\{\|Sx\|_K : x \in H \text{ and } \|x\|_H = 1\right\}$.

(7) S is *bounded* if $\|S\| < \infty$. A linear operator is bounded if and only if it is *continuous*, i.e., if $x_n \to x$ implies $Sx_n \to Sx$.

(8) The *adjoint* of S is the unique operator $S^*: K \to H$ such that $\langle Sx, y\rangle_K = \langle x, S^*y\rangle_H$ for all $x \in H$ and $y \in K$. It is easy to show that $\|S^*\| = \|S\|$.

(9) A bijective operator has an *inverse* $S^{-1}: K \to H$ defined by setting $S^{-1}y = x$ if $Sx = y$.

(10) We say S is *invertible*, or a *topological isomorphism*, if S is linear, bijective, continuous, and S^{-1} is continuous. In this case $\|S^{-1}\|^{-1}\,\|x\|_H \leq \|Sx\|_K \leq \|S\|\,\|x\|_H$ for all $x \in H$.

(11) S is an *isometry*, or *norm-preserving*, if $\|Sx\|_K = \|x\|_H$ for all $x \in H$. A linear map S is an isometry if and only if $\langle Sx, Sy\rangle_K = \langle x, y\rangle_H$ for all $x, y \in H$.

(12) A *unitary* map is a linear bijective isometry.

DEFINITION 1.2.2. Assume H is a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, and that $S, T: H \to H$.

(1) S is *self-adjoint* if $S = S^*$, i.e., if $\langle Sx, y\rangle = \langle x, Sy\rangle$ for all $x, y \in H$.

(2) S is *positive*, denoted $S \geq 0$, if $\langle Sx, x\rangle \geq 0$ for all $x \in H$. All positive operators are self-adjoint.

(3) We say that $S \geq T$ if $S - T \geq 0$.

(4) We denote by $L(H)$ the set of all bounded linear operators $S: H \to H$.

1.3. Translation, modulation, and dilation.

DEFINITION 1.3.1. Given a function f we define the following operators.

Translation:	$T_a f(x) = f(x-a)$,	for $a \in \mathbf{R}$;
Modulation:	$E_a f(x) = e^{2\pi i a x} f(x)$,	for $a \in \mathbf{R}$;
Dilation:	$D_a f(x) = \|a\|^{-1/2} f(x/a)$,	for $a \in \mathbf{R}\backslash\{0\}$.

Each of these is a unitary operator from $L^2(\mathbf{R})$ onto itself, and we have:

$$\begin{aligned} T_a E_b f(x) &= e^{2\pi i b(x-a)} f(x-a); \\ E_b T_a f(x) &= e^{2\pi i b x} f(x-a); \\ T_b D_a f(x) &= |a|^{-1/2} f(\tfrac{x-b}{a}); \\ D_a T_b f(x) &= |a|^{-1/2} f(\tfrac{x}{a} - b); \\ E_b D_a f(x) &= e^{2\pi i b x} |a|^{-1/2} f(\tfrac{x}{a}); \\ D_a E_b f(x) &= e^{2\pi i b x/a} |a|^{-1/2} f(\tfrac{x}{a}); \end{aligned}$$

$$\langle f, T_a g\rangle = \langle T_{-a} f, g\rangle; \qquad \langle f, E_a g\rangle = \langle E_{-a} f, g\rangle; \qquad \langle f, D_a g\rangle = \langle D_{1/a} f, g\rangle.$$

We also use the symbol E_a by itself to refer to the *exponential function* $E_a(x) = e^{2\pi i a x}$. The *two-dimensional exponentials* are $E_{(a,b)}(x,y) = e^{2\pi i a x} e^{2\pi i b y}$.

1.4. Fourier transforms.

DEFINITION 1.4.1. The *Fourier transform* of a function $f \in L^1(\mathbf{R})$ is $\hat{f}(\gamma) = \int f(x)\, e^{-2\pi i \gamma x}\, dx$, for $\gamma \in \hat{\mathbf{R}}$. We also set $\check{f}(\gamma) = \hat{f}(-\gamma) = \int f(x)\, e^{2\pi i \gamma x}\, dx$.

We define the Fourier transform of functions $f \in L^2(\mathbf{R})$ as follows. By real analysis techniques we can find functions $f_n \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ such that $f_n \to f$ in $L^2(\mathbf{R})$. The Fourier transform of each f_n is defined as above, and we can show that $\hat{f}_n \in L^2(\hat{\mathbf{R}})$ and $\hat{f}_n$ converges in $L^2(\hat{\mathbf{R}})$ to some function, which we call $\hat{f}$. One way of choosing the f_n is to set $f_n = f \cdot \chi_{[-n,n]}$. Then $\hat{f}$ is the limit $\hat{f}(\gamma) = \lim_{n\to\infty} \int_{-n}^{n} f(x)\, e^{-2\pi i \gamma x}\, dx$, where this limit is in the Hilbert space $L^2(\hat{\mathbf{R}})$, *not* a usual pointwise limit.

We have the following formulas:

$$(T_a f)^\wedge = E_{-a}\hat{f}; \qquad (E_a f)^\wedge = T_a \hat{f}; \qquad (D_a f)^\wedge = D_{1/a}\hat{f}.$$

Also, if $f, g \in L^2(\mathbf{R})$ then we have the *Plancherel formula* $\|f\|_2 = \|\hat{f}\|_2 = \|\check{f}\|_2$ and the *Parseval formula* $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle = \langle \check{f}, \check{g}\rangle$.

DEFINITION 1.4.2. If f, g are complex-valued functions defined on $\mathbf{R}$, their *convolution* $f * g$ is the function $(f*g)(x) = \int f(x-y)\, g(y)\, dy$, provided that the integral exists. If $f \in L^p(\mathbf{R})$, where $1 \le p \le \infty$, and $g \in L^1(\mathbf{R})$ then $f * g$ exists a.e. and $f * g \in L^p(\mathbf{R})$, with $\|f*g\|_p \le \|f\|_p\, \|g\|_1$.

DEFINITION 1.4.3. A sequence of functions $\{\rho_n\}_{n=1}^\infty$ is an *approximate identity* if:

(1) $\sup_n \|\rho_n\|_1 = \sup_n \int |\rho_n(x)|\, dx < \infty$,
(2) $\int \rho_n(x)\, dx = 1$ for all n,
(3) for every $\delta > 0$ we have $\lim_{n\to\infty} \int_{|x|>\delta} |\rho_n(x)|\, dx = 0$.

If $\{\rho_n\}_{n=1}^\infty$ is an approximate identity and $1 \le p < \infty$, then $\lim_{n\to\infty} \|f*\rho_n - f\|_p = 0$ for every $f \in L^p(\mathbf{R})$. If $\rho \in L^1(\mathbf{R})$ with $\int \rho(x)\, dx = 1$ and we define $\rho_n(x) = n\, \rho(nx)$, then $\{\rho_n\}_{n=1}^\infty$ is an approximate identity. Thus there are nearly as many examples of approximate identities as there are integrable functions. This makes it easy, in most cases, to find approximate identities that satisfy any additional conditions we might require.

Example 1.4.4. Let $\varphi \in L^1(\mathbf{R})$ be such that $\hat{\varphi} \in L^1(\hat{\mathbf{R}})$ and $\int \varphi(x)\,dx = 1$ (i.e., $\hat{\varphi}(0) = 1$). For example, take $\varphi(x) = e^{-\pi x^2}$, in which case $\hat{\varphi}(\gamma) = e^{-\pi\gamma^2}$, or take

$$\varphi(x) \;=\; \begin{cases} 0, & \text{if } x \le -\frac{1}{2} \text{ or } x \ge \frac{1}{2}, \\ 2x+1, & \text{if } -\frac{1}{2} < x < 0, \\ 1-2x, & \text{if } 0 \le x < \frac{1}{2}, \end{cases}$$

in which case $\hat{\varphi}(\gamma) = (\sin^2 \pi\gamma)/(\pi\gamma)^2$. If we let $\varphi_n(x) = n\,\varphi(nx)$ then $\{\varphi_n\}_{n=1}^{\infty}$ is an approximate identity with the property that $\hat{\varphi}_n \in L^1(\hat{\mathbf{R}})$ for every n. If we define ρ_n by $\check{\rho}_n = \varphi_n$, then $\{\check{\rho}\}_{n=1}^{\infty}$ is the approximate identity used in Theorem 3.2.8.

Example 1.4.5. Let $\varphi \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ be such that $\int \varphi(x)\,dx = 1$ and $\varphi(x) = \varphi(-x)$ (either of the functions given in Example 1.4.4 will do). Letting $\varphi_n(x) = n\,\varphi(nx)$, the approximate identity $\{\varphi_n\}_{n=1}^{\infty}$ satisfies $\varphi_n \in L^2(\mathbf{R})$ and $\varphi_n(x) = \varphi_n(-x)$ for every n. This shows the existence of the approximate identity needed for Theorem 3.3.9.

1.5. Compactly supported functions. We often will deal with functions supported in a finite interval. Let $I \subset \mathbf{R}$ be any interval of length $1/b$, and set $L^2(I) = \{f \in L^2(\mathbf{R}) : \operatorname{supp}(f) \subset I\}$. This is a closed subspace of $L^2(\mathbf{R})$, so is itself a Hilbert space with norm and inner product from $L^2(\mathbf{R})$. Moreover, the set of exponentials $\{b^{1/2} E_{mb}\chi_I\}_{m\in\mathbf{Z}}$ is an orthonormal basis for $L^2(I)$ (we will usually be slightly sloppy and assume that the exponential is automatically taken with support in the desired interval, writing E_{mb} instead of $E_{mb}\chi_I$). Therefore, for each $f \in L^2(I)$ we have $\sum |\langle f, E_{mb}\rangle|^2 = b^{-1}\int |f(x)|^2\,dx$ and $\sum \langle f, E_{mb}\rangle E_{mb} = b^{-1} f$. This representation of f in terms of exponentials is the *Fourier series* expansion of f.

2. Frames in Hilbert Spaces. Given a Banach space (such as $L^p(\mathbf{R})$), it is often advantageous to find a *basis* for the space, i.e., a fixed set of vectors $\{g_n\}$ such that any vector f in the space can be written $f = \sum c_n g_n$ for some unique choice of scalars c_n. For most of the spaces encountered in ordinary analysis we know that bases exist, but usually we need more than mere existence. For example, we may want the g_n to be easily generated in some way or to satisfy some special properties, the c_n be easy to compute, etc. These conditions can be difficult to satisfy simultaneously.

If the space we are working with is a Hilbert space (such as $L^2(\mathbf{R})$) then we know that it actually possesses an *orthonormal basis*, a set of vectors that in addition to being a basis is mutually orthogonal. Much effort has been expended in the literature in finding orthonormal bases for various Hilbert spaces which satisfy additional properties to suit some problem. However, the requirements of orthogonality and the basis property are very stringent, making it difficult as a rule to find a good orthonormal basis.

As an alternative to orthonormal bases, we present in this section a generalization known as *frames*. We show that if $\{g_n\}$ is a frame then we can write $f = \sum c_n g_n$ where the scalars c_n are known. However, we do not require the g_n to be orthogonal nor the c_n to be unique, yet we still retain good control on the behavior of the c_n and the sum. An advantage of frames is that the requirements are not as restrictive as orthonormal bases, which often allows us the freedom to impose whatever extra conditions we require. We will see in §§4 and 5 that this freedom allows us to construct frames for $L^2(\mathbf{R})$ of a very specific type, namely, the Weyl–Heisenberg and affine frames discussed in §0, whose frame elements are easily generated from a single fixed function.

Although we will describe frames in Hilbert spaces only, we must emphasize that Gröchenig has extended the notion to a large class of general Banach spaces, and that this extension is nontrivial. For details on this, see [31], where he also discusses how to derive generalized Weyl–Heisenberg or affine frames, i.e., frames arising as the orbit of a single function under a square-integrable group representation, in general spaces. We discuss this work briefly in §3.4. We mention also that both authors have been interested in frame decompositions in both the Banach and Hilbert space settings: [54] contains results on various kinds of stability of Weyl–Heisenberg frames in a general setting, while [36] concentrates on Hilbert space results.

We assume in this section that H is a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$.

2.1. Definitions and general results. The first definition of frames was in [18], where much of the general theory was laid out.

DEFINITION 2.1.1 ([18]). A sequence $\{x_n\}$ in a Hilbert space H is a *frame* if there exist numbers $A, B > 0$ such that for all $x \in H$ we have

$$A\|x\|^2 \;\le\; \sum_n |\langle x, x_n\rangle|^2 \;\le\; B\|x\|^2.$$

The numbers A, B are called the *frame bounds*. The frame is *tight* if $A = B$. The frame is *exact* if it ceases to be a frame whenever any single element is deleted from the sequence.

From the Plancherel theorem we see that every orthonormal basis is a tight exact frame with $A = B = 1$. For orthonormal bases, the Plancherel theorem is equivalent to the basis property, which gives a decomposition of the Hilbert space. We will see that the pseudo-Plancherel theorem satisfied by frames also implies a decomposition, although the representations induced need not be unique.

Note that since $\sum |\langle x, x_n\rangle|^2$ is a series of positive real numbers it converges absolutely, hence unconditionally. That is, every rearrangement of the sum also converges, and converges to the same value. Therefore every rearrangement of a frame is also a frame, and all sums involving frames actually converge unconditionally. Also, frames are clearly complete since if $x \in H$ and $\langle x, x_n\rangle = 0$ for all n, then $A\|x\|^2 \le \sum |\langle x, x_n\rangle|^2 = 0$, so $x = 0$.

The following example shows that tightness and exactness are not related.

Example 2.1.2. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H.

(1) $\{e_1, e_1, e_2, e_2, e_3, e_3, \cdots\}$ is a tight inexact frame with bounds $A = B = 2$, but is not an orthonormal basis, although it contains one.

(2) $\{e_1, e_2/2, e_3/3, \cdots\}$ is a complete orthogonal sequence, but not a frame.

(3) $\{e_1, e_2/\sqrt{2}, e_2/\sqrt{2}, e_3/\sqrt{3}, e_3/\sqrt{3}, e_3/\sqrt{3}, \cdots\}$ is a tight inexact frame with bounds $A = B = 1$, and no nonredundant subsequence is a frame.

(4) $\{2e_1, e_2, e_3, \cdots\}$ is a nontight exact frame with bounds $A = 1, B = 2$.

We let I denote the identity operator on H, i.e., $Ix = x$ for all $x \in H$.

THEOREM 2.1.3 [18]. *Given a sequence $\{x_n\}$ in a Hilbert space H, the following two statements are equivalent:*

(1) *$\{x_n\}$ is a frame with bounds A, B.*

(2) *$Sx = \sum\langle x, x_n\rangle x_n$ is a bounded linear operator with $AI \le S \le BI$, called the* frame operator *for $\{x_n\}$.*

Proof. (2) $\Rightarrow$ (1). If (2) holds then $\langle AIx, x\rangle \le \langle Sx, x\rangle \le \langle BIx, x\rangle$ for all x, but $\langle Ix, x\rangle = \|x\|^2$ and $\langle Sx, x\rangle = \sum |\langle x, x_n\rangle|^2$.

(1) $\Rightarrow$ (2). Fix $x \in H$, and let $s_N = \sum_{-N}^{N}\langle x, x_n\rangle x_n$. Recall that in a Hilbert space the norm of any $z \in H$ is given by $\|z\| = \sup\{|\langle z, y\rangle| : y \in H \text{ with } \|y\| = 1\}$.

For $M \leq N$ we therefore have by the Cauchy–Schwarz inequality for series that

$$\begin{aligned} \|s_N - s_M\|^2 &= \sup_{\|y\|=1} |\langle s_N - s_M, y\rangle|^2 \\ &= \sup_{\|y\|=1} \left| \sum_{M<|n|\leq N} \langle x, x_n\rangle \langle x_n, y\rangle \right|^2 \\ &\leq \sup_{\|y\|=1} \left(\sum_{M<|n|\leq N} |\langle x, x_n\rangle|^2 \right) \left(\sum_{M<|n|\leq N} |\langle x_n, y\rangle|^2 \right) \\ &\leq \sup_{\|y\|=1} \left(\sum_{M<|n|\leq N} |\langle x, x_n\rangle|^2 \right) B\|y\|^2 \\ &= B \sum_{M<|n|\leq N} |\langle x, x_n\rangle|^2 \\ &\to 0 \quad \text{as } M, N \to \infty. \end{aligned}$$

Thus $\{s_N\}$ is a Cauchy sequence in H, so must converge, so Sx is a well-defined element of H. By using the fact that $\|Sx\|^2 = \sup_{\|y\|=1} |\langle Sx, y\rangle|^2$, a calculation similar to the one above shows $\|S\| \leq B$, so S is bounded. Finally, the relations $AI \leq S \leq BI$ follow from the definition of frames. □

COROLLARY 2.1.4 [18].

(1) *S is invertible and $B^{-1}I \leq S^{-1} \leq A^{-1}I$.*

(2) *$\{S^{-1}x_n\}$ is a frame with bounds B^{-1}, A^{-1}, called the* dual frame *of $\{x_n\}$.*

(3) *Every $x \in H$ can be written $x = \sum \langle x, S^{-1}x_n\rangle x_n = \sum \langle x, x_n\rangle S^{-1}x_n$.*

Proof. (1) Since $AI \leq S \leq BI$ we have $\|I - B^{-1}S\| \leq \|\frac{B-A}{B}I\| = \frac{B-A}{B} < 1$. Elementary Hilbert space results imply immediately that $B^{-1}S$, and therefore S, is invertible. Since $\langle S^{-1}x, x\rangle = \langle S^{-1}x, S(S^{-1}x)\rangle \geq A\|S^{-1}x\|^2 \geq 0$ we see that S^{-1} is a positive operator. Also, S^{-1} commutes with both I and S, so we can multiply through by S^{-1} in the equation $AI \leq S \leq BI$ to obtain $B^{-1}I \leq S^{-1} \leq A^{-1}I$ (cf. [37, p. 269]).

(2) Since S^{-1} is positive it is self-adjoint. Therefore,

$$\sum_n \langle x, S^{-1}x_n\rangle S^{-1}x_n = S^{-1}\left(\sum_n \langle S^{-1}x, x_n\rangle x_n \right) = S^{-1}S(S^{-1}x) = S^{-1}x.$$

The result now follows from part (1) and Theorem 2.1.3, part (2).

(3) This follows by expanding $x = S(S^{-1}x)$ and $x = S^{-1}(Sx)$. □

Note that in the case of tight frames Corollary 2.1.4 reduces to $S = AI$, $S^{-1} = A^{-1}I$, and $x = A^{-1}\sum \langle x, x_n\rangle x_n$.

PROPOSITION 2.1.5 [18]. *Given a frame $\{x_n\}$ and given $x \in H$ let $a_n = \langle x, S^{-1}x_n\rangle$, so $x = \sum a_n x_n$. If it is possible to find other scalars c_n such that $x = \sum c_n x_n$ then $\sum |c_n|^2 = \sum |a_n|^2 + \sum |a_n - c_n|^2$.*

Proof. Note that $\langle x_n, S^{-1}x\rangle = \langle S^{-1}x_n, x\rangle = \overline{a}_n$. Substituting $x = \sum a_n x_n$ and $x = \sum c_n x_n$ into the first term of the inner product $\langle x, S^{-1}x\rangle$, we obtain $\sum |a_n|^2 = \langle x, S^{-1}x\rangle = \sum c_n \overline{a}_n$. Hence,

$$\begin{aligned} \sum_n |a_n|^2 + \sum_n |a_n - c_n|^2 &= \sum_n |a_n|^2 + \sum_n \big(|a_n|^2 - a_n\overline{c}_n - \overline{a}_n c_n + |c_n|^2\big) \\ &= \sum_n |c_n|^2. \qquad \square \end{aligned}$$

THEOREM 2.1.6 [18]. *The removal of a vector from a frame leaves either a frame or an incomplete set. In particular,*

$$\langle x_m, S^{-1}x_m\rangle \neq 1 \Rightarrow \{x_n\}_{n\neq m} \text{ is a frame;}$$
$$\langle x_m, S^{-1}x_m\rangle = 1 \Rightarrow \{x_n\}_{n\neq m} \text{ is incomplete.}$$

Proof. Fix m, and define $a_n = \langle x_m, S^{-1}x_n\rangle = \langle S^{-1}x_m, x_n\rangle$. We know that $x_m = \sum a_n x_n$, but we also have $x_m = \sum c_n x_n$ where $c_n = \delta_{mn}$. By Proposition 2.1.5, we therefore have

$$\begin{aligned} 1 = \sum_n |c_n|^2 &= \sum_n |a_n|^2 + \sum_n |a_n - c_n|^2 \\ &= |a_m|^2 + \sum_{n\neq m} |a_n|^2 + |a_m - 1|^2 + \sum_{n\neq m} |a_n|^2. \end{aligned}$$

Suppose now that $a_m = 1$. Then $\sum_{n\neq m} |a_n|^2 = 0$, so $a_n = \langle S^{-1}x_m, x_n\rangle = 0$ for $n \neq m$. That is, $S^{-1}x_m$ is orthogonal to x_n for every $n \neq m$. But $S^{-1}x_m \neq 0$ since $\langle S^{-1}x_m, x_m\rangle = a_m = 1$, so $\{x_n\}_{n\neq m}$ is incomplete in this case.

On the other hand, if $a_m \neq 1$ then $x_m = \frac{1}{1-a_m}\sum_{n\neq m} a_n x_n$, so for $x \in H$ we have

$$|\langle x, x_m\rangle|^2 = \left|\frac{1}{1-a_m}\sum_{n\neq m} a_n \langle x, x_n\rangle\right|^2 \leq C \sum_{n\neq m} |\langle x, x_n\rangle|^2,$$

where $C = |1-a_m|^{-2}\sum_{n\neq m} |a_n|^2$. Therefore,

$$\sum_n |\langle x, x_n\rangle|^2 = |\langle x, x_m\rangle|^2 + \sum_{n\neq m} |\langle x, x_n\rangle|^2 \leq (1+C) \sum_{n\neq m} |\langle x, x_n\rangle|^2,$$

from which it follows that $\{x_n\}_{n\neq m}$ is a frame with bounds $A/(1+C), B$. □

The proof of Theorem 2.1.6 shows that if $\langle x_m, S^{-1}x_m\rangle = 1$ then $\langle x_m, S^{-1}x_n\rangle = \langle S^{-1}x_m, x_n\rangle = 0$ for $n \neq m$. We therefore have the following corollary.

COROLLARY 2.1.7 [18]. *If $\{x_n\}$ is an exact frame, then $\{x_n\}$ and $\{S^{-1}x_n\}$ are* biorthonormal, *i.e.,* $\langle x_m, S^{-1}x_n\rangle = \delta_{mn}$.

2.2. Frames and bases. We have shown in Corollary 2.1.4 that frames provide decompositions of H, i.e., every $x \in H$ can be written $x = \sum c_n x_n$. We now consider whether these representations are unique.

DEFINITION 2.2.1. A sequence $\{\varphi_n\}$ in a Hilbert space H is a *basis* for H if for every $x \in H$ there exist *unique* scalars c_n such that $x = \sum c_n \varphi_n$. The basis is *bounded* if $0 < \inf \|\varphi_n\| \leq \sup \|\varphi_n\| < \infty$. It is *unconditional* if the series $\sum c_n \varphi_n$ converges unconditionally for every x, i.e., every permutation of the series converges.

In finite-dimensional spaces, a series converges unconditionally if and only if it converges absolutely. In infinite-dimensional spaces, absolute convergence still implies unconditional convergence but the reverse need not be true. In Hilbert spaces, all bounded unconditional bases are *equivalent* to orthonormal bases. That is, if $\{\varphi_n\}$ is a bounded unconditional basis, then there is an orthonormal basis $\{e_n\}$ and a topological isomorphism $U: H \to H$ such that $\varphi_n = Ue_n$ for all n [56].

We see immediately that an inexact frame cannot be a basis, for by definition there is then an m such that $\{x_n\}_{n\neq m}$ is a frame, and hence complete, while no subset of a basis can be complete. In fact, if we define $a_n = \langle x_m, S^{-1}x_n\rangle$, then $x_m = \sum a_n x_n$ by Corollary 2.1.4, but we also have $x_m = \sum c_n x_n$ where $c_n = \delta_{mn}$. By Theorem 2.1.6 we must have $a_m \neq 1$, so these are two different representations of x_m. On the other hand, we do have the following characterization of exact frames.

THEOREM 2.2.2 [56], [36]. *A sequence $\{x_n\}$ in a Hilbert space H is an exact frame for H if and only if it is a bounded unconditional basis for H.*

Proof. $\Rightarrow$. Assume $\{x_n\}$ is an exact frame with bounds A, B. Then $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthonormal, so for m fixed we have

$$A\,\|S^{-1}x_m\|^2 \;\le\; \sum_n |\langle S^{-1}x_m, x_n\rangle|^2 \;=\; |\langle S^{-1}x_m, x_m\rangle|^2 \;\le\; \|S^{-1}x_m\|^2\|x_m\|^2$$

and

$$\|x_m\|^4 \;=\; |\langle x_m, x_m\rangle|^2 \;\le\; \sum_n |\langle x_m, x_n\rangle|^2 \;\le\; B\,\|x_m\|^2.$$

Thus $A \le \|x_m\|^2 \le B$, so $\{x_n\}$ is bounded in norm. By Corollary 2.1.4 we have $x = \sum \langle x, S^{-1}x_n\rangle x_n$ for all $x \in H$, and we must show that this representation is unique. But if $x = \sum c_n x_n$ then $\langle x, S^{-1}x_m\rangle = \sum c_n \langle x_n, S^{-1}x_m\rangle = c_m$. Thus $\{x_n\}$ is a basis for H, and since the sums converge unconditionally we conclude that the basis is unconditional.

$\Leftarrow$. Assume $\{x_n\}$ is a bounded unconditional basis for H. Then there is an orthonormal basis $\{e_n\}$ and a topological isomorphism $U\colon H \to H$ such that $Ue_n = x_n$ for all n. Given $x \in H$ we therefore have

$$\sum_n |\langle x, x_n\rangle|^2 \;=\; \sum_n |\langle x, Ue_n\rangle|^2 \;=\; \sum_n |\langle U^*x, e_n\rangle|^2 \;=\; \|U^*x\|^2,$$

where U^* is the adjoint operator to U. But $\|{U^*}^{-1}\|^{-1}\,\|x\| \;\le\; \|U^*x\| \;\le\; \|U^*\|\,\|x\|$, so $\{x_n\}$ forms a frame. It is clearly exact since the removal of any vector from a basis leaves an incomplete set. $\square$

3. Continuous Coherent State Operators. The use of a generalized Fourier integral to convey simultaneous time and frequency information in a signal goes back at least to D. Gabor in 1946. In [29] he defines a windowed Fourier transform operator, using a Gaussian window. Much later, A. Grossmann and J. Morlet defined an affine coherent state integral operator which is now often called the wavelet transform. In [34], they prove certain continuity properties of this operator and present a formal inversion formula for it. We report these results in §3.3.

It was realized in [34] that the unitarity of the wavelet transform was a consequence of the theory of group representations. That the same is true of the Gabor transform is mentioned in [14] and [15]. This connection to group representations was exploited in a beautiful and significant way by Feichtinger and Gröchenig in [21]–[24], [31] to obtain discrete expansions of vectors in a large class of Banach spaces called coorbit spaces.

In this section we present some of the above-mentioned results on coherent state integral operators from the perspective of group representations. The value of such a perspective is that it demonstrates the deep connection between Gabor and wavelet transforms, which follows from the fact that each of these transforms arises from the representation of certain topological groups on $L^2(\mathbf{R})$. While this approach is necessary for the deeper understanding of the theory, it is of limited use in the practical study of coherent state expansions of $L^2(\mathbf{R})$. Therefore, we prove specific results directly, and use group representations to tie the results together at an abstract level.

The two group representations involved in the Gabor and wavelet transforms are as follows. Gabor transforms come from the representation of the Weyl–Heisenberg group, which is the set $\mathbf{T} \times \mathbf{R} \times \hat{\mathbf{R}}$, acting on $L^2(\mathbf{R})$ by $W(t,a,b)f(x) = t \cdot e^{2\pi i b(x-a)}$ $f(x-a)$. Letting $t = 1$ in this formula we see that $W(1,a,b)$ has the effect of shifting the function by a on the time axis, and by b on the frequency axis. The wavelet transform, on the other hand, comes from the representation of the affine, or $ax+b$, group, which can be thought of as the group of translations and dilations of $\mathbf{R}$. This group can be identified with the set $\mathbf{R} \times \mathbf{R}$, and acts on $L^2(\mathbf{R})$ by $U(u,v)f(x) = e^{-u/2}f(e^{-u}x-v)$. This action involves first the translation by v of f, then the $L^2(\mathbf{R})$-isometric dilation of the result.

In §3.1 we define the basic representation theory concepts needed in §3.

In §3.2 we obtain integral representations of functions in $L^2(\mathbf{R})$ by means of a Weyl–Heisenberg coherent state integral operator, and mention the relationship of this operator to the Wigner distribution and radar ambiguity function.

Section 3.3 is analogous to §3.2 in that we here obtain integral representations of functions in $L^2(\mathbf{R})$ by means of an affine coherent state integral operator. Both of the integral representations defined in §§3.2 and 3.3 can be thought of as generalizations of the representation of a function by its Fourier integral.

In §3.4 we sketch the theory of Feichtinger and Gröchenig, in which expansions of vectors in general Banach spaces are obtained by discretizing coherent state integral operators, such as the ones described in §§3.2 and 3.3. These expansions can be thought of as generalized Fourier series.

3.1. Background on group representations. In this section we let G denote a *locally compact group*, i.e., G is a locally compact topological space equipped with a group operation, $\cdot$, such that the mappings $(x,y) \mapsto x \cdot y$ from $G \times G$ into G, and $x \mapsto x^{-1}$ from G into G, are continuous. We refer the reader to [49] and [51, Chap. 3] for the precise definitions of these terms. We let μ be a *measure* on G (called a positive integral in [49]), and let $L^2(G)$ denote the Hilbert space of μ-square-integrable functions on G, i.e., $L^2(G) = \{F\colon G \to \mathbf{C} : \|F\|_{L^2(G)} = \left(\int_G |F(x)|^2\,d\mu(x)\right)^{1/2} < \infty\}$ with inner product $\langle F_1, F_2\rangle = \int_G F_1(x)\,\overline{F_2(x)}\,d\mu(x)$.

DEFINITION 3.1.1 [49]. A measure μ on a group G is said to be *left-invariant* provided that for every integrable function f on G and every $y \in G$ we have $\int_G f(y \cdot x)\,d\mu(x) = \int_G f(x)\,d\mu(x)$. It is a fact from the theory of measures that a left-invariant measure on G, known as *left Haar measure*, exists and is unique up to a constant multiple. Similar remarks hold for *right-invariant* measures and *right Haar measure*. We assume that a normalization of these Haar measures has been chosen and will refer to the resulting unique measures as *the* left and right Haar measures of G. If the left Haar measure is also the right Haar measure then G is said to be *unimodular*.

DEFINITION 3.1.2. Let H be a Hilbert space.

(1) A *representation* π of G on H is a mapping $\pi\colon G \to L(H)$ such that $\pi(x \cdot y) = \pi(x)\pi(y)$ for every $x, y \in G$.

(2) A vector $g \in H$ is *admissible* if $\int_G |\langle g, \pi(x)g\rangle|^2\,d\mu(x) < \infty$, where μ is the left Haar measure on G.

(3) A vector $g \in H$ is *cyclic* if $\operatorname{span}\{\pi(x)g\}_{x\in G}$ is dense in H, or equivalently, if the only $f \in H$ such that $\langle f, \pi(x)g\rangle = 0$ for all $x \in G$ is $f = 0$.

(4) π is *unitary* if the map $\pi(x)\colon H \to H$ is unitary for each $x \in G$.

(5) π is *irreducible* if every $g \in H\backslash\{0\}$ is cyclic.

(6) π is *square-integrable* if π is irreducible and there exists an admissible $g \in H\backslash\{0\}$.

3.2. Continuous Gabor transforms. Throughout this section we let $\mathbf{H} = \mathbf{T}\times\mathbf{R}\times\hat{\mathbf{R}}$ denote the Weyl–Heisenberg group, with group operation defined in Remark 3.2.2 and Haar measure in Proposition 3.2.3.

DEFINITION 3.2.1. We define a representation W of $\mathbf{H}$ on $L^2(\mathbf{R})$ by:

$$W(t,a,b)f(x) \;=\; t\,e^{2\pi ib(x-a)}f(x-a) \;=\; t\cdot T_aE_bf(x),$$

where $(t,a,b)\in\mathbf{H}$ and $f\in L^2(\mathbf{R})$.

Remark 3.2.2. Let (t_1,a_1,b_1), $(t_2,a_2,b_2)\in\mathbf{H}$. Then,

$$\begin{aligned}
W(t_1,a_1,b_1)W(t_2,a_2,b_2)f(x) \;&=\; W(t_1,a_1,b_1)\Big(t_2e^{2\pi ib_2(x-a_2)}f(x-a_2)\Big)\\
&=\; t_1e^{2\pi ib_1(x-a_1)}t_2e^{2\pi ib_2(x-a_1-a_2)}f(x-a_1-a_2)\\
&=\; t_1t_2e^{2\pi ib_1a_2}e^{2\pi i(b_1+b_2)(x-a_1-a_2)}f(x-a_1-a_2)\\
&=\; W(t_1t_2e^{2\pi ib_1a_2},a_1+a_2,b_1+b_2)f(x).
\end{aligned}$$

This means that in order for W to be a representation the group operation on $\mathbf{H}$ must be

$$(t_1,a_1,b_1)\cdot(t_2,a_2,b_2) \;=\; (t_1t_2e^{2\pi ib_1a_2},a_1+a_2,b_1+b_2).$$

It is easy to check that this is in fact a group operation, i.e., it is associative, with identity element $(1,0,0)$ and inverses $(t,a,b)^{-1} = (t^{-1}e^{2\pi iab},-a,-b)$. Since each $W(t,a,b)$ is a unitary operator on $L^2(\mathbf{R})$, we see that W is a unitary representation of $\mathbf{H}$ on $L^2(\mathbf{R})$.

PROPOSITION 3.2.3. *The product measure $dt\,da\,db$ is the left and right Haar measure on $\mathbf{H}$. In particular, $\mathbf{H}$ is unimodular.*

Proof. The left-invariance follows from the calculation

$$\begin{aligned}
&\int_{\hat{\mathbf{R}}}\int_{\mathbf{R}}\int_{\mathbf{T}} F((x,y,z)\cdot(t,a,b))dt\,da\,db\\
&= \int_{\hat{\mathbf{R}}}\int_{\mathbf{R}}\int_{\mathbf{T}} F(xte^{2\pi iza},a+y,b+z)\,dt\,da\,db\\
&= \int_{\hat{\mathbf{R}}}\int_{\mathbf{R}}\int_0^1 F(e^{2\pi i(s+x'+az)},a+y,b+z)\,ds\,da\,db\\
&= \int_{\hat{\mathbf{R}}}\int_{\mathbf{R}}\int_0^1 F(e^{2\pi iu},a+y,b+z)\,du\,da\,db\\
&= \int_{\hat{\mathbf{R}}}\int_{\mathbf{R}}\int_{\mathbf{T}} F(t,v,w)\,dt\,dv\,dw,
\end{aligned}$$

where we have written t as $e^{2\pi is}$ and x as $e^{2\pi ix'}$, used the periodicity of the exponential function and made the obvious substitutions. The right-invariance is similar. □

PROPOSITION 3.2.4 [15]. *If $f,g\in L^2(\mathbf{R})$ then*

$$\int_{\hat{\mathbf{R}}}\int_{\mathbf{R}}\int_{\mathbf{T}} |\langle f,W(t,a,b)g\rangle|^2\,dt\,da\,db \;=\; \|f\|_2^2\,\|g\|_2^2.$$

Proof. The left-hand side is equal to

$$\begin{aligned}
&\int_{\hat{\mathbf{R}}}\int_{\mathbf{R}}\int_{\mathbf{T}}\left|\int_{\mathbf{R}} \bar{t}\,f(x)\,e^{-2\pi ib(x-a)}\,\overline{g(x-a)}\,dx\right|^2 dt\,da\,db\\
&= \int_{\hat{\mathbf{R}}}\int_{\mathbf{R}}\left|\int_{\mathbf{R}} f(x)\,\overline{g(x-a)}\,e^{-2\pi ibx}\,dx\right|^2 da\,db\\
&= \int_{\mathbf{R}}\left(\int_{\hat{\mathbf{R}}}\left|\left(f\cdot T_a\bar{g}\right)^\wedge(b)\right|^2 db\right) da
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbf{R}} \left(\int_{\mathbf{R}} |(f \cdot T_a \bar{g})(x)|^2 \, dx \right) da && \text{(by Plancherel's formula)} \\
&= \int_{\mathbf{R}} |f(x)|^2 \int_{\mathbf{R}} |g(x-a)|^2 \, da \, dx \\
&= \int_{\mathbf{R}} |f(x)|^2 \int_{\mathbf{R}} |g(a)|^2 \, da \, dx \\
&= \left(\int_{\mathbf{R}} |f(x)|^2 \, dx \right) \left(\int_{\mathbf{R}} |g(a)|^2 \, da \right). \qquad \square
\end{aligned}
$$

COROLLARY 3.2.5. *W is a unitary and square-integrable representation of $\mathbf{H}$ on $L^2(\mathbf{R})$ and every $g \in L^2(\mathbf{R})$ is admissible.*

Proof. That every $g \in L^2(\mathbf{R})$ is admissible follows immediately from Proposition 3.2.4 by taking $f = g$. Now suppose $g \in L^2(\mathbf{R})\backslash\{0\}$ is fixed and we assume $f \in L^2(\mathbf{R})$ is such that $\langle f, W(t,a,b)g \rangle = 0$ for all $(t,a,b) \in \mathbf{H}$. Then $\|f\|_2 \, \|g\|_2 = 0$ by Proposition 3.2.4, so $f = 0$. Therefore W is irreducible as desired. $\square$

Observe that in the proof of Proposition 3.2.4 we show that

$$\int_{\hat{\mathbf{R}}} \int_{\mathbf{R}} \int_{\mathbf{T}} |\langle f, W(t,a,b)g \rangle|^2 \, dt \, da \, db \;=\; \int_{\hat{\mathbf{R}}} \int_{\mathbf{R}} |\langle f, W(1,a,b)g \rangle|^2 \, da \, db.$$

This suggests that we lose nothing by ignoring the toral component of the group representation W, and leads to the following definition.

DEFINITION 3.2.6. Given $g \in L^2(\mathbf{R})\backslash\{0\}$, the *$\Psi$-transform of g* is the operator Ψ_g on $L^2(\mathbf{R})$ defined by

$$\Psi_g f(a,b) \;=\; \langle f, W(1,a,b)g \rangle \;=\; \langle f, T_a E_b g \rangle.$$

By Proposition 3.2.4 and the preceding remark, Ψ_g maps $L^2(\mathbf{R})$ into $L^2(\mathbf{R} \times \hat{\mathbf{R}})$ and is a multiple of an isometry. Moreover, by the Cauchy–Schwarz inequality we have $|\Psi_g f(a,b)| = |\langle f, T_a E_b g \rangle| \le \|f\|_2 \, \|T_a E_b g\|_2 = \|f\|_2 \, \|g\|_2$, so $\Psi_g f$ is a bounded function of a, b. Since there are functions in $L^2(\mathbf{R} \times \hat{\mathbf{R}})$ that are not bounded, Ψ_g is not surjective.

Example 3.2.7. If we set $a = 0$ and $g = \chi_{[-N,N]}$ then $\Psi_g f(0,b) = \int_{-N}^{N} f(x) \, e^{-2\pi i b x} \, dx \to \hat{f}(b)$ in $L^2(\mathbf{R})$ as $N \to \infty$. In this sense the Ψ-transform is a generalization of the ordinary L^2-Fourier transform (cf. [6]). We would therefore like to have an inversion formula for the Ψ-transform analogous to that for the ordinary Fourier transform. Specifically, we would like to make rigorous sense of the formal inversion formula

$$f(u) \;=\; \int_{\hat{\mathbf{R}}} \int_{\mathbf{R}} \Psi_g f(a,b) \, T_a E_b g(u) \, da \, db.$$

As written, it is not clear that the expression on the right side even exists as an absolutely convergent integral. However, we can prove the following inversion formulas.

THEOREM 3.2.8 [6], [54]. *Let $\rho_n \in L^1(\hat{\mathbf{R}})$ be such that $\{\check{\rho}_n\}_{n=1}^{\infty}$ is an approximate identity.*

(1) *If $g \in L^2(\mathbf{R}) \cap L^1(\mathbf{R}) \setminus \{0\}$ then $\lim_{n\to\infty} \|f - f_n\|_2 = 0$ for all $f \in L^2(\mathbf{R})$, where*

$$f_n(u) \;=\; \frac{1}{\|g\|_2^2} \int_{\hat{\mathbf{R}}} \int_{\mathbf{R}} \Psi_g f(a,b) \, T_a E_b g(u) \, \rho_n(b) \, da \, db.$$

(2) *If $g \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R}) \setminus \{0\}$ then $\lim_{n\to\infty} \|f - f_n\|_1 = 0$ for all $f \in L^1(\mathbf{R})$, where f_n is as in* (1).

Proof. We prove only (1) since (2) is similar. First observe that an approximate identity with the required properties exists by Example 1.4.4. We have:

$$\begin{aligned}
&\int_{\hat{\mathbf{R}}}\int_{\mathbf{R}} \Psi_g f(a,b)\, T_a E_b g(u)\, \rho_n(b)\, da\, db \\
&= \int_{\hat{\mathbf{R}}}\int_{\mathbf{R}} \left(\int_{\mathbf{R}} f(x)\, \overline{g(x-a)}\, e^{-2\pi i x b}\, dx\right) g(u-a)\, e^{2\pi i u b}\, \rho_n(b)\, da\, db \\
&= \int_{\mathbf{R}} f(x) \left(\int_{\hat{\mathbf{R}}} \rho_n(b)\, e^{2\pi i b(x-u)}\, db\right) \left(\int_{\mathbf{R}} \overline{g(x-a)}\, g(u-a)\, da\right) dx \\
&= \int_{\mathbf{R}} f(x)\, \check{\rho}_n(x-u) \int_{\mathbf{R}} \overline{g((x-u)+a)}\, g(a)\, da\, dx \\
&= (f * \check{\rho}_n G)(u),
\end{aligned}$$

where $G(x) = \int \overline{g(x+a)}\, g(a)\, da$. Now, $|G(x)| \le \|g\|_2^2$ for every x and G is continuous at 0. By standard approximate identity techniques it is easy then to show that $\lim_{n\to\infty} \|f * \check{\rho}_n G - G(0)\cdot f\|_2 = 0$, from which the result follows. The reader can check that the assumptions $g \in L^2(\mathbf{R}) \cap L^1(\mathbf{R})$ and $\rho_n \in L^1(\mathbf{R})$ guarantee that all changes in the order of integration are justified. ∎

The Ψ-transform is closely related to the Wigner distribution and the ambiguity function. The Wigner distribution was introduced in 1932 by Wigner in connection with quantum mechanics, and the ambiguity function was introduced by P. M. Woodward in the early 1950s for radar analysis. We have space only to mention the connection here, and refer to [2], [11] or any standard reference for details.

Definition 3.2.9. Given $f, g \in L^2(\mathbf{R})$, the (*cross-*)*Wigner distribution* of f and g is

$$W_{f,g}(a,b) = \int_{\mathbf{R}} e^{-2\pi i b x} f(a+\tfrac{x}{2})\, \overline{g(a-\tfrac{x}{2})}\, dx.$$

The (*cross-*)*ambiguity function* is

$$A_{f,g}(a,b) = \int_{\mathbf{R}} e^{-2\pi i b x} f(x+\tfrac{a}{2})\, \overline{g(x-\tfrac{a}{2})}\, dx.$$

We have $A_{f,g}(a,b) = e^{-\pi i a b}\, \Psi_g f(a,b)$ and $W_{f,g}(a,b) = 2\, e^{-4\pi i a b}\, \Psi_{g_-} f(2a,2b)$, where $g_-(x) = g(-x)$.

3.3. Continuous wavelet transforms. In this section we let $\mathbf{A} = \mathbf{R} \times \mathbf{R}$ denote the affine group equipped with the group operation given in Remark 3.3.3 and the Haar measure given in Proposition 3.3.4.

Definition 3.3.1. We define

$$\begin{aligned}
H^2_+(\mathbf{R}) &= \{f \in L^2(\mathbf{R}) : \operatorname{supp}(\hat{f}) \subset [0,\infty)\}, \\
H^2_-(\mathbf{R}) &= \{f \in L^2(\mathbf{R}) : \operatorname{supp}(\hat{f}) \subset (-\infty,0]\}.
\end{aligned}$$

These are closed subspaces of $L^2(\mathbf{R})$, and therefore are Hilbert spaces with inner products and norms from $L^2(\mathbf{R})$. By the Plancherel formula, we have

$$\|f\|_{H^2_+} = \left(\int_0^\infty |\hat{f}(\gamma)|^2\, d\gamma\right)^{1/2} \quad \text{and} \quad \|f\|_{H^2_-} = \left(\int_{-\infty}^0 |\hat{f}(\gamma)|^2\, d\gamma\right)^{1/2}.$$

Moreover, $H^2_+(\mathbf{R})$ and $H^2_-(\mathbf{R})$ are orthogonal complements in $L^2(\mathbf{R})$. That is, $f \in H^2_+(\mathbf{R})$ if and only if $\langle f, g\rangle = 0$ for all $g \in H^2_-(\mathbf{R})$, and similarly for $H^2_-(\mathbf{R})$. We can identify $H^2_+(\mathbf{R})$ or $H^2_-(\mathbf{R})$ with the space of all real-valued functions in $L^2(\mathbf{R})$. For, if $f \in L^2(\mathbf{R})$ is real-valued then $\hat{f}(\gamma) = \overline{\hat{f}(-\gamma)}$ for all $\gamma \in \hat{\mathbf{R}}$, so the values of $\hat{f}$ on the positive or negative frequency axis completely determine $\hat{f}$, and hence f. Also,

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx \;=\; \int_{-\infty}^{\infty} |\hat{f}(\gamma)|^2\,d\gamma \;=\; 2\int_{0}^{\infty} |\hat{f}(\gamma)|^2\,d\gamma \;=\; 2\int_{-\infty}^{0} |\hat{f}(\gamma)|^2\,d\gamma.$$

Definition 3.3.2. We define a representation U of $\mathbf{A}$ on $L^2(\mathbf{R})$ by:

$$U(u,v)f(x) \;=\; e^{-u/2} f(e^{-u}x - v) \;=\; D_{e^u} T_v f(x)$$

where $(u,v) \in \mathbf{A}$ and $f \in L^2(\mathbf{R})$.

The following remark defines the group operation on $\mathbf{A}$ and shows that U is a representation, which is clearly unitary since $U(u,v)$ is a unitary map of $L^2(\mathbf{R})$ onto itself for each $(u,v) \in \mathbf{A}$. However, $U(u,v)$ can also be considered as a mapping on the smaller space $H^2_+(\mathbf{R})$, and it is clearly a unitary map of $H^2_+(\mathbf{R})$ onto itself, with similar remarks for $H^2_-(\mathbf{R})$. In other words, we can consider U as a unitary representation of $\mathbf{A}$ on $L^2(\mathbf{R}), H^2_+(\mathbf{R})$, or $H^2_-(\mathbf{R})$.

Remark 3.3.3. If $f \in L^2(\mathbf{R})$ and (u_1,v_1), $(u_2,v_2) \in \mathbf{A}$ then

$$\begin{aligned} U(u_1,v_1)U(u_2,v_2)f(x) &= U(u_1,v_1)\big(e^{-u_2/2} f(e^{-u_2}x - v_2)\big) \\ &= e^{-u_1/2}e^{-u_2/2} f\big(e^{-u_2}(e^{-u_1}x - v_1) - v_2\big) \\ &= e^{-(u_1+u_2)/2} f(e^{-(u_1+u_2)}x - e^{-u_2}v_1 - v_2) \\ &= U(u_1+u_2, e^{-u_2}v_1 + v_2)f(x). \end{aligned}$$

Thus if U is to be a group representation then the group operation on $\mathbf{A}$ must be

$$(u_1,v_1)\cdot(u_2,v_2) \;=\; (u_1+u_2, e^{-u_2}v_1 + v_2).$$

It is easy to verify that this is a group operation, with identity $(0,0)$ and inverses $(u,v)^{-1} = (-u, -ve^u)$.

Proposition 3.3.4. *The left Haar measure on* $\mathbf{A}$ *is the product measure* $du\,dv$*, while the right Haar measure is* $e^u du\,dv$*. Thus* $\mathbf{A}$ *is not unimodular.*

Proof. The right-invariance of $e^u du\,dv$ follows from

$$\begin{aligned} \int_{\mathbf{R}}\int_{\mathbf{R}} F\left((u,v)\cdot(x,y)\right) e^u\,du\,dv &= \int_{\mathbf{R}}\int_{\mathbf{R}} F(u+x, e^{-x}v+y)\,e^u\,du\,dv \\ &= \int_{\mathbf{R}}\int_{\mathbf{R}} F(s, e^{-x}v+y)\,e^{s-x}\,ds\,dv \\ &= \int_{\mathbf{R}}\int_{\mathbf{R}} F(s,t)\,e^s\,ds\,dt. \end{aligned}$$

The left-invariance of $du\,dv$ is similar. ∎

Theorem 3.3.5 [34]. *If* $f,g \in L^2(\mathbf{R})$ *then*

$$\begin{aligned} &\int_{\mathbf{R}}\int_{\mathbf{R}} |\langle f, U(u,v)g\rangle|^2\,du\,dv \\ &\qquad = \int_0^{\infty} |\hat{f}(\omega)|^2\,d\omega \int_0^{\infty} \frac{|\hat{g}(\gamma)|^2}{|\gamma|}\,d\gamma \;+\; \int_{-\infty}^{0} |\hat{f}(\omega)|^2\,d\omega \int_{-\infty}^{0} \frac{|\hat{g}(\gamma)|^2}{|\gamma|}\,d\gamma. \end{aligned}$$

Proof. The left-hand side is

$$\begin{aligned} &\int_{\mathbf{R}}\int_{\mathbf{R}} |\langle f, D_{e^u}T_v g\rangle|^2\,du\,dv \\ &= \int_{\mathbf{R}}\int_{\mathbf{R}} |\langle D_{e^{-u}} f, T_v g\rangle|^2\,du\,dv \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbf{R}}\int_{\mathbf{R}} |\langle D_{e^u}\hat{f}, E_{-v}\hat{g}\rangle|^2\,du\,dv \qquad \text{(by Parseval's formula)}\\
&= \int_{\mathbf{R}}\int_{\mathbf{R}} \left|\int_{\hat{\mathbf{R}}} e^{-u/2}\,\hat{f}(e^{-u}\gamma)\,\overline{\hat{g}(\gamma)}\,e^{2\pi i\gamma v}\,d\gamma\right|^2 dv\,du\\
&= \int_{\mathbf{R}}\int_{\mathbf{R}} \big|\big(D_{e^u}\hat{f}\cdot\bar{\hat{g}}\big)^{\vee}(v)\big|^2\,dv\,du\\
&= \int_{\mathbf{R}}\int_{\hat{\mathbf{R}}} \big|\big(D_{e^u}\hat{f}\cdot\bar{\hat{g}}\big)(\gamma)\big|^2\,d\gamma\,du \qquad \text{(by Plancherel's formula)}\\
&= \int_{\mathbf{R}}\int_{\hat{\mathbf{R}}} e^{-u}\,|\hat{f}(e^{-u}\gamma)|^2\,|\hat{g}(\gamma)|^2\,d\gamma\,du\\
&= \int_{\hat{\mathbf{R}}} |\hat{f}(\xi)|^2 \int_{\mathbf{R}} |\hat{g}(e^u\xi)|^2\,du\,d\xi\\
&= \int_0^\infty |\hat{f}(\xi)|^2 \int_{\mathbf{R}} |\hat{g}(e^u\xi)|^2\,du\,d\xi \;+\; \int_{-\infty}^0 |\hat{f}(\xi)|^2 \int_{\mathbf{R}} |\hat{g}(e^u\xi)|^2\,du\,d\xi\\
&= \int_0^\infty |\hat{f}(\xi)|^2\,d\xi\cdot\int_0^\infty \frac{|\hat{g}(\omega)|^2}{\omega}\,d\omega \;+\; \int_{-\infty}^0 |\hat{f}(\xi)|^2\,d\xi\cdot\int_{-\infty}^0 \frac{|\hat{g}(\omega)|^2}{-\omega}\,d\omega. \qquad \square
\end{aligned}
$$

COROLLARY 3.3.6.

(1) *U is a unitary, square-integrable representation of $\mathbf{A}$ on $H^2_+(\mathbf{R})$ and $H^2_-(\mathbf{R})$.*

(2) *U is a unitary representation of $\mathbf{A}$ on $L^2(\mathbf{R})$ which possesses admissible and cyclic elements, but is not irreducible.*

Proof. (1) We know that U is a unitary representation of $\mathbf{A}$ on $H^2_+(\mathbf{R})$, and from Theorem 3.3.5 any $g \in H^2_+(\mathbf{R})$ with $\int_0^\infty |\hat{g}(\gamma)|^2/|\gamma|\,d\gamma < \infty$ is admissible. So, we need only show that every $g \in H^2_+(\mathbf{R})\backslash\{0\}$ is cyclic. Assume $f \in H^2_+(\mathbf{R})$ satisfies $\langle f, U(u,v)g\rangle = 0$ for every $(u,v) \in \mathbf{A}$. Since $\hat{f}(\gamma) = \hat{g}(\gamma) = 0$ for $\gamma < 0$, we have by Theorem 3.3.5 that

$$\int_0^\infty |\hat{f}(\gamma)|^2\,d\gamma \int_0^\infty \frac{|\hat{g}(\gamma)|^2}{|\gamma|}\,d\gamma \;=\; \int_{\mathbf{R}}\int_{\mathbf{R}} |\langle f, U(u,v)g\rangle|^2\,du\,dv \;=\; 0.$$

Since $g \neq 0$, this implies $\hat{f}(\gamma) = 0$ for a.e. $\gamma \geq 0$, whence $f = 0$, and therefore g is cyclic. A similar proof works for $H^2_-(\mathbf{R})$.

(2) By Theorem 3.3.5, any $g \in L^2(\mathbf{R})$ with $\int_{-\infty}^\infty |\hat{g}(\gamma)|^2/|\gamma|\,d\gamma < \infty$ is admissible. We must show now that there exists some $g \in L^2(\mathbf{R})\backslash\{0\}$ which is cyclic for U, but that not every such g is cyclic. So, let $g \in L^2(\mathbf{R})\backslash\{0\}$ be any function such that $\int_0^\infty |\hat{g}(\gamma)|^2/|\gamma|\,d\gamma = \int_{-\infty}^0 |\hat{g}(\gamma)|^2/|\gamma|\,d\gamma < \infty$, and call this number c_g (for example, any function whose Fourier transform is even and vanishes on a neighborhood of the origin will do). Now suppose that $f \in L^2(\mathbf{R})$ and $\langle f, U(u,v)g\rangle = 0$ for all $(u,v) \in \mathbf{A}$. Then by Theorem 3.3.5,

$$0 \;=\; \int_{\mathbf{R}}\int_{\mathbf{R}} |\langle f, U(u,v)g\rangle|^2\,du\,dv \;=\; c_g\int_0^\infty |\hat{f}(\gamma)|^2\,d\gamma \;+\; c_g\int_{-\infty}^0 |\hat{f}(\gamma)|^2\,d\gamma,$$

so $\hat{f} \equiv 0$. Therefore $f = 0$, so g is cyclic.

We now construct a g which is not cyclic. These are also easy to find; for example, take any nonzero $g \in H^2_+(\mathbf{R})$. Then $U(u,v)g \in H^2_+(\mathbf{R})$ for all u, v, so $\operatorname{span}\{U(u,v)g\}_{(u,v)\in\mathbf{A}} \subset H^2_+(\mathbf{R})$, and therefore cannot be dense in $L^2(\mathbf{R})$. Alternatively, note that if $f \in H^2_-(\mathbf{R})$ then $\langle f, U(u,v)g\rangle = 0$ for all $(u,v) \in \mathbf{A}$, even though f need not be identically zero. In any case, this g is not cyclic, so U is not irreducible. $\square$

DEFINITION 3.3.7. Given an admissible $g \in L^2(\mathbf{R})$, the Φ*-transform of* g is the operator Φ_g given by $\Phi_g f(u,v) = \langle f, U(u,v)g\rangle = \langle f, D_{e^u} T_v g\rangle$.

From Theorem 3.3.5 we immediately obtain the following corollary.

COROLLARY 3.3.8 [34]. *Given an admissible* $g \in L^2(\mathbf{R})\backslash\{0\}$ *define*

$$c_g^+ \;=\; \int_0^\infty \frac{|\hat{g}(\gamma)|^2}{|\gamma|} d\gamma \qquad \textit{and} \qquad c_g^- \;=\; \int_{-\infty}^0 \frac{|\hat{g}(\gamma)|^2}{|\gamma|}\, d\gamma.$$

(1) *If* $g \in H^2_+(\mathbf{R})$ *then* $\Phi_g\colon H^2_+(\mathbf{R}) \to L^2(\mathbf{R}^2)$ *is a multiple of an isometry, with* $\|\Phi_g\| = c_g^+$.

(2) *If* $g \in H^2_-(\mathbf{R})$ *then* $\Phi_g\colon H^2_-(\mathbf{R}) \to L^2(\mathbf{R}^2)$ *is a multiple of an isometry, with* $\|\Phi_g\| = c_g^-$.

(3) *If* $g \in L^2(\mathbf{R})$ *with* $c_g^+ = c_g^- = c_g$ *then* $\Phi_g\colon L^2(\mathbf{R}) \to L^2(\mathbf{R}^2)$ *is a multiple of an isometry, with* $\|\Phi_g\| = c_g$.

As with the Ψ-transform, we would like to obtain an inversion formula for the Φ-transform. Ideally, we would like to say that if $g \in L^2(\mathbf{R})$ is admissible with $c_g^+ = c_g^- = 1$ then for every $f \in L^2(\mathbf{R})$,

$$f(x) \;=\; \int_{\mathbf{R}}\int_{\mathbf{R}} \Phi_g f(u,v)\, D_{e^u} T_v g(x)\, du\, dv.$$

Unfortunately, it is not clear that the above integral exists in general. The following theorem gives a rigorous interpretation of this formula.

THEOREM 3.3.9 [34]. *Suppose* $g \in L^2(\mathbf{R})$ *is admissible with* $c_g^+ = c_g^- = 1$. *Let* $\{\rho_n\}_{n=1}^\infty$ *be an approximate identity such that each* $\rho_n \in L^2(\mathbf{R})$ *and* $\rho_n(x) = \rho_n(-x)$ *for all* x. *Then* $\lim_{n\to\infty} \|f - f_n\|_2 = 0$ *for all* $f \in L^2(\mathbf{R})$, *where*

$$f_n(x) \;=\; \int_{\mathbf{R}}\int_{\mathbf{R}} \Phi_g f(u,v)\, (\rho_n * D_{e^u} T_v g)(x)\, du\, dv.$$

Proof. An approximate identity with the required properties exists by Example 1.4.5. Now,

$$\begin{aligned}
(f*\rho_n)(x) \;&=\; \int_{\mathbf{R}} f(t)\, \rho_n(x-t)\, dt \\
&=\; \langle f, \overline{T_x\rho_n}\rangle \qquad\qquad \text{(since } \rho_n \text{ is even)} \\
&=\; \langle \Phi_g f, \Phi_g(\overline{T_x\rho_n})\rangle \qquad\qquad \text{(since } \Phi_g \text{ is an isometry)} \\
&=\; \int_{\mathbf{R}}\int_{\mathbf{R}} \Phi_g f(u,v)\, \langle D_{e^u}T_v g, \overline{T_x\rho_n}\rangle\, du\, dv \\
&=\; \int_{\mathbf{R}}\int_{\mathbf{R}} \Phi_g f(u,v)\, (\rho_n * D_{e^u}T_v g)(x)\, du\, dv.
\end{aligned}$$

But $\{\rho_n\}$ is an approximate identity, so $\lim_{n\to\infty} \|f * \rho_n - f\|_2 = 0$. □

3.4. Feichtinger–Gröchenig theory. In this section we describe the theory of Feichtinger and Gröchenig, which produces coherent state decompositions of a large class of Banach spaces in a way that generalizes the notion of a frame in a Hilbert space. We cannot give a complete or rigorous exposition as this would take many pages and go beyond the scope of this paper, but it is an important contribution to the theory of coherent state expansions and so should be mentioned. We begin with the following well-known theorem, whose proof can be found in [35]. Throughout this section we let H be a Hilbert space, G a topological group with left Haar measure μ, and π a representation of G on H.

THEOREM 3.4.1. *If π is square-integrable then there exists a unique self-adjoint positive operator C: Domain$(C) \to H$ such that:*

(1) *Domain$(C) = \{g \in H : g$ is admissible$\}$,*

(2) *for any admissible $g_1, g_2 \in H$ and any $f_1, f_2 \in H$,*

$$\int_G \langle f_1, \pi(x)g_1\rangle \, \langle \pi(x)g_2, f_2\rangle \, d\mu(x) \;=\; \langle f_1, f_2\rangle\langle Cg_2, Cg_1\rangle.$$

Moreover, if G is unimodular then C is a multiple of the identity.

DEFINITION 3.4.2. Let $g \in H\backslash\{0\}$ be admissible. For $f \in H$ we let $V_g f$ be the complex-valued function on G given by $V_g f(x) = \langle f, \pi(x)g\rangle$. Following [23], we call $V_g f$ the *voice transform* of f with respect to g.

If we take $f_1 = f_2 = g_1 = g_2 = g$ in Theorem 3.4.1 then we have

$$\int_G |V_g g(x)|^2 \, d\mu(x) \;=\; \int_G |\langle g, \pi(x)g\rangle|^2 \, d\mu(x) \;=\; \|g\|^2 \, \|Cg\|^2.$$

If we take $f_1 = f_2 = f$ and $g_1 = g_2 = g$ then

$$\int_G |V_g f(x)|^2 \, d\mu(x) \;=\; \|f\|^2 \, \|Cg\|^2 \;=\; \frac{\|f\|^2}{\|g\|^2} \int_G |V_g g(x)|^2 \, d\mu(x).$$

Thus V_g maps H into $L^2(G)$ and is a multiple of an isometry.

If π in Theorem 3.4.1 is the representation W of $\mathbf{H}$ on $L^2(\mathbf{R})$ given in §3.2 then the operator C is the identity and we see that Theorem 3.4.1 implies Proposition 3.2.4.

If π in Theorem 3.4.1 is the representation U of $\mathbf{A}$ on $H^2_+(\mathbf{R})$ given in §3.3 then the operator C is given by $(Cg)^\wedge(\gamma) = \hat{g}(\gamma)/\gamma^{1/2}$, with similar remarks for $H^2_-(\mathbf{R})$. Thus Theorem 3.4.1 implies parts (1) and (2) of Corollary 3.3.9. However, it does not necessarily imply part (3) since U is not a square-integrable representation of $\mathbf{A}$ on $L^2(\mathbf{R})$.

We now describe the Feichtinger–Gröchenig theory. Assume that π is an irreducible, unitary representation of G on H which is *integrable*, i.e., there is a $g \in H\backslash\{0\}$ such that $\int_G |V_g g(x)| \, d\mu(x) = \int_G |\langle g, \pi(x)g\rangle| \, d\mu(x) < \infty$, and which is *continuous*, i.e., $\pi(x)g$ is a continuous map of G into H for all $x \in G$. Let $H_0 = \{g \in H : V_g g \in L^1(G)\}$, and let $H_0' \supset H$ be the dual of H_0. We can then extend formula (2) of Theorem 3.4.1 to hold for all $g_1, g_2 \in H_0$ and $f_1, f_2 \in H_0'$. This gives us a *reproducing formula*: if $g \in H_0\backslash\{0\}$ and $\|Cg\| = 1$ then from Theorem 3.4.1 part (2),

$$\begin{aligned}
\int_G V_g f(x) \, V_g g(x^{-1}y) \, d\mu(x) \;&=\; \int_G \langle f, \pi(x)g\rangle \, \langle g, \pi(x^{-1}y)g\rangle \, d\mu(x) \\
&=\; \int_G \langle f, \pi(x)g\rangle \, \langle \pi(x)g, \pi(y)g\rangle \, d\mu(x) \\
&=\; \langle f, \pi(y)g\rangle \\
&=\; V_g f(y).
\end{aligned}$$

Note that the integral operator on the left-hand side is a convolution operator on G.

Now, for certain spaces Y of functions on G for which the above convolution operator is defined and continuous for $g \in H_0$, we define the *coorbit space* $\mathrm{Co}(Y) = \{f \in H_0' : V_g f \in Y\}$ (which is independent of the choice of $g \in H_0$), and place on $\mathrm{Co}(Y)$ the norm $\|f\|_{\mathrm{Co}(Y)} = \|V_g f\|_Y$. At the same time, we define an appropriate sequence space Y_d corresponding to Y (for example, if $Y = L^p(G)$ then $Y_d = \ell^p(Z)$). Let $S = \{F \in Y : F = V_g f \text{ for some} f \in \mathrm{Co}(Y)\}$.

In the next step, we approximate the above convolution operator (which is the identity on S) by a discrete operator, similar to a Riemann sum. For example, let $\Psi = \{\psi_i\}$ be a collection of functions on G that satisfy:

(1) $\sup_i \|\psi_i\|_\infty < \infty$,

(2) there is an open set $O \subset G$ with compact closure and points $x_i \in G$ such that $\operatorname{supp}(\psi_i) \subset x_i O$ for each i,

(3) $\sum_i \psi_i(x) \equiv 1$,

(4) $\sup_{z\in G} \#\{i \in I : z \in x_i Q\} < \infty$ for each compact set $Q \subset G$.

We call such a Ψ a *bounded uniform partition of unity.* Define the operator T_Ψ on Y, associated to a particular bounded uniform partition of unity Ψ, by

$$T_\Psi F(y) \;=\; \sum_i \langle F, \psi_i\rangle \, V_g g(x_i^{-1} y).$$

It can be shown that there exist compact neighborhoods U and V of the identity in G such that the following hold: for *any* collection of points $\{x_i\} \subset G$ which is U-dense (i.e., $\cup x_i U = G$) and V-separated (i.e., $x_i V \cap x_j V = \emptyset$ if $i \neq j$), and *any* bounded uniform partition of unity Ψ associated to $\{x_i\}$, there are constants $A, B > 0$ such that $A\|F\|_Y \leq \|\{\langle F, \psi_i\rangle\}\|_{Y_d} \leq B\|F\|_Y$ for all $F \in Y$, and, when restricted to S, the operator T_Ψ is continuous and continuously invertible. Thus for each $f \in \mathrm{Co}(Y)$ we can write

$$\begin{aligned}
\langle f, \pi(y)g\rangle &= V_g f(y) \\
&= T_\Psi(T_\Psi^{-1} V_g f)(y) \\
&= \sum_i \langle T_\Psi^{-1} V_g f, \psi_i\rangle \, V_g g(x_i^{-1} y) \\
&= \sum_i \langle T_\Psi^{-1} V_g f, \psi_i\rangle \, \langle \pi(x_i) g, \pi(y) g\rangle \\
&= \Big\langle \sum_i \langle T_\Psi^{-1} V_g f, \psi_i\rangle \, \pi(x_i) g,\; \pi(y) g \Big\rangle.
\end{aligned}$$

We conclude that $f = \sum \lambda_i(f)\, \pi(x_i) g$, where $\lambda_i(f) = \langle T_\Psi^{-1} V_g f, \psi_i\rangle$, and that for some constants $A_0, B_0 > 0$ we have $A_0\|f\|_{\mathrm{Co}(Y)} \leq \|\{\lambda_i(f)\}\|_{Y_d} \leq B_0\|F\|_{\mathrm{Co}(Y)}$. Thus this is a generalization of frames to Banach spaces other than Hilbert spaces, for we have seen that a frame $\{x_n\}$ allows us to write $x = \sum \langle x, S^{-1}x_n\rangle x_n$, with $B^{-1}\|x\|^2 \leq \sum |\langle x, S^{-1}x_n\rangle|^2 \leq A^{-1}\|x\|^2$, where $A, B > 0$ are the frame bounds. This is like a Feichtinger–Gröchenig-type decomposition with $\lambda_n(x) = \langle x, S^{-1}x_n\rangle$ and $Y_d = \ell^2(\mathbf{Z})$. In fact, it can be shown that finding Feichtinger–Gröchenig-type decompositions for Hilbert spaces is equivalent to finding frames for Hilbert spaces.

4. Weyl–Heisenberg Frames. As we pointed out at the end of §3, it is possible to discretize general coherent state integral operators and thereby obtain expansions of functions in Banach spaces in terms of a lattice of coherent states. This is the contribution of Feichtinger and Gröchenig, and it is a generalization of previous direct expansions. We examine two of these direct expansions in this and the next section.

In this section, we deal with the case of the Weyl–Heisenberg coherent state integral operator defined in §3.2, which we called the Ψ-transform. We use the theory of frames presented in §2 to obtain expansions of functions in $L^2(\mathbf{R})$ in terms of a discrete lattice of W–H coherent states.

In §4.1 we prove the existence of W–H frames for $L^2(\mathbf{R})$. The idea of using Hilbert space frames to obtain decompositions of $L^2(\mathbf{R})$ is due to A. Grossmann, and most of the results in this section are the work of Daubechies, Grossmann, and Meyer and can be found in the fundamental papers [15] and [14]. These results and proofs are mostly based on specific $L^2(\mathbf{R})$ methods, which greatly improve the general Feichtinger–Gröchenig results but have limited applicability outside of $L^2(\mathbf{R})$. Generalizations and new results incorporating both $L^2(\mathbf{R})$ and Feichtinger–Gröchenig methods due to Walnut are in [54].

Section 4.2 deals with continuity properties of the frame operator, which the reader will recognize as a formal discretization of the Ψ-transform. The results in this section are due to Walnut and are treated more extensively in [54].

In §4.3 we introduce the Zak transform and use it to prove more results on the existence of frames and also to shed some light on the interdependence of the lattice parameters and the mother wavelet. These results are due to many groups, including Zak, Daubechies, and especially Janssen. New results due to Heil are in [36].

We give some examples of W–H frames in §4.4.

4.1. Existence of Weyl–Heisenberg frames.

DEFINITION 4.1.1. Given $g \in L^2(\mathbf{R})$ and $a, b > 0$, we say that (g, a, b) *generates a W–H frame* for $L^2(\mathbf{R})$ if $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ is a frame for $L^2(\mathbf{R})$. The function g is referred to as the *mother wavelet*, *analyzing wavelet*, or *fiducial vector*. The numbers a, b are the *frame parameters*, with a being the *shift parameter* and b the *modulation parameter*.

It is clear that $\{E_{mb}T_{na}g\}$ is a frame for $L^2(\mathbf{R})$ if and only if $\{T_{na}E_{mb}g\}$ is a frame. We switch freely between these two formats, depending on which is most convenient in a given situation.

THEOREM 4.1.2 [15]. *Let $g \in L^2(\mathbf{R})$ and $a, b > 0$ be such that:*

(1) *there exist constants A, B such that $0 < A \le \sum_n |g(x - na)|^2 \le B < \infty$ a.e.,*

(2) *g has compact support, with $\operatorname{supp}(g) \subset I \subset \mathbf{R}$, where I is some interval of length $1/b$.*

Then (g, a, b) generates a W–H frame for $L^2(\mathbf{R})$ with frame bounds $b^{-1}A, b^{-1}B$.

Proof. Fix n, and observe that the function $f \cdot T_{na}\bar{g}$ is supported in $I_n = I + na = \{x + na : x \in I\}$, an interval of length $1/b$. Now, it follows from condition (1) that g is bounded, so $f \cdot T_{na}\bar{g} \in L^2(I_n)$. But the collection of functions $\{b^{1/2}E_{mb}\}_{m\in\mathbf{Z}}$ is an orthonormal basis for $L^2(I_n)$, so $\sum_m |\langle f \cdot T_{na}\bar{g}, E_{mb}\rangle|^2 = b^{-1} \int |f(x)|^2\, |g(x - na)|^2\, dx$. Therefore,

$$\begin{aligned}\sum_{m,n} |\langle f, E_{mb}T_{na}g\rangle|^2 &= \sum_{m,n} |\langle f \cdot T_{na}\bar{g}, E_{mb}\rangle|^2 \\ &= b^{-1} \sum_n \int_{\mathbf{R}} |f(x)|^2\, |g(x - na)|^2\, dx \\ &= b^{-1} \int_{\mathbf{R}} |f(x)|^2 \sum_n |g(x - na)|^2\, dx.\end{aligned}$$

The result now follows by using condition (1). $\square$

COROLLARY 4.1.3 [15]. *Suppose g is a continuous function supported on an interval I of length $L > 0$ which does not vanish in the interior of I. Then (g, a, b) generates a frame for $L^2(\mathbf{R})$ for any $0 < a < L$ and $0 < b \le 1/L$.*

Proof. Since $1/b \ge L$ we see that the support of g is contained in an interval of length $1/b$. Define $G(x) = \sum |g(x - na)|^2$. The result will follow from Theorem 4.1.2 if we show that G is bounded both above and below. Since g is compactly supported, the sum defining G is actually a finite sum, with at most $1/ab$ terms, and therefore G

is bounded above since g is. Now let J be the subinterval of I with the same center but with length a. Given $x \in \mathbf{R}$ there is always an $n \in \mathbf{Z}$ such that $x - na \in J$, so $\inf_{x\in\mathbf{R}} G(x) \ge \inf_{x\in J} |g(x)|^2 > 0$. $\square$

PROPOSITION 4.1.4 [14]. *Whether g has compact support or not, it is necessary that condition* (1) *of Theorem* 4.1.2 *hold in order that (g, a, b) generate a frame. In particular, g must be bounded.*

Proof. Let $G(x) = \sum |g(x-na)|^2$, and assume ess $\inf_{x\in\mathbf{R}} G(x) = 0$. Given $\delta > 0$ we can then find a set $\Delta \subset I \subset \mathbf{R}$, where I is an interval of length $1/b$, such that $|\Delta| > 0$ and $G(x) \le b\,\delta$ on Δ. If we set $f = \chi_\Delta$ then $\|f\|_2^2 = |\Delta|$, and, as in the proof of Theorem 4.1.2,

$$\begin{aligned}\sum_{m,n} |\langle f, E_{mb}T_{na}g\rangle|^2 \;&=\; b^{-1} \sum_n \int_{\mathbf{R}} |f(x)|^2\, |g(x-na)|^2\, dx \\ &=\; b^{-1} \int_\Delta \sum_n |g(x-na)|^2\, dx \\ &\le\; b^{-1}\, |\Delta|\, b\,\delta \\ &=\; \delta\, \|f\|_2^2.\end{aligned}$$

Since δ was arbitrary, (g, a, b) cannot generate a frame. A similar proof shows that G must be bounded above. $\square$

Since $(E_{mb}T_{na}g)^\wedge = T_{mb}E_{-na}\hat{g}$ and the Fourier transform is a unitary map of $L^2(\mathbf{R})$ onto $L^2(\hat{\mathbf{R}})$, we see that (g, a, b) generates a W–H frame for $L^2(\mathbf{R})$ if and only if $(\hat{g}, b, a)$ generates a W–H frame for $L^2(\hat{\mathbf{R}})$. From Proposition 4.1.4 we therefore have that both g and $\hat{g}$ must be bounded in this case.

It is easy to see that if g satisfies condition (2) of Theorem 4.1.2 and if $ab > 1$ then ess $\inf_{x\in\mathbf{R}} \sum |g(x-na)|^2 = 0$, so g cannot generate a frame. In fact, the set $\{E_{mb}T_{na}g\}$ is not even complete, since $\cup\,\mathrm{supp}(T_{na}g)$ does not cover $\mathbf{R}$, and therefore any function supported in the complement of $\cup\,\mathrm{supp}(T_{na}g)$ will be orthogonal to every $E_{mb}T_{na}g$. As we will mention in §4.3, it can actually be shown that if $ab > 1$ then $\{E_{mb}T_{na}g\}$ can never be complete in $L^2(\mathbf{R})$ for *any* $g \in L^2(\mathbf{R})$.

The following theorem on the existence of W–H frames in $L^2(\mathbf{R})$ with non-compactly supported mother wavelets is due to Daubechies, with a proof using the Poisson summation formula appearing in [14]. The proof below avoids the use of the Poisson summation formula and generalizes slightly the condition on the mother wavelet found in [14].

THEOREM 4.1.5 [54]. *Let $g \in L^2(\mathbf{R})$ and $a > 0$ be such that:*

(1) *there exist constants A, B such that* $0 < A \le \sum_n |g(x-na)|^2 \le B < \infty$ *a.e.*,

(2) $\lim_{b\to 0} \sum_{k\ne 0} \beta(k/b) = 0$, *where*

$$\beta(s) \;=\; \operatorname*{ess\,sup}_{x\in\mathbf{R}} \left| \sum_n g(x-na)\,\overline{g(x-s-na)} \right| \;=\; \left\| \sum_n T_{na}g \cdot T_{na+s}\bar{g} \right\|_\infty.$$

Then there exists $b_0 > 0$ such that (g, a, b) generates a W–H frame for $L^2(\mathbf{R})$ for each $0 < b < b_0$.

Proof. First assume that f is continuous and compactly supported. This will guarantee that all subsequent interchanges of summation and integration are fully justified. For fixed n consider the $1/b$-periodic function given by

$$F_n(t) \;=\; \sum_k f(t-k/b)\,\overline{g(t-na-k/b)}.$$

Now, $F_n \in L^1[0,1/b]$ since both f and g are bounded, and

$$\int_{\mathbf{R}} f(t)\,\overline{g(t-na)}\,e^{-2\pi imbt}\,dt \;=\; \int_0^{1/b} F_n(t)\,e^{-2\pi imbt}\,dt.$$

Since $\{b^{1/2}E_{mb}\}_{m\in\mathbf{Z}}$ is an orthonormal basis for $L^2[0,1/b]$ we have by the Plancherel formula that

$$\sum_m \left|\int_0^{1/b} F_n(t)\,e^{-2\pi imbt}\,dt\right|^2 \;=\; b^{-1}\int_0^{1/b} |F_n(t)|^2\,dt.$$

Therefore,

$$\begin{aligned}
&\sum_n\sum_m |\langle f, E_{mb}T_{na}g\rangle|^2\\
&= \sum_n\sum_m \left|\int_{\mathbf{R}} f(t)\,\overline{g(t-na)}\,e^{-2\pi imbt}\,dt\right|^2\\
&= b^{-1}\sum_n \int_0^{1/b} \left|\sum_k f(t-k/b)\,\overline{g(t-na-k/b)}\right|^2 dt\\
&= b^{-1}\sum_n \int_0^{1/b} \sum_l \overline{f(t-l/b)}\,g(t-na-l/b)\cdot\sum_k f(t-k/b)\,\overline{g(t-na-k/b)}\,dt\\
&= b^{-1}\sum_n\sum_l \int_0^{1/b} \overline{f(t-l/b)}\,g(t-na-l/b)\cdot\sum_k f(t-k/b)\,\overline{g(t-na-k/b)}\,dt\\
&= b^{-1}\sum_n \int_{\mathbf{R}} \overline{f(t)}\,g(t-na)\cdot\sum_k f(t-k/b)\,\overline{g(t-na-k/b)}\,dt\\
&= b^{-1}\sum_k \int_{\mathbf{R}} \overline{f(t)}\,f(t-k/b)\cdot\sum_n g(t-na)\,\overline{g(t-na-k/b)}\,dt\\
&= b^{-1}\int_{\mathbf{R}} |f(t)|^2\cdot\sum_n |g(t-na)|^2\,dt\\
&\qquad +\; b^{-1}\sum_{k\neq 0}\int_{\mathbf{R}} \overline{f(t)}\,f(t-k/b)\cdot\sum_n g(t-na)\,\overline{g(t-na-k/b)}\,dt\\
&= (*).
\end{aligned}$$

But by the Cauchy–Schwarz inequality we have

$$(*) \;\le\; b^{-1}B\,\|f\|_2^2 \;+\; b^{-1}\sum_{k\neq 0}\beta(k/b)\int_{\mathbf{R}}\overline{f(t)}\,f(t-k/b)\,dt \;\le\; B_0(b)\,\|f\|_2^2$$

and

$$(*) \;\ge\; b^{-1}A\,\|f\|_2^2 \;-\; b^{-1}\sum_{k\neq 0}\beta(k/b)\int_{\mathbf{R}}\overline{f(t)}\,f(t-k/b)\,dt \;\ge\; A_0(b)\,\|f\|_2^2,$$

where

$$A_0(b) = b^{-1}A \;-\; b^{-1}\sum_{k\neq 0}\beta(k/b) \qquad\text{and}\qquad B_0(b) = b^{-1}B \;+\; b^{-1}\sum_{k\neq 0}\beta(k/b).$$

By condition (2), there is a $b_0 > 0$ such that $A_0(b) > 0$ and $B_0(b) < \infty$ for all $0 < b < b_0$.

Now let $f \in L^2(\mathbf{R})$ be arbitrary. Then we can find a sequence of continuous, compactly supported functions f_j such that $f_j \to f$ in $L^2(\mathbf{R})$ as $j \to \infty$. By the above results we have $A_0(b)\,\|f_j\|_2^2 \le \sum |\langle f_j, E_{mb}T_{na}g\rangle|^2 \le B_0(b)\,\|f_j\|_2^2$. It is not hard to show that these inequalities hold in the limit as $j \to \infty$, whence (g, a, b) generates a W–H frame with bounds $A_0(b), B_0(b)$ for all $0 < b < b_0$. □

As Theorem 4.1.5 is stated, the value of b_0 for which (g, a, b) generates a frame for all $0 < b < b_0$ is dependent on both g and a. In fact, there is nothing to prevent the value of b_0 from going to zero as a goes to zero. Using different methods, the Feichtinger–Gröchenig theory implies that (for certain g) there is a rectangle of pairs of (a, b), depending only on g, for which (g, a, b) will generate a frame. The same has been shown in [54] with less stringent conditions on g, in particular it is only required there that $|g|$ be bounded above and below on an interval and that $g \in W(L^\infty, L^1)$, an important space defined as follows.

Definition 4.1.6. Given a function g we say that $g \in W(L^\infty, L^1)$ if for some $a > 0$,

$$\|g\|_{W,a} \;=\; \sum_n \|g \cdot \chi_{[an,a(n+1))}\|_\infty \;=\; \sum_n \|T_{na}g \cdot \chi_{[0,a)}\|_\infty \;<\; \infty.$$

$W(L^\infty, L^1)$ is easily seen to be a Banach space. The subset of $W(L^\infty, L^1)$ consisting of continuous functions was first studied by Wiener in [55]. Analogously defined spaces $W(L^p, L^q)$ are known as *mixed-norm* or *amalgam spaces*. An excellent survey article on these spaces is [28]. More general spaces $W(B, C)$ of distributions which, roughly speaking, are "locally in B" and "globally in C" were first studied by Feichtinger [19], who gave them the name *Wiener-type spaces*. Wiener-type spaces are used by Feichtinger to define *modulation spaces* which constitute a large class of non-Hilbert spaces which admit W–H coherent state expansions [20], [23], [24], [31].

Proposition 4.1.7.

(1) *If $\|g\|_{W,a}$ is finite for some a then it is finite for all a.*

(2) *If $0 < a \le b$ then $\|g\|_{W,b} \le 2\,\|g\|_{W,a}$.*

(3) *For any $a > 0$ and $b \in \mathbf{R}$ we have $\|T_b g\|_{W,a} \le 2\,\|g\|_{W,a}$.*

Proof. (1) Given a, b, let $P_n = [an, a(n+1))$ and $Q_n = [bn, b(n+1))$, and let $\{I_j\}$ be the collection of all nonempty intersections of elements from $\{P_n\}$ and $\{Q_n\}$. It is clear that the number of I_j that can be contained in a given P_n is bounded independently of n, and we call this bound M. Therefore,

$$\sum_j \|g \cdot \chi_{I_j}\|_\infty \;\le\; \sum_n \sum_{I_j \subset P_n} \|g \cdot \chi_{P_n}\|_\infty \;\le\; M \sum_n \|g \cdot \chi_{P_n}\|_\infty.$$

Also,

$$\sum_n \|g \cdot \chi_{Q_n}\|_\infty \;=\; \sum_n \|g \cdot \chi_{I_{j(n)}}\|_\infty \;\le\; \sum_j \|g \cdot \chi_{I_j}\|_\infty,$$

where $j(n)$ is such that $I_{j(n)} \subset Q_n$ and $\|g \cdot \chi_{Q_n}\|_\infty = \|g \cdot \chi_{I_{j(n)}}\|_\infty$. This implies $\|g\|_{W,b} \le M\|g\|_{W,a}$, and an analogous argument gives the opposite inequality.

(2) Keeping the notation above, the assumption $a \le b$ implies that $M = 2$.

(3) The proof is similar to (1). □

THEOREM 4.1.8 [54]. *Given $g \in W(L^\infty, L^1)$ such that g satisfies condition* (1) *of Theorem* 4.1.5 *for some a. Then there is a $b_0 > 0$ such that (g, a, b) generates a W–H frame for $L^2(\mathbf{R})$ for all $0 < b \le b_0$.*

Proof. By Theorem 4.1.5 we need only show $\lim_{b\to 0} \sum_{k\neq 0} \beta(k/b) = 0$. Without loss of generality, we consider $b \le 1/a$. Now, we first claim that given functions f, h we always have

$$\sum_k \Big\| \sum_n |T_{na} f|\, |T_{na+k/b} h| \Big\|_\infty \;\le\; 4\, \|f\|_{W,a} \|h\|_{W,a}.$$

To see this, note that $\sum_n |T_{na} f|\, |T_{na+k/b} h|$ is a-periodic for each k, so

$$\begin{aligned}
\sum_k \Big\| \sum_n |T_{na} f|\, |T_{na+k/b} h| \Big\|_\infty &= \sum_k \Big\| \sum_n |T_{na} f|\, |T_{na+k/b} h|\, \chi_{[0,a)} \Big\|_\infty \\
&\le \sum_k \sum_n \|T_{na} f \cdot \chi_{[0,a)}\|_\infty \, \|T_{na+k/b} h \cdot \chi_{[0,a)}\|_\infty \\
&\le \sum_n \|T_{na} f \cdot \chi_{[0,a)}\|_\infty \sum_k \|T_{k/b}(T_{na} h) \cdot \chi_{[0,1/b)}\|_\infty \\
&\le \sum_n \|T_{na} f \cdot \chi_{[0,a)}\|_\infty \, \|T_{na} h\|_{W,1/b} \\
&\le 2\, \|h\|_{W,1/b} \sum_n \|T_{na} f \cdot \chi_{[0,a)}\|_\infty \\
&\le 4\, \|h\|_{W,a}\, \|f\|_{W,a}.
\end{aligned}$$

Now fix $\epsilon > 0$ and let N be so large that $\sum_{|n|\ge N} \|g \cdot \chi_{[an, a(n+1))}\|_\infty < \epsilon$. Let $g_0 = g \cdot \chi_{[-aN, aN]}$ and $g_1 = g - g_0$, so $\|g_1\|_{W,a} < \epsilon$. If $b \ge 2aN$ then

$$\begin{aligned}
\sum_{k\neq 0} \beta(k/b) &= \sum_{k\neq 0} \Big\| \sum_n T_{na} g \cdot T_{na+k/b} \bar{g} \Big\|_\infty \\
&\le \sum_{k\neq 0} \Big\| \sum_n |T_{na} g|\, |T_{na+k/b} g| \Big\|_\infty \\
&= \sum_{k\neq 0} \Big\| \sum_n |T_{na} g_0 + T_{na} g_1|\, |T_{na+k/b} g_0 + T_{na+k/b} g_1| \Big\|_\infty \\
&\le \sum_{k\neq 0} \Big\| \sum_n |T_{na} g_0|\, |T_{na+k/b} g_0| \Big\|_\infty \;+\; \sum_k \Big\| \sum_n |T_{na} g_0|\, |T_{na+k/b} g_1| \Big\|_\infty \\
&\qquad + \sum_k \Big\| \sum_n |T_{na} g_1|\, |T_{na+k/b} g_0| \Big\|_\infty \;+\; \sum_k \Big\| \sum_n |T_{na} g_1|\, |T_{na+k/b} g_1| \Big\|_\infty \\
&\le 0 \;+\; 8\, \|g_0\|_{W,a} \|g_1\|_{W,a} \;+\; 4\, \|g_1\|_{W,a}^2 \\
&\le 8\epsilon\, \|g\|_{W,a} \;+\; 4\epsilon^2,
\end{aligned}$$

from which the result follows. □

4.2. The Weyl–Heisenberg frame operator. Assume (g, a, b) generates a W–H frame for $L^2(\mathbf{R})$. From Theorem 2.1.3 the *frame operator* corresponding to (g, a, b) is

$$Sf \;=\; \sum_{m,n} \langle f, E_{mb} T_{na} g \rangle E_{mb} T_{na} g,$$

and we know that S is a topological isomorphism of $L^2(\mathbf{R})$ onto itself. We also know (cf. Corollary 2.1.4) that the *dual frame* of $\{E_{mb}T_{na}g\}$ is $\{S^{-1}(E_{mb}T_{na}g)\}$. A straightforward calculation gives $S^{-1}(E_{mb}T_{na}g) = E_{mb}T_{na}S^{-1}g$, without any assumptions about the support of g, so the dual is actually $\{E_{mb}T_{na}S^{-1}g\}$, another W–H frame with mother wavelet $S^{-1}g$.

The following theorem shows that the sum defining Sf above actually converges for quite general g, whether or not g generates a W–H frame.

THEOREM 4.2.1 [54]. *Given $a, b > 0$ and $g \in W(L^\infty, L^1)$. For $k \in \mathbf{Z}$ define $G_k = \sum_n T_{na}g \cdot T_{na+k/b}\bar{g}$. Then the sum Sf converges for each $f \in L^2(\mathbf{R})$, and is given by*

$$Sf \;=\; b^{-1} \sum_k T_{k/b}f \cdot G_k.$$

Proof. First observe that the sum defining G_k converges pointwise for each k since $g \in W(L^\infty, L^1)$. Fix $f \in L^2(\mathbf{R})$. Then for arbitrary $h \in L^2(\mathbf{R})$ we can, using the Fourier series arguments of Theorem 4.1.5, compute that $\big\langle b^{-1} \sum T_{k/b}f \cdot G_k,\, h\big\rangle = \sum \langle f, E_{mb}T_{na}g\rangle\langle E_{mb}T_{na}g, h\rangle = \langle Sf, h\rangle$. Since this is true for all h, the result follows. $\square$

Note that if g is compactly supported in an interval of length $1/b$ then $G_k \equiv 0$ for $k \neq 0$. Thus $Sf(x) = f(x)\, b^{-1}G_0(x) = f(x)\, b^{-1} \sum |g(x - na)|^2$. Moreover, if g satisfies condition (1) of Theorem 4.1.2 then S is invertible (as is expected since (g, a, b) then generates a frame), and $S^{-1}f = bf/G_0$. The following theorem shows that under certain conditions a formula can be given for $S^{-1}f$ when g is not compactly supported.

THEOREM 4.2.2 [54]. *Given $a, b > 0$ and $g \in W(L^\infty, L^1)$, and suppose g satisfies condition* (1) *of Theorem* 4.1.5. *Let G_k be as in Theorem* 4.2.1, *and define formally the following functions:*

(1) $G_0^{(0)} \equiv 1$; $G_j^{(0)} \equiv 0$ *for* $j \neq 0$;

(2) $G_j^{(1)} \equiv 0$; $G_j^{(1)} = -G_j/G_0$ *for* $j \neq 0$;

(3) $G_j^{(m)} = \sum_n T_{n/b}G_{j-n}^{(1)} \cdot G_n^{(m-1)}$;

(4) $H_k = \sum_{m=0}^{\infty} G_k^{(m)}$.

Suppose that $\sum \|G_j^{(1)}\|_\infty < 1$. Then the sums defining $G_j^{(m)}$ and H_k converge absolutely and uniformly, and $\sum \|H_k\|_\infty < \infty$. Moreover, S is invertible, and $S^{-1}f = b\big(\sum_k T_{k/b}f \cdot H_k\big)/G_0$.

Proof. Claim (1). $\sum \|G_j^{(m)}\|_\infty \leq \big(\sum \|G_j^{(1)}\|_\infty\big)^m$ for all $m \geq 0$.

This is obvious for $m = 0$ and 1, so suppose it holds for some $m \geq 1$. Then

$$\begin{aligned}
\sum_j \|G_j^{(m+1)}\|_\infty \;&=\; \sum_j \bigg\| \sum_n T_{n/b}G_{j-n}^{(1)} \cdot G_n^{(m)} \bigg\|_\infty \\
&\leq\; \sum_j \sum_n \|G_{j-n}^{(1)}\|_\infty \|G_n^{(m)}\|_\infty \\
&=\; \sum_j \|G_j^{(1)}\|_\infty \sum_n \|G_n^{(m)}\|_\infty \\
&\leq\; \sum_j \|G_j^{(1)}\|_\infty \bigg(\sum_n \|G_n^{(1)}\|_\infty \bigg)^m
\end{aligned}$$

$$= \left(\sum_n \|G_n^{(1)}\|_\infty\right)^{m+1},$$

so the claim holds for $m+1$, and therefore for all m.

It follows from Claim (1) that the sums defining $G_j^{(m)}$ and H_k converge absolutely and uniformly and that $\sum \|H_k\|_\infty < \infty$. Define now $S_0 f = b(Sf)/G_0$ for $f \in L^2(\mathbf{R})$. Using Theorem 4.2.1 it is easy to see that $\|f - S_0 f\|_2 = \|\sum T_{j/b} f \cdot G_j^{(1)}\|_2 \leq \|f\|_2 \sum \|G_j^{(1)}\|_\infty$. By assumption we therefore have $\|I - S_0\| \leq \sum \|G_j^{(1)}\|_\infty < 1$, which implies that S_0 is invertible and $S_0^{-1} f = \sum_0^\infty (I - S_0)^m f$ for all $f \in L^2(\mathbf{R})$.

Claim (2). $(I - S_0)^m f = \sum_k T_{k/b} f \cdot G_k^{(m)}$ for all f which are continuous and compactly supported.

By the formula for Sf given in Theorem 4.2.1, it is easy to see that the claim holds for $m = 0$ and 1. Assume it holds for some $m \geq 1$. Then

$$\begin{aligned}
(I - S_0)^{m+1} f &= (I - S_0)^m (I - S_0) f \\
&= \sum_k T_{k/b}((I - S_0) f) \cdot G_k^{(m)} \\
&= \sum_k T_{k/b}\left(\sum_j T_{j/b} f \cdot G_k^{(1)}\right) \cdot G_k^{(m)} \\
&= \sum_k \sum_j T_{(j+k)/b} f \cdot T_{k/b} G_j^{(1)} \cdot G_k^{(m)} \\
&= \sum_k \sum_n T_{n/b} f \cdot T_{k/b} G_{n-k}^{(1)} \cdot G_k^{(m)} \\
&= \sum_n T_{n/b} f \sum_k T_{k/b} G_{n-k}^{(1)} \cdot G_k^{(m)} \\
&= \sum_n T_{n/b} f \cdot G_n^{(m+1)}.
\end{aligned}$$

The interchanges in summation are justified by Claim (1) and the compactness of the support of f.

Therefore, for f continuous and compactly supported, we have

$$S_0^{-1} f = \sum_{m=0}^\infty \sum_k T_{k/b} f \cdot G_k^{(m)} = \sum_k T_{k/b} f \sum_{m=0}^\infty G_k^{(m)} = \sum_k T_{k/b} f \cdot H_k.$$

Now let $f \in L^2(\mathbf{R})$ be arbitrary, and let $\{f_j\}$ be a sequence of continuous, compactly supported functions converging in $L^2(\mathbf{R})$ to f. Since $\sum \|H_k\|_\infty < \infty$, we know that $\sum_k T_{k/b} f \cdot H_k$ converges in $L^2(\mathbf{R})$. Moreover, for each j we have $S_0^{-1} f_j = \sum_k T_{k/b} f_j \cdot H_k$. The result now follows by taking the limit as $j \to \infty$. □

4.3. The Zak transform. The Zak transform has been used explicitly and implicitly in numerous mathematical and applied articles. J. Zak studied the operator beginning in the 1960s, in connection with solid state physics [3], [9], [10], [57], and called it the *kq*-representation. It has also been called the Weil–Brezin map [1], [2], and some have claimed its history extends as far back as Gauss [53]. Some of the most important new results have been made by A. J. E. M. Janssen [8], [38]. We recommend the article [39] for a survey of the properties of the Zak transform.

DEFINITION 4.3.1. The *Zak transform* of a function f is (formally)

$$Zf(t,\omega) \;=\; a^{1/2} \sum_{k\in\mathbf{Z}} f(ta+ka)\, e^{2\pi ik\omega}$$

for $(t,\omega) \in \mathbf{R}\times\hat{\mathbf{R}}$, and where $a>0$ is a fixed parameter.

Zf is defined pointwise at least for continuous functions with compact support. We show in Theorem 4.3.2 that the series defining Zf converges in an L^2-norm sense for $f \in L^2(\mathbf{R})$. Formally, we have the *quasiperiodicity relations*

$$Zf(t+1,\omega) \;=\; e^{-2\pi i\omega} Zf(t,\omega) \qquad \text{and} \qquad Zf(t,\omega+1) \;=\; Zf(t,\omega).$$

Therefore, the values of $Zf(t,\omega)$ for $(t,\omega)\in\mathbf{R}\times\hat{\mathbf{R}}$ are completely determined by its values in the unit square $Q=[0,1)\times[0,1)$. We define $L^2(Q) = \big\{F\colon Q\to\mathbf{C} : \|F\|_2 = \big(\int_0^1\int_0^1 |F(t,\omega)|^2\,d\omega\,dt\big)^{1/2} < \infty\big\}$. This is a Hilbert space with inner product $\langle F,G\rangle = \int_0^1\int_0^1 F(t,\omega)\,\overline{G(t,\omega)}\,d\omega\,dt$. An orthonormal basis for $L^2(Q)$ is given by the set of two-dimensional exponentials, $\{E_{(m,n)}\}_{m,n\in\mathbf{Z}}$.

THEOREM 4.3.2 [38]. *The Zak transform is a unitary map of $L^2(\mathbf{R})$ onto $L^2(Q)$.*

Proof. For simplicity, take $a=1$. Given $f\in L^2(\mathbf{R})$ and $k\in\mathbf{Z}$ set $F_k(t,\omega) = f(t+k)\,e^{2\pi ik\omega}$. Since

$$\|F_k\|_2^2 \;=\; \int_0^1\int_0^1 |f(t+k)\,e^{2\pi ik\omega}|^2\,d\omega\,dt \;=\; \int_0^1 |f(t+k)|^2\,dt \;<\; \infty,$$

we see that $F_k \in L^2(Q)$. Moreover, these functions are orthogonal: if $j\neq k$ then

$$\langle F_j,F_k\rangle \;=\; \int_0^1 f(t+j)\,\overline{f(t+k)} \left(\int_0^1 e^{2\pi i(j-k)\omega}\,d\omega\right) dt \;=\; 0.$$

This orthogonality allows us to write $\|\sum F_k\|_2^2 = \sum \|F_k\|_2^2 = \|f\|_2^2$. Thus $Zf = \sum F_k$ is well-defined, linear, and norm-preserving. For $m,n\in\mathbf{Z}$ define $\varphi_{mn}(x) = T_nE_m\chi_{[0,1)}(x) = e^{2\pi i(x-n)m}\chi_{[n,n+1)}(x)$. Certainly $\varphi_{mn}\in L^2(\mathbf{R})$, and in fact $\{\varphi_{mn}\}$ forms an orthonormal basis for $L^2(\mathbf{R})$. We compute $Z\varphi_{mn}(t,\omega) = \sum e^{2\pi i(t+k-n)m}\chi_{[n,n+1)}(t+k)e^{2\pi ik\omega}$. The only nonzero term in this series is $k=n$, so $Z\varphi_{mn}(t,\omega) = e^{2\pi itm}e^{2\pi in\omega} = E_{(m,n)}(t,\omega)$. Thus Z maps the orthonormal basis $\{\varphi_{mn}\}$ onto the orthonormal basis $\{E_{(m,n)}\}$, which shows that Z is surjective and completes the proof. ☐

The unitary nature of the Zak transform allows us to translate conditions on frames for $L^2(\mathbf{R})$ into conditions in $L^2(Q)$, where things are frequently easier to deal with. Precisely, a set of functions $\{f_i\}$ is complete/a frame/an exact frame/an orthonormal basis for $L^2(\mathbf{R})$ if and only if the same is true for $\{Zf_i\}$ in $L^2(Q)$. This property is especially useful for analyzing W–H frames when $ab=1$. In this case, an easy computation gives $Z(T_{na}E_{mb}g) = E_{(m,n)}Zg$, which places great restrictions on the form Zg can take if (g,a,b) is to generate a frame.

THEOREM 4.3.3 [14], [38], [36]. *Given $a,b>0$ with $ab=1$ and $g\in L^2(\mathbf{R})$.*

(1) *$\{T_{na}E_{mb}g\}$ is complete in $L^2(\mathbf{R})$ if and only if $Zg\neq 0$ a.e.*

(2) *The following statements are equivalent:*

  (a) *$0<A\le |Zg|^2\le B<\infty$ a.e.*

  (b) *(g,a,b) generates a frame for $L^2(\mathbf{R})$ with frame bounds A,B.*

  (c) *(g,a,b) generates an exact frame for $L^2(\mathbf{R})$ with frame bounds A,B.*

(3) *(g,a,b) generates an orthonormal basis for $L^2(\mathbf{R})$ if and only if $|Zg|=1$ a.e.*

Proof. We prove only (2) since the others are similar.

(2a) $\Rightarrow$ (2b). Assume (2a) holds. It suffices to show that $\{E_{(m,n)}Zg\}$ is a frame for $L^2(Q)$. Given any $F \in L^2(Q)$ we have $F \cdot \overline{Zg} \in L^2(Q)$ since Zg is bounded. But $\{E_{(m,n)}\}$ is an orthonormal basis for $L^2(Q)$, so

$$\sum_{m,n} |\langle F, E_{(m,n)}Zg\rangle|^2 \;=\; \sum_{m,n} |\langle F \cdot \overline{Zg}, E_{(m,n)}\rangle|^2 \;=\; \|F \cdot \overline{Zg}\|_2^2.$$

But $A\|F\|_2^2 \le \|F \cdot \overline{Zg}\|_2^2 \le B\|F\|_2^2$, so (2b) holds.

(2b) $\Rightarrow$ (2a). Assume (2b) holds, so $\{E_{(m,n)}Zg\}$ is a frame for $L^2(Q)$, i.e., $A\|F\|_2^2 \le \sum |\langle F, E_{(m,n)}Zg\rangle|^2 \le B\|F\|_2^2$ for $F \in L^2(Q)$. But $\sum |\langle F, E_{(m,n)}Zg\rangle|^2 = \|F \cdot \overline{Zg}\|_2^2$ as before, so $A\|F\|_2^2 \le \|F \cdot \overline{Zg}\|_2^2 \le B\|F\|_2^2$ for all $F \in L^2(Q)$, which implies easily that $A \le |Zg|^2 \le B$ a.e.

(2b) $\Rightarrow$ (2c). Assume (2b) holds, so $\{E_{(m,n)}Zg\}$ is a frame for $L^2(Q)$. By (2a) $\Leftrightarrow$ (2b) we know Zg is bounded above and below, so the mapping $UF = F \cdot Zg$ is a topological isomorphism of $L^2(Q)$ onto itself. Since $\{E_{(m,n)}Zg\}$ is obtained from the orthonormal basis $\{E_{(m,n)}\}$ by the topological isomorphism U, we have from §2.2 that $\{E_{(m,n)}Zg\}$ is a bounded unconditional basis, hence an exact frame.

Alternatively, we can prove the exactness directly. To do this, we need only show that $\{E_{(m,n)}Zg\}_{(m,n)\ne(k,l)}$ is incomplete for every k, l. Since the calculations are all the same, we assume $k = l = 0$. Since Zg is bounded both above and below we can define $F = 1/\overline{Zg} \in L^2(Q)$. But for $(m,n) \ne (0,0)$ we have $\langle F, E_{(m,n)}Zg\rangle = \langle F \cdot \overline{Zg}, E_{(m,n)}\rangle = \langle 1, E_{(m,n)}\rangle = 0$. Thus F is orthogonal to every $E_{(m,n)}Zg$ with $(m,n) \ne (0,0)$, but we have $F \ne 0$. Hence $\{E_{(m,n)}Zg\}_{(m,n)\ne(0,0)}$ is incomplete. □

The preceding results give us hope that we can find good W–H orthonormal bases for $L^2(\mathbf{R})$, since all we need do is find some nice function whose Zak transform has absolute value 1. It is natural to consider functions whose Zak transform is continuous first, but the quasiperiodicity introduces interesting complications. For example, the function whose Zak transform is 1 on the unit cube Q does not have a continuous Zak transform since by quasiperiodicity it possesses jump discontinuities on the lines $t = k$ for integers k.

THEOREM 4.3.4 [38]. *Let $f \in L^2(\mathbf{R})$ be such that Zf is continuous on $\mathbf{R} \times \hat{\mathbf{R}}$. Then Zf has a zero.*

Proof. Suppose $F = Zf$ was continuous but nonvanishing. Then by [50, Lemma VI.1.7] there is a continuous real-valued φ such that $F(t,\omega) = |F(t,\omega)|\, e^{i\varphi(t,\omega)}$ for $(t,\omega) \in [0,1] \times [0,1]$. Now, $F(t,1) = F(t,0)$ and $F(1,\omega) = e^{-2\pi i\omega}F(0,\omega)$, so we must have $e^{i\varphi(t,1)} = e^{i\varphi(t,0)}$ and $e^{i\varphi(1,\omega)} = e^{i(\varphi(0,\omega)-2\pi\omega)}$. Therefore, for each t and ω there are integers l_t and k_ω such that $\varphi(t,1) = \varphi(t,0) + 2\pi l_t$ and $\varphi(1,\omega) = \varphi(0,\omega) - 2\pi\omega + 2\pi k_\omega$. But the functions $\varphi(t,1) - \varphi(t,0)$ and $\varphi(1,\omega) - \varphi(0,\omega) + 2\pi\omega$ are *continuous* functions of t and ω, respectively, so all the integers l_t must be equal to one and the same integer l, and all the k_ω must equal the single integer k. Therefore,

$$\begin{aligned} 0 \;&=\; \varphi(0,0) - \varphi(1,0) + \varphi(1,0) - \varphi(1,1) \\ &\qquad + \varphi(1,1) - \varphi(0,1) + \varphi(0,1) - \varphi(0,0) \\ &=\; -2\pi k - 2\pi l - 2\pi + 2\pi k + 2\pi l \\ &=\; -2\pi, \end{aligned}$$

a contradiction. □

Example 4.3.5 ([3], [15]). The Zak transform of the Gaussian function $g(x) = e^{-rx^2}$ is continuous and has a single zero in Q. Therefore the Weyl–Heisenberg states

$\{T_{na}E_{mb}g\}$ for $ab = 1$ are complete in $L^2(\mathbf{R})$ but do not form a frame. However, it can be shown that g does generate a frame for other values of ab (cf. Example 4.4.4).

Example 4.3.6. Let $a = b = 1$ and $g = \chi_{[0,1)}$. The Zak transform of g is $Zg(t,\omega) \equiv 1$ for $(t,\omega) \in Q$. Therefore g generates a W–H orthonormal basis for $L^2(\mathbf{R})$.

While no function whose Zak transform is continuous can generate a W–H frame when $ab = 1$, we have not yet shown that this excludes "nice" functions from being mother wavelets. The following theorem, due to R. Balian [4] (and independently to F. Low [43]), shows that, in fact, if (g, a, b) generates a frame when $ab = 1$ then either g is not smooth or does not decay very fast. An elegant proof by G. Battle for the orthonormal basis case, based on the Heisenberg uncertainty principle, is in [5]. See also the results by Daubechies and Janssen in [16] and by Benedetto, Heil, and Walnut in [7].

THEOREM 4.3.7. *Given $g \in L^2(\mathbf{R})$ and $a, b > 0$ with $ab = 1$. If (g, a, b) generates a W–H frame, then either $xg(x) \notin L^2(\mathbf{R})$ or $\gamma\hat{g}(\gamma) \notin L^2(\hat{\mathbf{R}})$.*

In summary, W–H frames with $ab = 1$ are bases for $L^2(\mathbf{R})$ but have unpleasant mother wavelets. It can be shown that all W–H frames with $ab < 1$ are inexact, and that it is impossible to construct a W–H frame when $ab > 1$ (cf. [14]). Thus $ab = 1$ is a "critical value" for W–H frames. The Zak transform is especially suited for the $ab = 1$ case, but can be used to prove some of these other results when ab is rational, while ab irrational have generally required other methods (cf. [52]).

4.4. Examples.

Example 4.4.1. The function $g = \chi_{[0,1)}$ generates a W–H frame for all $0 < a, b \leq 1$. This follows immediately from Theorem 4.1.2 and the fact that $\sum |g(x - na)|^2$ is a step function for all values of $0 < a, b \leq 1$.

Example 4.4.2. Let $g = \chi_{[0,1)}$. Then by Example 4.4.1, $\hat{g}$ must generate a frame for $L^2(\mathbf{R})$ for all $0 < a, b \leq 1$. Since $\hat{g}(\gamma) = (\sin \pi\gamma)/(\pi\gamma)$, this is an example of a noncompactly supported mother wavelet.

Example 4.4.3. Define $g(x) = (\cos \pi x) \cdot \chi_{[-1/2,1/2]}(x)$. Then g generates a frame for all $0 < a < 1$ and $0 < b \leq 1$, which is tight for $a = \frac{1}{2}$ and $b = 1$. It is easy to generalize this example to obtain tight frames for any $a, b > 0$ such that $ab < 1$, and in fact to do so with mother wavelets of arbitrary smoothness. For example, let $\varphi \in C^\infty(\mathbf{R})$ be any real-valued function with $\mathrm{supp}(\varphi)$ contained in an interval I of length $1/b$ and which is nonzero in the interior of I. By Corollary 4.1.3, φ then generates a frame for $L^2(\mathbf{R})$, and $G(x) = \sum |\varphi(x - na)|^2$ is bounded above and below. If we define $g(x) = b^{1/2}\, \varphi(x)\, G(x)^{-1/2}$ then $\mathrm{supp}(g) \subset I$ and $\sum |g(x - na)|^2 \equiv b$, so (g, a, b) generates a tight frame with bounds $A = B = 1$. Contrast this situation to the case $ab = 1$, where it is impossible to construct mother wavelets which are both smooth and compactly supported (Theorem 4.3.7).

Example 4.4.4. Let $g(x) = 1/(1+x^2)$. Since $g \in W(L^\infty, L^1)$ and is continuous and positive, $\sum |g(x-na)|^2$ is bounded above and below for all $a > 0$. Given a, we therefore have by Theorem 4.1.8 that (g, a, b) generates a W–H frame for all sufficiently small b. The same remarks apply to the Gaussian function $g(x) = e^{-rx^2}$. In [5], Daubechies estimates numerically the largest valid value of b for certain values of a. In particular, the Gaussian does not generate a frame for $b = 1/a$ (cf. Example 4.3.5).

5. Affine Frames. The expansion of functions by means of wavelets has been more extensively studied than its Weyl–Heisenberg counterpart. The first such expansions were obtained by Frazier and Jawerth in [25], where they decompose elements of

Besov spaces (an example of which is $L^2(\mathbf{R})$) by means of dilations and translations of functions with compactly supported Fourier transforms. These ideas led eventually to the decomposition theory of Feichtinger and Gröchenig.

In parallel with these developments, Daubechies, Grossmann, and Meyer combined the theory of the continuous wavelet transform (§3) with the theory of frames (§2), obtaining in [15] affine frames for $L^2(\mathbf{R})$. These ideas were developed further in [14]. Later, Daubechies and Meyer discovered wavelet orthonormal bases for $L^2(\mathbf{R})$ in which the mother wavelet has certain types of smoothness properties (see [12], [48]). These results were given an elegant interpretation by Mallat in the context of image processing. Together with Meyer, he developed the concept of *multiresolution analysis.* This idea has been exploited very sucessfully by Mallat to develop image processing algorithms, which are proving to have applications to edge detection problems. To do this, it is necessary to define a multiresolution analysis for $L^2(\mathbf{R}^2)$. This is not difficult to do and for details the reader may consult [12], [44], [48].

The wavelet orthonormal basis of Meyer is actually an unconditional basis for virtually all of the functional spaces used in modern analysis. Consequently, wavelet techniques have found applications in operator theory, which accounts in part for their current popularity. In addition, wavelet techniques have been applied to problems in numerical analysis, signal processing, and seismic analysis.

The results in this section are adapted mostly from [14] and [15].

In §5.1 we show the existence of affine coherent state frames for $L^2(\mathbf{R})$, $H^2_+(\mathbf{R})$ and $H^2_-(\mathbf{R})$. We give conditions on the lattice and the mother wavelet which guarantee the existence of a frame for each of these spaces.

In §5.2 we prove results about the affine frame operator.

In §5.3 we give some examples of affine frames and in §5.4 we discuss the affine orthonormal basis discovered by Meyer which is generated by a smooth mother wavelet with compactly supported Fourier transform. Multiresolution analysis is also outlined in this section.

5.1. Existence of affine frames.

Definition 5.1.1. Given $g \in H^2_+(\mathbf{R})$, $a > 1$, and $b > 0$, we say that (g, a, b) *generates an affine frame* for $H^2_+(\mathbf{R})$ if $\{D_{a^n} T_{mb} g\}_{m,n \in \mathbf{Z}}$ is a frame for $H^2_+(\mathbf{R})$. The function g is referred to as the *mother wavelet*, *analyzing wavelet*, or *fiducial vector.* The numbers a, b are the *frame parameters*, a being the *dilation parameter*, and b the *shift parameter.*

We make similar definitions for $H^2_-(\mathbf{R})$ and $L^2(\mathbf{R})$, and remark that it is sometimes necessary to take two mother wavelets in order to form a frame for $L^2(\mathbf{R})$ (cf. Theorem 5.1.3).

Theorem 5.1.2 [15]. *Let $g \in L^2(\mathbf{R})$ be such that* $\operatorname{supp}(\hat{g}) \subset [l, L]$*, where $0 \leq l < L < \infty$, and let $a > 1$ and $b > 0$ be such that:*

(1) *there exist A, B such that $0 < A \leq \sum_n |\hat{g}(a^n\gamma)|^2 \leq B < \infty$ for a.e. $\gamma \geq 0$,*
(2) $(L - l) \leq 1/b$.

Then for all $f \in L^2(\mathbf{R})$,

$$b^{-1} A \int_0^\infty |\hat{f}(\gamma)|^2 \, d\gamma \;\leq\; \sum_{m,n} |\langle f, D_{a^n} T_{mb} g \rangle|^2 \;\leq\; b^{-1} B \int_0^\infty |\hat{f}(\gamma)|^2 \, d\gamma.$$

In particular, $\{D_{a^n} T_{mb} g\}$ is a frame for $H^2_+(\mathbf{R})$ with bounds $b^{-1}A, b^{-1}B$.

Proof. Given $n \in \mathbf{Z}$, the function $D_{a^n}\hat{f} \cdot \bar{\hat{g}}$ is supported in $I = [l, l + 1/b]$, an interval of length $1/b$. Now, $\hat{g}$ is bounded, so $D_{a^n}\hat{f} \cdot \bar{\hat{g}} \in L^2(I)$. But $\{b^{1/2}E_{mb}\}_{m\in\mathbf{Z}}$ is an orthonormal basis for $L^2(I)$, so $\sum_m |\langle D_{a^n}\hat{f} \cdot \bar{\hat{g}}, E_{mb}\rangle|^2 = b^{-1}\int |D_{a^n}\hat{f}(\gamma) \cdot \overline{\hat{g}(\gamma)}|^2\, d\gamma$. Noting that $\operatorname{supp}(\hat{g}) \subset I \subset [0,\infty)$, and making a change of variable, we have

$$\int_{\hat{\mathbf{R}}} |D_{a^n}\hat{f}(\gamma) \cdot \overline{\hat{g}(\gamma)}|^2\, d\gamma \;=\; \int_0^\infty a^{-n}|\hat{f}(a^{-n}\gamma)|^2\, |\hat{g}(\gamma)|^2\, d\gamma \;=\; \int_0^\infty |\hat{f}(\gamma)|^2\, |\hat{g}(a^n\gamma)|^2\, d\gamma.$$

Finally, $\sum_{m,n} |\langle f, D_{a^n}T_{mb}g\rangle|^2 = \sum_{m,n} |\langle \hat{f}, D_{a^{-n}}E_{-mb}\hat{g}\rangle|^2 = \sum_{m,n} |\langle D_{a^n}\hat{f} \cdot \bar{\hat{g}}, E_{mb}\rangle|^2$, so the result follows. $\square$

A similar theorem holds for $H^2_-(\mathbf{R})$. By taking a mother wavelet from $H^2_+(\mathbf{R})$ and one from $H^2_-(\mathbf{R})$ we can obtain a frame for $L^2(\mathbf{R})$, as the following theorem shows.

THEOREM 5.1.3 [15]. *Let $g_1, g_2 \in L^2(\mathbf{R})$ be such that* $\operatorname{supp}(\hat{g}_1) \subset [-L, -l]$ *and* $\operatorname{supp}(\hat{g}_2) \subset [l, L]$, *where $0 \le l < L < \infty$, and let $a > 1$, $b > 0$ be such that:*

(1) *there exist A, B such that*

$$0 < A \le \sum_n |\hat{g}_1(a^n\gamma)|^2 \le B < \infty \qquad \textit{for a.e. } \gamma \le 0,$$

$$0 < A \le \sum_n |\hat{g}_2(a^n\gamma)|^2 \le B < \infty \qquad \textit{for a.e. } \gamma \ge 0,$$

(2) $(L - l) \le 1/b$.

Then $\{D_{a^n}T_{mb}g_1, D_{a^n}T_{mb}g_2\}$ is a frame for $L^2(\mathbf{R})$ with bounds $b^{-1}A, b^{-1}B$.

The following corollary supplies examples of functions that satisfy the hypotheses of Theorem 5.1.3.

COROLLARY 5.1.4 [15]. *Let $g_1, g_2 \in L^2(\mathbf{R})$ be such that:*

(1) $\operatorname{supp}(\hat{g}_1) \subset [-L, -l]$ *and* $\operatorname{supp}(\hat{g}_2) \subset [l, L]$, *where $0 < l < L < \infty$,*

(2) *$\hat{g}_1$ and $\hat{g}_2$ are continuous and do not vanish on $(-L, -l)$ and (l, L), respectively.*

Then $\{D_{a^n}T_{mb}g_1, D_{a^n}T_{mb}g_2\}$ is a frame for $L^2(\mathbf{R})$ for all $1 < a < L/l$ and $0 < b \le 1/(L - l)$.

The proof is very similar to that of Corollary 4.1.3.

We would like to obtain an affine frame for $L^2(\mathbf{R})$ that requires only one mother wavelet, instead of two as in Theorem 5.1.3. We may hope that if g_1, g_2 satisfy the hypotheses of Theorem 5.1.3 then $(g_1 + g_2, a, b)$ generates an affine frame for $L^2(\mathbf{R})$, but this is not true in general. For example, take $a = 2$, $b = 1$, $\hat{g}_1 = \chi_{(-2,-1]}$, and $\hat{g}_2 = \chi_{[1,2)}$. Then $\{D_{2^n}T_m g_1, D_{2^n}T_m g_2\}$ is an orthonormal basis for $L^2(\mathbf{R})$, but $\{D_{2^n}T_m(g_1 + g_2)\}$ is not complete since $\hat{f} = \chi_{(-2,-1]} - \chi_{[1,2)}$ is orthogonal to every $D_{2^n}T_m(g_1 + g_2)$. However, we can prove that if we take b small enough then $g_1 + g_2$ will generate a frame:

THEOREM 5.1.5. *Let $g_1, g_2 \in L^2(\mathbf{R})$ be as in Theorem 5.1.3. If $2L < 1/b$ then $(g_1 + g_2, a, b)$ generates an affine frame for $L^2(\mathbf{R})$.*

Proof. Let $g = g_1 + g_2$. Then $\operatorname{supp}(D_{a^n}\hat{f} \cdot \bar{\hat{g}}) \subset [-L, L] \subset [-\frac{1}{2b}, \frac{1}{2b}]$, so $D_{a^n}\hat{f} \cdot \bar{\hat{g}} \in L^2[-\frac{1}{2b}, \frac{1}{2b}]$. But $\{b^{1/2}E_{mb}\}$ is an orthonormal basis for $L^2[-\frac{1}{2b}, \frac{1}{2b}]$, so

$$\begin{aligned}
&\sum_{m,n} |\langle f, D_{a^n}T_{mb}g\rangle|^2 \\
&\quad = \sum_{m,n} |\langle D_{a^n}\hat{f} \cdot \bar{\hat{g}}, E_{-mb}\rangle|^2
\end{aligned}$$

$$= b^{-1}\sum_n \int_{\mathbf{R}} a^{-n}|\hat{f}(a^{-n}\gamma)|^2\,|\hat{g}(\gamma)|^2\,d\gamma$$
$$= \int_{-\infty}^{0} |\hat{f}(\gamma)|^2\cdot b^{-1}\sum_n |\hat{g}_1(a^n\gamma)|^2\,d\gamma \quad + \quad \int_0^{\infty} |\hat{f}(\gamma)|^2\cdot b^{-1}\sum_n |\hat{g}_2(a^n\gamma)|^2\,d\gamma,$$

from which the result follows. □

In analogy with Theorem 4.1.5, the following theorem gives a condition on g whose Fourier transforms are not necessarily compactly supported so that (g, a, b) generates an affine frame for $L^2(\mathbf{R})$ for some frame parameters.

THEOREM 5.1.6 [14]. *Let $g \in L^2(\mathbf{R})$ and $a > 1$ be such that:*

(1) *there exist A, B such that $0 < A \le \sum_n |\hat{g}(a^n\gamma)|^2 \le B < \infty$ for a.e. $\gamma \in \hat{\mathbf{R}}$,*
(2) $\lim_{b\to 0}\sum_{k\neq 0} \beta(k/b)^{1/2}\beta(-k/b)^{1/2} = 0$, *where*

$$\beta(s) = \operatorname*{ess\,sup}_{|\gamma|\in[1,a]} \sum_n |\hat{g}(a^n\gamma)\,\hat{g}(a^n\gamma - s)|.$$

Then there exists $b_0 > 0$ such that (g, a, b) generates an affine frame for $L^2(\mathbf{R})$ for each $0 < b < b_0$.

Proof. For fixed n consider the a^n/b-periodic function defined by

$$F_n(\gamma) \;=\; \sum_k \hat{f}(\gamma - a^n k/b)\,\overline{\hat{g}(a^{-n}\gamma - k/b)}.$$

Since f and g are in $L^2(\mathbf{R})$ we have $F_n \in L^1[0, a^n/b]$ and

$$\int_{\hat{\mathbf{R}}} \hat{f}(\gamma)\,\overline{\hat{g}(a^{-n}\gamma)}\,e^{2\pi i m a^{-n} b\gamma}\,d\gamma \;=\; \int_0^{a^n/b} F_n(\gamma)e^{2\pi i m a^{-n} b\gamma}\,d\gamma.$$

Now, $\{a^{-n/2}b^{1/2}E_{ma^{-n}b}\}_{m\in\mathbf{Z}}$ is an orthonormal basis for $L^2[0, a^n/b]$, so

$$\sum_m \left|\int_0^{a^n/b} F_n(\gamma)\,e^{2\pi i m a^{-n} b\gamma}\,d\gamma\right|^2 \;=\; \frac{a^n}{b}\int_0^{a^n/b} |F_n(\gamma)|^2\,d\gamma.$$

Therefore,

$$\sum_n\sum_m |\langle f, D_{a^n}T_{mb}g\rangle|^2$$
$$= \sum_n\sum_m |\langle \hat{f}, D_{a^{-n}}E_{-mb}\hat{g}\rangle|^2$$
$$= \sum_n\sum_m |\langle \hat{f}, E_{-ma^n b}D_{a^{-n}}\hat{g}\rangle|^2$$
$$= \sum_n a^{-n}\sum_m \left|\int_{\hat{\mathbf{R}}} \hat{f}(\gamma)\,\overline{\hat{g}(a^{-n}\gamma)}\,e^{2\pi i m a^{-n} b\gamma}\,d\gamma\right|^2$$
$$= \sum_n a^{-n}\frac{a^n}{b}\int_0^{a^n/b} \left|\sum_k \hat{f}(\gamma - a^n k/b)\,\overline{\hat{g}(a^{-n}\gamma - k/b)}\right|^2 d\gamma$$
$$= b^{-1}\sum_n \int_0^{a^n/b} \sum_l \hat{f}(\gamma - a^n l/b)\,\overline{\hat{g}(a^{-n}\gamma - l/b)}$$

$$
\begin{aligned}
&\qquad\qquad \cdot \sum_k \overline{\hat{f}(\gamma - a^n k/b)}\, \hat{g}(a^{-n}\gamma - k/b)\, d\gamma \\
&= b^{-1} \sum_n \sum_l \int_0^{a^n/b} \hat{f}(\gamma - a^n l/b)\, \overline{\hat{g}(a^{-n}\gamma - l/b)} \\
&\qquad\qquad \cdot \sum_k \overline{\hat{f}(\gamma - a^n k/b)}\, \hat{g}(a^{-n}\gamma - k/b)\, d\gamma \\
&= b^{-1} \sum_n \int_{\hat{\mathbf{R}}} \hat{f}(\gamma)\, \overline{\hat{g}(a^{-n}\gamma)} \cdot \sum_k \overline{\hat{f}(\gamma - a^n k/b)}\, \hat{g}(a^{-n}\gamma - k/b)\, d\gamma \\
&= b^{-1} \sum_k \sum_n \int_{\hat{\mathbf{R}}} \hat{f}(\gamma)\, \overline{\hat{f}(\gamma - a^n k/b)}\, \overline{\hat{g}(a^{-n}\gamma)}\, \hat{g}(a^{-n}\gamma - k/b)\, d\gamma \\
&= b^{-1} \int_{\hat{\mathbf{R}}} |\hat{f}(\gamma)|^2 \cdot \sum_n |\hat{g}(a^n\gamma)|^2\, d\gamma \\
&\qquad + b^{-1} \sum_{k\neq 0} \sum_n \int_{\hat{\mathbf{R}}} \hat{f}(\gamma)\, \overline{\hat{f}(\gamma - a^n k/b)}\, \overline{\hat{g}(a^{-n}\gamma)}\, \hat{g}(a^{-n}\gamma - k/b)\, d\gamma \\
&= (*).
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality twice, we find that

$$
\begin{aligned}
(*) \;&\le\; b^{-1} \int_{\hat{\mathbf{R}}} |\hat{f}(\gamma)|^2 \sum_n |\hat{g}(a^n\gamma)|^2\, d\gamma \\
&\qquad + b^{-1} \sum_{k\neq 0} \sum_n \int_{\hat{\mathbf{R}}} |\hat{f}(\gamma)| \Big(|\hat{g}(a^{-n}\gamma)|\, |\hat{g}(a^{-n}\gamma - k/b)| \Big)^{1/2} \\
&\qquad\qquad \cdot |\hat{f}(\gamma - a^n k/b)| \Big(|\hat{g}(a^{-n}\gamma)|\, |\hat{g}(a^{-n}\gamma - k/b)| \Big)^{1/2} d\gamma \\
&\le\; b^{-1} B\, \|\hat{f}\|_2^2 \;+\; b^{-1} \sum_{k\neq 0} \sum_n \left(\int_{\hat{\mathbf{R}}} |\hat{f}(\gamma)|^2\, |\hat{g}(a^{-n}\gamma)|\, |\hat{g}(a^{-n}\gamma - k/b)|\, d\gamma \right)^{1/2} \\
&\qquad\qquad \cdot \left(\int_{\hat{\mathbf{R}}} |\hat{f}(\gamma)|^2\, |\hat{g}(a^{-n}\gamma)|\, |\hat{g}(a^{-n}\gamma + k/b)|\, d\gamma \right)^{1/2} \\
&\le\; \left(b^{-1} B \;+\; b^{-1} \sum_{k\neq 0} \beta(k/b)^{1/2} \beta(-k/b)^{1/2} \right) \|\hat{f}\|_2^2.
\end{aligned}
$$

A similar calculation shows

$$
(*) \;\ge\; \left(b^{-1} A \;-\; b^{-1} \sum_{k\neq 0} \beta(k/b)^{1/2} \beta(-k/b)^{1/2} \right) \|\hat{f}\|_2^2,
$$

so the conclusion follows from condition (2). □

When $a = 2$ an improved version of Theorem 5.1.6 (due to P. Tchamitchian) holds in which the function β is replaced by a new function β_1 which takes into account possible cancellations which may arise from the phase portion of $\hat{g}$ and which are lost in the function β. This theorem is useful in analyzing the Meyer wavelet (§5.4). The proof can be found in [14].

THEOREM 5.1.7. *Let $g \in L^2(\mathbf{R})$ and $a = 2$ satisfy the hypotheses of Theorem* 5.1.6, *and let $b > 0$. If $\{D_{2^n}T_{mb}g\}$ is a frame for $L^2(\mathbf{R})$ with frame bounds A', B' then*

$$A' \;\geq\; b^{-1}\biggl(A \;-\; 2\sum_{l=0}^{\infty}\beta_1\left(\tfrac{2l+1}{b}\right)^{1/2}\beta_1\left(-\tfrac{2l+1}{b}\right)^{1/2}\biggr)$$

and

$$B' \;\leq\; b^{-1}\biggl(B \;+\; 2\sum_{l=0}^{\infty}\beta_1\bigl(\tfrac{2l+1}{b}\bigr)^{1/2}\beta_1\bigl(-\tfrac{2l+1}{b}\bigr)^{1/2}\biggr),$$

where

$$\beta_1(s) \;=\; \operatorname*{ess\,sup}_{\gamma\in\hat{\mathbf{R}}}\sum_m\Bigl|\sum_{j\geq 0}\hat{g}(2^{m+j}\gamma)\;\overline{\hat{g}\bigl(2^j(2^m\gamma+s)\bigr)}\Bigr|$$

and A, B are as in Theorem 5.1.6.

5.2. The affine frame operator. We will look at the frame operators associated with various affine frames. As usual, we let S denote the frame operator, so that, for example, the frame operator associated with the frame of Theorem 5.1.2 is $Sf = \sum\langle f, D_{a^n}T_{mb}g\rangle D_{a^n}T_{mb}g$ for $f \in H^2_+(\mathbf{R})$.

THEOREM 5.2.1 [14].

(1) *If g, a, b satisfy the hypotheses of Theorem* 5.1.2 *then $Sf = (\hat{f}\cdot H)^\vee$ and $S^{-1}f = (\hat{f}/H)^\vee$ for $f \in H^2_+(\mathbf{R})$, where*

$$H(\gamma) \;=\; \begin{cases} 0, & \gamma < 0,\\ b^{-1}\sum|\hat{g}(a^n\gamma)|^2, & \gamma > 0.\end{cases}$$

(2) *If g_1, g_2, a, b satisfy the hypotheses of Theorem* 5.1.3 *then $Sf = (\hat{f}\cdot H)^\vee$ and $S^{-1}f = (\hat{f}/H)^\vee$ for $f \in L^2(\mathbf{R})$, where*

$$H(\gamma) \;=\; \begin{cases} b^{-1}\sum|\hat{g}_1(a^n\gamma)|^2, & \gamma < 0,\\ b^{-1}\sum|\hat{g}_2(a^n\gamma)|^2, & \gamma > 0.\end{cases}$$

(3) *If g_1, g_2, a, b satisfy the hypotheses of Theorem* 5.1.5 *then $Sf = (\hat{f}\cdot H)^\vee$ and $S^{-1}f = (\hat{f}/H)^\vee$ for $f \in L^2(\mathbf{R})$, where*

$$H(\gamma) \;=\; b^{-1}\sum_n|(g_1+g_2)^\wedge(a^n\gamma)|^2.$$

Proof. We prove only (1) since the others are similar. As in the proof of Theorem 5.1.2, we have $D_{a^n}\hat{f}\cdot\bar{\hat{g}} \in L^2(I)$, where I is an interval of length 1/b. Since $\{b^{1/2}E_{mb}\}_{m\in\mathbf{Z}}$ is an orthonormal basis for $L^2(I)$ we have $\sum_m\langle D_{a^n}\hat{f}\cdot\bar{\hat{g}}, E_{mb}\rangle E_{mb} = b^{-1}D_{a^n}\hat{f}\cdot\bar{\hat{g}}$. Therefore,

$$\begin{aligned}(Sf)^\wedge(\gamma) \;&=\; \sum_{m,n}\langle f, D_{a^n}T_{mb}g\rangle(D_{a^n}T_{mb}g)^\wedge(\gamma)\\ &=\; \sum_{m,n}\langle\hat{f}, D_{a^{-n}}E_{-mb}\hat{g}\rangle D_{a^{-n}}E_{-mb}\hat{g}(\gamma)\\ &=\; \sum_n D_{a^{-n}}\biggl(\sum_m\langle D_{a^n}\hat{f}\cdot\bar{\hat{g}}, E_{mb}\rangle E_{mb}(\gamma)\cdot\hat{g}(\gamma)\biggr)\\ &=\; \sum_n D_{a^{-n}}\left(b^{-1}D_{a^n}\hat{f}(\gamma)\cdot\overline{\hat{g}(\gamma)}\cdot\hat{g}(\gamma)\right)\end{aligned}$$

$$= b^{-1}\sum_n D_{a^{-n}}\left(a^{-n/2}\hat{f}(a^{-n}\gamma)\cdot|\hat{g}(\gamma)|^2\right)$$
$$= b^{-1}\sum_n \hat{f}(\gamma)\cdot|\hat{g}(a^n\gamma)|^2. \qquad \square$$

In §4 we showed that the dual frame of a W–H frame is another W–H frame, generated by $S^{-1}g$. This is not true in the case of affine frames, but there is some simplification of the dual frame. In particular, it is easy to check that $S(D_{a^n}f) = D_{a^n}(Sf)$, so $S^{-1}(D_{a^n}f) = D_{a^n}(S^{-1}f)$.

5.3. Examples.

Example 5.3.1 ([14]). It is easy to construct, for any $a > 1$ and $b > 0$, a function $g \in L^2(\mathbf{R})$ such that:

(1) $\operatorname{supp}(\hat{g}) \subset [l, L]$, where $0 < l \le L < \infty$ and $L - l = 1/b$,

(2) $\int |\hat{g}(\gamma)|^2/|\gamma|\, d\gamma = 1$,

(3) $\sum |\hat{g}(a^n\gamma)|^2 = \begin{cases} 1/(b\ln a), & \text{if } \gamma \ge 0, \\ 0, & \text{otherwise.}\end{cases}$

Set $g_1 = g$ and $g_2 = \bar{g}$. Since $\widehat{\bar{g}}(\gamma) = \overline{\hat{g}(-\gamma)}$ it follows from Theorem 5.1.3 that $\{D_{a^n}T_{mb}g_1, D_{a^n}T_{mb}g_2\}$ is a tight frame for $L^2(\mathbf{R})$ with bound $1/(b\ln a)$. To construct g, let $v \in C^\infty(\hat{\mathbf{R}})$ be such that $v(\gamma) = 0$ for $\gamma \le 0$, $0 \le v(\gamma) \le 1$ for $0 < \gamma < 1$, and $v(\gamma) = 1$ for $\gamma \ge 1$. Set $l = 1/(b(a^2-1))$ and $L = a^2 l$ and define

$$\hat{g}(\gamma) = (\ln a)^{-1/2}\begin{cases} 0, & \text{if } \gamma \le l \text{ or } \gamma > a^2 l, \\ \sin\frac{\pi}{2}v\left(\frac{\gamma - l}{l(a-1)}\right), & \text{if } l < \gamma \le al, \\ \cos\frac{\pi}{2}v\left(\frac{\gamma - al}{al(a-1)}\right), & \text{if } al < \gamma \le a^2 l.\end{cases}$$

Example 5.3.2. Using Theorem 2.1.6, the frame in Example 5.3.1 is easily seen to be inexact, hence not a basis. If we let $\hat{g}_1 = \chi_{(-2,-1]}$ and $\hat{g}_2 = \chi_{[1,2)}$ and take $a = 2$ and $b = 1$ then $\{D_{2^n}T_m g_1, D_{2^n}T_m g_2\}$ is an affine orthonormal basis for $L^2(\mathbf{R})$.

Example 5.3.3. A well-known affine orthonormal basis for $L^2(\mathbf{R})$ generated by a single mother wavelet is the *Haar system*. Here we take $a = 2$, $b = 1$, and $g = \chi_{[0,1/2]} - \chi_{[1/2,1]}$. The elements of this basis are not smooth. In §5.4, in contrast, we discuss the Meyer wavelet, a C^∞ function which generates an affine orthonormal basis for $L^2(\mathbf{R})$. The fact that the Haar system is an orthonormal basis can be seen directly, although it does not follow from the theorems in §5.1. It can also be demonstrated using the multiresolution analysis techniques of §5.4.

5.4. The Meyer wavelet. As we mentioned in §4.3, a W–H frame forms a basis for $L^2(\mathbf{R})$ if and only if $ab = 1$. Moreover, W–H frames for this critical value are composed of functions which are either not smooth or do not decay quickly. Y. Meyer showed that a different situation holds for the affine case when (in 1985) he exhibited a C^∞ function with compactly supported Fourier transform which generates an affine orthonormal basis for $L^2(\mathbf{R})$ [46]. We give the construction below, along with the definition of *multiresolution analysis*, developed by S. Mallat and Meyer from ideas of Mallat [44], [47]. In this section we take $a = 2$ and $b = 1$.

DEFINITION 5.4.1 [14]. The *Meyer wavelet* is the function $\psi \in L^2(\mathbf{R})$ defined by $\hat{\psi}(\gamma) = e^{i\gamma/2}\omega(|\gamma|)$, where

$$\omega(\gamma) = \begin{cases} 0, & \text{if } \gamma \le \frac{1}{3} \text{ or } \gamma \ge \frac{4}{3}, \\ \sin\frac{\pi}{2}v(3\gamma - 1), & \text{if } \frac{1}{3} \le \gamma \le \frac{2}{3}, \\ \cos\frac{\pi}{2}v(\frac{3\gamma}{2} - 1), & \text{if } \frac{2}{3} \le \gamma \le \frac{4}{3},\end{cases}$$

and $v \in C^\infty(\hat{\mathbf{R}})$ is such that $v(\gamma) = 0$ for $\gamma \le 0$, $v(\gamma) = 1$ for $\gamma \ge 1$, $0 \le v(\gamma) \le 1$ for $\gamma \in [0,1]$, and $v(\gamma) + v(1-\gamma) = 1$ for $\gamma \in [0,1]$.

With some computation, it can be verified that $\|\psi\|_2 = 1$, $\sum |\hat{\psi}(2^n\gamma)|^2 \equiv 1$, and $\beta_1(k) = 0$ for every odd $k \in \mathbf{Z}$, where β_1 is as in Theorem 5.1.7. It follows from Theorem 5.1.7 that $\{D_{2^n}T_m\psi\}$ is a tight frame for $L^2(\mathbf{R})$ with frame bounds $A = B = 1$. The frame operator S is therefore the identity, which together with the fact that $\|D_{2^n}T_m\psi\|_2 = 1$ for all m, n implies by Theorem 2.1.6 that the frame is exact, and so Corollary 2.1.7 implies that $\{D_{2^n}T_m\psi\}$ is an orthonormal basis. This fact of orthonormality, as we have presented it, seems almost miraculous, depending on fortunate cancellations in the calculations. Multiresolution analysis puts this miracle into place as part of a larger whole.

DEFINITION 5.4.4 [12]. A *multiresolution analysis* for $L^2(\mathbf{R})$ consists of

(1) Closed subspaces $V_n \subset L^2(\mathbf{R})$ for $n \in \mathbf{Z}$ satisfying
- (a) $V_n \supset V_{n+1}$,
- (b) $\cap V_n = \{0\}$,
- (c) $\cup V_n$ is dense is $L^2(\mathbf{R})$,
- (d) $V_{n+1} = D_2 V_n = \{D_2 f : f \in V_n\}$,

(2) A function $\varphi \in V_0$ such that $\{T_m\varphi\}_{m\in\mathbf{Z}}$ is an orthonormal basis for V_0.

For example, one multiresolution analysis can be obtained by setting $V_0 = \{f \in L^2(\mathbf{R}) : f$ is constant on each $[m, m+1)\}$, $V_n = D_{2^n}V_0$, and $\varphi = \chi_{[0,1)}$. Each multiresolution analysis turns out to generate an affine orthonormal basis for $L^2(\mathbf{R})$, the one just given generating the Haar system [12]. Another multiresolution analysis turns out to have the Meyer wavelet as the mother wavelet of its affine orthonormal basis.

A sketch of how multiresolution analyses generate orthonormal bases is as follows. Since V_n is contained in V_{n-1}, we can define W_n to be the orthogonal complement of V_n in V_{n-1}. One shows the existence of a function $\psi \in W_0$ such that $\{T_m\psi\}_m$ is an orthonormal basis for W_0. It is easy to see that $W_{n+1} = D_2W_n$, so $\{D_{2^n}T_m\psi\}_m$ is an orthonormal basis for W_n. Finally, $L^2(\mathbf{R}) = \bigoplus W_n$, so $\{D_{2^n}T_m\psi\}_{m,n}$ must form an orthonormal basis for $L^2(\mathbf{R})$.

Multiresolution analysis is proving to be an important tool both in pure mathematics and in signal and image-processing applications. Unfortunately, we do not have the space to explore this beautiful topic further, but recommend the article [12] for excellent analysis and applications.

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ERRATA

Note: This errata listing is not included in the published version of this paper.

Page 634, Example 1.4.4. The definition of φ is incorrect. This "hat function" should be supported on $[-1, 1]$, not $[-1/2, 1/2]$, i.e., φ should be defined as

$$\varphi(x) \;=\; \max\{1 - |x|, 0\} \;=\; \begin{cases} 0, & \text{if } x \le -1 \text{ or } x \ge 1, \\ x + 1, & \text{if } -1 < x < 0, \\ 1 - x, & \text{if } 0 \le x < 1. \end{cases}$$

With this definition, it does follow that $\int \varphi(x)\, dx = 1$ and that $\hat{\varphi}(\gamma) = (\sin^2 \pi\gamma)/(\pi\gamma)^2$.

Page 635. In the fourth paragraph of Section 2.1, the statement "Therefore every rearrangement of a frame is also a frame" is correct, but the remainder of that sentence, "and all sums involving frames actually converge unconditionally" is incorrect. The correct statement is that the sums using the *canonical* expansions determined by the frame converges unconditionally. In particular, the series $Sx = \sum \langle x, x_n \rangle\, x_n$ defining the frame operator converges unconditionally, as do the inversion formulas $x = \sum \langle x, S^{-1}x_n \rangle\, x_n = \sum \langle x, x_n \rangle\, S^{-1}x_n$. However, these expansions of x need not be unique, and there may exist other expansions $x = \sum c_n x_n$ which converge conditionally. Examples of frames for which this situation occurs were given in [36]; see also [CH] and [Hol].

ERRATA REFERENCES

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