On the geography of slopes of fibrations

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Abstract The slopes of fibrations of genus $g \geq 2$ have strict lower (resp., upper) bound, namely $\lambda_m(g)$ (resp., $\lambda_M(g)$). In this paper, we show that if $g \neq 3$, then each rational number $r \in [\lambda_m(g), \lambda_M(g)]$ can occur as the slope of some fibration of genus g. Similar result is also true for $g = 3$ and $r \in [\lambda_m(3), 9]$.

1 Introduction

Let X be a minimal nonsingular projective surface of general type over $\mathbb{C}$, and its Chern numbers be $c_1^2(X), c_2(X), \chi(\mathcal{O}_X)$ with $c_1^2(X) + c_2(X) = 12\chi(\mathcal{O}_X)$. The geography problem of surfaces asks: for which pair of integers (a, b) does there exist X such that $c_1^2(X) = a$ and $c_2(X) = b$? We refer to [BHPV04, VII §8] for general existence results. One of them, due to Sommese [So84], states that every rational point in $[1/5, 3]$ can occur as the slope $c_1^2(X)/c_2(X)$ of some surface X.

It is natural to study the geography problem for fibrations, which is an open problem in [AK02, §1.1].

Let S be a nonsingular projective surface, and C be a nonsingular projective curve of genus b. Let $f: S \to C$ be a relatively minimal fibration of genus g. One has the following well-known relative invariants

$$\begin{cases} \chi_f = \chi(\mathcal{O}_S) - (g-1)(b-1), \\ e_f = c_2(S) - 4(g-1)(b-1), \\ K_f^2 = c_1^2(S) - 8(g-1)(b-1). \end{cases} \tag{1.1}$$

When $g \geq 2$, $K_f^2 = 0$ iff $\chi_f = 0$ iff f is a locally trivial fibration. For a locally non-trivial fibration f, the slope of f is defined as

$$\lambda_f = K_f^2/\chi_f.$$

The following geography problem of fibrations has been considered by Xiao, and he obtained a partial result when $g = 2$ (see [Xi85, Theorem 2.9] and [Xi92, Theorem 4.3.5]).

Question 1.1. *Let $g \geq 2$ be an integer. For what kinds of non-negative integers (χ, K, b), can we find a fibration $f: S \to C$ of genus g with $\chi_f = \chi$, $K_f^2 = K$ and $g(C) = b$?*

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Xiao pointed out that the existence of fibrations with high slope is interesting for the difficulty of construction, and raised an open problem in [Xi92] to find examples of hyperelliptic fibrations with maximal slopes. Though the latter problem is solved ([Mo98, LT13]), the existence of fibrations with high slopes is still confusing.

In this note, we will focus on Question 1.1 for slopes of fibrations.

Question 1.2. *Given $r > 0$ and $g \geq 2$. Is there a fibration $f\colon S \to C$ of genus g with $\lambda_f = r$?*

It is known that ([Xi87, CH88]) $\lambda_f \geq \lambda_m(g) := \frac{4g-4}{g}$ for any $g \geq 2$, and the equality can be reached. If $g \geq 3$, then $\lambda_f \leq 12$. Moreover, the equality holds here if and only if f is a Kodaira fibration, that is, f has no singular fibers but it has variable moduli. See [GD-H91, Za95] for explicit Kodaira fibrations. If f is hyperelliptic, Xiao ([Xi92]) showed that $\lambda_f \leq \lambda_M^h(g)$, where

$$\lambda_M^h(g) = \begin{cases} 12 - \frac{8g+4}{g^2}, & \text{if } g \text{ is even,} \\ 12 - \frac{8g+4}{g^2-1}, & \text{if } g \text{ is odd.} \end{cases}$$

Let

$$\lambda_M(g) = \begin{cases} 7, & \text{if } g = 2, \\ 12, & \text{if } g \geq 3, \end{cases}$$

then our main result is as follows.

Theorem 1.3. *If $g \neq 3$, then, for each rational number $r \in [\lambda_m(g), \lambda_M(g)]$, there exists a fibration of genus g with slope r.*

Now we outline the proof of the above theorem. Firstly, we take a semistable fibration $f\colon S \to C$ with slope λ_f such that f is composed by a fibration $\varphi\colon P \to C$ and a generically finite morphism $\Pi\colon S \dashrightarrow P$ of degree 2 (which may be a rational map, see Sect. 3 for details). Secondly, after taking a cyclic cover $\pi_1\colon C_1 \to C$ of degree d branching over non-critical points of φ, we get a pullback fibration $f_1\colon S_1 \to C_1$ (resp., $\varphi_1\colon P_1 \to C_1$) of f (resp., φ) with respect to π_1. Thirdly, after adding k general fibers of φ_1 to the branch locus of $\Pi_1\colon S_1 \dashrightarrow P_1$, we construct a modified general finite map $\Pi_1'\colon S_1' \to P_1$, and a fibration $f_1'\colon S_1' \to C_1$.

$$\begin{tikzcd}
 & & S_1 \arrow[r] \arrow[dl, dashed, "\Pi_1"'] \arrow[dd, "f_1"] & S \arrow[dd, "f"] \arrow[dr, dashed, "\Pi"] & \\
S_1' \arrow[r, dashed, "\Pi_1'"] \arrow[dr, "f_1'"'] & P_1 \arrow[dr, "\varphi_1"] & & & P \arrow[dl, "\varphi"] \\
 & C_1 \arrow[r, equal] & C_1 \arrow[r, "\pi_1"] & C &
\end{tikzcd}$$

Finally, we will show that the slopes of f_1' can take every rational number in some interval by varying d and k. It is shown that ([Ta10]) relative invariants are the sum of modular invariants and the Chern numbers of singular fibers, see Section 2. We will use these formulas to calculate relative invariants of f_1', which is convenient. The relation between modular invariants of f_1' with those of f is the motivation of our construction.

Our method is also available when Π is cyclic.

But for $g = 3$, we can not give a complete answer due to the lack of Kodaira fibration of genus 3 with generically cyclic cover Π. However, we have the following partial result by the same method.

Theorem 1.4. *If $g = 3$, then, for each rational number $r \in [\lambda_m(3), 9]$, there exists a fibration of genus 3 with slope r.*

Note that, for a fibration f of genus 3 coming from a cyclic triple cover over a ruled surface, [CT06] shows $\lambda_f \leq 9$. We will give an example (Example 2.6) to show that this upper bound is sharp in this case.

Our proof is based on that of Sommese's result, see [So84]. The difference is that we construct a fibred surface with new *singular* fibers by modifying branch locus. The latter trick has been used by Xiao [Xi92] and Chen [Ch87].

By our construction, it is easy to see that lots of the fibrations we obtain are not semistable. So it is interesting to consider the geography problem of slopes of semistable fibrations, and we raise the following conjecture.

Conjecture 1.5. *Let $g \geq 2$ be an integer. For each rational number $r \in [\lambda_m(g), \lambda_M(g)]$, there exists a semistable fibration of genus g with slope r.*

2 Preliminaries

2.1 The Chern numbers of singular fibers

Denote by F a singular fiber of $f\colon S \to C$ over $p \in C$. Let $\pi\colon \widetilde{C} \to C$ be a semistable reduction of F, i.e., π is ramified over $p = f(F)$ and some non-critical points of f, and the fibers of the pullback fibration $\tilde{f}\colon \widetilde{S} \to \widetilde{C}$ over $\pi^{-1}(p)$ are semistable. Here $\tilde{f}\colon \widetilde{S} \to \widetilde{C}$ is the unique relatively minimal birational model of $X \times_C \widetilde{C} \to \widetilde{C}$.

Definition 2.1 ([Ta10], Definition 1.1)**.** The Chern numbers of F are defined as follows,

$$c_1^2(F) = K_f^2 - \frac{1}{d}K_{\tilde{f}}^2, \quad c_2(F) = e_f - \frac{1}{d}e_{\tilde{f}}, \quad \chi_F = \chi_f - \frac{1}{d}\chi_{\tilde{f}}, \tag{2.1}$$

where d is the degree of π.

Note that $\frac{1}{d}K_{\tilde{f}}^2$, $\frac{1}{d}e_{\tilde{f}}$, and $\frac{1}{d}\chi_{\tilde{f}}$ are independent of the choice of the semistable reduction π, so the above definition is well-defined (see [Ta96, Lemma 2.3] or [Ta10] for more details).

If $\chi_F \neq 0$, then we define

$$\lambda_F \triangleq c_1^2(F)/\chi_F.$$

In order to compute these invariants, we introduce some numerical invariants of the singularity of a curve. Let B be a curve on S which is a nonzero effective divisor.

Definition 2.2. A *partial resolution* of the singularities of B is a sequence of blowing-ups $\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_r\colon \widehat{S} \to S$,

$$(\widehat{S}, \sigma^*B) = (S_r, B_r) \overset{\sigma_r}{\to} S_{r-1} \overset{\sigma_{r-1}}{\to} \cdots \overset{\sigma_2}{\to} (S_1, B_1) \overset{\sigma_1}{\to} (S_0, B_0) = (S, B),$$

satisfying the following conditions:

(i) $B_{r,\mathrm{red}}$ has at worst ordinary double points as its singularities;

(ii) $B_i = \sigma_i^* B_{i-1}$ is the total transform of B_{i-1};

(iii) σ_i is the blowing-up of S_{i-1} at a singular point $(B_{i-1,\mathrm{red}}, p_{i-1})$ which is not an ordinary double points.

Now let $\sigma\colon \hat{S} \to S$ be the partial resolution of the singularities of F. Then $\sigma^* F$ is called the *normal crossing model* of F. A (-1)-curve in a fiber is called *redundant* if it meets the other components in at most two points. It is obvious that a redundant (-1)-curve can be contracted without introducing singularities worse than ordinary double points. After a finite number of such contractions, we will finally obtain a normal crossing fiber $\overline{F}$ containing no redundant (-1)-curves. We call $\overline{F}$ the *minimal normal crossing model* of F (see [LuT13, Definition 2.2]). So $\overline{F}$ is determined uniquely by F.

In Definition 2.2, we denote by m_i the multiplicity of $B_{i,\mathrm{red}}$ at p_i. If $q \in B_{r,\mathrm{red}}$ is a double point, and the two local components of (B_r, q) have multiplicities a_q and b_q, then we define $[a_q, b_q] := \frac{\gcd(a_q,b_q)^2}{a_q b_q}$. In particular, if B_{red} has only one singular point $p = p_0$, then we define

$$\alpha_p = \sum_{i=1}^{r} (m_i - 2)^2, \quad \beta_p = \sum_{q \in B_r} [a_q, b_q],$$

where q runs over all of the double points of $B_{r,\mathrm{red}}$. These two invariants are independent of the resolution. Let

$$\alpha_B := \sum_{p \in B} \alpha_p, \quad \beta_B := \sum_{p \in B} \beta_p.$$

Definition 2.3. Let $\overline{F}$ be the minimal normal crossing model of F, and let $G(\overline{F})$ be the dual graph of $\overline{F}$. A *H-J branch* of rational curves in $G(\overline{F})$ is the following subgraph.

$$\overset{n_1}{\underset{-e_1}{\circ}} \!-\!\!-\!\! \overset{n_2}{\underset{-e_2}{\circ}} \!-\!\!-\! \quad \cdots \quad \!-\!\!-\!\! \overset{n_r}{\underset{-e_r}{\circ}} \!-\!\!-\!\! \overset{n_{r+1}}{\bullet}$$

In the above dual graph, $\overset{n_i}{\underset{-e_i}{\circ}}$ denotes a smooth rational curve Γ_i with $\Gamma_i^2 = -e_i$ whose multiplicity in $\overline{F}$ is n_i. Moreover, $\bullet$ denotes either an irrational curve, or a smooth rational curve meeting at 3 or more points with the other components. For convenience, in the following context, we omit the subscript $(-e)$ whenever $e = 2$.

Let β_F^- be the total contribution of all the H-J branches in $G(\overline{F})$ to β_F, and $\beta_F^+ := \beta_F - \beta_F^-$. Denote by μ_F the sum of the Milnor numbers of the singularities of F_{red}. Let $N_F = g - p_a(F_{\mathrm{red}})$. By [Ta96, Ta10], we know that

$$\begin{cases} c_1^2(F) = 4N_F + F_{\mathrm{red}}^2 + \alpha_F - \beta_F^-, \\ c_2(F) = 2N_F + \mu_F - \beta_F^+, \\ 12\chi_F = 6N_F + F_{\mathrm{red}}^2 + \alpha_F + \mu_F - \beta_F. \end{cases} \tag{2.2}$$

The following examples will be used in our proof.

Example 2.4. Let F_h (resp., F_g) be a singular fiber of a fibration of genus g with the following dual graph. Here we assume that all the irreducible curves in F_h (resp., F_g) have smooth support.

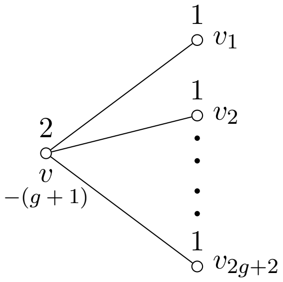


Figure 1: Hyperelliptic fiber F_h

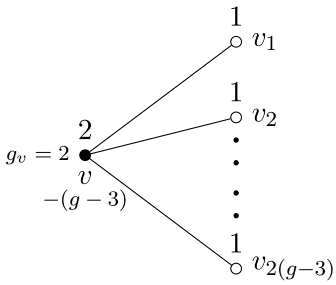


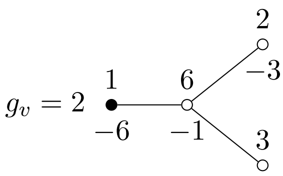
Figure 2: Fiber F_g of genus $g \geq 4$

Then F_h and F_g are both minimal normal crossing. One can obtain their Chern numbers as in Table 1. For similar detailed computation, we refer to [LT13].

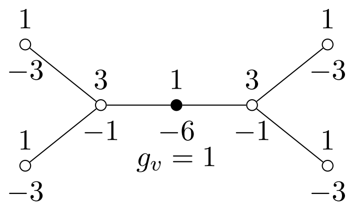
Fibers	N_F	F_{red}^2	β_F^-	β_F^+	α_F	μ_F	$c_1^2(F)$	χ_F	λ_F
F_h	g	$-(g+1)$	$g+1$	0	0	$2g+2$	$2g-2$	$\frac{g}{2}$	$\frac{4g-4}{g}$
F_g	$g-2$	$-(g-3)$	$g-3$	0	0	$2(g-3)$	$2g-2$	$\frac{g-2}{2}$	$\frac{4g-4}{g-2}$

Table 1: The Chern numbers of F_h and F_g

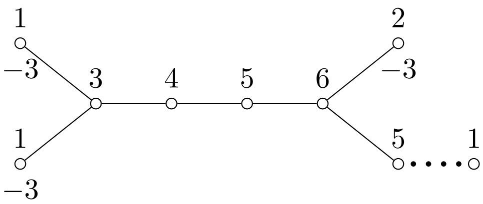
Example 2.5. Let F_0 (resp., F_1, F_∞, F) be a singular fiber of a fibration of genus 3, and $\overline{F}_0$ (resp., $\overline{F}_1$, $\overline{F}_\infty$, $\overline{F}$) be its minimal normal crossing model. Suppose the dual graphs of these fibers are as follows.



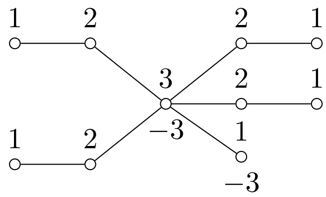
(a): $\overline{F}_0$



(b): $\overline{F}_1$



(c): $\overline{F}_\infty$



(d): $\overline{F}$

From a straightforward computation, we obtain the Chern numbers of these singular fibers as in Table 2.

2.2 Modular invariants

The fibration $f\colon S \to C$ induces a moduli map $J\colon C \to \overline{\mathcal{M}}_g$ from the base curve C to the moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus g. Let λ be the Hodge divisor class of

Fibers	N_F	F_{red}^2	β_F^-	β_F^+	α_F	μ_F	$c_1^2(F)$	$c_2(F)$	χ_F
F_0	0	0	$\frac{5}{6}$	$\frac{1}{6}$	1	2	$\frac{1}{6}$	$\frac{11}{6}$	$\frac{1}{6}$
F_1	0	0	$\frac{4}{3}$	$\frac{2}{3}$	2	8	$\frac{2}{3}$	$\frac{22}{3}$	$\frac{2}{3}$
F_∞	3	-5	$\frac{11}{6}$	$\frac{1}{6}$	0	11	$\frac{31}{6}$	$\frac{101}{6}$	$\frac{11}{6}$
F	3	-4	3	0	0	9	5	15	$\frac{5}{3}$

Table 2: The Chern numbers of singular fibers for $g = 3$

$\overline{\mathcal{M}}_g$, δ be the boundary divisor class, and $\kappa = 12\lambda - \delta$. Then there are three fundamental *modular invariants* of f defined as follows (see [Ta10]),

$$\kappa(f) = \deg J^*\kappa, \quad \lambda(f) = \deg J^*\lambda, \quad \delta(f) = \deg J^*\delta.$$

If f is semistable, then

$$\kappa(f) = K_f^2, \quad \delta(f) = e_f, \quad \lambda(f) = \chi_f. \tag{2.3}$$

The modular invariants satisfy the *base change property*, i.e., if $\tilde{f}$ is the pullback fibration of f with respect to π of degree d, then

$$\kappa(\tilde{f}) = d \cdot \kappa(f), \quad \delta(\tilde{f}) = d \cdot \delta(f), \quad \lambda(\tilde{f}) = d \cdot \lambda(f). \tag{2.4}$$

Let $F_1, \cdots, F_s$ be all the singular fibers of f. It is proved in [Ta94, Ta96] that

$$\begin{cases} K_f^2 = \kappa(f) + \sum_{i=1}^s c_1^2(F_i), \\ e_f = \delta(f) + \sum_{i=1}^s c_2(F_i), \\ \chi_f = \lambda(f) + \sum_{i=1}^s \chi_{F_i}. \end{cases} \tag{2.5}$$

Example 2.6. By using results in Example 2.5, we will find a semistable fibration f of genus 3 with slope $\lambda_f = 9$, which derives from a cyclic covering over a ruled surface (see Sect. 3 for details).

Let $\{C_\lambda\}_{\lambda \in \mathbb{P}^1}$ be a pencil of curves in $\mathbb{P}^2$ defined by $F - \lambda G = 0$, where

$$F = Y^3 Z + X^2(X^2 - Z^2), \quad G = Z^2(X^2 - Z^2).$$

After blowing-up the base points of $\{C_\lambda\}$, we obtain a relatively minimal fibration $h\colon S_0 \to \mathbb{P}^1$, where S_0 is a rational surface with 16 exceptional curves and

$$c_1^2(S_0) = -7, \quad \chi(\mathcal{O}_{S_0}) = 1, \quad c_2(S_0) = 19.$$

By (1.1), we have

$$K_h^2 = 9, \quad \chi_h = 3, \quad e_h = 27.$$

All the singular fibers of h are F_0, F_1, F_∞ where $F_\lambda := h^{-1}(\lambda)$. The dual graphs of the minimal normal crossing models $\overline{F}_i$'s are as in Figures (a)–(c) in Example 2.5 respectively.

Thus, by (2.5) and Table 2, we have

$$\kappa(h) = 3, \quad \delta(h) = 1, \quad \lambda(h) = \frac{1}{3}.$$

It is easy to see that h is also a fibration of genus 3 from a cyclic triple covering over a ruled surface

$$\varphi_0\colon \mathbb{P}^1\times\mathbb{P}^1 \longrightarrow \mathbb{P}^1,\quad (x,\lambda)\to\lambda,$$

defined by

$$y^3=(\lambda-x^2)(x^2-1).$$

After a semistable reduction, the pullback semistable fibration $f\colon S\to C$ is a cyclic triple covering over a ruled surface. By (2.3) and (2.4), the slope of f is

$$\lambda_f=\frac{\kappa(h)}{\lambda(h)}=9.$$

2.3 Resolution of cyclic coverings

In our paper, a cyclic covering of degree n is a finite covering $\pi\colon Y\to X$ on a smooth algebraic surface X such that the function field $\mathbb{C}(Y)$ of Y is a cyclic extension of $\mathbb{C}(X)$ of degree n. The branch locus R of π satisfies $R\equiv nL$ for some effective divisor L on X.

For reader's convenience, we will recall some basic facts on cyclic coverings. We refer to [BHPV04, Ta91] for more details.

By a sequence of blowing-ups $\sigma\colon\overline{X}\to X$ at the singular points of R, the pullback of R in $\overline{X}$ is a normal crossing divisor. The normalization $\overline{Y}$ of $\overline{X}\times_X Y$ is a cyclic covering $\bar\pi\colon\overline{Y}\to\overline{X}$ whose branch locus $\overline{R}$ is a normal crossing divisor contained in the pullback of R in $\overline{X}$. Thus $\overline{Y}$ admits at worst Hirzebruch-Jung singularities which can be resolved directly by the method of Hirzebruch, where the resolution is denoted by $h\colon Y'\to\overline{Y}$. Finally, we get the resolution morphism $\sigma':=\bar\sigma\circ h\colon Y'\to Y$ which is called the *Jung's resolution* (see [PP11] or [LuT14]). Thus $\pi':=\bar\pi\circ h\colon Y'\to\overline{X}$ is a generically cyclic covering.

$$\begin{tikzcd}
Y' \arrow[r, "h"] \arrow[dr, "\pi'"'] & \overline{Y} \arrow[r, "\bar\sigma"] \arrow[d, "\bar\pi"'] & Y \arrow[d, "\pi"] \\
 & \overline{X} \arrow[r, "\sigma"] & X
\end{tikzcd}$$

If $n=2$ or 3, then π is totally ramified on R and there is a *canonical resolution* $\sigma\colon\overline{X}\to X$ such that the branch locus $\overline{R}$ of $\bar\pi$ is smooth and hence $\overline{Y}$ is a smooth surface, i.e., $Y'=\overline{Y}$ (see [BHPV04, III, Sect. 7] and [Ta91, II]). However, h is not usually an identity map whenever $n>3$.

The branch locus R of a cyclic covering $\pi\colon Y\to X$ can be written as follows:

$$R=A_1+2A_2+\cdots+(n-1)A_{n-1},$$

where each A_i is a reduced effective divisor and contains no component of other A_j's.

Let $\varphi\colon X\to C$ be a fibration of genus g_0. The cyclic covering π gives a fibration $\bar\varphi=\varphi\circ\sigma\circ\pi'\colon Y'\to C$ of genus g (not necessarily relatively minimal). Let Γ be a smooth fiber of φ satisfying that

(i) Γ is a component of A_1;

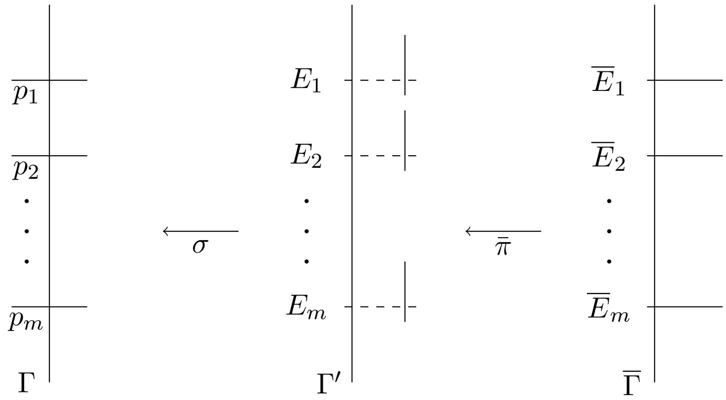
(ii) Γ meets transversely with other irreducible components of R at $p_1, \ldots, p_m$. It's easy to see that $m = (\sum_{i=1}^{n-1} A_i)\Gamma$.

The pulling-back $F := \pi'^*(\sigma^*(\Gamma))$ of Γ is the corresponding fiber of $\bar{\varphi}$.

Now we will illustrate how to obtain the fiber F by the following examples. Since we only use cyclic coverings of degree $n \leq 3$ in the proofs of our theorems, such cyclic coverings will be investigated in detail firstly.

Example 2.7. We assume that $n = 2$. In this case, R is reduced. Since $R \equiv 2L$ and $\Gamma^2 = 0$, $m = R\Gamma$ is even.

To illustrate the canonical resolution, a thin line is used to represent a component contained in R and a dashed line is used to represent a component not contained in R.



From the canonical resolution, one has

$$\sigma^*\Gamma = \Gamma' + E_1 + \cdots + E_m,$$

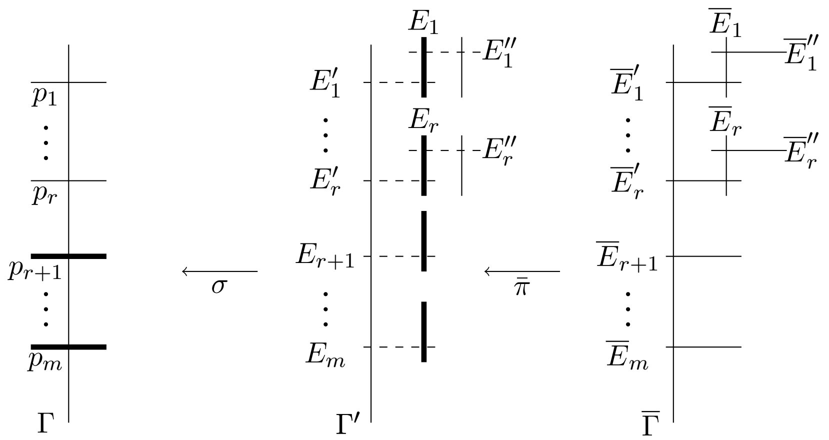
where Γ' is the strict transform of Γ with $(\Gamma')^2 = -m$ and E_i's are (-1)-curves respectively. Note that $\bar{\pi}^*\Gamma' = 2\overline{\Gamma}$ and $\bar{\pi}^*E_i = \overline{E}_i$ where $\overline{\Gamma}$ is a smooth curve of genus g_0 with $\overline{\Gamma}^2 = -\frac{m}{2}$ and $\overline{E}_i$'s are (-2)-curves respectively. Therefore we get

$$F = 2\overline{\Gamma} + \sum_{i=1}^{m} \overline{E}_i.$$

From Hurwitz formula, $g = 2g_0 - 1 + \frac{m}{2}$.

Example 2.8. Assume that $n = 3$. In this case, $R = A_1 + 2A_2$. We assume that Γ meets transversely with other components of A_1 (resp., A_2) at $p_1, \ldots, p_r$ (resp., $p_{r+1}, \ldots, p_m$). One has $r + m \equiv 0 \pmod 3$ by $R \equiv 3L$ and $\Gamma^2 = 0$.

In the following canonical resolution, a thin (resp., thick) line is used to represent a component contained in A_1 (resp., A_2) and a dashed line is used to represent a component not contained in R.



We have

$$\sigma^*\Gamma = \Gamma' + \sum_{i=1}^{r}\left(E_i + 2E_i' + E_i''\right) + \sum_{i=r+1}^{m} E_i,$$

where Γ' is the strict transform of Γ with $(\Gamma')^2 = -m-r$, $E_1, \ldots, E_r$ are (-3)-curves, and other exceptional curves are all (-1)-curves respectively.

One has $\bar\pi^*\Gamma' = 3\overline{\Gamma}$, $\bar\pi^*E_i = 3\overline{E}_i$ $(i = 1, \ldots, r)$, $\bar\pi^*E_i = \overline{E}_i$ $(i = r+1, \ldots, m)$, $\bar\pi^*E_i' = \overline{E}_i'$ and $\bar\pi^*E_i'' = \overline{E}_i''$ where $\overline{\Gamma}$ is a smooth curve of genus g_0 with $\overline{\Gamma}^2 = -\frac{m+r}{3}$, $\overline{E}_i$'s are (-1)-curves $(i = 1, \ldots, r)$ and other components of F are all (-3)-curves respectively. Therefore we get

$$F = 3\overline{\Gamma} + \sum_{i=1}^{r}\left(3\overline{E}_i + 2\overline{E}_i' + \overline{E}_i''\right) + \sum_{i=r+1}^{m} \overline{E}_i.$$

The minimal normal-crossing model of F can be obtained by contracting $\overline{E}_1, \ldots, \overline{E}_r$. From Hurwitz formula, $g = 3g_0 - 2 + m$.

Example 2.9. In the case of $n > 3$, since Γ intersects with R transversely, $\Gamma' := \sigma^*\Gamma$ is also the strict transform of Γ in $\overline{X}$. So Γ' is a smooth fiber of the fibration $\varphi \circ \sigma \colon \overline{X} \to C$ of genus g_0 which passes through m nodes of $\overline{R}$ (i.e., the pullbacks of the intersections p_i's). Therefore $F = n\overline{\Gamma} + V$ where $\overline{\Gamma}$ is the pulling-back component of Γ' in Y' and V is an effective divisor supported on the exceptional set of the resolution h.

We can describe V more precisely.

Let p be an intersection of A_i and Γ', then $\bar\pi$ is defined locally by $z^n = xy^i$ in a neighbourhood of p where x (resp., y) is the local coordinate of A_i (resp., Γ'). Thus $\bar\pi$ is totally ramified over p and the exceptional set of $\bar\pi^{-1}(p)$ in Y' is a Hirzebruch-Jung chain with a unique component B_p meeting transversely with $\overline{\Gamma}$. One can find a unique effective divisor Θ_p supported on this H-J chain satisfying that $\Theta_p B_p = -n$ and $\Theta_p B = 0$ for other component B of the H-J chain (see [BHPV04, III, Sect. 5]).

Therefore, from Zariski Lemma, one has $V = \sum_p \Theta_p$ where p runs over all the nodes of $\overline{R}$ lying on Γ'.

3 Construction and slopes

Let $\varphi\colon P \to C$ be a fibration and $\Pi\colon S_0 \to P$ be a cyclic cover with branch locus R. By the Jung's resolution σ, one can get a smooth surface $\overline{S}$ with a generically finite cover $\overline{\Pi}\colon \overline{S} \to \overline{P}$ where $\rho\colon \overline{P} \to P$ is a sequence of blowing-ups at the singular points of R. Thus one has a fibration $\bar{f} = \varphi \circ \rho \circ \overline{\Pi} : \overline{S} \to C$. For convenience, the relatively minimal model $f\colon S \to C$ of $\bar{f}$ is said to be a *covering over* φ *with respect to* Π.

$$\begin{tikzcd}
\overline{S} \arrow[r, equal] \arrow[dd, "1:1"'] \arrow[ddr, "\bar{f}"] & \overline{S} \arrow[r, "\sigma"] \arrow[d, "\overline{\Pi}"'] & S_0 \arrow[d, "\Pi"] \\
 & \overline{P} \arrow[r, "\rho"] \arrow[d] & P \arrow[d, "\varphi"] \\
S \arrow[r, "f"] & C \arrow[r, equal] & C
\end{tikzcd}$$

After a base change $\pi\colon \widetilde{C} \to C$, we have the pullback fibrations $\tilde{f}\colon \widetilde{S} \to \widetilde{C}$ (relatively minimal) and $\tilde{\varphi}\colon \widetilde{P} \to \widetilde{C}$. In fact, $\tilde{f}$ is a covering of $\tilde{\varphi}$ w.r.t. the covering $\widetilde{\Pi}\colon \widetilde{P} \times_P S_0 \to \widetilde{P}$ whose branch locus is $\tilde{\pi}^* R$. For convenience, we say that $(\tilde{f}, \tilde{\varphi}, \widetilde{\Pi})$ is the pullback of (f, φ, Π) by the base change π.

$$\begin{tikzcd}
\widetilde{S} \arrow[r, dashed, "1:1"] \arrow[ddr, "\tilde{f}"'] & \widetilde{P} \times_P S_0 \arrow[r, dashed] \arrow[d, "\widetilde{\Pi}"] & S_0 \arrow[d, "\Pi"'] \\
 & \widetilde{P} \arrow[r, dashed, "\tilde{\pi}"] \arrow[d, "\tilde{\varphi}"] & P \arrow[d, "\varphi"] \\
 & \widetilde{C} \arrow[r, "\pi"] & C
\end{tikzcd}$$

Lemma 3.1. *Assume that* Π *is a cyclic cover of degree* n *with branch locus* $R \equiv nL$ *for some divisor* L *on* P. *Let* A *be a very ample divisor on* C *of degree* e *and* $q_1 + \cdots + q_{ne}$ *be a sufficiently general element of* $|nA|$. *Then we can get a cyclic cover* $\Pi'\colon S_0' \to P$ *whose branch locus is*

$$R' = R + \varphi^*(q_1 + \cdots + q_{ne})$$

and a relatively minimal fibration $f'\colon S' \to C$ *w.r.t.* Π'. *Specially,* Π, Π' *coincide outside* $\cup_{i=1}^{ne} \varphi^{-1}(q_i)$.

Furthermore, after a base change $\pi\colon \tilde{C} \to C$ *of degree* n *which is totally ramified over* $q_1, \ldots, q_{ne}$*, the pullback fibrations of* f, f' *coincide.*

Proof. The existence of such a cyclic cover is from the obvious fact that $R' \equiv n(L + \varphi^*(A))$.

In order to prove the second part, it's enough to show that the pullback fibrations of f, f' come from the same cover. In fact, the branch locus of $\widetilde{\Pi}'$ is

$$\tilde{\pi}^* R' = \tilde{\pi}^* R + \sum_{i=1}^{ne} \tilde{\varphi}^* \pi^*(q_i) = \tilde{\pi}^* R + n \sum_{i=1}^{ne} \tilde{\varphi}^*(\tilde{q}_i),$$

where $\tilde{q}_i = \pi^{-1}(q_i)$. So these two covers coincide by a normalization. $\square$

Lemma 3.2. *Under the assumptions and notations in Lemma 3.1, we have that*

(1) the modular invariants of f and f' coincide respectively, i.e.,

$$\kappa(f') = \kappa(f), \qquad \lambda(f') = \lambda(f), \qquad \delta(f') = \delta(f);$$

(2) the Chern numbers of F_i's coincide with each other respectively, where $F_i = f'^*(q_i)$ $(i = 1, \dots, ne)$, i.e.,

$$c_1^2(F_i) = c_1^2(F_j), \qquad c_2(F_i) = c_2(F_j), \qquad \chi_{F_i} = \chi_{F_j} \quad (i \neq j).$$

For convenience, let F be one of the singular fibers F_i's. Therefore, the slope of f' is

$$\lambda_{f'} = \frac{\kappa(f) + ne \cdot c_1^2(F)}{\lambda(f) + ne \cdot \chi_F}. \tag{3.1}$$

Proof. Let $\pi\colon \widetilde{C} \to C$ be the cyclic cover of degree n totally ramified over $q_1, \dots, q_{ne}$. Denote by $\tilde{f}\colon \widetilde{S} \to \widetilde{C}$ (resp., $\tilde{f}'\colon \widetilde{S}' \to \widetilde{C}$ or $\tilde{\varphi}\colon \widetilde{P} \to \widetilde{C}$) the pullback fibration of f (resp., f' or φ) with respect to π. We assume that they are relatively minimal.

(1) By Lemma 3.1, the two fibrations $\tilde{f}$ and $\tilde{f}'$ are the same. So

$$\kappa(\tilde{f}) = \kappa(\tilde{f}'), \quad \lambda(\tilde{f}) = \lambda(\tilde{f}'), \quad \delta(\tilde{f}) = \delta(\tilde{f}'),$$

and thus, by (2.4),

$$\kappa(f') = \kappa(f), \qquad \lambda(f') = \lambda(f), \qquad \delta(f') = \delta(f).$$

(2) Given a general point $q \in C$. The fiber $\Gamma = \varphi^*(q)$ of φ is exactly the one in Example 2.7 (resp., Example 2.8 or Example 2.9) for $n = 2$ (resp., $n = 3$ or $n > 3$). Hence the Chern numbers of the fiber $F = f'^*(q)$ is determined uniquely by (2.2).

Combining (1), (2) and (2.5), we obtain (3.1). □

Remark 3.3. In the proof of Lemma 3.2 (2), if $(n, g_0, m) = (2, 0, 2(g+2))$ (resp., $(n, g_0, m) = (2, 2, 2(g-3))$), then the dual graph of F_i is just Figure 1 (resp., Figure 2). Similarly, if $(n, g_0, r, m) = (3, 0, 4, 5)$, then the dual graph of F_i is Figure (d) in Example 2.5.

Corollary 3.4. *Under the notations and assumptions in Lemma 3.1, denote by F one of the singular fibers of f' over q_i's. If f is semistable of genus g and $\lambda_F < \lambda_f$, then for each rational number $r \in (\lambda_F, \lambda_f]$, there is a fibration of genus g with slope r.*

Proof. Take a base change $\pi\colon \widetilde{C} \to C$ of degree d ramified over non-critical points of f. Let $(\tilde{f}, \tilde{\varphi}, \widetilde{\Pi})$ be the pullback of (f, φ, Π) w.r.t. π. Note that $\tilde{f}$ is semistable and $\widetilde{\Pi}$ is also a cyclic cover of degree n.

By (2.4), one has

$$\frac{\kappa(\tilde{f})}{\kappa(f)} = \frac{\lambda(\tilde{f})}{\lambda(f)} = d. \tag{3.2}$$

So $\lambda_{\tilde{f}} = \lambda_f$.

Applying Lemma 3.2 and (3.2) to $(\tilde{f}, \tilde{\varphi}, \widetilde{\Pi})$, one gets a new triple $(f', \tilde{\varphi}, \Pi')$. The slope of f' is

$$\lambda_{f'} = \frac{d \cdot \kappa(f) + ne \cdot c_1^2(F)}{d \cdot \lambda(f) + ne \cdot \chi_F}. \tag{3.3}$$

For $r \in (\lambda_F, \lambda_f)$, one can find d, e such that

$$\frac{d}{ne} = \frac{\chi_F}{\chi_f} \cdot \frac{r - \lambda_f}{\lambda_f - r}.$$

Thus the slope of f' in (3.3) is r. □

4 The proof of Theorem 1.3

Proof of Theorem 1.3. We prove the theorem by two steps.

Step 1. Firstly, we consider the case of hyperelliptic fibrations of genus $g \geq 2$. That is, for each $g \geq 2$ and $r \in \mathbb{Q} \cap [\lambda_m(g), \lambda_M^h(g)]$, we will show that there exists a hyperelliptic fibration of genus g with slope r.

By [LT13], we can find a semistable hyperelliptic fibration $f\colon S \to C$ of genus g with slope $\lambda_f = \lambda_M^h(g)$. The fibration f is a covering over a ruled surface $\varphi\colon P \to C$ w.r.t. a double cover $\Pi\colon S_0 \to P$. Thus we may assume that $r \in \mathbb{Q} \cap (\lambda_m(g), \lambda_M^h(g))$.

Let F_h be the new singular fiber as in Lemma 3.2 (2). The dual graph of F_h is Figure 1 by Remark 3.3. From Example 2.4, one gets $\lambda_{F_h} = \lambda_m(g) < \lambda_M^h(g)$. Therefore the result for hyperelliptic fibrations is obtained by Corollary 3.4.

Step 2. Since each fibration of genus 2 is hyperelliptic, we only need to consider $g \geq 4$. In this case, [GD-H91] gives a Kodaira fibration $f\colon S \to C$ of genus g which is a covering over a fibration $\varphi\colon P \to C$ of genus 2 w.r.t. a double cover $\Pi\colon S_0 \to P$ whose branch locus consists of $2(g-3)$ sections of φ.

Let F_g be the new singular fiber in Lemma 3.2 (2) in this case. The dual graph of F_g is Figure 2, see Remark 3.3. Similarly as above, there exists a fibration of genus g with slope r, for each $r \in \mathbb{Q} \cap (\lambda_{F_g}, \lambda_M(g))$.

From Example 2.4, one has

$$\lambda_m(g) < \lambda_{F_g} = \frac{4g-4}{g-2} < \lambda_M^h(g).$$

Thus

$$[\lambda_m(g), \lambda_M^h(g)] \cup (\lambda_{F_g}, \lambda_M(g)] = [\lambda_m(g), \lambda_M(g)].$$

Now we finish the proof by combining with Step 1. □

5 The geography of slopes for $g = 3$

The construction of Kodaira fibration in [GD-H91] does not work for $g = 3$. Zaal [Za95] gave another construction: the obtained Kodaira fibration $f\colon S \to C$ of genus 3 satisfies that S is a quadruple cover over a ruled surface. Unfortunately, this quadruple cover is not cyclic.

The result in this case we obtain is Theorem 1.4, whose proof is as follows.

Proof of Theorem 1.4. We take a semistable fibration f of genus 3 with slope $\lambda_f = 9$ as in Example 2.6. Let F be the new singular fiber in Lemma 3.2 (2) in this case. The dual graph of F is Figure (d) in Example 2.5 by Remark 3.3. Since f derives from a cyclic covering over a ruled surface, by Corollary 3.4, there exists a fibration of genus 3 with slope r for each $r \in \mathbb{Q} \cap (\lambda_F, 9)$.

From Example 2.5, one can see that

$$\lambda_m(3) = \frac{8}{3} < \lambda_F = 3 < \lambda_M^h(3) = 8.5.$$

Then we know that

$$[\lambda_m(3), \lambda_M^h(g)] \cup (\lambda_F, 9] = [\lambda_m(3), 9].$$

Now we finish the proof by combining with Step 1 in the proof of Theorem 1.3. $\square$

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References

[AK02] T. Ashikaga, K. Konno: *Global and local properties of pencils of algebraic curves*, Algebraic Geometry 2000, Azumino, Advanced Studies in pure mathematics **36** (2002), 1-49.

[BHPV04] W. P. Bath, K. Hulek, C. Peters, A. Van de Ven: *Compact complex surfaces*, secnond ed., Springer-Verlag, 2004.

[Ch87] Z. Chen: *On the geography of surfaces (simply connected minimal surfaces with positive index)*, Math. Ann. **277** (1987), 141-164.

[CT06] Z. Chen, S.-L. Tan: *Upper bounds on the slope of a genus 3 fibration,* in Recent Progress on Some Problems in Several Complex Variables and Partial Differential Equations, Contemporary Math. **400** (2006), 65-88.

[CH88] M. Cornalba, J. Harris: *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, Ann. Scient. Ec. Norm. Sup. **21** (1988), 455-475.

[GD-H91] G. Gonzalez Diez, W. Harvey: *On complete curves in moduli space, I and II*, Math. Proc. Camb. Phil. Soc. **110** (1991), 461-466, 467-472.

[LT13] X.-L. Liu, S.-L. Tan: *Families of hyperelliptic curves with maximal slopes*, Sci. China Math. **56** (9) (2013), 1743-1750.

[LuT13] J. Lu, S.-L. Tan: *Inequalities between the Chern numbers of a singular fiber in a family of algebraic curves*, Trans. Amer. Math. Soc. **365** (2013), 3373-3396.

[LuT14] J. Lu, S.-L. Tan: *On surface singularities of multiplicity three*, Methods and Applications of Analysis **21** (4) (2014), 457-480.

[Mo98] A. Moriwaki: *Relative Bogomolov's inequality and the cone of positive divisors on the moduli space of stable curves*, J. Amer. Math. Soc. **11** (1998), 569-600.

[PP11] P. Popescu-Pampu: *Introduction to Jung's method of resolution of singularities*, Contemp. Math. **538** (2011), 401–432.

[So84] A. J. Sommese: *On the density of ratios of Chern numbers of algebraic surfaces*, Math. Ann. **268** (1984), 207-221.

[Ta91] S.-L. Tan: *Galois triple covers of surfaces*, Sci. China Ser. A **34** (8) (1991), 935-942.

[Ta94] S.-L. Tan: *On the base changes of pencils of curves, I*, Manus. Math. **84** (1994), 225–244.

[Ta96] S.-L. Tan: *On the base changes of pencils of curves, II*, Math. Z. **222** (1996), 655–676.

[Ta10] S.-L. Tan: *Chern numbers of a singular fiber, modular invariants and isotrivial families of curves*, Acta Math. Viet. **35** (1) (2010), 159-172.

[Xi85] G. Xiao: *Surface fibrées en courbes degenre deux*, Lect. Notes in Math., 1137 (1985), Springer-Verlag.

[Xi87] G. Xiao: *Fibred algebraic surfaces with low slope*, Math. Ann. **276** (1987), 449-466.

[Xi92] G. Xiao: *The fibrations of algbraic surfaces*, Shanghai Scientific & Technical Publishers, 1992 (in Chinese).

[Za95] C. Zaal: *Explicit complete curves in the moduli space of curves of genus three*, Geom. Dedicata **56** (1995), 185-196.

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