II. FOURIER TRANSFORM ON $L^1(\mathbb{R})$

In this chapter we will discuss the Fourier transform of Lebesgue integrable functions defined on $\mathbb{R}$. To fix the notation, we denote

$$L^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} |f(t)|dt < \infty\}.$$

The norm $||\cdot||_{L^1(\mathbb{R})}$ on $L^1(\mathbb{R})$ is defined as

$$||f||_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} |f(t)|dt.$$

For convenience, we usually write $||\cdot||_1$ in stead of $||\cdot||_{L^1(\mathbb{R})}$. It can be checked that $||\cdot||_{L^1(\mathbb{R})}$ satisfies conditions in Definition 2 of last Chapter, hence it is indeed a norm.

Similarly to the case of $L^2([-\pi,\pi])$, we can define **Cauchy Sequence** in $L^1(\mathbb{R})$ and **Convergence of sequence of functions** in $L^1(\mathbb{R})$. We will let the reader to work out the details of these definition. Also similar to $L^2([-\pi,\pi])$, every Cauchy sequence in $L^1(\mathbb{R})$ converges to some function in $L^1(\mathbb{R})$, though this is a fact beyond the scope of our course.

Definition 1. *For any function $f \in L^1(\mathbb{R})$, we define*

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx.$$

For any function $f \in L^1(\mathbb{R})$, $\mathcal{F}(f)$ is a well defined function, since for any $\xi \in \mathbb{R}$,

$$\mid \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx \mid \leq \int_{-\infty}^{\infty} \mid f(x)e^{-i\xi x} \mid dx = \int_{-\infty}^{\infty} \mid f(x) \mid dx.$$

The function $\mathcal{F}(f)$ is called the Fourier transform of f. For convenience, we often write $\hat{f}$ is stead of $\mathcal{F}(f)$.

Theorem 1. *Let $f, g \in L^1(\mathbb{R})$. Then*

(a) $\widehat{(f+g)}(\xi) = \hat{f}(\xi) + \hat{g}(\xi)$.

(b) For any $c \in \mathbb{C}$, $\widehat{(cf)}(\xi) = c(\hat{f})(\xi)$.

(c) Let $\overline{f}$ be the complex conjugate of f, then $\widehat{\overline{f}}(\xi) = \overline{\hat{f}(-\xi)}$.

(d) For any fixed real number y, let function f_y be defined as $f_y(x) = f(x-y)$, then $\hat{f}_y(\xi) = \hat{f}(\xi)e^{-i\xi y}$.

(e) For any fixed real number $\lambda > 0$, let function φ be defined as $\varphi(x) = \lambda f(\lambda x)$, then $\hat{\varphi}(\xi) = \hat{f}(\frac{\xi}{\lambda})$.

Remark The theorem above contains some of the most frequently used properties of Fourier transform. The proof is routine, we leave the details for the reader. Less used, is the fact that the Fourier transform of any $f \in L^1(\mathbb{R})$ is a bounded uniformly continuous function. To show this fact, we need some big theorem in the theory of Lebesque integration.

Lebesque Dominant Convergence Theorem. *Let $f, g \in L^1(\mathbb{R})$. Suppose that the sequence $\{f_n\}_{n=1}^{\infty} \subset L^1(\mathbb{R})$ satisfies the following two conditions:*

(a)$|f_n(x)| \leq g(x)$ for every $n \in \mathbb{N}$ and almost every $x \in \mathbb{R}$.

(b)For almost every $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Then $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{\infty}^{\infty} f(x)dx$.

Theorem 2. *Let $f \in L^1(\mathbb{R})$. Then*

(a) $|\hat{f}(\xi)| \leq ||f||_1$ for any $\xi \in \mathbb{R}$.

(b) $\hat{f}$ is uniformly continuous on $\mathbb{R}$.

Proof. (a) We compute

$$|\hat{f}(\xi)| =| \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx |\leq \int_{-\infty}^{\infty} | f(x)e^{-i\xi x} | dx = \int_{-\infty}^{\infty} | f(x) | dx = ||f||_1.$$

(b) For any real number η, ξ, let us look at $|\hat{f}(\xi+\eta) - \hat{f}(\xi)|$, and hope that for any $\varepsilon > 0$, we can find a $\delta > 0$ independent of ξ, such that for any $|\eta| < \delta$, and any ξ, we always have $|\hat{f}(\xi+\eta) - \hat{f}(\xi)| < \varepsilon$. We compute

$$|\hat{f}(\xi+\eta) - \hat{f}(\xi)| =| \int_{-\infty}^{\infty} f(x)e^{-i\xi+\eta x} - f(x)e^{-i\xi x}dx |$$

$$\leq \int_{-\infty}^{\infty} | f(x)e^{-i\xi+\eta x} - f(x)e^{-i\xi x} | dx = \int_{-\infty}^{\infty} | f(x) | \cdot | e^{-\eta x} - 1 | dx.$$

Since the right-hand side above is a function with variable η, we only need to show that $\lim_{\eta\to 0} \int_{-\infty}^{\infty} | f(x) | \cdot | e^{-\eta x} - 1 | dx = 0$. Thus it is enough to show that for any sequence $\{\eta_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} \eta_n = 0$, $\lim_{n\to\infty} \int_{-\infty}^{\infty} | f(x) | \cdot | e^{-\eta_n x} - 1 | dx = 0$.

Now Lebesgue Dominant Convergence Theorem comes into play, since for any such $\{\eta_n\}_{n=1}^{\infty}$, clearly $\lim_{n\to\infty} \mid f(x) \mid \cdot \mid e^{-\eta_n x} - 1 \mid = 0$ for any $x \in \mathbb{R}$. Also $\mid f(x) \mid \cdot \mid e^{-\eta_n x} - 1 \mid \leq 2 \mid f(x) \mid$ and certainly $2 \mid f(x) \mid \in L^1(\mathbb{R})$ since $f(x) \in L^1(\mathbb{R})$. It then follows that $\lim_{n\to\infty} \int_{-\infty}^{\infty} \mid f(x) \mid \cdot \mid e^{-\eta_n x} - 1 \mid dx = 0$. $\square$

Next we introduce the concept of convolution of two functions $f, g \in L^1(\mathbb{R})$, we need **Fubini's Theorem.** The following is a not quite correct version of this theorem. But is will do for our course.

Fubini's Theorem. *Suppose $\int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} \mid f(x,y) \mid dx\}dy < \infty$. Then for almost every fixed $x \in \mathbb{R}$, $f(x,y) \in L^1(\mathbb{R})$ as a function in y; For almost every fixed $y \in \mathbb{R}$, $f(x,y) \in L^1(\mathbb{R})$ as a function in x. Furthermore,*

$$\int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} f(x,y)dx\}dy = \int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} f(x,y)dy\}dx.$$

Remark Suppose $f, g \in L^1(\mathbb{R})$, then for every fixed y, $f(x-y)g(y)$ is the product of the function $f(x-y) \in L^1(\mathbb{R})$ with a constant $g(y)$, thus for fixed y, $f(x-y)g(y) \in L^1(\mathbb{R})$ (as a function in x). So $\int_{-\infty}^{\infty} \mid f(x-y)g(y) \mid dx$ is well defined for each fixed y, namely it is a function of y. Now let us integrate:

$$\int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} \mid f(x-y)g(y) \mid dx\}dy = \int_{-\infty}^{\infty} \mid g(y) \mid \{\int_{-\infty}^{\infty} \mid f(x-y) \mid dx\}dy$$

$$= \int_{-\infty}^{\infty} \mid g(y) \mid \{\int_{-\infty}^{\infty} \mid f(x) \mid dx\}dy = \int_{-\infty}^{\infty} \mid g(y) \mid ||f||_1 dy = ||f||_1 ||g||_1 < \infty.$$

Now according to Fubini's theorem, for almost every x, $f(x-y)g(y) \in L^1(\mathbb{R})$ as a function of y, hence $\int_{-\infty}^{\infty} f(x-y)g(y)dy$ is well defined. We denote $h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$, and we will try to prove that $h \in L^1(\mathbb{R})$. Indeed,

$$\int_{-\infty}^{\infty} \mid h(x) \mid dx = \int_{-\infty}^{\infty} \mid \int_{-\infty}^{\infty} f(x-y)g(y)dy \mid dx \leq \int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} \mid f(x-y)g(y) \mid dy\}dx$$

$$= \int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} \mid f(x-y)g(y) \mid dx\}dy = ||f||_1 ||g||_1 < \infty.$$

The second to the last equal sign above is a consequence of Fubini's theorem. Usually we use $f * g$ to denote the function h above.

Definition 2. *Let $f, g \in L^1(\mathbb{R})$. Then $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$ is well defined and $f * g \in L^1(\mathbb{R})$. We call $f * g$ the convolution of f and g.*

We list a few basic properties of convolution.

Theorem 3. *Let $f, g \in L^1(\mathbb{R})$. Then*

(a) $f * g = g * f$.

(b) $\widehat{f * g}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$.

The proof of this theorem is routine with the aid of Fubini's theorem. We leave it to the reader. Next we introduce the Inverse Fourier transform.

Definition 3. *Let $f \in L^1(\mathbb{R})$ be a function such that $\hat{f} \in L^1(\mathbb{R})$. Then Inverse Fourier transform of $\hat{f}$ is defined as*

$$\mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi.$$

Our main goal in this chapter is to prove the fact that under certain conditions, we can take Inverse Fourier transform of $\hat{f}$ to get back to function f, but **Not Always!** First, let us introduce a special class of functions in $L^1(\mathbb{R})$.

Definition 4. *The function e^{-x^2} is called the* **Gaussian function**. *For any $\alpha > 0$, denote*

$$G_\alpha(x) = \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{x^2}{4\alpha}}.$$

We call the collection of G_α's the Gaussian family.

Remark Using the fact that $\int_{-\infty}^{\infty} G(x) = \sqrt{\pi}$, it can be checked that each function in the Gaussian family is in $L^1(\mathbb{R})$. In order to find the Fourier transform of these functions, we rely on the following facts which can be obtained through computations not too lengthy, and it is left to the reader.

Lemma 1. *For any $a > 0$,*

$$\int_{-\infty}^{\infty} e^{-ax^2} e^{-i\xi x} dx = \sqrt{\frac{\pi}{a}} e^{-\frac{\xi^2}{4a}}.$$

Lemma 2. *For any $\alpha > 0$, let G_α be defined above. Then $\int_{-\infty}^{\infty} G_\alpha(x) dx = 1$, $\widehat{G_\alpha}(\xi) = e^{-\alpha\xi^2}$.*

Relying on these two lemmas and some more trivial facts about Gaussian family, we reach the following two important technical results. We will only prove the first of two propositions below since the second one involves only simple computation.

Proposition 1. *Let $f \in L^1(\mathbb{R})$. $f(x)$ is continuous at $x = t$. Then*

$$\lim_{\alpha \to 0^+} (f * G_\alpha)(t) = f(t).$$

Proof. We want to show that for any $\varepsilon > 0$, there is a $\delta > 0$, such that for any $0 < \alpha < \delta$, we always have $\mid (f * G_\alpha)(t) - f(t) \mid < \varepsilon$.

Since $\int_{-\infty}^{\infty} G_\alpha(x)dx = 1$, we see that for any $\alpha > 0$, $\mid (f * G_\alpha)(t) - f(t) \mid$

$$=\mid \int_{-\infty}^{\infty} f(t-x)G_\alpha(x)dx - \int_{-\infty}^{\infty} f(t)G_\alpha(x)dx \mid=\mid \int_{-\infty}^{\infty} (f(t-x) - f(t))G_\alpha(x)dx \mid .$$

Because $f(x)$ is continuous at $x = t$, so for this $\varepsilon > 0$, there exists $\eta > 0$, such that for all $|x| < \eta$, we have $|f(t-x) - f(t)| < \frac{1}{3}\varepsilon$. Thus, for any $\alpha > 0$,

$$\mid (f*G_\alpha)(t)-f(t) \mid\leq \int_{-\eta}^{\eta} \mid f(t-x)-f(t) \mid G_\alpha(x)dx+\int_{|x|>\eta} \mid f(t-x)-f(t) \mid G_\alpha(x)dx$$

$$\leq \int_{-\eta}^{\eta} \mid f(t-x)-f(t) \mid G_\alpha(x)dx+\int_{|x|>\eta} \mid f(t-x) \mid G_\alpha(x)dx+\int_{|x|>\eta} \mid f(t) \mid G_\alpha(x)dx$$

$$\leq \frac{1}{3}\varepsilon + \max_{|x|>\eta} G_\alpha(x) \int_{|x|>\eta} \mid f(t-x) \mid dx+ \mid f(t) \mid \int_{|x|>\eta} G_\alpha(x)dx$$

$$\leq \frac{1}{3}\varepsilon + G_\alpha(\eta)||f||_1+ \mid f(t) \mid \int_{|x|>\frac{\eta}{\sqrt{\alpha}}} G_1(x)dx.$$

With η fixed, now it should be easy to choose a desirable δ. We will let the reader to finish the proof. $\square$

Proposition 2. *Let $f, g \in L^1(\mathbb{R})$. Then*

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty} \hat{f}(x)g(x)dx.$$

We need one more technical lemma before the major theorem of the chapter. The proof of following lemma, like that of above proposition, is only computations and we leave it for the reader.

Lemma 3. *For fixed $t \in \mathbb{R}$ and $\alpha > 0$, define $g_{t,\alpha}(y) = \frac{1}{2\pi}e^{iyt}e^{-\alpha y^2}$, then*

$$\widehat{g_{t,\alpha}}(x) = G_\alpha(t-x).$$

Finally, the main theorem of this chapter.

Theorem 4. *Let $f \in L^1(\mathbb{R})$ be a function such that $\hat{f} \in L^1(\mathbb{R})$. If $f(x)$ is continuous at $x = t$. Then*

$$f(t) = (\mathcal{F}^{-1}\hat{f})(t).$$

Proof. Note that by Proposition 1, $f(t) = \lim_{\alpha\to 0^+}(f * G_\alpha)(t)$. So we only need to show

$$(\mathcal{F}^{-1}\hat{f})(t) = \lim_{\alpha\to 0^+}(f * G_\alpha)(t).$$

Now by Theorem 3, Lemma 3, Proposition 2, and the definition of the function $g_{x,\alpha}$, in that order we have

$$(f * G_\alpha)(t) = \int_{-\infty}^{\infty} f(x)G_\alpha(t-x)dx = \int_{-\infty}^{\infty} f(x)\widehat{g_{t,\alpha}}(x)dx$$

$$= \int_{-\infty}^{\infty} \hat{f}(x)g_{t,\alpha}(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x)e^{ixt}e^{-\alpha x^2}dx.$$

So the reach our goal, it is enough to show that for any sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers with $\lim_{n\to\infty}\alpha_n = 0$,

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x)e^{ixt}e^{-\alpha_n x^2}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x)e^{ixt}dx.$$

Now Lebesque Dominant Convergence Theorem comes to rescue. Indeed, since for almost all x,

$$\lim_{n\to\infty}\hat{f}(x)e^{ixt}e^{-\alpha_n x^2} = \hat{f}(x)e^{ixt}.$$

Also for almost all x, $\mid \hat{f}(x)e^{ixt}e^{-\alpha_n x^2} \mid \leq \mid \hat{f}(x) \mid$ and $\mid \hat{f}(x) \mid \in L^1(\mathbb{R})$, the conclusion follows. $\square$