LEFT CELLS OF THE WEIGHTED COXETER GROUP $(\widetilde{B}_n, \widetilde{\ell})$

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Abstract. The affine Coxeter group $(\widetilde{B}_n, S)$ can be realized as the fixed point set of the affine Weyl group $(\widetilde{D}_m, \widetilde{S})$, $m \in \{2n, 2n+1\}$, under a certain group automorphism $\alpha_{m,n}$ with $\alpha_{m,n}(\widetilde{S}) = \widetilde{S}$. Let $\widetilde{\ell}$ be the length function of $\widetilde{D}_m$. We study the properties of the left cells of the weighted Coxeter group $(\widetilde{B}_n, \widetilde{\ell})$ and give an explicit description for all the left cells L of $(\widetilde{B}_n, \widetilde{\ell})$ with $a(L) \leqslant 6$.

§0. Introduction.

0.1. In his book [6], Lusztig introduced a weighted Coxeter group (W, L), which is, by definition, a Coxeter system (W, S) together with a weight function $L : W \longrightarrow \mathbb{Z}$. Lusztig proposed a bundle of conjectures, intending to generalize many results on cells of W in the equal parameter case to the unequal parameters case. The most successful part for such a generalization is when (W, L) is in a certain *quasi-split case*, that is, W can be realized as the fixed point set of a finite or an affine Coxeter system $(\widetilde{W}, \widetilde{S})$ under a group automorphism α with $\alpha(\widetilde{S}) = \widetilde{S}$, the weight function L is the restriction to W of the length function $\widetilde{\ell}$ of $\widetilde{W}$. The weighted Coxeter group $(\widetilde{W}, \widetilde{\ell})$ is called in a *split case* (see [6, Chapter 16], [5], [2], [3], [10]).

Key words and phrases. Affine Coxeter group; weighted Coxeter group; quasi-split case; left cells.

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0.2. In a weighted Coxeter group (W, L), Lusztig introduced the preorders $\underset{L}{\leqslant}$, $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$, which generate the equivalence relations $\underset{L}{\sim}$, $\underset{R}{\sim}$, $\underset{LR}{\sim}$, the corresponding equivalence classes are called *left cells*, *right cells* and *two-sided cells*. Lusztig defined a function $a : W \longrightarrow \mathbb{N} := \{0, 1, 2, ...\}$ in terms of the structural coefficients of the associated Hecke algebra, which was proved to be constant on any cell of (W, L) whenever (W, L) is either in a split case or in a quasi-split case.

0.3. In the present paper, we consider the affine Coxeter group $W = \widetilde{B}_n$ with $n \geqslant 3$ in certain quasi-split cases where $\widetilde{W}$ is the affine Coxeter group $\widetilde{D}_m$ for some $m \in \{2n, 2n+1\}$. It is our aim to describe all the cells in the weighted Coxeter group $(\widetilde{B}_n, \widetilde{\ell})$. But this is a big and difficult task. As a starting part of the whole project, the present paper is to describe all the left cells L of $\widetilde{B}_n$ with $a(L) \leqslant 6$ by applying a result in [8, Theorem B]. We introduce the star operations on the elements and also on the left cells of $\widetilde{B}_n$ (see Theorem 2.5 and Corollary 2.6). We prove the left-connectedness for the left cells of $\widetilde{B}_n$ with $a(L) \leqslant 6$ (see Theorem 2.10). We describe the minimum elements of L with respect to the weak order $\leqslant_w$ in terms of ideals of certain digraphs, by which we obtain the numbers of those left cells L with $a(L) = i$ for each $i \leqslant 6$ (see Tables 1-2 in 4.5 and 4.10).

0.4. The contents of the paper are organized as follows. We introduce some concepts, notation and known results on the groups $\widetilde{D}_m$ and $\widetilde{B}_n$ in Section 1. Then we give some general discussion on the cells of $\widetilde{B}_n$ in Section 2. In Sections 3 and 4, we describe all the left cells L of $\widetilde{B}_n$ with $a(L) = i$ for any $0 \leqslant i \leqslant 6$.

§1. Some known results on the groups $\widetilde{D}_m$ and $\widetilde{B}_n$.

In this section, we introduce some concepts, notation and known results on the groups $\widetilde{D}_m$ and $\widetilde{B}_n$ for later use.

1.1. For any $i \leqslant j$ in the integer set $\mathbb{Z}$, denote by $[i, j]$ the set $\{i, i+1, ..., j\}$ and denote $[0, j]$, $[1, j]$ simply by $(j]$, $[j]$ respectively. Let $\widetilde{S}_m = \{t_i \mid i \in (m]\}$ be the Coxeter generator set of the affine Coxeter group $\widetilde{D}_m$, $m \in \{2n, 2n+1\}$, such that $o(t_0t_2) = o(t_it_{i+1}) = o(t_{m-2}t_m) = 3$ for any $i \in [m-2]$, where the notation $o(x)$ stands for the order of an element x. Let $\widetilde{\ell}$ be the length function and $\widetilde{\leqslant}$ the Bruhat-Chevalley order on the Coxeter system $(\widetilde{D}_m, \widetilde{S}_m)$. Then the affine Coxeter group $\widetilde{B}_n$ can be regarded as the fixed point set of the group $\widetilde{D}_m$ under the automorphism $\alpha_{m,n}$ determined by $\alpha_{m,n}(t_i) = t_{m-i}$, $i \in (m]$. Let $s_i = t_i t_{m-i}$ for any $i \in (n-1]$, $s_n = t_n$ if $m = 2n$ and $s_n = t_n t_{n+1} t_n$ if $m = 2n+1$. Then $S = \{s_0, s_1, ..., s_n\}$ forms the Coxeter generator set of the group $\widetilde{B}_n$ satisfying that $o(s_0 s_2) = o(s_i s_{i+1}) = 3$ for $i \in [n-2]$ and $o(s_{n-1} s_n) = 4$. Denote $S_0 = S - \{s_n\}$.

In the present paper, when we mention the group $\widetilde{B}_n$, we always assume $n \geqslant 3$ and that $\widetilde{B}_n$ is in the quasi-split case with the weight function $\widetilde{\ell}$.

1.2. Denote by $\widetilde{a}$, a the functions a on $\widetilde{D}_m$, $\widetilde{B}_n$ respectively. For any $w \in \widetilde{D}_m$, denote $\widetilde{\mathcal{L}}(w) = \{t \in \widetilde{S}_m \mid tw \mathrel{\widetilde{<}} w\}$ and $\widetilde{\mathcal{R}}(w) = \{t \in \widetilde{S}_m \mid wt \mathrel{\widetilde{<}} w\}$. Also, let ℓ be the length function and $\leqslant$ the Bruhat-Chevalley order on the Coxeter system $(\widetilde{B}_n, S)$. For any $w \in \widetilde{B}_n$, denote $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ and $\mathcal{R}(w) = \{s \in S \mid ws < w\}$. Let $w, y \in \widetilde{B}_n$ and $w', y' \in \widetilde{D}_m$. Then the following facts can be checked easily:

(1.2.1) $y \mathrel{\widetilde{\leqslant}} w$ if and only if $y \leqslant w$.

(1.2.2) Let $i \in (n]$. Then $t_i \in \widetilde{\mathcal{L}}(w)$ if and only if $s_i \in \mathcal{L}(w)$ if and only if $t_{m-i} \in \widetilde{\mathcal{L}}(w)$; $t_i \in \widetilde{\mathcal{R}}(w)$ if and only if $s_i \in \mathcal{R}(w)$ if and only if $t_{m-i} \in \widetilde{\mathcal{R}}(w)$.

Kazhdan-Lusztig in [4] and Lusztig in [6] proved the following results:

(1.2.3) $w \underset{L}{\leqslant} y$ implies $\mathcal{R}(w) \supseteq \mathcal{R}(y)$, hence $w \underset{L}{\sim} y$ implies $\mathcal{R}(w) = \mathcal{R}(y)$; $w \underset{R}{\leqslant} y$ implies $\mathcal{L}(w) \supseteq \mathcal{L}(y)$, hence $w \underset{R}{\sim} y$ implies $\mathcal{L}(w) = \mathcal{L}(y)$ (see [6, Lemma 8.6]). $w' \underset{L}{\leqslant} y'$ implies $\widetilde{\mathcal{R}}(w') \supseteq \widetilde{\mathcal{R}}(y')$, hence $w' \underset{L}{\sim} y'$ implies $\widetilde{\mathcal{R}}(w') = \widetilde{\mathcal{R}}(y')$; $w' \underset{R}{\leqslant} y'$ implies $\widetilde{\mathcal{L}}(w') \supseteq \widetilde{\mathcal{L}}(y')$, hence $w' \underset{R}{\sim} y'$ implies $\widetilde{\mathcal{L}}(w') = \widetilde{\mathcal{L}}(y')$ (see [4, Proposition 2.4]).

(1.2.4) $w \underset{LR}{\leqslant} y$ implies $a(w) \geqslant a(y)$. Hence $w \underset{LR}{\sim} y$ implies $a(w) = a(y)$; $w' \underset{LR}{\leqslant} y'$ implies $\widetilde{a}(w') \geqslant \widetilde{a}(y')$. Hence $w' \underset{LR}{\sim} y'$ implies $\widetilde{a}(w') = \widetilde{a}(y')$ (see [6, Chapters 14-16]).

(1.2.5) $\widetilde{a}(w) = a(w)$ (see [6, Lemma 16.5]);

(1.2.6) $a(w) = a(y)$ and $w \underset{L}{\leqslant} y$ (respectively, $w \underset{R}{\leqslant} y$) implies $w \underset{L}{\sim} y$ (respectively, $w \underset{R}{\sim} y$); $\widetilde{a}(w') = \widetilde{a}(y')$ and $w' \underset{L}{\leqslant} y'$ (respectively, $w' \underset{R}{\leqslant} y'$) implies $w' \underset{L}{\sim} y'$ (respectively, $w' \underset{R}{\sim} y'$) (see [6, Chapters 14-16]);

(1.2.7) $w \underset{L}{\sim} y$ (respectively, $w \underset{R}{\sim} y$) in $\widetilde{D}_m$ if and only if $w \underset{L}{\sim} y$ (respectively, $w \underset{R}{\sim} y$) in $\widetilde{B}_n$ (see [6, Lemma 16.14]);

1.3. Let (W, S) be a finite or affine Coxeter group with the length function ℓ and the function a. For any $J \subset S$ with the subgroup W_J of W generated by J being finite, let w_J be the longest element of W_J. For any $x, y, z \in W$, we use the notation $z = x \cdot y$ to indicate $z = xy$ and $\ell(z) = \ell(x) + \ell(y)$. Let $W_{(i)} = \{w \in W \mid a(w) = i\}$. Let Δ_i be the set of all the left cells of W in $W_{(i)}$ for any $i \in \mathbb{N}$. Let $\mathcal{L}_i = \{w \in W_{(i)} \mid a(sw) < i, \ \forall\ s \in \mathcal{L}(w)\}$.

A *weak order* $\leqslant_w$ on the Coxeter system (W, S) is a partial order defined by $x \leqslant_w y$ if $\ell(y) = \ell(x) + \ell(yx^{-1})$. Clearly, the relation $x \leqslant_w y$ implies $x \leqslant y$ but not the converse. We see by (1.2.5) that any $x \in \mathcal{L}_i$ is a minimum element in the left cell of W containing it under $\leqslant_w$ and that any minimum element in any $L \in \Delta_i$ with respect to $\leqslant_w$ is in $\mathcal{L}_i$.

The following result is a special case of [8, Theorem B].

Proposition 1.4. *Let $w \in \widetilde{D}_m$ be in $\widetilde{W}_{(i)}$, $i \in (6]$. Then there is an expression $w = x \cdot w_J \cdot y$ for some $x, y \in \widetilde{D}_m$ and some $J \subset \widetilde{S}_m$ with $\widetilde{\ell}(w_J) = i$. Moreover, if $w \in \mathcal{L}_i$, then $x = 1$ in the above expression.*

1.5. For any $s, t \in \widetilde{S}_m$ with $o(st) = 3$, denote by $\widetilde{\mathcal{D}}_L(s, t)$ (respectively, $\widetilde{\mathcal{D}}_R(s, t)$) the set of all $w \in \widetilde{D}_m$ such that $\widetilde{\mathcal{L}}(w) \cap \{s, t\}$ (respectively, $\widetilde{\mathcal{R}}(w) \cap \{s, t\}$) contains exactly one element . When $w \in \widetilde{\mathcal{D}}_L(s, t)$, exactly one (denote it by ${}^{\tilde{*}}w$) of sw, tw is also in $\widetilde{\mathcal{D}}_L(s, t)$, call the transformation $w \mapsto {}^{\tilde{*}}w$ a *left $\{s, t\}$-star operation* (or a *left star operation* in short) on w. Similarly, we can define a *right $\{s, t\}$-star operation* (or a *right star operation* in short) on any $w \in \widetilde{\mathcal{D}}_R(s, t)$ with $w^{\tilde{*}}$ the resulting element.

The following results are well known.

Proposition 1.6. (see [4]) *Let $x, y \in \widetilde{D}_m$. Let $s, t \in \widetilde{S}_m$ satisfy $o(st) = 3$.*

(1) If $x \in \widetilde{\mathcal{D}}_L(s, t))$ then ${}^{\tilde{*}}x \underset{L}{\sim} x$; if $x \in \widetilde{\mathcal{D}}_R(s, t)$ then $x^{\tilde{*}} \underset{R}{\sim} x$.

(2) Assume $x, y \in \widetilde{\mathcal{D}}_R(s, t)$. Then $x \underset{L}{\sim} y$ if and only if $x^{\tilde{*}} \underset{L}{\sim} y^{\tilde{*}}$.

(3) Assume $x, y \in \widetilde{\mathcal{D}}_L(s, t)$. Then $x \underset{R}{\sim} y$ if and only if ${}^{\tilde{*}}x \underset{R}{\sim} {}^{\tilde{*}}y$.

1.7. We see by (1.2.3) that for any left cell L of $\widetilde{D}_m$, if there exists some $w \in L$ satisfying $w \in \widetilde{\mathcal{D}}_R(s, t)$ for some $s, t \in \widetilde{S}_m$ with $o(st) = 3$, then $L \subseteq \widetilde{\mathcal{D}}_R(s, t)$. In this case, we see by Proposition 1.6 (2) that the set $L^{\tilde{*}} := \{y^{\tilde{*}} \mid y \in L\}$ forms a left cell of $\widetilde{D}_m$, called *the left cell obtained from L by a right $\{s,t\}$-star operation.* Similarly, by Proposition 1.6 (3), we can define a right cell ${}^{\tilde{*}}R$ of $\widetilde{D}_m$ obtained by a left $\{s,t\}$-star operation on a right cell R in the case where $R \subseteq \widetilde{\mathcal{D}}_L(s,t)$.

We say that two elements $x, y \in \widetilde{D}_m$ have the same *right generalized τ-invariants*, if for any sequence $x_0 = x, x_1, ..., x_r$ in $\widetilde{D}_m$ with x_i obtained from x_{i-1} by a right star operation for every $i \in [r]$, there exists the corresponding sequence $y_0 = y, y_1, ..., y_r$ in $\widetilde{D}_m$ with y_i obtained from y_{i-1} by a right star operation for every $i \in [r]$ such that $\widetilde{\mathcal{R}}(x_j) = \widetilde{\mathcal{R}}(y_j)$ for any $j \in (r]$ and if the condition holds when interchanging the roles of x, y.

The following result is well known.

Lemma 1.8. (see [7, Proposition 4.2]) *If $x \underset{L}{\sim} y$ in $\widetilde{D}_m$ then x, y have the same right generalized τ-invariants.*

1.9. We introduce a certain *direct graph* (or a *digraph* in short) $\Gamma = (N, A)$, where N is a node set, and $A = A_0 \cup A_1$, A_0 is an arrow set, each arrow is an ordered node pair in N, i.e., $x \longrightarrow y$ with $x \neq y$ in N, call x the *tail* and y the *head* of the arrow $x \longrightarrow y$, and the set A_1 either is empty or consists of a single unordered node pair with a double-edge connected, i.e., $x =\!= y$ with $x \neq y$ in N. Call $z \in N$ a *source* (respectively, a *sink*) of Γ, if z is the tail (respectively, the head) of any arrow of Γ incident to z. By an *ideal* of a digraph $\Gamma = (N, A)$, we mean a sub-digraph $\Gamma' = (N', A')$ of Γ such that N' contains all the sources in N and that for any $x \longrightarrow y$ (respectively, $x =\!= y$) in A, the condition $y \in N'$ implies $x \in N'$.

Example 1.10. Observe the digraphs in Figure 1.

(1) The digraph (a) consists of 15 nodes and 19 arrows. The the nodes 0, 3, 5 in the leftmost column are sources and the nodes 6, 0 at right and top-right places are sinks. The digraph (b) is an ideal of (a). Note that two nodes of a digraph in the different places are regarded as different nodes though they might have the same label. For example, there are three different nodes with the same label 5 in the digraph (a).

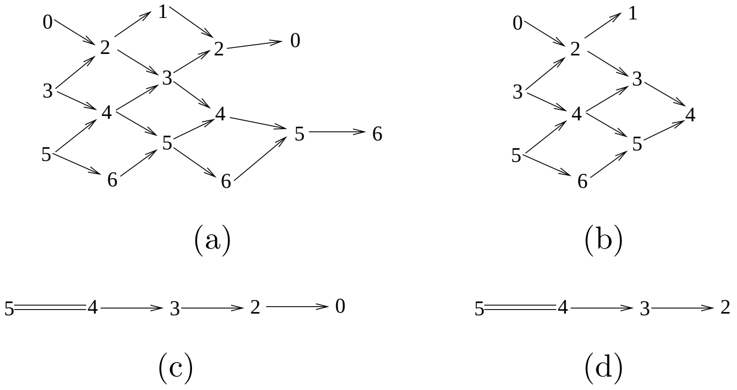

Figure 1

(2) The digraph (c) consists of 5 nodes and 3 arrows and 1 double edge, where the nodes 5, 4 are sources and the nodes 5, 0 are sinks (hence the node 5 is both a source and a sink). The digraph (d) is an ideal of the digraph (c).

1.11. To each $w \in \mathcal{L}_i$ with $i \in [6]$, we associate a digraph $\Gamma_w = (N_w, A_w)$ as follows. Let $w = v_1 v_2 \cdots v_r$ be a reduced expression of w with $v_i \in S$ (hence $\ell(w) = r$). We define a digraph $\Gamma'_w = (N'_w, A'_w)$, where N'_w consists of all the nodes $\mathbf{v}_i$ with $i \in [r]$, and A'_w consists of all arrows $\mathbf{v}_i \longrightarrow \mathbf{v}_j$, where $i < j$ and $o(v_i v_j) > 2$ and that there is no sequence $i_0 = i < i_1 < \cdots < i_m = j$ with $m > 1$ and $o(v_{i_{p-1}} v_{i_p}) > 2$ for every $p \in [m]$. If Γ'_w contains some sub-digraph $\mathbf{v}_i \longrightarrow \mathbf{v}_j \longrightarrow \mathbf{v}_k$ with $i < j < k$ in $[r]$ such that $o(v_i v_j) = 3$ and $v_i = v_k$ and that there is no q with $i < q < k$ and $q \neq j$ satisfying $o(v_i v_q) \geqslant 3$ (respectively, $\mathbf{v}_i \longrightarrow \mathbf{v}_j \longrightarrow \mathbf{v}_k \longrightarrow \mathbf{v}_l$ with $i < j < k < l$ in $[r]$ such that $o(v_i v_j) = 4$ and $v_i = v_k$ and $v_j = v_l$ and that there is no q, p with $i < q < k$ and $j < p < l$ and $q \neq j$ and $p \neq k$ satisfying $o(v_i v_q), o(v_j v_p) \geqslant 3$), then by the condition of $w \in \mathcal{L}_i$, $i \in [6]$, such a sub-digraph, when it exists, must be unique with $\{v_i, v_j\}$ its source set. In particular, the sub-digraph $\mathbf{v}_i \longrightarrow \mathbf{v}_j \longrightarrow \mathbf{v}_k \longrightarrow \mathbf{v}_l$ occurs only if $m = 2n$ and hence $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2, 1)$. In this case, we define a digraph Γ_w which is obtained from Γ'_w by replacing the sub-digraph $\mathbf{v}_i \longrightarrow \mathbf{v}_j \longrightarrow \mathbf{v}_k$ (respectively, $\mathbf{v}_i \longrightarrow \mathbf{v}_j \longrightarrow \mathbf{v}_k \longrightarrow \mathbf{v}_l$) by $\mathbf{v}_i =\!= \mathbf{v}_j$ and by replacing all the arrows $\mathbf{v}_k \longrightarrow \mathbf{v}_m$ ($m \neq l$), $\mathbf{v}_l \longrightarrow \mathbf{v}_p$ by $\mathbf{v}_i \longrightarrow \mathbf{v}_m$, $\mathbf{v}_j \longrightarrow \mathbf{v}_p$ respectively. Note that $\mathbf{v}_j \longrightarrow \mathbf{v}_k$ (respectively, $\mathbf{v}_k \longrightarrow \mathbf{v}_l$) is the only arrow in Γ'_w with $\mathbf{v}_k$ (respectively, $\mathbf{v}_l$) its head. If Γ'_w contains neither a sub-digraph $\mathbf{v}_i \longrightarrow \mathbf{v}_j \longrightarrow \mathbf{v}_k$ nor $\mathbf{v}_i \longrightarrow \mathbf{v}_j \longrightarrow \mathbf{v}_k \longrightarrow \mathbf{v}_l$

as above, then we define Γ_w to be Γ'_w.

Example 1.12. Let $w = s_1s_2s_1s_3s_4$ and $y = s_5s_4s_5s_4s_3s_2$ be in $\widetilde{B}_5$. Then $w, y \in \mathcal{L}_6$. The corresponding digraphs Γ'_x and Γ_x with $x \in \{w, y\}$ are as follows.

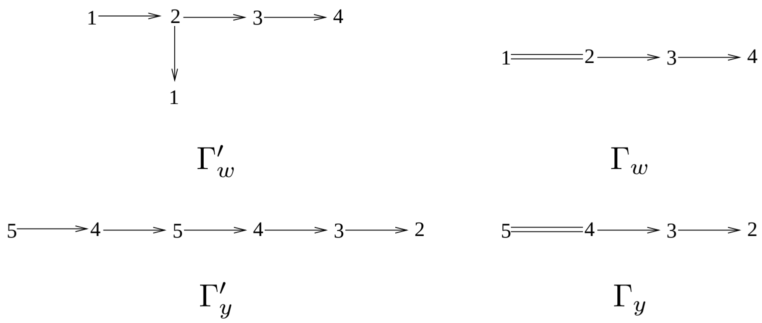

Figure 2

1.13. Note that for any $w \in \mathcal{L}_i$, $i \in [6]$, the element $s \in S$ is in $\mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) if and only if there is a node labeled by s is a source (respectively, a sink) in Γ_w. In particular, when Γ_w contains a double edge, the set $\mathcal{L}(w)$ always consists of two elements of S corresponding to the nodes joined by the double edge. In the above example, we have $\mathcal{L}(w) = \{s_1, s_2\}$ and $\mathcal{L}(y) = \{s_4, s_5\}$ and $\mathcal{R}(w) = \{s_1, s_4\}$ and $\mathcal{R}(y) = \{s_2, s_5\}$. The digraph Γ_w depends only on the element $w \in \mathcal{L}_i$ but not on a particular choice of a reduced expression of w.

We shall describe all the left cells of $\widetilde{B}_n$ in the set $W_{(i)}$ for any $i \in (6]$. Each of those left cells contains a minimum elements w with Γ_w occurring as an ideal of the digraph defined above.

§2. Some results on $\widetilde{B}_n$.

In the present section, we study the properties for the cells of the group $\widetilde{B}_n$. Some of them will be used in the subsequent sections.

2.1. Let W be any finite or affine Coxeter group. For any $i \in \mathbb{N}$, let $F(i)$ be the set of all $w \in W$ with $a(w) = i$ such that $a(sw), a(wt) < i$ for any $s \in \mathcal{L}(w)$ and any $t \in \mathcal{R}(w)$. Clearly, the inclusion $F(i) \subseteq \mathcal{L}_i$ holds in general. In [9, Algorithm 3.4], we constructed an affective algorithm for finding the set $\mathcal{L}_i$ from $F(i)$ whenever the set $F(i)$ is known explicitly. Though such an algorithm was originally constructed for W in a split case, it is apparently also applicable for W in a quasi-split case by the results in [6, Chapters 14-16].

By Proposition 1.4 and by regarding $\widetilde{B}_n$ as the fixed point set of $\widetilde{D}_m$ under the automorphism $\alpha_{m,n}$, the sets $F(i)$, $i \in (6]$, can be described as follows.

Lemma 2.2. *For any $i \in (6]$, the subset $F(i)$ of $\widetilde{B}_n$ consists of all the elements w_J with $J \subset S$ and $\widetilde{\ell}(w_J) = i$, and some additional elements in the following cases:*
(1) Assume $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3)$. Then $s_n s_{n-1} s_n \in F(4)$. When $n \geqslant 4$, we have $\{s_k s_n s_{n-1} s_n \mid k \in (n-3]\} \subseteq F(6)$.
(2) Assume $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,1)$. Then $s_{n-1} s_n s_{n-1} \in F(3)$. When $n \geqslant 4$, we have $\{s_k s_{n-1} s_n s_{n-1} \mid k \in (n-3]\} \subseteq F(5)$.

2.3. For any $s, t \in S$ with $o(st) \in \{3,4\}$, denote by $\mathcal{D}_L(s,t)$ (respectively, $\mathcal{D}_R(s,t)$) the set of all $w \in \widetilde{B}_n$ such that $\mathcal{L}(w) \cap \{s,t\}$ (respectively, $\mathcal{R}(w) \cap \{s,t\}$) contains exactly one element. When $w \in \mathcal{D}_L(s,t)$, exactly one (denote it by ${}^*w$) of sw, tw is also in $\mathcal{D}_L(s,t)$ unless that $o(st) = 4$ and $w = xy$ for some $y \in \widetilde{B}_n$ and some $x \in W_{\{s,t\}}$ with $\mathcal{L}(y) \cap \{s,t\} = \emptyset$ and $\ell(x) = 2$. In this exceptional case, we have $\{sw, tw\} \subseteq \mathcal{D}_L(s,t)$ (denote by ${}^*w_1$, ${}^*w_2$ with $\{{}^*w_1, {}^*w_2\} = \{sw, tw\}$ and $\ell({}^*w_1) < \ell({}^*w_2)$). In any of the above cases, we say that the transformation $w \mapsto w' \in \{{}^*w, {}^*w_1, {}^*w_2\}$ is a *left $\{s,t\}$-star operation* (or a *left star operation* in short) on w if $\widetilde{\ell}(w'w^{-1}) = \min\{\widetilde{\ell}(s), \widetilde{\ell}(t)\}$. Clearly, the resulting element is unique whenever it exists.

Similarly, we can define the elements w^*, w_1^*, w_2^* from w and a right $\{s,t\}$-star operation on w accordingly whenever $w \in \mathcal{D}_R(s,t)$.

Regarding $\widetilde{B}_n$ as the fixed points set of $\widetilde{D}_m$ under the automorphism $\alpha_{m,n}$, we have the following results.

Lemma 2.4. *Let $x, y \in \widetilde{B}_n$.*

(1) If y can be obtained from x by a left (respectively, right) $\{s,t\}$-star operation in $\widetilde{B}_n$ for some $s, t \in S$ with $o(st) = 3$ then y can be obtained from x by successively applying two left (respectively, right) star operations in $\widetilde{D}_m$.

(2) If y can be obtained from x by a left (respectively, right) $\{s_{n-1}, s_n\}$-star operation in $\widetilde{B}_n$, then y can be obtained from x by successively applying some left (respectively, right) star operations in $\widetilde{D}_m$ if and only if $\widetilde{\ell}(yx^{-1}) = \min\{\widetilde{\ell}(s_{n-1}), \widetilde{\ell}(s_n)\}$

(respectively, $\widetilde{\ell}(x^{-1}y) = \min\{\widetilde{\ell}(s_{n-1}), \widetilde{\ell}(s_n)\}$*).*

Proof. The results can be checked directly in a case-by-case way. □

Theorem 2.5. *Let* $x, y \in \widetilde{B}_n$*. Let* $s, t \in S$ *satisfy* $o(st) \in \{3, 4\}$*.*

(1) If $x', x'' \in \widetilde{B}_n$ *be obtained from* x *by a left, respectively, right, star operation, then* $x' \underset{L}{\sim} x$ *and* $x'' \underset{R}{\sim} x$*.*

(2) Assume $x \underset{L}{\sim} y$*.*

(2a) If $x \in \mathcal{D}_R(s, t)$ *then* $y \in \mathcal{D}_R(s, t)$*;*

(2b) In the case of (2a), if there exists some $x'' \in \widetilde{B}_n$ *which is obtained from* x *by a right* $\{s, t\}$*-star operation, then there exists some* $y'' \in \widetilde{B}_n$ *which is obtained from* y *by a right* $\{s, t\}$*-star operation;*

(2c) $x'' \underset{L}{\sim} y''$ *in the case of (2b).*

(3) Assume $x \underset{R}{\sim} y$*.*

(3a) If $x \in \mathcal{D}_L(s, t)$ *then* $y \in \mathcal{D}_L(s, t)$*;*

(3b) In the case of (3a), if there exists some $x' \in \widetilde{B}_n$ *which is obtained from* x *by a left* $\{s, t\}$*-star operation, then there exists some* $y' \in \widetilde{B}_n$ *which is obtained from* y *by a left* $\{s, t\}$*-star operation;*

(3c) $x' \underset{R}{\sim} y'$ *in the case of (3b).*

Proof. (1) follows by Lemma 2.4 and (1.2.7). For the remaining part of the lemma, we need only to show (2), then (3) is just another side version of (2) and hence can be shown similarly. Now assume $x \underset{L}{\sim} y$. Then $\mathcal{R}(y) = \mathcal{R}(x)$ by (1.2.3). This implies (2a). Written $x = x_0x_1$ and $y = y_0y_1$ with $\mathcal{R}(x_0) \cap \{s, t\} = \mathcal{R}(y_0) \cap \{s, t\} = \emptyset$ and $x_1, y_1 \in W_{\{s,t\}}$. Under the assumption of (2b), we need only to deal with the case where $o(st) = 4$ and $\widetilde{\ell}(x^{-1}x'') = \min\{\widetilde{\ell}(s), \widetilde{\ell}(t)\}$. In this case, we have to show that there exists some $y'' \in \mathcal{D}_R(s, t)$ with $\widetilde{\ell}(y^{-1}y'') = \min\{\widetilde{\ell}(s), \widetilde{\ell}(t)\}$. For otherwise, assume $\widetilde{\ell}(s) > \widetilde{\ell}(t)$ for the sake of definiteness, then by the assumption $x \underset{L}{\sim} y$ in $\widetilde{B}_n$ and (1.2.3), we would be in one of the following two cases:

(a) $(s, t) = (s_{n-1}, s_n)$ with $(\widetilde{\ell}(s_{n-1}), \widetilde{\ell}(s_n)) = (2, 1)$; either $y_1 = s_n$ and $x_1 \in \{s_ns_{n-1}s_n, s_{n-1}s_n\}$, or $y_1 = s_{n-1}s_ns_{n-1}$ and $x_1 \in \{s_ns_{n-1}, s_{n-1}\}$;

(b) $(s, t) = (s_n, s_{n-1})$ with $(\widetilde{\ell}(s_{n-1}), \widetilde{\ell}(s_n)) = (2, 3)$; either $y_1 = s_{n-1}$ and $x_1 \in \{s_ns_{n-1}, s_{n-1}s_ns_{n-1}\}$, or $y_1 = s_ns_{n-1}s_n$ and $x_1 \in \{s_n, s_{n-1}s_n\}$.

We can observe that in any of the cases (a)-(b), x, y would have different right generalized τ-invariants regarding as elements of $\widetilde{D}_m$, contradicting Lemma 1.8 by (1.2.7). Let us take $y_1 = s_n$ and $x_1 = s_{n-1}s_n$ in the case (a) as an example to illustrate our observation. In this example, we have $y = y_0 t_n$ and $x = x_0 t_{n-1} t_{n+1} t_n$. Applying a right $\{t_{n-1}, t_n\}$-star operation on both y and x in $\widetilde{D}_m$, we get the resulting elements $y^{\tilde{*}} = y_0 t_n t_{n-1}$ and $x^{\tilde{*}} = x_0 t_{n-1} t_{n+1}$. Since $\widetilde{\mathcal{R}}(y^{\tilde{*}}) \cap \{t_{n-1}, t_{n+1}\} = \{t_{n-1}\}$ and $\{t_{n-1}, t_{n+1}\} \subseteq \widetilde{\mathcal{R}}(x^{\tilde{*}})$, this implies that x, y have different right generalized τ-invariants in $\widetilde{D}_m$. So (2) is proved. Finally, y'' is obtained from y by the same sequence of right star operations as x'' from x regarding as elements of $\widetilde{D}_m$. In $\widetilde{B}_n$, we get $x'' \underset{L}{\sim} y''$ from $x \underset{L}{\sim} y$ by (1.2.7) and Proposition 1.6 (2). $\square$

Corollary 2.6. *In the setup of Theorem 2.5, we have*

(1) $x \underset{L}{\sim} y$ if and only if $x'' \underset{L}{\sim} y''$;

(2) $x \underset{R}{\sim} y$ if and only if $x' \underset{R}{\sim} y'$.

Proof. (1) follows by Theorem 2.5 and the fact that a right $\{s, t\}$-star operation on the set $\mathcal{D}_R(s, t)$ is involutive. Then (2) is just another side version of (1). $\square$

2.7. We see by (1.2.3) that if a left cell L of $\widetilde{B}_n$ satisfies $L \cap \mathcal{D}_R(s, t) \neq \emptyset$ for some $s, t \in S$ with $o(st) \in \{3, 4\}$, then $L \subseteq \mathcal{D}_R(s, t)$.

In this case, suppose that there exists some $x'' \in \widetilde{B}_n$ which is obtained from some $x \in L$ by a right $\{s, t\}$-star operation. Let L^* be the set of all $y'' \in \widetilde{B}_n$ with y'' obtained from some $y \in L$ by a right $\{s, t\}$-star operation. Then L^* forms a left cell of $\widetilde{B}_n$ by Theorem 2.5 and Corollary 2.6 (1). Call L^* *the left cell obtained from L by a right $\{s, t\}$-star operation.*

Similarly, by (1.2.3), Theorem 2.5 and Corollary 2.6 (2), we can define the right cell ${}^*R$ of $\widetilde{B}_n$ obtained by a left $\{s, t\}$-star operation on a right cell R in the case where $R \subseteq \mathcal{D}_L(s, t)$ for some $s, t \in S$ with $o(st) \in \{3, 4\}$ and there exists some $x' \in \widetilde{B}_n$ obtained from some $x \in R$ by a left $\{s, t\}$-star operation.

We say that two elements $x, y \in \widetilde{B}_n$ have the same *right generalized τ-invariants in $\widetilde{B}_n$*, if for any sequence $x_0 = x, x_1, ..., x_r$ in $\widetilde{B}_n$ with x_i obtained from x_{i-1} by a right star operation for every $i \in [r]$, there exists the corresponding sequence

$y_0 = y, y_1, ..., y_r$ in $\widetilde{B}_n$ with y_i obtained from y_{i-1} by a right star operation for every $i \in [r]$ such that $\mathcal{R}(x_j) = \mathcal{R}(y_j)$ for any $j \in (r]$ and if the condition holds when interchanging the roles of x, y.

By Lemma 2.4, we see that if $x, y \in \widetilde{B}_n$ have the same right generalized τ-invariants in $\widetilde{D}_m$ then they have the same right generalized τ-invariants in $\widetilde{B}_n$. So by (1.2.7) and Lemma 1.8, we get the following result.

Lemma 2.8. (Comparing with Lemma 1.8) *If $x \underset{L}{\sim} y$ in $\widetilde{B}_n$ then x, y have the same right generalized τ-invariants in $\widetilde{B}_n$.*

As a result in applying [9, Algorithm 3.4] on the subsets $F(i)$, $i \in (6]$, of $\widetilde{B}_n$, we observe the following phenomenon by Lemma 2.8.

Lemma 2.9. *Any left cell L of $\widetilde{B}_n$ with $a(L) \in (6]$ has a unique minimum element (denoted by w_L) with respect to $\leqslant_w$ except for the case where $a(L) = 6$ and there is some minimum $w \in L$ satisfying $\mathcal{L}(w) = \{s_i, s_j\}$ with $o(s_i s_j) = 3$ (call it the exceptional case for L).*

A subset K of a Coxeter group (W, S) is *left-connected*, if, for any $x, y \in K$, there exists a sequence $x_0 = x, x_1, ..., x_r = y$ in K with some $r \geqslant 0$ such that $x_{i-1}x_i^{-1} \in S$ for every $i \in [r]$. Lusztig conjectured in [1] that any left cell of an affine Coxeter group is left-connected.

Theorem 2.10. *Any left cell L of $\widetilde{B}_n$ with $a(L) \leqslant 6$ is left-connected.*

Proof. Given a left cell L of $\widetilde{B}_n$. Any $x \in L$ can be written in the form $x = x' \cdot x''$ for some $x', x'' \in \widetilde{B}_n$ with x'' a minimum element of L with respect to $\leqslant_w$. Let $x' = s_1 s_2 \cdots s_r$ be a reduced expression of x' with $s_i \in S$ and let $x_i = s_i s_{i+1} \cdots s_r x''$ for $i \in [r]$. Then $x_0 = x, x_1, ..., x_r, x_{r+1} = x''$ is a sequence of elements in L by (1.2.4) and (1.2.6). So we see by Lemma 2.9 that any left cell L of $\widetilde{B}_n$ with $a(L) \leqslant 6$ is left-connected if L is not in the exceptional case. The result for L in the exceptional case will be proved in Lemma 3.5 later. □

§3. Left cells L of $\widetilde{B}_n$ with $a(L) = 6$.

In the present section, we consider the left cells of $\widetilde{B}_n$ in the set $W_{(6)}$.

3.1. By [6, Lemma 16.5], Proposition 1.4 and Lemma 2.2, we see that any $w \in \widetilde{B}_n$ is in $\mathcal{L}_6$ if and only if the following conditions (1)-(3) on w hold:

(1) One of the following cases holds:

(1a) $J := \mathcal{L}(w) \subseteq S_0$ and $\ell(w_J) = 3$;

(1b) $\mathcal{L}(w) = \{s_{n-1}, s_n\}$ and $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2, 1)$;

(1c) $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2, 3)$ and $w = s_k s_n s_{n-1} s_n \cdot y$ for some $y \in \widetilde{B}_n$ and some $k \in (n-3]$ with $n \geqslant 4$.

(2) $w \neq x' \cdot w_I \cdot y'$ for any $x', y' \in \widetilde{B}_n$ and any $I \subset S$ with either $\widetilde{\ell}(w_I) = 6$ and $x' \neq 1$, or $\widetilde{\ell}(w_I) > 6$.

(3) $w \neq x' \cdot s_k s_n s_{n-1} s_n \cdot y'$ for any $x', y' \in \widetilde{B}_n$ and any $k \in (n-3]$ with $n \geqslant 4$ and $x' \neq 1$ if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2, 3)$.

3.2. Let Ξ be the set of all subsets J of S_0 such that $|J| = 3$ and $\ell(w_J) = 3$. Let L and L' be two left cells of $\widetilde{B}_n$ with $a(L) = a(L') = 6$ such that there are some $w \in L \cap \mathcal{L}_6$ and $w' \in L' \cap \mathcal{L}_6$ with $\mathcal{L}(w) \in \Xi$. By Lemma 2.9, we have $L = L'$ if and only if $w = w'$.

Let Ξ_1 be the set of all $J \in \Xi$ such that the equations $o(ts) = o(ts') = 3$ can't hold simultaneously for any $t \in S - J$ and any $s \neq s'$ in J. Let Ξ_2 be the set of all $J \in \Xi$ such that there exists some $t \in J$ satisfying that for any $s \in J - \{t\}$, there exists some $r \in S - J$ with $o(sr) = o(rt) = 3$. Let $\Xi_3 = \Xi - (\Xi_1 \cup \Xi_2)$. Then $\Xi = \Xi_1 \dot{\cup} \Xi_2 \dot{\cup} \Xi_3$. For any $i \in \{1, 2, 3\}$, let Ξ_{i1} be the set of all $J \in \Xi_i$ with $s_{n-1} \in J$ and let $\Xi_{i2} = \Xi_i - \Xi_{i1}$. Let Ξ_{311} be the set of all $J \in \Xi_{31}$ such that $s_{n-3} \in J$ and let $\Xi_{312} = \Xi_{31} - \Xi_{311}$.

Let Ξ'_1 be the set of all $J = \{s, s'\} \subseteq S_0$ with $o(ss') = 3$ and let $\Xi'_2 = \{\{s_{n-1}, s_n\}\}$.

Let $\Xi'' = \{s_k s_n s_{n-1} s_n \mid k \in (n-3]\}$ if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2, 3)$ and $n \geqslant 4$. Let $\Xi'' = \emptyset$ if otherwise.

3.3. Let $\mathcal{L}'_{n,6,i}$ be the set of all $w \in \mathcal{L}_6$ with $\mathcal{L}(w) \in \Xi'_i$ for $i \in \{1, 2\}$. Let $\mathcal{L}''_{n,6}$ be the set of all $w \in \mathcal{L}_6$ such that $w = z \cdot y$ for some $z \in \Xi''$ and some $y \in \widetilde{B}_n$. Let $\mathcal{L}_{n,6,j}$ be the set of all $w \in \mathcal{L}_6$ with $\mathcal{L}(w) \in \Xi_j$ for $j \in \{1, 2, 3\}$. Let $\mathcal{L}_{n,6,jk}$ be the set of all $w \in \mathcal{L}_6$ with $\mathcal{L}(w) \in \Xi_{jk}$ for $j \in \{1, 2, 3\}$ and $k \in \{1, 2\}$. Let $\mathcal{L}_{n,6,31k}$ be the set of all $w \in \mathcal{L}_6$ with $\mathcal{L}(w) \in \Xi_{31k}$ for $k \in \{1, 2\}$.

3.4. Let $L'_{n,6,i}$ be the set of all left cells L of $\widetilde{B}_n$ such that L contains some elements in $\mathcal{L}'_{n,6,i}$ for $i \in \{1,2\}$. Then any L in $L'_{n,6,1}$ (respectively, $L'_{n,6,2}$) contains an element w in $\mathcal{L}'_{n,6,1}$ (respectively, $\mathcal{L}'_{n,6,2}$) with Γ_w an ideal of a digraph in (a)-(c) (respectively, (d)) of Figure 3.

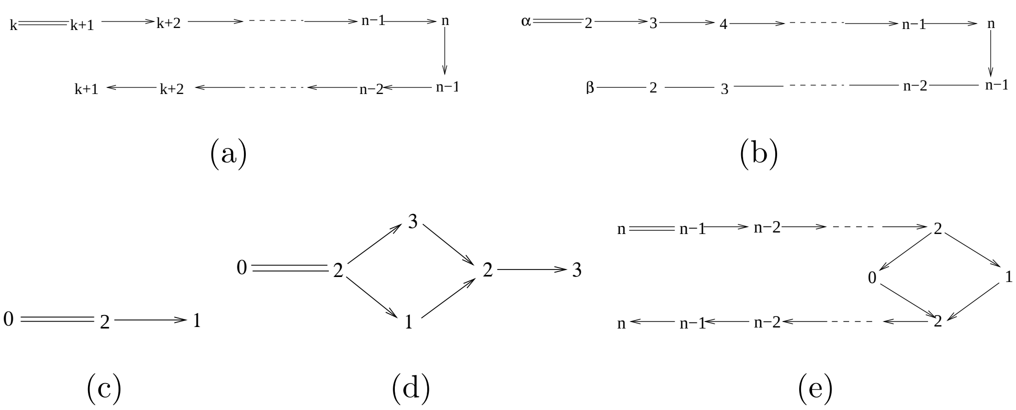

Figure 3

where $k \in [2, n-2]$ in (a); $\{\alpha, \beta\} = \{0, 1\}$ in (b)-(c).

Lemma 3.5. *Let $L \in L'_{n,6,i}$ and $l'_{n,6,i} = |L'_{n,6,i}|$ for $i \in \{1,2\}$.*

(1) L is left-connected.

(2) $l'_{3,6,1} = 12$ if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,1)$; $l'_{n,6,1} = n^2$ if either $n \geqslant 4$ or $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3)$.

(3) $l'_{n,6,2}$ is $2n$ if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,1)$ and 0 if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3)$.

Proof. When either $n \geqslant 4$ or $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3)$, the set $\mathcal{L}'_{n,6,1}$ consists of all the elements in (i)-(iv) below.

(i) $x_{k;i,j} := s_k s_{k+1} s_k s_{k+2} s_{k+3} \cdots s_{j-1} s_j \cdot s_{k-1} s_{k-2} \cdots s_{i+1} s_i$ and $y_{k;i,j} := x_{k;i,n} \cdot s_{n-1} s_{n-2} \cdots s_{j+1} s_j$ for $i \leqslant k < j$ in $[n]$ with $k \neq n-1$;

(ii) $x_{k;0,j} := s_k s_{k+1} s_k s_{k+2} s_{k+3} \cdots s_{j-1} s_j \cdot s_{k-1} s_{k-2} \cdots s_2 s_0$ and $y_{k;0,j} := x_{k;0,n} \cdot s_{n-1} s_{n-2} \cdots s_{j+1} s_j$ for $k < j$ in $[2, n]$ with $k \neq n-1$;

(iii) $x_{0;0,j} := s_0 s_2 s_0 s_3 s_4 \cdots s_{j-1} s_j$ and $y_{0;0,j} := x_{0;0,n} \cdot s_{n-1} s_{n-2} \cdots s_{j+1} s_j$ for $j \in [2, n]$;

(iv) $x_{\alpha;\alpha,\beta} := s_\alpha s_2 s_\alpha s_\beta$ and $y_{\alpha;\alpha,\beta} := s_\alpha s_2 s_\alpha s_3 s_4 \cdots s_{n-1} s_n s_{n-1} \cdots s_3 s_2 s_\beta$ for $\{\alpha, \beta\} = \{0, 1\}$.

When $(\widetilde{\ell}(S_0),\widetilde{\ell}(s_n))=(2,1)$, the set $\mathcal{L}'_{3,6,1}$ consists of all the elements in (i)-(iv) with $n=3$, and those in (v) below.

(v) $x_{\alpha;\alpha,\beta}z$ for $z\in\{s_3,s_3s_2,s_3s_2s_3\}$ and $\{\alpha,\beta\}=\{0,1\}$.

We have $\mathcal{L}'_{n,6,2}=\emptyset$ if $(\widetilde{\ell}(S_0),\widetilde{\ell}(s_n))=(2,3)$ and $\mathcal{L}'_{n,6,2}=\{z_{n,a},z'_{n,0},w_{n,b}\mid a\in(n-1],b\in[2,n]\}$ if $(\widetilde{\ell}(S_0),\widetilde{\ell}(s_n))=(2,1)$, where

$$\begin{aligned} z_{n,a} &:= s_ns_{n-1}s_ns_{n-1}s_{n-2}\cdots s_{a+1}s_a,\\ z'_{n,0} &:= s_ns_{n-1}s_ns_{n-1}s_{n-2}\cdots s_3s_2s_0,\\ w_{n,b} &:= z_{n,0}\cdot s_2s_3\cdots s_{b-1}s_b. \end{aligned}$$

For any $i\leqslant k<j-1$, the element $x_{k+1;i,j}$ (respectively, $y_{k+1;i,j}$, $x_{k+1;0,j}$, $y_{k+1;0,j}$) can be obtained from $x_{k;i,j}$ (respectively, $y_{k;i,j}$, $x_{k;0,j}$, $y_{k;0,j}$) by a left $\{s_{k+1},s_{k+2}\}$-star operation followed by a left $\{s_k,s_{k+1}\}$-star operation. Hence $x_{k+1;i,j}\underset{L}{\sim}x_{k;i,j}$ and $y_{k+1;i,j}\underset{L}{\sim}y_{k;i,j}$ and $x_{k+1;0,j}\underset{L}{\sim}x_{k;0,j}$ and $y_{k+1;0,j}\underset{L}{\sim}y_{k;0,j}$ by Theorem 2.5 (1). Also, $x_{\alpha;\alpha,\beta}$ can be obtained from $x_{\beta;\beta,\alpha}$ by a left $\{s_2,s_\alpha\}$-star operation followed by a left $\{s_2,s_\beta\}$-star operation; $x_{0;0,j}$, $y_{0;0,j}$, can be obtained from $x_{2;0,j}$, $y_{2;0,j}$, respectively by a left $\{s_0,s_2\}$-star operation followed by a left $\{s_2,s_3\}$-star operation. Hence $x_{\alpha;\alpha,\beta}\underset{L}{\sim}x_{\beta;\beta,\alpha}$ and $x_{0;0,j}\underset{L}{\sim}x_{2;0,j}$ and $y_{0;0,j}\underset{L}{\sim}y_{2;0,j}$ by Theorem 2.5 (1).

When $n=3$ and $(\widetilde{\ell}(S_0),\widetilde{\ell}(s_n))=(2,1)$, for any $z\in I:=\{s_3,s_3s_2,s_3s_2s_3\}$ and $\{\alpha,\beta\}=\{0,1\}$, the element $x_{\alpha;\alpha,\beta}z$ can be obtained from $x_{\beta;\beta,\alpha}z$ by a left $\{s_\alpha,s_2\}$-star operation followed by a left $\{s_2,s_\beta\}$-star operation.

Note that $x_{i;i,n}=y_{i;i,n}$ for any $i\in(n-2]$. Let $(\widetilde{\ell}(S_0),\widetilde{\ell}(s_n))=(2,c)$ be with $c\in\{1,3\}$. For any $n\geqslant 3$, define the sets

$$E'_n=\{x_{i;i,j},y_{i';i',j'},y_{1;1,0}\mid i<j\ \text{ in } (n],i\neq n-1;i'<j'\ \text{ in } (n-1]\}.$$

$$E_{n,c}=\begin{cases} E'_n, & \text{if } (n,c)\neq(3,1),\\ E'_3\cup\{x_{0;0,1}z\mid z\in I\}, & \text{if } (n,c)=(3,1). \end{cases}$$

Assume $(\widetilde{\ell}(S_0),\widetilde{\ell}(s_n))=(2,c)$ with $c\in\{1,3\}$. From the above discussion, for any $L\in L'_{n,6,1}$, we see that L contains a minimum element in the set $E_{n,c}$ and that for any $x\in L$, there exists a sequence $x_0=x,x_1,...,x_r$ in L with some $r\geqslant 0$

such that $x_{i-1}x_i^{-1} \in S$ for every $i \in [r]$ and $x_r \in E_{n,c}$. The map $w \mapsto \Gamma_w$ gives rise to a bijection from the set $E_{n,c}$, $(n,c) \neq (3,1)$, (respectively, $E_{3,1}$; $\mathcal{L}'_{n,6,2}$) to the set of ideals of the digraphs (a)-(c) (respectively, (b), (d); (e)) in Figure 3. We see from the digraphs in Figure 3 that any two different elements in the set $E_{n,3}$ (respectively, $E_{n,1} \cup \mathcal{L}'_{n,6,2}$) have different right generalized τ-invariants and hence belong to different left cells of $\widetilde{B}_n$ by Lemma 2.8. This implies that any $L \in L'_{n,6,i}$, $i \in \{1,2\}$, is left-connected, (1) follows.

Hence (2)-(3) follows just by enumerating the sets $E_{n,c}$ and $\mathcal{L}'_{n,6,2}$. □

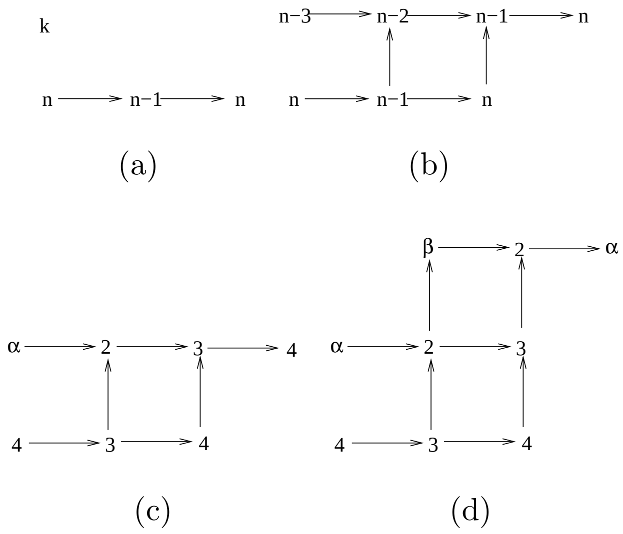

Figure 4.

where $k \in (n-4]$ in (a); $n > 4$ in (a)-(b); $\{\alpha, \beta\} = \{0, 1\}$ and $n = 4$ in (c)-(d).

3.6. Now assume $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3)$ and $n \geqslant 4$. Let $L''_{n,6}$ be the set of all left cells L of $\widetilde{B}_n$ with $L \cap \mathcal{L}''_{n,6} \neq \emptyset$. If $n > 4$ then any $L \in L''_{n,6}$ contains some $w \in \mathcal{L}''_{n,6}$ with Γ_w either the digraph (a) or an ideal of the digraph (b) in Figure 4, where Γ_w contains $n \longrightarrow n-1 \longrightarrow n$ as its sub-digraph. If $n = 4$ then any $L \in L''_{n,6}$ contains some $w \in \mathcal{L}''_{n,6}$ with Γ_w an ideal of one of the digraphs (c)-(d) in Figure 4, where Γ_w contains $4 \longrightarrow 3 \longrightarrow 4$ as its sub-digraph.

Lemma 3.7. *When* $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3)$ *and* $n \geqslant 4$, *let* $l''_{n,6} = |L''_{n,6}|$. *Then*

$$l''_{n,6} = \begin{cases} n+1, & \textit{if } n > 4, \\ 16, & \textit{if } n = 4, \\ 0, & \textit{if } n = 3. \end{cases}$$

Proof. By Lemmas 2.2 and 2.8, this follows by counting the ideals Γ' of the digraphs in Figure 4, where Γ' contains $n \longrightarrow n-1 \longrightarrow n$ as its sub-digraph. $\square$

3.8. Let $L_{n,6,j}$ (respectively, $L_{6,1i}$) be the set of all left cells L of $\widetilde{B}_n$ such that L contains some elements in $\mathcal{L}_{n,6,j}$ for $j \in \{1,2,3\}$ (respectively, in $\mathcal{L}_{6,1i}$ for $i \in \{1,2\}$). Then any L in $L_{6,11}$ (respectively, $L_{6,12}$) contains an element w in $\mathcal{L}_{6,11}$ (respectively, $\mathcal{L}_{6,12}$) with Γ_w the digraph (a) (respectively, an ideal of the digraph (b)) in Figure 5.

(a) (b)

Figure 5.

where i, j, k in (a) satisfy $\{s_i, s_j, s_k\} \in \Xi_1$ and $n-1 \notin \{i,j,k\}$; i, j in (b) satisfy $\{s_i, s_j, t_{n-1}\} \in \Xi_1$.

Lemma 3.9. *Let $l_{n,6,1} = |L_{n,6,1}|$. Then $l_{n,6,1} = 0$ unless $n > 7$. When $n > 7$, we have*

$$l_{n,6,1} = \begin{cases} \frac{1}{6}n(n-5)(n-7), & \textit{if } (\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,1), \\ \frac{1}{6}(n-2)(n-6)(n-7), & \textit{if } (\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3). \end{cases}$$

Proof. When $n \leqslant 7$, we have $L_{n,6,1} = \emptyset$ since $\Xi_1 = \emptyset$. Now assume $n > 7$. Consider a set $E = \{s_i, s_j, s_k\} \in \Xi_1$. We may assume the node s_j locates between s_i and s_k in the Coxeter graph of $\widetilde{B}_n$. Hence $j \in [4, n-4]$. When s_j is fixed, the number of the choices of the set $\{s_i, s_k\}$ is $(j-2)(n-3-j)$. This implies that $|\Xi_1| = \sum_{j=4}^{n-4}(j-2)(n-3-j) = \frac{1}{6}(n-2)(n-6)(n-7)$. Next consider

$E = \{s_i, s_j, s_{n-1}\} \in \Xi_{11}$. When s_j is fixed, the number of the choices of s_i is $j-2$. So $|\Xi_{11}| = \sum_{j=4}^{n-4}(j-2) = \frac{1}{2}(n-4)(n-7)$.

Any left cell L of $\widetilde{B}_n$ in $L_{n,6,1}$ contains a unique minimum element (denote it by w_L) by Lemma 2.9. Let L and L' be two left cells of $\widetilde{B}_n$ in $L_{n,6,1}$ with $\mathcal{L}(w_L) = E$ and $\mathcal{L}(w_{L'}) = E'$. If either $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3)$, or $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,1)$ and $s_{n-1} \notin E$ then $L = L'$ if and only if $E = E'$. On the other hand, if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,1)$ and $s_{n-1} \in E$, then there exist exactly two left cells L satisfying $\mathcal{L}(w_L) = E$ (see the digraphs (a) and (b) in Figure 5).

Based on the above observation, we have $l_{n,6,1} = |\Xi_1| = \frac{1}{6}(n-2)(n-6)(n-7)$ if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3)$, and $l_{n,6,1} = |\Xi_1| + |\Xi_{11}| = \frac{1}{6}n(n-5)(n-7)$ if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,1)$. $\square$

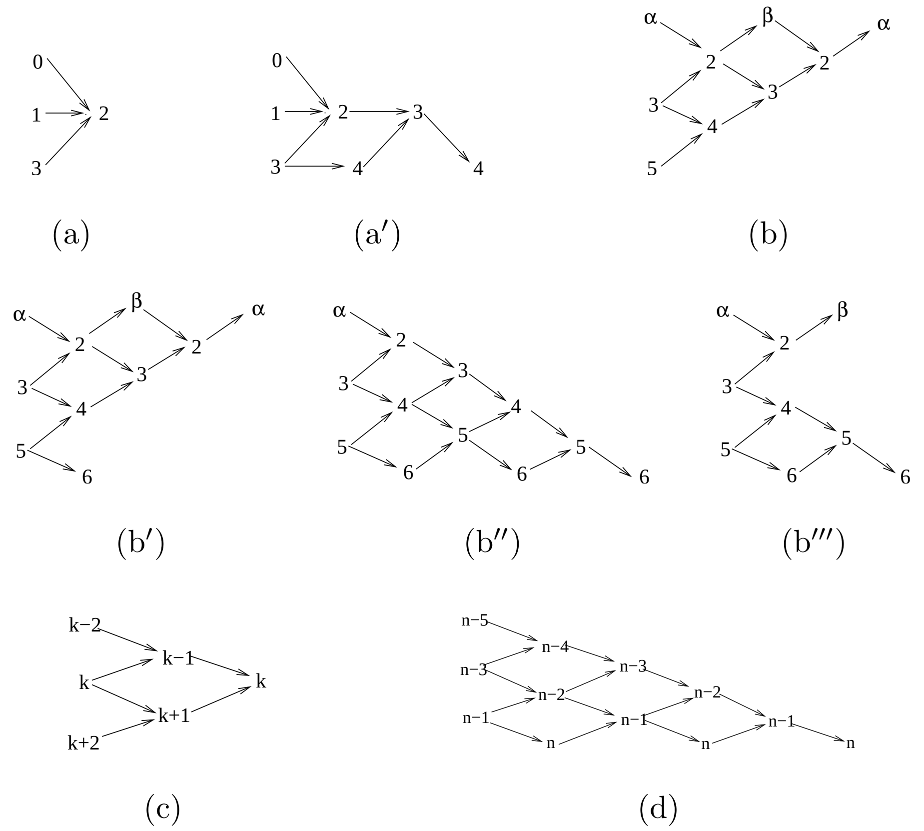

Figure 6

where $\{\alpha, \beta\} = \{0, 1\}$ in (b)-(b‴); $k \in [4, n-3]$ in (c).

Lemma 3.10. *Let $l_{n,6,2} = |L_{n,6,2}|$. Then $l_{3,6,2} = 0$ and $l_{4,6,2} = 6$ and $l_{5,6,2} = 2$. On the other hand, we have*

$$l_{n,6,2} = \begin{cases} 5n-8, & \text{if } (\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3) \text{ and } n > 5, \\ 5n+7, & \text{if } (\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,1) \text{ and } n > 6, \\ 66, & \text{if } (\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,1) \text{ and } n = 6. \end{cases}$$

Proof. Any $L \in L_{n,6,2}$ has a unique minimum element (denote it by w_L) by Lemma 2.9. The equation $l_{3,6,2} = 0$ is obvious. Then the equation $l_{4,6,2} = 6$ (respectively, $l_{5,6,2} = 2$) holds by the fact that the digraph Γ_w corresponding to any w in $\mathcal{L}_{4,6,2}$ (respectively, $\mathcal{L}_{5,6,2}$) is an ideal of the digraph (a′) (respectively, (a)) in Figure 6. Now assume $n > 5$ and observe the digraphs in Figure 6. The number l_1 of $L \in L_{n,6,2}$ with $\mathcal{L}(w_L) = \{s_0, s_1, s_3\}$ is 2 by counting the ideals of the digraph (a). Let l_2 be the number of $L \in L_{n,6,2}$ with $\mathcal{L}(w_L) \in \{\{s_0, s_3, s_5\}, \{s_1, s_3, s_5\}\}$. Then $l_2 = 20$ if either $n > 6$, or $n = 6$ and $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_6)) = (2,3)$ by counting the ideals of the digraph (b), and $l_2 = 64$ if $n = 6$ and $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_6)) = (2,1)$ by counting the ideals of the digraphs (b′), (b″), (b‴) (Note that the repeated ones should be taken off). When $n > 6$, the number l_3 of $L \in L_{n,6,2}$ with $\mathcal{L}(w_L) \in \{\{s_{k-2}, s_k, s_{k+2}\} \mid k \in [4, n-3]\}$ is $5(n-6)$ if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_6)) = (2,3)$ by counting the ideals of the digraph (c) and $5(n-3)$ if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_6)) = (2,1)$ by counting the ideals of the digraphs (c), (d). We have $l_{n,6,2} = l_1 + l_2 + l_3$. Therefore $l_{n,6,2} = 5n - 8$ if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3)$. If $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,1)$, then $l_{6,6,2} = 66$ and $l_{n,6,2} = 5n+7$ for $n > 6$. □

Lemma 3.11. *Let $l_{n,6,3} = |L_{n,6,3}|$. Then $l_{3,6,3} = l_{4,6,3} = 0$. When $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3)$, we have $l_{5,6,3} = 3$, $l_{6,6,3} = 7$ and $l_{n,6,3} = \frac{1}{2}(5n^2 - 35n + 44)$ if $n \geqslant 7$. When $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,1)$, we have $l_{5,6,3} = 8$, $l_{6,6,3} = 13$ and $l_{n,6,3} = \frac{1}{2}(5n^2 - 21n - 12)$ if $n \geqslant 7$.*

Proof. We see from Figure 7 that $l_{3,6,3} = l_{4,6,3} = 0$. If $n \in \{5, 6\}$ and if L is a left cell of $\widetilde{B}_n$ containing some $w \in \mathcal{L}_6$ with $\mathcal{L}(w) \in \Xi_3$, then the digraph Γ_w must be an ideal of one of the digraphs (c), (g) in Figure 7. This implies that $l_{5,6,3} = 3$, $l_{6,6,3} = 7$ if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3)$ and that $l_{5,6,3} = 8$, $l_{6,6,3} = 13$

if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,1)$. Now assume $n \geqslant 7$. If $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3)$ then $l_{n,6,3} = 2(n-5)(n-6) + 6(n-6) + \frac{1}{2}(n-4)(n+1) = \frac{1}{2}(5n^2 - 35n + 44)$ by counting the ideals of the digraphs (a)-(c) in Figure 7. If $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,1)$ then $l_{n,6,3} = \frac{1}{2}(5n^2 - 35n + 44) + 6 + 2(n-7) + 4(n-5) + n = \frac{1}{2}(5n^2 - 21n - 12)$ by counting the ideals of the digraphs (a)-(g) in Figure 7. $\square$

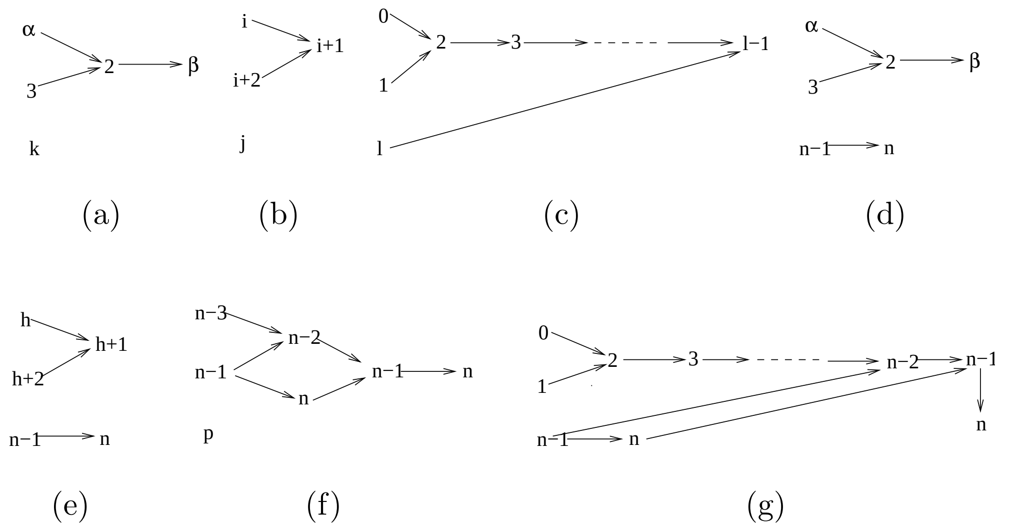

Figure 7

where $k \in [6, n-1]$ in (a); $j \in (i-3] \cup [i+5, n-1]$ and $(j,i) \neq (0,3)$ in (b) with $i \in [2, n-3]$ if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3)$ and with $i \in [2, n-4]$, $j \neq n-1$ if $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,1)$; $l \in [4, n-1]$ in (c); $n \geqslant 7$ in (d); $h \in [2, n-6]$ in (e); $p \in (n-6]$ in (f); $\{\alpha, \beta\} = \{0, 1\}$ in (a) and (d).

When $(L(S_0), L(s_n)) = (2, c)$, $c \in \{1, 3\}$, denote by $m_{2,c}(n,k)$ the number of left cells L of $\widetilde{B}_n$ with $a(L) = k$.

Theorem 3.12.

$$m_{2,1}(n,6) = \begin{cases} \binom{n+2}{3} + (n+1)^2, & \textit{if } n \geqslant 7, \\ 127, & \textit{if } n = 6, \\ 45, & \textit{if } n = 5, \\ 30, & \textit{if } n = 4, \\ 18, & \textit{if } n = 3. \end{cases}$$

$$m_{2,3}(n,6)=\begin{cases}\binom{n+1}{3}+n^2+1, & \textit{if } n\geqslant 6,\\ 36, & \textit{if } n=5,\\ 34, & \textit{if } n=4,\\ 9, & \textit{if } n=3.\end{cases}$$

Proof. We have $l_{n,6}=l'_{n,6,1}+l''_{n,6}+\sum_{i=1}^{3}l_{n,6,i}$ if $(\widetilde{\ell}(S_0),\widetilde{\ell}(s_n))=(2,3)$ and $l_{n,6}=\sum_{j=1}^{2}l'_{n,6,j}+\sum_{i=1}^{3}l_{n,6,i}$ if $(\widetilde{\ell}(S_0),\widetilde{\ell}(s_n))=(2,1)$. So our result follows by Lemmas 3.5, 3.7, 3.9, 3.10 and 3.11. $\square$

§4. The numbers $m_{2,c}(n,a)$ for $c\in\{1,3\}$ and $a\leqslant 5$.

We assume that $(L(S_0),L(s_n))$ is $(2,1)$ in 4.1-4.5 and $(2,3)$ in 4.6-4.10.

4.1. Now we consider the left cells of $\widetilde{B}_n$ in the set $W_{(5)}$. Let $E=\{s_is_js_n, s_ks_{n-1}s_ns_{n-1}\mid i,j\in(n-2], o(s_is_j)=2, k\in(n-3]\}$. By Lemma 2.2, we see that any $w\in\widetilde{B}_n$ is in $\mathcal{L}_5$ if and only if the following conditions hold:

(1) $w=z\cdot y$ for some $y\in\widetilde{B}_n$ and some $z\in E$;

(2) $w\neq x\cdot z\cdot y$ for any $x,y\in\widetilde{B}_n$ and any $z\in E$ with $x\neq 1$;

(3) $w\neq x'\cdot w_I\cdot y'$ for any $x',y'\in\widetilde{B}_n$ and any $I\subset S$ with $\widetilde{\ell}(w_I)\geqslant 6$.

Hence the ideals of the following digraphs correspond to the elements in $\mathcal{L}_5$.

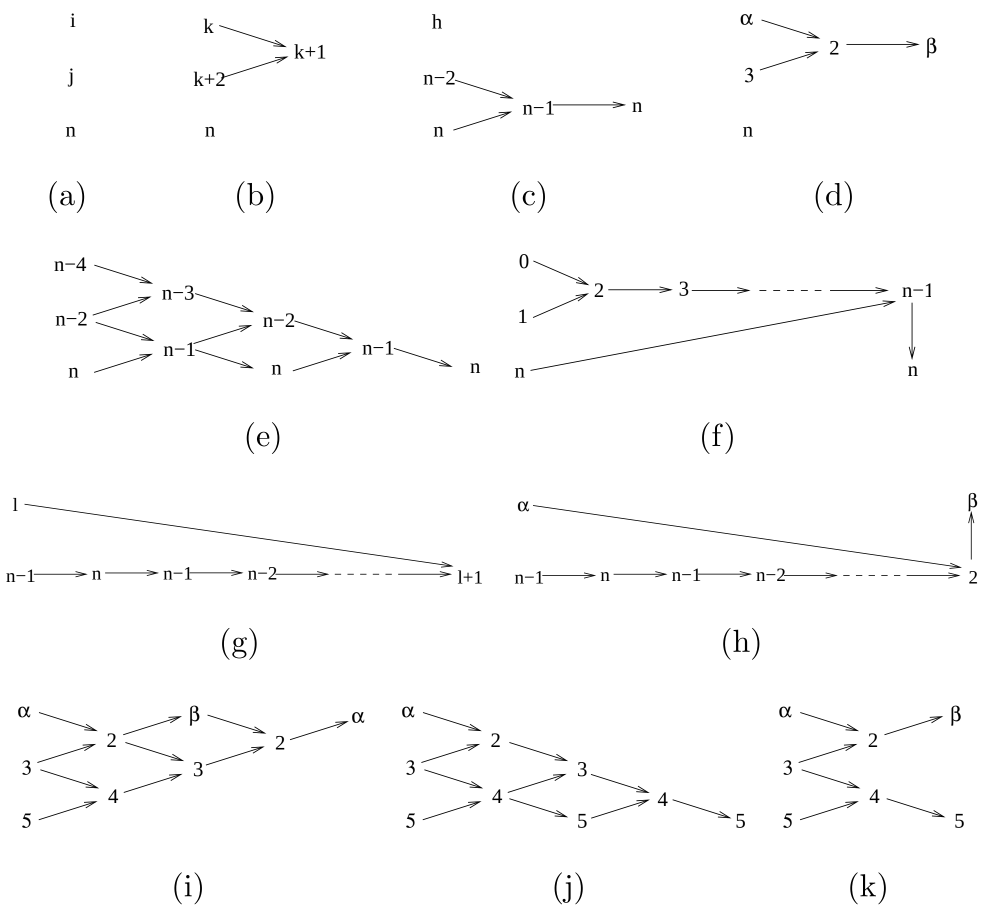

Figure 8.

where $i, j \in (n-3]$ with $j - i \geqslant 3$ and $(i, j) \neq (0, 3)$ in (a); $k \in [2, n-5]$ in (b); $h \in (n-5]$ in (c); $l \in [2, n-3]$ in (g); $\{\alpha, \beta\} = \{0, 1\}$ in (d), (h)-(k); also, $n > 5$ in (a)-(e), $n = 5$ in (i)-(k), $n \geqslant 5$ in (g), $n \geqslant 4$ in (h) and $n \geqslant 3$ in (f). In Figure 8, we see that the number of digraphs is $\frac{1}{2}(n-3)(n-6)$ in (a); $n-6$ in (b); $n-4$ in (c), (g); 2 in each of (d), (h), (i), (j), (k); 1 in each of (e), (f). Hence by counting the ideals of the digraphs in Figure 8 (where a considered ideal in any of the digraphs (g), (h) is required to contain the sub-digraph $n-1 \longrightarrow n \longrightarrow n-1$) and by Lemma 2.9, we get

$$m_{2,1}(n,5)=\begin{cases} n^2+n+1, & \text{if } n>5,\\ 47, & \text{if } n=5,\\ 10, & \text{if } n=4,\\ 3, & \text{if } n=3.\end{cases}$$

4.2. By Lemma 2.2, any $w\in\widetilde{B}_n$ is in $\mathcal{L}_4$ if and only if the following conditions hold:

(1) $w=s_is_j\cdot y$ for some $y\in\widetilde{B}_n$ and some $i,j\in(n-1]$ with $o(s_is_j)=2$;

(2) $w\neq x'\cdot w_I\cdot y'$ for any $x',y'\in\widetilde{B}_n$ and any $I\subseteq S$ with either $\widetilde{\ell}(w_I)>4$ or $\widetilde{\ell}(w_I)=4$ and $x'\neq 1$.

Observe the digraphs in Figure 9.

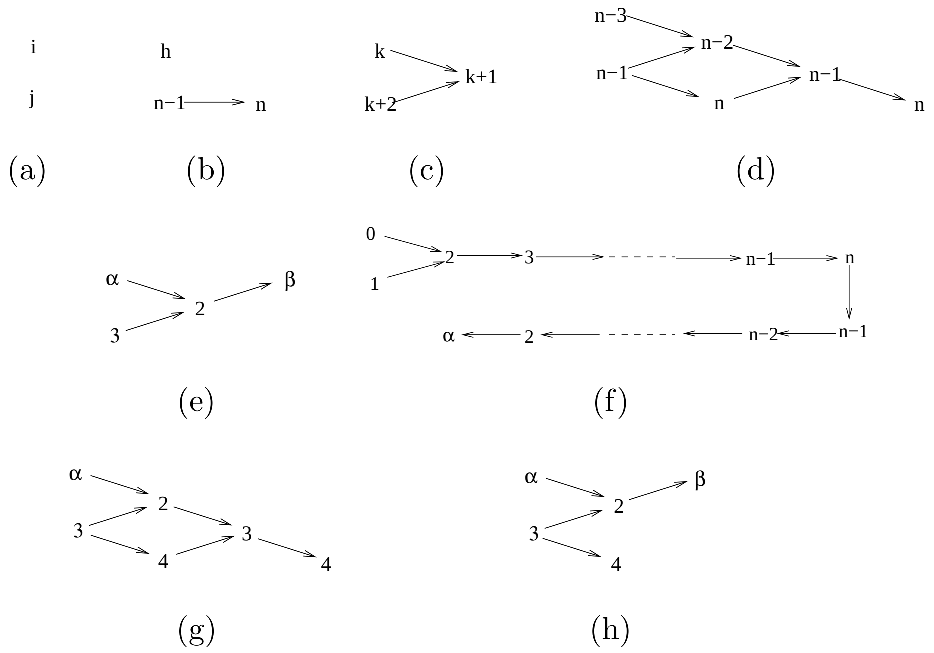

Figure 9.

where $i,j\in(n-2]$ with $j-i\geqslant 3$ and $(i,j)\neq(0,3)$ in (a); $h\in(n-4]$ and $n>4$ in (b), (e); $k\in[2,n-4]$ in (c); $\{\alpha,\beta\}=\{0,1\}$ in (e)-(h); $n=4$ in (g)-(h).

When $n>4$ (respectively, $n=4$; $n=3$), each left cell L of $\widetilde{B}_n$ with $a(L)=4$ contains an element $w\in\mathcal{L}_4$ with Γ_w an ideal in one of the digraphs (a)-(f) (respectively, (f)-(h); (f)). By Lemma 2.9, we get

$$m_{2,1}(n,4) = \begin{cases} \binom{n+1}{2} + 2n + 1, & \text{if } n > 4, \\ 24, & \text{if } n = 4, \\ 6, & \text{if } n = 3. \end{cases}$$

4.3. Let $E = \{s_k s_n, s_{n-1}s_n s_{n-1} \mid k \in (n-2]\}$. Then by Lemma 2.2, any $w \in \widetilde{B}_n$ is in $\mathcal{L}_3$ if and only if the following conditions hold.

(1) $w = z \cdot y$ for some $y \in \widetilde{B}_n$ and some $z \in E$;

(2) $w \neq x \cdot z \cdot y$ for any $x, y \in \widetilde{B}_n$ and any $z \in E$ with $x \neq 1$;

(3) $w \neq x' \cdot w_I \cdot y'$ for any $x', y' \in \widetilde{B}_n$ and any $I \subseteq S$ with $\widetilde{\ell}(w_I) > 3$.

Observe the digraphs in Figure 10.

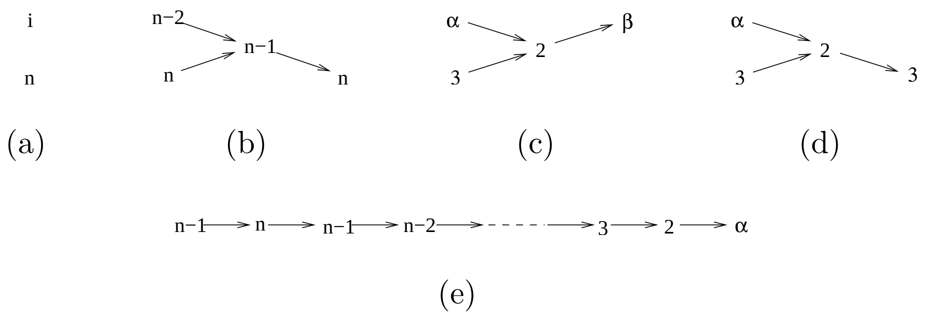

Figure 10.

where $i \in (n-3]$ and $n > 3$ in (a); $\{\alpha, \beta\} = \{0, 1\}$ in (c)-(e); Γ_w contains $n-1 \longrightarrow n \longrightarrow n-1$ as its sub-digraph when being an ideal of the digraph (e).

When $n > 3$ (respectively, $n = 3$), each left cell L of $\widetilde{B}_n$ with $a(L) = 3$ contains an element $w \in \mathcal{L}_3$ with Γ_w an ideal of one of the digraphs (a), (b), (e) (respectively, (c)-(e)) in Figure 10. By Lemma 2.9, we get

$$m_{2,1}(n,3) = \begin{cases} 2n+1, & \text{if } n > 3, \\ 11, & \text{if } n = 3. \end{cases}$$

4.4. We see that $\mathcal{L}_2 = S_0 \cup \{s_{n-1}s_n\}$ is a representative set for the left cells L of $\widetilde{B}_n$ with $a(L) = 2$ and that there is a unique left cell L' of $\widetilde{B}_n$ with $a(L') = 1$, where $s_n \in L'$.

4.5. The results in 4.1-4.4 and in the first part of Theorem 3.12 are summarized in Table 1.

a	$m_{2,1}(n,a)$, $n>6$	$m_{2,1}(6,a)$	$m_{2,1}(5,a)$	$m_{2,1}(4,a)$	$m_{2,1}(3,a)$
6	$\binom{n+2}{3}+(n+1)^2$	127	45	30	18
5	n^2+n+1	6^2+6+1	47	10	3
4	$\binom{n+1}{2}+2n+1$	$\binom{6+1}{2}+2\cdot 6+1$	$\binom{5+1}{2}+2\cdot 5+1$	24	6
3	$2n+1$	$2\cdot 6+1$	$2\cdot 5+1$	$2\cdot 4+1$	11
2	$n+1$	$6+1$	$5+1$	$4+1$	$3+1$
1	1	1	1	1	1

Table 1.

4.6. In the remaining part of the section, we assume $(\widetilde{\ell}(S_0), \widetilde{\ell}(s_n)) = (2,3)$.

Let $E = \{s_k s_n \mid k \in (n-2]\}$. By Lemma 2.2, any $w \in \widetilde{B}_n$ is in $\mathcal{L}_5$ if and only if the following conditions hold.

(1) $w = z \cdot y$ for some $y \in \widetilde{B}_n$ and some $z \in E$;

(2) $w \neq x \cdot z \cdot y$ for any $x, y \in \widetilde{B}_n$ and any $z \in E$ with $x \neq 1$;

(3) $w \neq x' \cdot w_I \cdot y'$ for any $x', y' \in \widetilde{B}_n$ and any $I \subseteq S$ with $\widetilde{\ell}(w_I) > 5$.

Observe the digraphs in Figure 11.

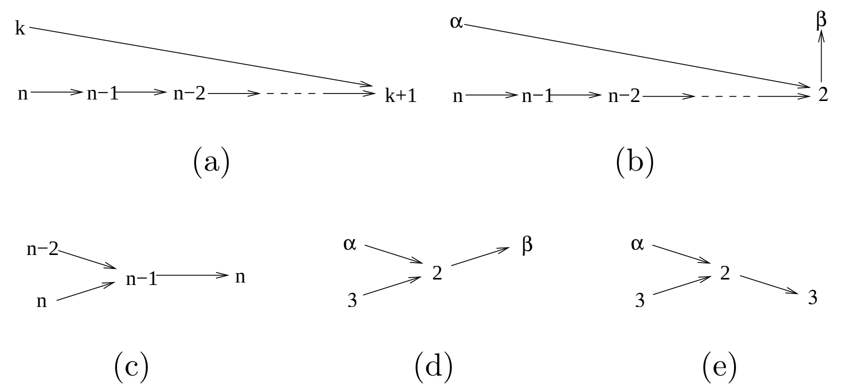

Figure 11.

where $k \in [2, n-3]$ in (a) and $\{\alpha, \beta\} = \{0, 1\}$ in (b), (d)-(e); $n = 3$ in (d)-(e).

When $n > 4$ (respectively, $n = 4$; $n = 3$), each left cell L of $\widetilde{B}_n$ with $a(L) = 5$ contains an element $w \in \mathcal{L}_5$ with Γ_w is an ideal of one of the digraphs (a)-(c) (respectively, (b)-(c); (d)) in Figure 11. By Lemma 2.9, we get

$$m_{2,3}(n,5) = \begin{cases} \binom{n+1}{2} + 1, & \text{if } n > 3, \\ 8, & \text{if } n = 3. \end{cases}$$

4.7. Let $E = \{s_i s_j \mid i, j \in (n-1], o(s_i s_j) = 2\}$. By Lemma 2.2, any $w \in \widetilde{B}_n$ is in $\mathcal{L}_4$ if and only if the following conditions hold.

(1) Either $\mathcal{L}(w) = \{s_i, s_j\} \subseteq S_0$ with $o(s_i s_j) = 2$, or $w = s_n s_{n-1} s_n$;

(2) $w \notin \{x \cdot y, s_n s_{n-1} s_n \cdot x \mid x, y \in \widetilde{B}_n, x \neq 1, \mathcal{L}(y) \in E\}$;

(3) $w \neq x' \cdot w_I \cdot y'$ for any $x', y' \in \widetilde{B}_n$ and any $I \subseteq S$ with $\widetilde{\ell}(w_I) > 4$.

Observe the digraphs in Figure 12.

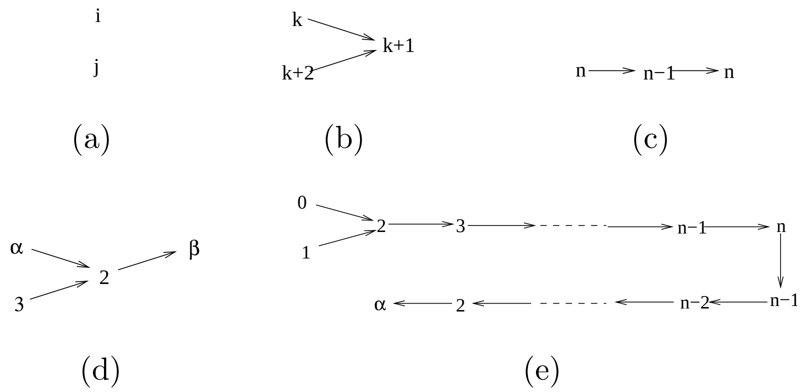

Figure 12.

where $i, j \in (n-1]$ with $j - i \geqslant 3$ and $(i.j) \neq (0,3)$ in (a); $k \in [2, n-3]$ in (b); $\{\alpha, \beta\} = \{0, 1\}$ in (d)-(e); $n > 3$ in (d).

When $n > 4$ (respectively, $n = 4$; $n = 3$), each left cell L of $\widetilde{B}_n$ with $a(L) = 4$ contains an element $w \in \mathcal{L}_4$ with Γ_w an ideal of one of the digraphs (a)-(e) (respectively, (c)-(e); (c), (e)) in Figure 12. By Lemma 2.9, we get

$$m_{2,3}(n,4) = \begin{cases} \binom{n+2}{2}, & \text{if } n > 3, \\ 7, & \text{if } n = 3. \end{cases}$$

4.8. By Lemma 2.2, any $w \in \widetilde{B}_n$ is in $\mathcal{L}_3$ if and only if the following conditions hold.

(1) $\mathcal{L}(w)=\{s_n\}$;

(2) $w\neq x'\cdot w_I\cdot y'$ for any $x',y'\in\widetilde{B}_n$ and any $I\subseteq S$ with either $\widetilde{\ell}(w_I)>3$, or $I=\{s_n\}$ and $x'\neq 1$.

Observe the digraph in Figure 13.

$$n\longrightarrow n-1\longrightarrow n-2\longrightarrow ----\longrightarrow 2\longrightarrow \alpha$$

Figure 13.

where $\alpha\in\{0,1\}$. Each left cell L of $\widetilde{B}_n$ with $a(L)=3$ contains an element $w\in\mathcal{L}_3$ with Γ_w an ideal of the digraph in Figure 13. By Lemma 2.9, we get

$$m_{2,3}(n,3)=n+1.$$

4.9. We see that $\mathcal{L}_2=S_0$ is a representative set for the left cells L of $\widetilde{B}_n$ with $a(L)=2$ and that there is no left cell L of $\widetilde{B}_n$ satisfying $a(L)=1$.

4.10. The results in 4.6-4.9 and in the second part of Theorem 3.12 are summarized in Table 2.

a	$m_{2,3}(n,a)$, $n>5$	$m_{2,3}(5,a)$	$m_{2,3}(4,a)$	$m_{2,3}(3,a)$
6	$\binom{n+1}{3}+n^2+1$	36	34	9
5	$\binom{n+1}{2}+1$	$\binom{6}{2}+1$	$\binom{5}{2}+1$	8
4	$\binom{n+2}{2}$	$\binom{7}{2}$	$\binom{6}{2}$	7
3	$n+1$	6	5	4
2	n	5	4	3
1	0	0	0	0

Table 2.

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