A new extension of a “divergent” Ramanujan-type supercongruence

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Abstract. We give a new extension of a “divergent” Ramanujan-type supercongruence of Guillera and Zudilin by establishing a $q$-analogue of this result. Our proof makes use of the “creative microscoping” method, which was introduced by the second author and Zudilin in 2019. We also present a similar extension of the (L.2) supercongruence of Van Hamme in the modulus $p^2$ case.

Keywords: supercongruence; basic hypergeometric series; cyclotomic polynomials; creative microscoping.

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1 Introduction

Employing the Wilf–Zeilberger (WZ) method, Guillera and Zudilin [5] established the following supercongruence:

$$\sum_{k=0}^{(p-1)/2} (-1)^k \binom{1/3}{k/3} (3k + 1)2^{3k} \equiv p(-1)^{(p-1)/2} \pmod{p^3} \quad \text{for } p > 2. \quad (1.1)$$

Here and in what follows, $p$ is a prime, and $(a)_b = \Gamma(a+b)/\Gamma(a)$ denotes the Pochhammer symbol also for $b$ not being a non-negative integer. In the spirit of [19], the supercongruence (1.1) corresponds to a “divergent” Ramanujan-type series for $1/\pi$:

$$\sum_{k=0}^{\infty} (-1)^k \binom{1/3}{k/3} (3k + 1)2^{3k} \sim \frac{1}{\pi} \quad (1.2)$$

(see [5, (47)]). Note that the left-hand side of (1.2) has to be understood as the analytic continuation of the corresponding hypergeometric series. Namely, the formula (1.2) can be written as

$$\frac{1}{2\pi i} \int_{-\frac{1}{2}+\infty i}^{-\frac{1}{2}-\infty i} \frac{(\frac{1}{3})^s}{(1)^s} \Gamma(-s)(3s + 1)2^{3s} ds = \frac{1}{\pi}.\quad (1.2)$$

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See [4] for more divergent hypergeometric formulas for $1/\pi$ and $1/\pi^2$.

Sun [14, Conjecture 5.1(ii)] formulated a refinement of (1.1) modulo $p^4$, which was later confirmed by Chen, Xie, and He [2]. A $q$-analogue of (1.1) was obtained by the second author [6] using the $q$-WZ method. A further generalization of this $q$-analogue was conjectured by the second author and Zudilin in [9, Conjecture 4.6] where two particular cases were settled by themselves. The second author and Schlosser [8, Theorem 6.1] finally confirmed this conjecture.

In this paper, we give the following new generalization of (1.1): for $p > 2$ and $0 \leq s \leq (p - 1)/8$,

$$
\sum_{k=2s}^{(p-1)/2+2s} (-1)^k \frac{(1/2)^{k-s}(1/2)^{k+s}(1/2)^k}{(k-2s)!(k+2s)!} (3k + 1)2^{3k} \equiv p(-1)^{(p-1)/2} \pmod{p^3}.
$$

It is easy to see that the $s = 0$ case of (1.3) reduces to (1.1). We shall prove (1.3) by establishing the following $q$-congruence.

**Theorem 1.1.** Let $n > 1$ be an odd integer and let $0 \leq s \leq (n - 1)/8$. Then, modulo $\Phi_n(q)^3$,

$$
\sum_{k=2s}^{(n-1)/2+2s} (-1)^k [3k + 1] \frac{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}(q^2;q^2)_k}{(q;q)_{k-2s}(q;q)_{k+2s}(q;q)_k} \equiv (-q)^{(n-1)^2/4[n]}.
$$

Here and throughout the paper,

$$(a; q)_n = \begin{cases} 
(1 - a)(1 - aq) \cdots (1 - aq^{n-1}), & \text{if } n = 1, 2, \ldots, \\
1, & \text{if } n = 0, \\
\frac{1}{(1 - aq^{-1})(1 - q^{-2}) \cdots (1 - aq^n)} & \text{if } n = -1, -2, \ldots
\end{cases}
$$

is the $q$-shifted factorial, and $[n] = 1 + q + \cdots + q^{n-1}$ denotes the $q$-integer. For convenience, we will also adopt the compact $q$-notation for a product of $q$-shifted factorials: $(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_m; q)_n$. Moreover, $\Phi_n(q)$ represents the $n$-th cyclotomic polynomial in $q$, i.e.,

$$
\Phi_n(q) = \prod_{\gcd(k,n)=1} (q - \zeta^k),
$$

where $\zeta$ is an $n$-th primitive root of unity. It is not hard to see that (1.3) follows from (1.4) by taking $n = p$ and $q \to 1$.

Note that a similar $q$-supercongruence also with the parameter $s$ was given by the second author [7]. For some other important work on $q$-supercongruences, we refer the reader to [1,10–12,16–18,20].
The second author and Zudilin [9, Theorem 4.4] proved that, for any odd integer \( n > 1 \),
\[
\sum_{k=0}^{(n-1)/2} (-1)^k [6k + 1] \frac{(q; q^2)_k^3}{(q^4; q^4)_k^3} \equiv (-q)^{-((n-1)(n+5)/8 \lfloor n \rfloor)} \pmod{n \Phi_n(q^2)},
\]
which is a \( q \)-analogue of the (L.2) supercongruence of Van Hamme [15]:
\[
\sum_{k=0}^{(p-1)/2} (-1)^k \left( \frac{1}{2} \right)_k^3 \equiv p \left( \frac{-2}{p} \right) \pmod{p^3} \quad \text{for } p > 2,
\]
where \( \left( \frac{\cdot}{p} \right) \) is the Legendre symbol modulo \( p \).

In this paper, we shall give the following generalization of (1.5) in the modulus \( \Phi_n(q^2) \) case.

**Theorem 1.2.** Let \( n > 1 \) be an odd integer and let \( 0 \leq s \leq (n - 1)/4 \). Then, modulo \( \Phi_n(q^2) \),
\[
\sum_{k=s}^{(n-1)/2+s} (-1)^k [6k + 1] \frac{(q; q^2)_{k-2s}(q^2)_{k+2s}(q^2)_k}{(q^4; q^4)_{k-s}(q^4)_{k+s}(q^4)_k} \equiv (-q)^{-((n-1)(n+5)/8 \lfloor n \rfloor)}.
\]

Note that the \( q \)-congruence (1.6) does not hold \( \Phi_n(q^2) \) in general. Letting \( n = p \) and \( q \to 1 \) in (1.6), we obtain the following supercongruence: for \( p > 2 \) and \( 0 \leq s \leq (p - 1)/4 \),
\[
\sum_{k=s}^{(p-1)/2+s} (-1)^k \frac{\left( \frac{1}{2} \right)_{k-2s} \left( \frac{1}{2} \right)_{k+2s} \left( \frac{1}{2} \right)_k}{(k - s)!(k + s)!k!2^k} (6k + 1) \equiv p(-1)^{(p-1)/2} \pmod{p^2},
\]
which is an extension of (1.5) modulo \( p^2 \).

We shall prove Theorems 1.1 and 1.2 in Sections 2 and 3, respectively.

## 2 Proof of Theorem 1.1

Following Gasper and Rahman’s monograph [3], the basic hypergeometric series \( r+1 \phi_r \) with \( r + 1 \) upper parameters \( a_1, \ldots, a_{r+1} \), \( r \) lower parameters \( b_1, \ldots, b_r \), base \( q \), and argument \( z \) is defined as
\[
\phi_{r+1} \left[ \begin{array}{c} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{array} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k z^k}{(q, b_1, \ldots, b_r; q)_k}.
\]

Then Jackson’s \( q \)-analogue of Dougall’s \( \gamma F_6 \) summation formula (see [3, Appendix (II.22)]) can be written as
\[
\phi_{7} \left[ \begin{array}{c} a, qa^2, -qa^2, b, c, d, e, q^{-n} \\ a^2, -a^2, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{array} ; q, q \right]
\]

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be stated as follows:

We also need the following quadratic transformation formula due to Gasper and Rahman [3, (3.8.13)] can be stated as follows:

\[
\sum_{k=0}^{\infty} (a; q^2)_k (1 - aq^{3k})(d, aq/d; q^2)_k (b, c, aq/bc; q)_k q^k = (aq^2, bq, cq, aq/bc; q^2)_\infty \left[ \begin{array}{l} b, c, aq/bc \\ dq, aq^2/a \end{array} \right]_2 (q^2, q^2)_\infty^{3 \phi_2}.
\]

(2.2)

We also need the following quadratic transformation formula due to Gasper and Rahman (see [3, (3.8.14)]):

\[
\sum_{k=0}^{\infty} \frac{1 - acq^{3k}}{1 - ac} (aq; q^2)_k (f, a^2c^2q^{2n+1}/f, q^{-2n}; q^2)_k q^k = (acq; q)_2 (acq^2q/2b, abq/f; q^2)_n
\]

\[
\times 10 W_9(ac^2/b; f, ac/b, c, cq/b, eq^2/b, a^2c^2q^{2n+1}/f, q^{-2n}; q^2, q^2),
\]

(2.3)

where

\[
r + 3 W_{r+2}(a_0; a_1, a_2, \ldots, a_r; q, z) = \sum_{k=0}^{\infty} \frac{(1 - a_0q^2k)}{(1 - a_0)(a_0; q)_k (a_1; q)_k \cdots (a_r; q)_k} z^k.
\]

Lemma 2.1. Let \( n > 1 \) be an odd integer. Let \( a \) be an indeterminate and let \( s \geq 0 \). Then

\[
\sum_{k=2s}^{(n-1)/2+2s} (-1)^{k} (1 - q^{3k+1-n})(aq; q^2)_k \frac{(q^{1-n}; q^2)_k}{(1 - q^{-1})(q; q)_k} (q^{-1+n}/a; q^2)_k = 0.
\]

(2.4)

Proof. It is easy to see that the left-hand side of (2.4) can be written as

\[
\sum_{k=0}^{(n-1)/2} (-1)^{k+2s} (1 - q^{3k+6s+1-n})(aq; q^2)_k (q^{1-n}/a; q^2)_k q^k = (aq; q^2)_s (q^{-1+n}/a; q^2)^2_s
\]

\[
\times \sum_{k=0}^{(n-1)/2} (-1)^{k} (1 - q^{3k+6s+1-n})(aq^{2s+1}, q^{6s+1-n}, q^{4s+1-n}/a; q^2)_k
\]

(2.5)

If \( s \geq (n-1)/6 \), then either \((q^{-1+n}; q^2)_s = 0 \) or \((1 - q^{3k+6s+1-n})(q^{6s+1-n}; q^2)_k = 0 \), and so the right-hand side of (2.5) vanishes. If \( s < (n-1)/6 \), then taking \( a = q^{6s+1-n}, d = aq^{2s+1}, b \mapsto q/b, c \mapsto q/c \) in (2.2), we get

\[
\sum_{k=0}^{\infty} \frac{1 - q^{3k+6s+1-n}}{1 - q^{6s+1-n}} \frac{(q^{6s+1-n}, aq^{2s+1}, q^{4s+1-n}/a; q^2)_k (q/b, c, q^{6s-n}bc; q)_k q^k}{(q^{4s+1-n}/a, aq^{2s+1}, q; q)_k (q^{6s+2-n}b, q^{6s+2-n}c, q^3/bc; q^2)_k}.
\]

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\[ \frac{(q^{6s+3-n}, q^2/b, q^2/c, q^{6s+1-n}bc; q^2)_\infty}{(q, q^{6s+2-n}b, q^{6s+2-n}c, q^2/bc; q^2)_\infty} q^\phi_2 \left[ \frac{q/b, q/c, q^{6s-n}bc}{aq^{2s+2}, q^{4s+n}/a; q^2} \right]. \]

(2.6)

The left-hand side of (2.6) terminates at \( k = (n - 1)/2 - 3s \), while the right-hand side of (2.6) is equal to 0, because the numerator of the fraction before the \( 3\phi_2 \) series contains the factor \((q^{6s+3-n}; q^2)_\infty\). Thus, letting \( b, c \to 0 \) in (2.6), we immediately conclude that the right-hand side of (2.5) vanishes. \( \square \)

We now give a parametric version of Theorem 1.1.

**Theorem 2.2.** Let \( n > 1 \) be an odd integer and let \( 0 \leq s \leq (n - 1)/8 \). Then, modulo \( \Phi_n(q)(1 - aq^n)(a - q^n) \),

\[ \sum_{k=2s}^{(n-1)/2+2s} (-1)^k [3k + 1] (aq; q^2)^k (q; q)_{2s}(q; q^2)_{k+s}(q/a; q^2)_k (q; q)_{k-2s}(q/a; q)_{2s}(aq; q)_k = (-q)^{(n-1)/4}[n]. \]

(2.7)

**Proof.** Since \( q^n \equiv 1 \pmod{\Phi_n(q)} \), it follows from (2.4) that

\[ \sum_{k=2s}^{(n-1)/2+2s} (-1)^k [3k + 1] (aq; q^2)^k (q; q)_{2s}(q; q^2)_{k+s}(q/a; q^2)_k (q; q)_{k-2s}(q/a; q)_{2s}(aq; q)_k \equiv 0 \pmod{\Phi_n(q)}. \]

(2.8)

Namely, the \( q \)-congruence (2.7) is true modulo \( \Phi_n(q) \).

For \( a = q^{-n} \), the left-hand side of (2.7) can be written as

\[ \sum_{k=2s}^{(n-1)/2+2s} (-1)^k [3k + 1] (q^{1-n}; q^2)^k (q; q)_{2s}(q; q^2)_{k+s}(q^{1+n}; q^2)_k (q; q)_{k-2s}(q^{1+n}; q)_{2s}(q^{1-n}; q)_k \]

\[ = \sum_{k=0}^{(n-1)/2-s} (-1)^{k+2s}[3k + 6s + 1] (q^{1-n}; q^2)^k (q; q)_{k+s}(q; q^2)_{k+3s}(q^{1+n}; q^2)_{k+2s} (q; q)_{k}(q^{1+n}; q)_{k+4s}(q^{1-n}; q)_{k+2s} \]

\[ = (q^{1-n}; q^2)^s(q; q^2)^{3s}(q^{1+n}; q^2)_2s (q; q^{1+n}; q^{1-n}; q^2)_2s \times \sum_{k=0}^{(n-1)/2-s} (-1)^k [3k + 6s + 1] (q^{2s+1-n}; q^{6s+1}, q^{4s+1+n}; q^2)_k (q, q^{4s+1+n}, q^{2s+1-n}, q)_k. \]

(2.9)

Applying (2.3) in which the parameters \( a, b, c, f, \) and \( n \) are replaced by \( q^{6s+1}/b, q/c, b, q^{6s+1} \), and \( (n - 1)/2 - s \), respectively, we have

\[ \sum_{k=0}^{(n-1)/2-s} \frac{(1 - q^{3k+6s+1})(q^{6s+1}/b, q/c, bc; q)_k(q^{6s+1}, q^{4s+1+n}, q^{2s+1-n}; q)_k}{(1 - q^{6s+1})(bc, q^{6s+2}, q^{6s+3}/bc; q^2)_k(q, q^{2s+1-n}, q^{4s+1+n}; q)_k} q^{k} \]

\[ = \frac{(q^{6s+2}, q)_{n-2s-1}(bcq, q^2/bc; q^2)_{(n-1)/2-s}}{(q; q)_{n-2s-1}(q^{6s+3}/bc, bcq^{6s+2}; q^2)_{(n-1)/2-s}} \]

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\[ \times 10W_9(bcq^{6s}; q^{6s+1}, b, cq^{6s}, bc, bcq, q^{4s+1+n}, q^{1+2s-n}, q^2, q^2) \]
\[ = (q^{6s+2}; q)_{n-2s-1}(bcq, q^2/bc; q^2)_{(n-1)/2-s} \]
\[ x sW_7(bcq^{6s}; b, cq^{6s}, bc, q^{4s+1+n}, q^{1+2s-n}, q^2, q^2) \]
\[ = (q^{6s+2}; q)_{n-2s-1}(bcq, q^2/bc; q^2)_{(n-1)/2-s}(cq^{6s+2}, q^{6s+2}/b, q^2/c; q^2)_{(n-1)/2-s} \]
\[ = (q^{6s+2}; q)_{n-2s-1}(bcq; q^2)_{(n-1)/2-s}(cq^{6s+2}/b, q^2/c, q^2)_{(n-1)/2-s}, \]
where we have used the summation formula (2.1). Letting \( b, c \rightarrow 0 \) in the above identity leads to
\[ \sum_{k=0}^{(n-1)/2-s} (-1)^k \left[ \frac{3k + 6s + 1}{[q^{6s+1}, q^{4s+1+n}; q^2]_k} \right] = (-1)^{(n-1)/2-s}(q^{6s+3}, q^2)_{(n-1)/2-s}q^{(n-1-2s)/4}. \tag{2.10} \]

Substituting (2.10) into (2.9) and making some simplifications, we get
\[ \sum_{k=2s}^{(n-1)/2+2s} (-1)^k[3k + 1] \left( \frac{q^{1-n}; q^2}_{k-2s}(q^{2}; q^2)_{k+2s}(q^{1-n}; q^2)_k \right) = (-q)^{(n-1)/2}[n]. \]

This indicates that the \( q \)-congruence (2.7) holds modulo \( 1 - aq^n \).

Similarly, for \( a = q^n \), since \( 0 \leq s \leq (n-1)/8 \), the left-hand side of (2.7) is well-defined and can be written as
\[ \sum_{k=2s}^{(n-1)/2+2s} (-1)^k[3k + 1] \left( \frac{q^{1-n}; q^2}_{k-2s}(q^{2}; q^2)_{k+2s}(q^{1-n}; q^2)_k \right) \]
\[ = \sum_{k=0}^{(n-1)/2} (-1)^{k+2s}[3k + 6s + 1] \left( \frac{q^{1-n}; q^2}_{k+s}(q^{2}; q^2)_{k+3s}(q^{1-n}; q^2)_{k+2s} \right) \]
\[ = \frac{q^{1+n}; q^2}{(q^{-1}; q)_s(q^{1+n}; q^2)_{2s}} \]
\[ \times \sum_{k=0}^{(n-1)/2-2s} (-1)^k[3k + 6s + 1] \left( \frac{q^{6s+1+n}; q^{4s+1+n}; q^2}{{q^{6s+1+n}, q^{4s+1+n}; q^2})_k} \right), \tag{2.11} \]

Replacing \( a, b, c, f, \) and \( n \) in (2.3) by \( q^{6s+1}/b, q/c, b, q^{6s+1} \), and \( (n-1)/2 - 2s \), respectively, we get
\[ \sum_{k=0}^{(n-1)/2-2s} \left( \frac{1 - q^{3k+6s+1}}{1 - q^{6s+1}}(q^{6s+1}/b, q/c, bc; q)_k(q^{6s+1}, q^{4s+1-n}, q^{2s+1+n}; q^2)_k \right) \]
\[ \times \frac{q^{6s+1}/b, q/c, bc; q)_k(q^{6s+1}, q^{4s+1-n}, q^{2s+1+n}; q^2)_k}{(q^{6s+1}/b, q/c, bc; q)_k(q^{6s+1}, q^{4s+1-n}; q^2)_k} \]
Lemma 3.1. Proof of Theorem 1.1. Hence, letting
\[ (1 - q^{6s+2}; q_n - 4s - 1)(bcq, q^2/bc; q^2)_{(n-1)/2-2s} \]
\[ (q; q)_{n-4s-1}(q^{6s+3}/bc, bcq^{6s+2}; q^2)_{(n-1)/2-2s} \]
\times_{10} W_7(bcqe^{6s}; b, eq^{6s}, bc, bcq, q^{2s+1+n}, q^{1+4s-n}; q^2, q^2)
\[ = (q^{6s+2}; q)_{n-4s-1}(bcq, q^2/bc; q^2)_{(n-1)/2-2s} \]
\[ = (q; q)_{n-4s-1}(q^{6s+3}/bc, bcq^{6s+2}; q^2)_{(n-1)/2-2s} \]
\times_{8} W_7(bcqe^{6s}; b, eq^{6s}, bc, q^{2s+1+n}, q^{1+4s-n}; q^2, q^2)
\[ = (q^{6s+3}; q^2)_{(n-1)/2-2s}(bcq; q^2)_{(n-1)/2-2s}(q^{6s+2}/b, q^2/c; q^2)_{(n-1)/2-2s} \]
\[ (q; q^2)_{(n-1)/2-2s}(q^{6s+3}/bc; q^2)_{(n-1)/2-2s}(q^{6s+2}/c, q^2b; q^2)_{(n-1)/2-2s}. \]

Letting \( b, c \to 0 \) in the above identity implies that
\[ \sum_{k=0}^{(n-1)/2-2s} (-1)^k \frac{(3k + 6s + 1)(q^{2s+1+n}, q^{4s+1-n}; q^2)_k}{[6s + 1](q, q^{4s+1-n}, q^{2s+1+n}; q)_k} \]
\[ = (-1)^{(n-1)/2-2s} \frac{(q^{6s+3}; q^2)_{(n-1)/2-2s}}{(q; q^2)_{(n-1)/2-2s}} q^{(n-1-4s)^2/4}. \quad (2.12) \]

Substituting (2.12) into (2.11), we are led to
\[ \sum_{k=2s}^{(n-1)/2+2s} (-1)^k [3k + 1] \frac{(q^{1+n}; q^2)_{k-s}(q^2; q^2)_{k+s}(q^{1-n}; q^2)_k}{(q; q)_{k-2s}(q^{1-n}; q)_{k+2s}(q^{1+n}; q)_k} = (-q)^{(n-1)/4}[n]. \]

This means that the \( q \)-congruence (2.7) holds modulo \( a - q^n \).

Since \( \Phi_n(q) \), \( 1 - aq^n \), and \( a - q^n \) are pairwise relatively prime polynomials in \( q \), we complete the proof of (2.7). \( \square \)

Proof of Theorem 1.1. When \( a = 1 \) the denominators on both sides of (2.7) are coprime with \( \Phi_n(q) \), and the polynomial \( (1 - aq^n)(a - q^n) = (1 - q^n)^2 \) contains the factor \( \Phi_n(q)^2 \). Hence, letting \( a = 1 \) in (2.7), we obtain the desired \( q \)-supercongruence (1.4). \( \square \)

3 Proof of Theorem 1.2

We need the following summation formula of Rahman [13, (4.6)]:
\[ \sum_{k=0}^{\infty} \frac{(1 - aq^{3k})(a, d, q/d; q)_k(b; q^2)_k}{(1-a)(q^2, aq^2/d, adq; q^2)_k(aq/b; q)_k} q^{k+1} \frac{q^{(k+1)2}}{b^k} = \frac{(aq, aq^2, adq/b, aq^2/bd; q^2)_{\infty}}{(aq/b, aq^2/b, aq^2/d, adq; q^2)_{\infty}}. \quad (3.1) \]

which was already utilized by the second author and Zudilin [9] to prove (1.5).

Like before, we first give the following \( q \)-series identity.

Lemma 3.1. Let \( n > 1 \) be an odd integer. Let \( a \) be an indeterminate and let \( s \geq 0 \). Then
\[ \sum_{k=s}^{(n-1)/2+s} (-1)^k \frac{(1 - q^{6k+1-n})(aq; q^2)_{k-s}(q^{1-n}; q^2)_{k+2s}(q/a; q^2)_k}{(1-q^{1+n})(q^2; q^4)_{k-s}(q^{4-n}/a; q^4)_{k+s}(aq^{4-n}; q^4)_k} q^{3k^2-nk} = 0. \quad (3.2) \]
Proof. It is easy to see that the left-hand side of (3.2) can be written as

\[
\sum_{k=0}^{(n-1)/2} (-1)^{k+s} \frac{(1 - q^{6k+6s+1-n})(aq; q^2)_{k+s} (q^{-1}; q^2)^{k+3s} (q/a; q^2)^{k+s} q^{3(k+s)^2-n(k+s)}}{(1 - q^{1-n})(q^4; q^4)_{k} (q^{4+n}/a; q^4)^{k+2s} (aq^{1-n}; q^4)^{k+s}} q^{3^2 + 6ks - nk}.
\]

If \( s \geq (n - 1)/6 \), then either \((q^{1-n}; q^2)^{3s} = 0 \) or \((1 - q^{6k+6s+1-n})(q^{6s+1-n}; q^2)^{k} = 0 \), and so the right-hand side of (3.3) vanishes. If \( s < (n - 1)/6 \), then performing the parameter substitutions \( q \mapsto q^2, a = q^{6s+1-n}, d = aq^{-2s} \) in (3.1), we obtain

\[
\sum_{k=0}^{\infty} \frac{(1 - q^{6k+6s+1-n})(q^{6s+1-n}, q^{2s+1}/a; q^2)_{k} (b; q^4)^{k}}{(1 - q^{1-n})(q^4, q^{8s+4-n}/a, aq^{4s+4-n}; q^4)_{k} (q^{6s+3-n}/b; q^4)^{k}} b^k
\]

\[
= \frac{(q^{6s+3-n}, q^{6s+5-n}, aq^{4s+4-n}/b, q^{8s+4-n}/ab; q^4)^{\infty}}{(q^{6s+3-n}/b, q^{6s+5-n}/b, q^{8s+4-n}/a, aq^{4s+4-n}; q^4)^{\infty}}.
\] (3.4)

The left-hand side of (3.4) terminates at \( k = (n - 1)/2 - 3s \), while the right-hand side of (3.4) vanishes because the numerator has the factor \((q^{6s+3-n}, q^{6s+5-n}; q^4)^{\infty} \). Letting \( b \to \infty \) in (3.4), we see that the summation on the right-hand side of (3.3) vanishes. \( \square \)

We have the following parametric generalization of Theorem 1.2.

Theorem 3.2. Let \( n > 1 \) be an odd integer and let \( 0 \leq s \leq (n - 1)/4 \). Then, modulo \( \Phi_n(q)(a - q^n) \),

\[
\sum_{k=s}^{(n-1)/2+s} (-1)^k [6k + 1] (aq; q^2)_{k-2s} (q; q^2)_{k+2s} (q/a; q^2)_{k} / (q^4; q^4)^{k-s} (q^4/a; q^4)^{k+s} (aq^4; q^4)^{k} \equiv (-q)^{-(n-1)(n+5)/8}[n].
\] (3.5)

Proof. Replacing \( a \) and \( q \) by \( a^{-1} \) and \( q^{-1} \), respectively, the \( q \)-congruence (3.5) is equivalent to

\[
\sum_{k=s}^{(n-1)/2+s} (-1)^k [6k + 1] (aq; q^2)_{k-2s} (q; q^2)_{k+2s} (q/a; q^2)_{k} / (q^4; q^4)^{k-s} (q^4/a; q^4)^{k+s} (aq^4; q^4)^{k} q^{3k^2} \equiv (-q)^{-(n-1)(n-3)/4}[a^{-s}][n] \quad (\mod \Phi_n(q)(a - q^n)).
\] (3.6)

In view of \( q^n \equiv 1 \ (\mod \Phi_n(q)) \), from (3.2) we deduce that

\[
\sum_{k=s}^{(n-1)/2+s} (-1)^k [6k + 1] (aq; q^2)_{k-2s} (q; q^2)_{k+2s} (q/a; q^2)_{k} / (q^4; q^4)^{k-s} (q^4/a; q^4)^{k+s} (aq^4; q^4)^{k} q^{3k^2} \equiv 0 \ (\mod \Phi_n(q)).
\]

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Namely, the $q$-congruence (3.5) holds modulo $\Phi_n(q)$.

For $a = q^n$, the right-hand side of (3.6) can be written as

$$
\sum_{k=0}^{(n-1)/2} (1-k+s)(6k+6s+1)(q^{1+n};q^2)_{k-s}(q^2)_{k+3s}(q^{1-n};q^2)_{k+s} q^3(k+s)^2
$$

$$
= (-1)^s q^{3s}(q^{1+n};q^2)_{s}(q^2)_{3s}(q^{1-n};q^2)_{s}
$$

$$
\times \sum_{k=0}^{(n-1)/2} (1-k)(6k+6s+1)(q^{1+n-2s};q^{2s+1+n};q^2)_{k} q^{3k^2+6ks}.
$$

(3.7)

Setting $q \to q^2, a = q^{6s+1}, d = q^{n+1-2s}$ and $b \to \infty$ in (3.1), we get

$$
\sum_{k=0}^{(n-1)/2-s} (1-k)\frac{q^{6k+6s+1} - 1 - q^{6s+1}}{1 - q^{6s+1}}(q^{1+n-2s};q^{2s+1+n};q^2)_{k} q^{3k^2+6ks}
$$

$$
= \frac{(q^{6s+3};q^4)^{(n-1-2s)/4}}{(q^{6s+4-n};q^{4s+4+n};q^4)^{(n-1-2s)/4}}, \text{ if } n \equiv 2s + 1 \pmod{4},
$$

$$
\frac{(q^{6s+5};q^4)^{(n-1-2s)/4}}{(q^{6s+4-n};q^{4s+4+n};q^4)^{(n-1-2s)/4}}, \text{ if } n \equiv 2s + 3 \pmod{4}.
$$

Plugging the above identity into (3.7) and making some simplifications, we obtain

$$
\sum_{k=0}^{(n-1)/2} (1-k+s)(6k+6s+1)(q^{1+n};q^2)_{k-s}(q^2)_{k+3s}(q^{1-n};q^2)_{k+s} q^3(k+s)^2
$$

$$
= (-1)^{(n-1)(n-3)/8+4s} q^{-sn[n]}.
$$

This proves that (3.5) holds modulo $a-q^n$. Since $\Phi_n(q)$ and $a-q^n$ are coprime polynomials in $q$, we complete the proof of (3.5).

\textbf{Proof of Theorem 1.2}. When $a = 1$ the denominators on the two sides of (3.5) are coprime with $\Phi_n(q)$, and the polynomial $a-q^n = 1-q^n$ has the factor $\Phi_n(q)$. Therefore, letting $a = 1$ in (2.7), we are led to (1.6).

\textbf{4 Concluding remarks}

Guillera and Zudilin [5] also proved the following “divergent” Ramanujan-type supercongruence:

$$
\sum_{k=0}^{(p-1)/2} \frac{(1)^k}{k!^3} (3k+1)2^{2k} \equiv p \pmod{p^3} \quad \text{for } p > 2.
$$

(4.1)
However, we did not find any interesting generalization of (4.1) like (1.3), nor in the modulus $p^2$ case. Perhaps an interested reader may make some progress on this problem.

We believe that the following stronger version of Theorem 1.1 should be true.

**Conjecture 4.1.** The $q$-supercongruence (1.4) holds modulo $[n]\Phi_n(q)^2$.

Note that the $s = 0$ case of Conjecture 4.1 was already proved by the second author [6]. A natural way to prove Conjecture 4.1 is to establish the following $q$-congruence: for any odd integer $n > 1$ and non-negative integer $s$,

\[
\sum_{k=2s}^{N+2s} (-1)^k[3k + 1] \frac{(q; q^2)_{k-s}(q; q^2)_{k+s}(q; q^2)_k}{(q; q)_{k-2s}(q; q)_{k+2s}(q; q)_k} \equiv 0 \pmod{\Phi_n(q)}.
\]

where $N = (n-1)/2$ or $n-1$, and then use the technique of roots of unity in [9]. However, we cannot deduce this $q$-congruence for $N = (n-1)/2$ from (2.8) by taking $a = 1$ directly, since the denominators of the left-hand side of (2.8) might contain the factor $\Phi_n(q)$ when $a = 1$.

We also conjecture that Theorem 1.2 has the following generalization.

**Conjecture 4.2.** The $q$-supercongruence (1.6) holds modulo $[n]\Phi_n(q)$.

Similarly, to prove Conjecture 4.2, we need to build the $q$-congruence: for any odd integer $n > 1$ and non-negative integer $s$,

\[
\sum_{k=2s}^{N+2s} (-1)^k[3k + 1] \frac{(q; q^2)_{k-s}(q; q^2)_{k+s}(q; q^2)_k}{(q; q)_{k-2s}(q; q)_{k+2s}(q; q)_k} \equiv 0 \pmod{\Phi_n(q)}.
\]

where $N = (n-1)/2$ or $n-1$. But it seems difficult to accomplish this. We hope that an interested reader can settle these two conjectures.

5 Statements and Declarations

**Competing interests:** The authors have no relevant financial or non-financial interests to disclose.

**References**


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