ON THE ADJOINT CANONICAL DIVISOR OF A FOLIATION

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Dedicated to the memory of Professor Gang Xiao

ABSTRACT. In this paper, we describe the structure of the negative part of a Zariski decomposition of $K_X + K_{\mathcal{F}}$ for a relatively minimal foliation $(X, \mathcal{F})$ whenever $K_X + K_{\mathcal{F}}$ is pseudoeffective.

1. INTRODUCTION

For any semistable fibration $f : X \to \mathbb{P}^1$ of genus g on a smooth algebraic surface X, [TTZ05] gives a classical inequality

$$K_f^2 \geq 4g - 4$$

where $K_f = c_1(\omega_{X/\mathbb{P}^1})$ is the relative canonical divisor. This inequality is essentially from the key fact that both K_f and $K_X + K_f$ are nef.

Naturally, we are interested in the analogues of a foliation $\mathcal{F}$ on a smooth algebraic surface X. More precisely, we hope to investigate the *canonical divisor* $K_{\mathcal{F}}$ and the *adjoint canonical divisor* $K_X + K_{\mathcal{F}}$ of $\mathcal{F}$. More generally, one can define an ϵ-adjoint divisor $\epsilon K_X + K_{\mathcal{F}}$ $(0 < \epsilon \leq 1)$ which is studied in [SS23] for $\epsilon \ll 1$.

In particular, one has $K_{\mathcal{F}} = K_f$ for a foliation $\mathcal{F}$ generated by the above semistable fibration $f : X \to \mathbb{P}^1$. In this case, both $K_{\mathcal{F}}$ and $K_X + K_{\mathcal{F}}$ are nef. However, they are not necessarily nef for other foliations. Therefore we need to consider the Zariski decompositions of $K_{\mathcal{F}}$ and $K_X + K_{\mathcal{F}}$ respectively whenever they are pseudoeffective.

Miyaoka's rationality criterion says that $\mathcal{F}$ is a foliation by a rational curves if $K_{\mathcal{F}}$ is not pseudoeffective (see [Miy87] or [Bru15, Theorem 7.1]). If $K_{\mathcal{F}}$ is pseudoeffective, it has a Zariski decomposition

$$K_{\mathcal{F}} \overset{num}{=} P + N \tag{1.1}$$

where

1. N is a $\mathbb{Q}^+$-divisor and the intersection matrix of the irreducible components of N is negative definite;
2. P is a nef $\mathbb{Q}$-divisor and $PN = 0$ (see [Sak84, Corollary 7.5] or [Fuj79, Theorem 1.12]). P (resp., N) is called *positive* (resp., *negative*) part.

Furthermore, if $\mathcal{F}$ is relative minimal, then N is a disjoint union of maximal $\mathcal{F}$-chains and the integral part $\lfloor N \rfloor = 0$ (see [McQ00] or [Bru15, Theorem 8.1]).

In this paper, we shall study mainly the *adjoint canonical divisor* $K_X + K_{\mathcal{F}}$ of a relatively minimal foliation $\mathcal{F}$. We assume that $K_X + K_{\mathcal{F}}$ is pseudoeffective and denote a Zariski decomposition of $K_X + K_{\mathcal{F}}$ by

$$K_X + K_{\mathcal{F}} = \overline{P} + \overline{N}$$

where $\overline{P}$ (resp., $\overline{N}$) is the positive (resp., negative) part of $K_{\mathcal{F}}$. We hope to answer the following problem.

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[1] This work is supported by NSFC, Science and Technology Commission of Shanghai Municipality (No. 22DZ2229014).

[2] *2020 Mathematics Subject Classification.* 14C21, 14D06, 14H10, 32S65, 37F75

[3] *Key words and phrases.* foliation, fibration, canonical divisor, Chern number, Zariski decomposition.

Problem 1.1. *What is the structure of the negative part $\overline{N}$?*

For this purpose, we will consider a bimeromorphic morphism $\rho : (X, \mathcal{F}) \to (X_0, \mathcal{F}_0)$ onto a so-called *relatively minimal A-D-E model* $\mathcal{F}_0$ of $\mathcal{F}$ on a smooth algebraic surface X_0 (see Definition 3.1). The adjoint canonical divisor $K_{X_0} + K_{\mathcal{F}_0}$ is also pseudoeffective and has a Zariski decomposition

$$K_{X_0} + K_{\mathcal{F}_0} = P_0 + N_0 \tag{1.2}$$

with a positive part P_0 and a negative part N_0. One can see easily that

$$\overline{P} = \rho^* P_0, \quad \overline{N} = \rho^* N_0 + V \tag{1.3}$$

where V is a $\mathbb{Q}^+$-divisor supported on the exceptional set of ρ (see (3.1) and Theorem 3.9 for a precise expression). Therefore it's sufficient to determine the structure of N_0.

Our main result is as follows.

Theorem 1.2. *Let $(X, \mathcal{F})$ be a relative minimal foliation. If $K_X + K_{\mathcal{F}}$ is pseudoeffective, then $\mathcal{F}$ is minimal and the negative part $\overline{N}$ of the Zariski decomposition of $K_X + K_{\mathcal{F}}$ can be expressed as in* (1.3) *where the support of N_0 is a disjoint union of maximal simple $\mathcal{F}_0$-chains (see Definition 2.1) and the integral part $\lfloor N_0 \rfloor = 0$. Furthermore, $\rho^* N_0$ is disjoint from the exceptional set of ρ.*

Remark 1.3. *However, it is possible that V contains some curves which are not $\mathcal{F}$-invariant.*

An interesting question is when $K_X + K_{\mathcal{F}}$ is pseudoeffective. The following result provide an partial answer.

Theorem 1.4. *Let $(X, \mathcal{F})$ be a relatively minimal foliation with a non-zero pseudoeffective canonical divisor $K_{\mathcal{F}}$. Set*

$$\rho(X) := \frac{1}{2}(K_X + K_{\mathcal{F}})K_{\mathcal{F}}.$$

We have

$$h^0(K_X + K_{\mathcal{F}}) \geq \chi(\mathcal{O}_X) + \rho(X).$$

The equality holds if $\mathcal{F}$ is of general type, i.e., $P^2 > 0$ (see [Bru15, Ch 8., Sec.1]*).*

Therefore $K_X + K_{\mathcal{F}}$ is pseudoeffective if it satisfies one of the following conditions:

1. $\mathrm{kod}(X) \geq 0$*;*
2. $\mathrm{kod}(X) = -\infty$ *and* $\rho(X) \geq q(X)$*,*

where $q(X)$ is the irregularity of X.

Corollary 1.5. *For any relatively minimal foliation $(X, \mathcal{F})$ of general type with $h^0(K_{\mathcal{F}}) > 0$, we have*

$$q(X) \leq 1 + \rho(X).$$

For any foliation $(Y, \mathcal{G})$ with a minimal model $(X, \mathcal{F})$, we can define some invariants of $\mathcal{G}$ by the adjoint canonical divisor of $\mathcal{F}$:

1. *adjoint numerical Kodaira dimension*

$$\bar{\nu}(\mathcal{F}) = \begin{cases} 0, & \text{if } \overline{P} \overset{num}{=} 0. \\ 1, & \text{if } \overline{P} \overset{num}{\neq} 0 \text{ but } \overline{P}^2 = 0, \\ 2, & \text{if } \overline{P}^2 = 0. \end{cases}$$

   In order to be complete, we also set $\bar{\nu}(\mathcal{F}) = -\infty$ if $K_X + K_{\mathcal{F}}$ is not pseudoeffective;

2. *adjoint Kodaira dimension*

$$\bar{k}(\mathcal{F}) := \limsup_{n \to +\infty} \frac{\log h^0(X, n(K_X + K_{\mathcal{F}}))}{\log n};$$

3. *adjoint the first Chern numer* $\bar{c}_1^2(\mathcal{F}) := \overline{P}^2$.

Remark 1.6. [Tan23] *defines a biholomorphic invariant* $c_1^2(\mathcal{F})$ *for any foliation* $\mathcal{F}$ *and proves that* $c_1^2(\mathcal{F}) = P^2$ *for the positive part* P *in* (1.1) *whenever* $\mathcal{F}$ *is relatively minimal.*

As an application, one can investigate an algebraic foliation generated by a semistable fibration.

Corollary 1.7. *Let* $f : X \to B$ *be a non-trivial semistable fibration of genus* $g \geq 2$ *over a smooth algebraic curve* B *of genus* b *and* $\mathcal{F}$ *be the foliation induced by* f. *Then* $K_X + K_{\mathcal{F}}$ *is nef and* $\bar{\nu}(\mathcal{F}) \geq 1$. *We have*

$$h^0(K_X + K_{\mathcal{F}}) = \chi_f + K_f^2 + 3(g-1)(b-1) \tag{1.4}$$

and

$$\bar{c}_1^2(\mathcal{F}) = 4\left(K_f^2 + 4(g-1)(b-1)\right) \geq 0 \tag{1.5}$$

where $K_f = c_1(\omega_{X/B})$ *is the relative canonical divisor of* f *and*

$$\chi_f = \deg f_*\omega_{X/B} = \chi(\mathcal{O}_X) - (g-1)(b-1)$$

is a positive invariant (cf. [AK00, pp.6] *or* [BHPV04, Ch. III, Theorem 18.2]*).*

In particular, if $B \cong \mathbb{P}^1$, the non-negativity of $\bar{c}_1^2(\mathcal{F})$ is equivalent to the well-known inequality $K_f^2 \geq 4(g-1)$ in [TTZ05, Theorem 2.1]. They describe such fibrations satisfying $\bar{c}_1^2(\mathcal{F}) = 0$ which can be rephrased in the language of foliation theory as follows.

Corollary 1.8. *Let* $(X, \mathcal{F}, f)$ *be as in Corollary 1.7. Then* $\bar{\nu}(\mathcal{F}) = 1$ *iff* $B \cong \mathbb{P}^1$ *and* X *is the minimal resolution of the singularities of a double covering surface* $\pi : Z \to \mathbb{P}^1 \times C$ *ramified over a curve of numerical type* $2F_1 + (2g+2-4g(C))F_2$*, and fibration* f *is induced by the first projection* $pr_1 : \mathbb{P}^1 \times C \to \mathbb{P}^1$ *where* F_i *is a fiber of the i-th projection of* $\mathbb{P}^1 \times C$.

We will give an example for an algebraic foliation $\mathcal{F}$ with $\bar{\nu}(\mathcal{F}) = 0$ in Sec. 5.

There are some open problem on the adjoint canonical divisor $K_X + K_{\mathcal{F}}$.

Problem 1.9. *When is* $K_X + K_{\mathcal{F}}$ *pseudoeffective for a minimal foliation* $\mathcal{F}$*?*

Problem 1.10. *Is there a foliation* $\mathcal{F}$ *satisfying* $\bar{\nu}(\mathcal{F}) \neq \bar{k}(\mathcal{F})$*?*

Problem 1.11. *What is the relation between* $c_1^2(\mathcal{F})$ *and* $\bar{c}_1^2(\mathcal{F})$*?*

Problem 1.12. *How to give a classification of all foliations with adjoint numerical Kodaira dimensions* ≤ 1*?*

Problem 1.13. *Given a foliation* $\mathcal{F}$ *generated by a non-semistable fibration* $f : X \to \mathbb{P}^1$. *Is there an inequality similar to the classical inequality in* [TTZ05, Theorem 2.1] *by the non-negativity of* $c_1^2(\mathcal{F})$ *and* $\bar{c}_1^2(\mathcal{F})$*?*

Problem 1.14. *When does a minimal foliation has a unique relatively minimal A-D-E model up to a biholomorphic morphism?*

2. Preliminaries

2.1. **$\mathcal{F}$-invariant curves and singularities of a foliation $\mathcal{F}$.** We recall some definitions and basic facts about foliations on a surface (see [Bru15] or [CF18, Sec. 2] for more details).

Let X be a smooth algebraic surface with a tangent bundle T_X. A foliation $\mathcal{F}$ on X is given by a short exact sequence

$$0 \longrightarrow T_{\mathcal{F}} \longrightarrow T_X \longrightarrow \mathcal{I}_Z \otimes N_{\mathcal{F}} \longrightarrow 0$$

where $T_{\mathcal{F}}$ and $N_{\mathcal{F}}$ are line bundles and $\mathcal{I}_Z$ is an ideal sheaf supported on a finite set. $K_{\mathcal{F}} := c_1(T_{\mathcal{F}}^*)$ is called the canonical divisor of $\mathcal{F}$.

A curve $C \subseteq X$ is said to be $\mathcal{F}$-invariant if the inclusion $T_{\mathcal{F}}|_C \to T_X|_C$ factors through T_C where T_C is the tangent bundle of C.

An $\mathcal{F}$*-chain* Θ is a Hirzebruch-Jung string $\Theta = \Gamma_1 + \cdots + \Gamma_l$ consisting of $\mathcal{F}$-invariant curves Γ_i's satisfying that

(1) all singularities of $\mathcal{F}$ on Θ are reduced and non-degenerated;
(2) there is only one singularity of $\mathcal{F}$, says $p_l (\in \Gamma_l)$, on $\Theta - \{p_1, \ldots, p_{l-1}\}$ where $p_i = \Gamma_i \cap \Gamma_{i+1}$ $(i = 1, \ldots, l-1)$;
(3) Γ_1 has only one singularity p_1.

For convenience, Γ_1 is said to be *the first component* of Θ. More details can be found in [Bru15, Ch.8, Sec.2].

Definition 2.1. *A simple $\mathcal{F}$-chain is an $\mathcal{F}$-chain consisting of $\mathcal{F}$-invariant (-2)-curves. We say a simple $\mathcal{F}$-chain is maximal if it can not be contained other simple $\mathcal{F}$-chains.*

However, it's possible that a maximal simple $\mathcal{F}$-chain is contained in an $\mathcal{F}$-chain.

An $\mathcal{F}$-invariant (-1)-curve C is said to be $\mathcal{F}$*-exceptional* if the contraction of C to a point p produces a new foliation which has at p a regular point or a reduced singular point.

$\mathcal{F}$ is said to be reduced if all singularities of $\mathcal{F}$ are reduced. Furthermore, a reduced foliation is called *relatively minimal* if it has no $\mathcal{F}$-exceptional curve. Each foliation has a relatively minimal model (see [Bru15, Proposition 5.1]). A relatively minimal foliation $(X, \mathcal{F})$ is said to be minimal if any bimeromorphic map $f : (X, \mathcal{F}) \dashrightarrow (Y, \mathcal{G})$ sending $\mathcal{F}$ to a relatively minimal foliation $\mathcal{G}$ is in fact a biholomorphic map.

Consider a blowing-up $\sigma : (\widetilde{X}, \widetilde{\mathcal{F}}, E) \to (X, \mathcal{F}, p)$ centered at a singularity p of $\mathcal{F}$ with an exceptional curve $E \subset \widetilde{X}$ and a pulling-back foliation $\widetilde{\mathcal{F}}$. Let $a(p)$ be the vanishing order of $\mathcal{F}$ at p. One has

$$K_{\widetilde{\mathcal{F}}} = \sigma^* K_{\mathcal{F}} + (1 - l(p))\, E \tag{2.1}$$

where $l(p)$ is the *order* of $\mathcal{F}$ at p defined by

$$l(p) = \begin{cases} a(p), & \text{if } E \text{ is } \mathcal{F}\text{-invariant}, \\ a(p) + 1, & \text{otherwise.} \end{cases}$$

See [Bru15, Ch. 2, Sec. 3] for more details.

Let U be a neighborhood in X with a local coordinate (x, y) and

$$v = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y} \quad (a, b \in \mathbb{C}\{x, y\}) \tag{2.2}$$

be a local generator of $\mathcal{F}$ at a singularity $p = (0, 0)$. Let B be an $\mathcal{F}$-invariant branch passing through p. We take a minimal Puiseux's parametrization of B at p:

$$\varphi : \mathbb{D} \to B, \quad t \to (\varphi_x(t), \varphi_y(t)) \tag{2.3}$$

where $\varphi_x, \varphi_y \in \mathbb{C}\{t\}$ and $\mathbb{D}$ is a disk centered at $0 \in \mathbb{C}$. The multiplicity $\mu_p(\mathcal{F}, B)$ of $\mathcal{F}$ at B is defined by the order of $\varphi^*(B)$ at $t = 0$. More precisely, one has

$$\mu_p(\mathcal{F}, B) = \begin{cases} \nu_0(a(\varphi_x(t), \varphi_y(t))) - \nu_0(\varphi_x(t)) + 1 & \text{if } \varphi_x(t) \neq 0, \\ \nu_0(b(\varphi_x(t), \varphi_y(t))) - \nu_0(\varphi_y(t)) + 1 & \text{if } \varphi_y(t) \neq 0, \end{cases} \tag{2.4}$$

where a, b are as in (2.2) and $\nu_0(h)$ is the order of the zero $t = 0$ of $h \in \mathbb{C}\{t\}$ (see [Car94]). $\mu_p(\mathcal{F}, B) \geq 0$ the equality holds iff p is not a singularity of $\mathcal{F}$.

Remark 2.2. *For a smooth point p of B, $\mu_p(\mathcal{F}, B) = Z(\mathcal{F}, B, p)$ where $Z(\mathcal{F}, B, p)$ is the Gomez-Mont-Seade-Verjovsky index (cf.* [Bru15, Bru97, GSV91]*). If B is a smooth irreducible $\mathcal{F}$-invariant curve, we denote the sum of $\mu_p(\mathcal{F}, B)$'s for all $p \in B$ by $Z(\mathcal{F}, B)$.*

Let $\sigma : \widetilde{X} \to X$ be a blowing-up centered at $p \in B$ with an exceptional curve E and $\widetilde{B}$ be the strict transform of B with the only one point $\tilde{p} := E \cap \widetilde{B}$. From [Car94], one has

$$\mu_p(\mathcal{F}, B) = \mu_{\tilde{p}}(\widetilde{\mathcal{F}}, \widetilde{B}) + m_p(B)(l(p) - 1) \tag{2.5}$$

where $m_p(B)$ is the multiplicity of B at p and $\widetilde{\mathcal{F}}$ is the pulling-back foliation of $\mathcal{F}$.

Lemma 2.3. *$\mu_p(\mathcal{F}, B) = 1$ iff either p has a nonzero eigenvalue or p is a saddle-node with a strong separatrix B.*

In particular, in this case, if $m_p(B) \geq 2$, then p is a dicritical singularity with a local generator $v = x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$ for $\lambda \in \mathbb{Q}^+$ by choosing a proper coordinate.

Proof. Firstly, we consider the case for $m_p(B) = 1$. By choosing a suitable coordinate, we can take the minimal parametrization (2.3) as $\varphi(t) = (t, 0)$. Thus the local generator (2.2) of $\mathcal{F}$ can be taken as

$$v = (x^m w(x) + yu(x,y)) \frac{\partial}{\partial x} + yv(x,y)\frac{\partial}{\partial y}, \quad u, v \in \mathbb{C}\{x,y\},\ w \in \mathbb{C}\{x\},\ w(0) \neq 0.$$

By (2.4), we get $\mu_p(\mathcal{F}, B) = m$. In particular, $\mu_p(\mathcal{F}.B) = 1$ iff either the eigenvalue of p is nonzero or p is a saddle-node with a strong separatrix B.

So it's enough to consider the case for $m_p(B) \geq 2$ from the above discussion.

($\Longleftarrow$) Since $m_p(B) \geq 2$, p is a singularity of $\mathcal{F}$ with a local generator $v = x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}$ for $\lambda \in \mathbb{Q}^+$ by choosing a proper coordinate (see [Bru15, pp. 7-8]). Thus (2.4) gives $\mu_p(\mathcal{F}, B) = 1$.

($\Longrightarrow$) If $l(p) \geq 2$, then (2.5) implies that $l(p) = 2$ and $m_p(B) = 1$, a contradiction. So $l(p) = 1$. If the eigenvalue of p is nonzero, the proof is finished. If p is a saddle-node, then B is a strong separatrix by [Bru15, pp. 31] and so $m_p(B) = 1$, a contradiction.

In what follows, we assume p is a nilpotent singularity. By choosing a suitable coordinate, the local generator of $\mathcal{F}$ can be written as

$$v = (y + u(x,y)) \frac{\partial}{\partial x} + v(x,y)\frac{\partial}{\partial y}$$

where u, v are holomorphic functions which vanish at $(0,0)$ up to order 2. Consider the minimal parametrization (2.3) of B at p. In this case, $\varphi_x\varphi_y \neq 0$. From (2.4), one has

$$\nu_0(\varphi_y(t) + u(\varphi_x(t), \varphi_y(t))) = \nu_0(\varphi_x(t)), \quad \nu_0(v(\varphi_x(t), \varphi_y(t))) = \nu_0(\varphi_y(t)).$$

Hence

$$\nu_0(\varphi_y(t)) = \nu_0(\varphi_x(t)), \quad 2\nu_0(\varphi_x(t)) \leq \nu_0(\varphi_y(t)).$$

It implies that $\nu_0(\varphi_x(t)) = \nu_0(\varphi_y(t)) = 0$, a contradiction.

This proof is completed. □

Due to Camacho-Lins Neto-Sad's formula in nondicritical case ([CLS84, Theorem 1]), we have a modified version which can be rephrased as follows.

Lemma 2.4 (Camacho-Lins Neto-Sad's formula)**.** *Consider a sequence of blowing ups*

$$X_r \xrightarrow{\sigma_r} (X_{r-1}, p_{r-1}) \xrightarrow{\sigma_{r-1}} \cdots (X_1, p_1) \xrightarrow{\sigma_1} (X_0, p_0) := (X, p)$$

where σ_{i+1} is a blowing-up centered at a point $p_i \in X_i$ and $(X_r, \widetilde{\mathcal{F}})$ is the pulling-back foliation of $(X, \mathcal{F})$. Let E_i (resp., $\mathcal{E}_i$) in X_r be the strict (resp., total) transform of the exceptional curve of σ_i. Write $\mathcal{E}_1 = \sum_{i=1}^{r} n_i E_i$.

If each E_i is $\widetilde{\mathcal{F}}$-invariant, then the order $l(p)$ of the singularity p satisfies

$$1 + l(p) = \sum_{i=1}^{r} \sum_{q \in E_i} n_i \left(\mu_q(\widetilde{\mathcal{F}}, E_i) - \mu_q(\mathcal{E}_1)\right) \tag{2.6}$$

where $\mu_q(\mathcal{E}_1)$ is the Milnor's number of the support of $\mathcal{E}_1$ at q, namely,

$$\mu_q(\mathcal{E}_1) = \begin{cases} 1, & \textit{if } q \textit{ is a corner}, \\ 0, & \textit{else}. \end{cases}$$

Proof. It's similar to the proof of [CLS84, Theorem 1]. □

Recall the notation in Remark 2.2. We set

$$Z(\widetilde{\mathcal{F}}, E_i) := \sum_{q\in E_i} \mu_q(\widetilde{\mathcal{F}}, E_i).$$

The formula (2.6) is equivalent to

$$1 + l(p) = \sum_{i=1}^{r} n_i(Z(\widetilde{\mathcal{F}}, E_i) - k_i) \tag{2.7}$$

where k_i is the number of irreducible components of $\mathcal{E}_1$ meeting transversely with E_i.

2.2. **Cerveau-Lins Neto's formula.** Due to Cerveau-Lins Neto formula for a foliation on $\mathbb{P}^2$ (see [CLN91, pp. 885]), we have a generalized result as follows.

Lemma 2.5 (Cerveau-Lins Neto formula)**.** *For any irreducible $\mathcal{F}$-invariant curve C, we have*

$$2 - 2g(C) + K_{\mathcal{F}}C = \sum_{p\in C}\sum_{B\in C(p)} \mu_p(\mathcal{F}, B)$$

where $C(p)$ is the set of analytic branches of C passing through p and $g(C)$ is the geometric genus of C.

Proof. Let $\sigma : \widetilde{X} \to X$ be a blowing-up centered at $p \in C$ with an exceptional curve E. Let $C(p) = \{B_1, \ldots, B_k\}$ and $\widetilde{B}_i$ be the strict transform of B_i with the only one point $\tilde{p}_i := E\cap\widetilde{B}_i$. By (2.5),

$$\mu_p(\mathcal{F}, B_i) = \mu_{\tilde{p}_i}(\widetilde{\mathcal{F}}, \widetilde{B}_i) + m_i(l(p) - 1)$$

where m_i is the multiplicity of B_i at p and $\widetilde{\mathcal{F}}$ is the pulling-back foliation of $\mathcal{F}$.

Let $\widetilde{C}$ be the strict transform of C. By (2.1) and $\sigma^*C = \widetilde{C} + (\sum_{i=1}^{k} m_i)E$, we get

$$K_{\mathcal{F}}C = K_{\widetilde{\mathcal{F}}}\widetilde{C} + \left(\sum_{i=1}^{k} m_i\right)(l(p) - 1).$$

So

$$\sum_{p\in C}\sum_{B\in C(p)} \mu_p(\mathcal{F}, B) - K_{\mathcal{F}}C = \sum_{\tilde{p}\in\widetilde{C}}\sum_{\widetilde{B}\in\widetilde{C}(\tilde{p})} \mu_{\tilde{p}}(\widetilde{\mathcal{F}}, \widetilde{B}) - K_{\widetilde{\mathcal{F}}}\widetilde{C}.$$

Therefore, it's enough to consider the case that $\mathcal{F}$ is reduced and C is smooth. In this case, $C(p) = \{C\}$ and $\mu_p(\mathcal{F}, C) = Z(\mathcal{F}, C, p)$ for each singularity p of $\mathcal{F}$. From [Bru15, Ch. 2, Proposition 3],

$$2 - 2g(C) + K_{\mathcal{F}}C = \sum_{p\in C} Z(\mathcal{F}, C, p) = \sum_{p\in C} \mu_p(\mathcal{F}, C).$$

This proof is finished. □

As the applications of Cerveau-Lins Neto formula, one can obtain the following consequences which are essentially due to [McQ08, Lemma II. 3.2, Proposition III.1.2, Theorem IV.1.1].

Corollary 2.6. *For any irreducible $\mathcal{F}$-invariant curve C, we have $K_{\mathcal{F}}C \geq -2$. The equality holds iff*

1. *$C \cong \mathbb{P}^1$ and $C^2 = 0$;*
2. *there is no singularity of $\mathcal{F}$ on C;*
3. *$K_{\mathcal{F}}$ is not pseudo-effective and hence $\mathcal{F}$ is a foliation by rational curves.*

Proof. From Lemma 2.5, $K_{\mathcal{F}}C \geq 2g(C) - 2 \geq -2$. If $K_{\mathcal{F}}C = -2$, then $g(C) = 0$ and $\mu_p(\mathcal{F}, B) = 0$ for each $p \in C$ and $B \in C(p)$. So C contains no singularity of $\mathcal{F}$ and hence C is smooth. Thus $C \cong \mathbb{P}^1$. By Camacho-Sad formula ([CS82, Suw98]), $C^2 = 0$ and so C is nef. Thus $K_{\mathcal{F}}C < 0$ implies that $K_{\mathcal{F}}$ is not pseudo-effective. From [Miy87], $\mathcal{F}$ is a foliation induced by a family of rational curves.

Conversely, for an irreducible $\mathcal{F}$-invariant curve satisfying the above conditions (1) and (2), one can get $K_{\mathcal{F}}C = -2$ by Cerveau-Lins Neto formula. □

Corollary 2.7. *Assume that $K_{\mathcal{F}}$ is pseudo-effective. We have $K_{\mathcal{F}}C \geq -1$ for any irreducible $\mathcal{F}$-invariant curve C. The equality holds iff one of the following cases occurs.*

(1) *C is the first component of an $\mathcal{F}$-chain ;*
(2) *C is an $\mathcal{F}$-exceptional curve with only one singularity.*

Proof. By Corollary 2.6, $K_{\mathcal{F}}C \geq -1$. Assume that $K_{\mathcal{F}}C = -1$. From Lemma 2.5, $g(C) = 0$ and C has only one singularity p of $\mathcal{F}$ with $\mu_p(\mathcal{F}, B) = 1$ where B is a unique branch of C passing through p. Since $K_{\mathcal{F}}$ is pseudo-effective, $C^2 < 0$.

If C is smooth at p, then $C \cong \mathbb{P}^1$ by $g(C) = 0$. From Camacho-Sad formula, one can see that p is a reduced singularity with an eigenvalue $\lambda = C^2 < 0$. Namely, C occurs in one of the above cases.

It's enough to claim that C is smooth at p. If not, the local generator of $\mathcal{F}$ at p can be taken as $v = x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}$ ($\lambda \in \mathbb{Q}^+$) by Lemma 2.3. Consider the resolution of p as in [Bru15, pp. 7-8], denoted by $\hat{\pi} : (\widehat{X}, \widehat{\mathcal{F}}) \to (X, \mathcal{F})$. Let $\widehat{C}$ be the strict transform of C under $\hat{\pi}$. One can see that $\widehat{C}$ has no singularity of $\widehat{\mathcal{F}}$. So $K_{\widehat{\mathcal{F}}}\widehat{C} = -2$ by Cerveau-Lins Neto formula. Corollary 2.6 implies that $\widehat{C}^2 = 0$. So $C^2 \geq 0$, a contradiction.

Conversely, any $\mathcal{F}$-invariant curve C in case (1) or (2) satisfies $K_{\mathcal{F}}C = -1$ from Cerveau-Lins Neto formula. □

3. A-D-E singularities of foliations

3.1. **Relatively minimal A-D-E model of a foliation.** Let p be a singularity of $\mathcal{F}$ in a neighborhood U. From [Sei68] or [Bru15, Theorem 1.1], one has a minimal resolution of the singularity p:

$$(U_r, \mathcal{F}_r) \xrightarrow{\sigma_r} (U_{r-1}, \mathcal{F}_{r-1}, p_{r-1}) \xrightarrow{\sigma_{r-1}} \cdots (U_1, \mathcal{F}_1, p_1) \xrightarrow{\sigma_1} (U_0, \mathcal{F}_0, p_0) := (U, \mathcal{F}, p)$$

where σ_{i+1} is a blowing-up of a neighborhood U_i at the non-reduced singularity p_i of $\mathcal{F}_i$ with order l_i, $\mathcal{F}_{i+1} = \sigma_{i+1}^*\mathcal{F}_i$ is the pulling-back of the foliation $\mathcal{F}_i$ and $(U_r, \mathcal{F}_r)$ has at worst reduced singularities ($i = 0, \ldots, r-1$).

Definition 3.1. *For a given positive integer k, p is said to be a k-simple singularity of $\mathcal{F}$ if $l_i \leq k$ for $i = 0, 1 \ldots, r-1$. For convenience, a 2-simple singularity is also called an A-D-E singularity of $\mathcal{F}$.*

We say $\mathcal{F}$ is an A-D-E foliation if each singularity of $\mathcal{F}$ is an A-D-E singularity. $(X, \mathcal{F})$ is said to be a relatively minimal A-D-E foliation if it's an A-D-E foliation and any bimeromorphic morphism $(X, \mathcal{F}) \to (Y, \mathcal{G})$ onto an A-D-E foliation $(Y, \mathcal{G})$ is in fact a biholomorphism.

Example 3.2. *A 1-simple singularity p occurs in one of the following cases:*

(1) *p is a reduced singularity;*
(2) *p has a Poincaré-Dulac form $xdy - (ny + x^n)dx$ by choosing a suitable local coordinate.*

We will classify all A-D-E singularities of a foliation in Sec. 3.2. Here are some classical examples.

Example 3.3. *Take $v = \frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}$ in* (2.2) *for a given function $f \in \mathbb{C}\{x, y\}$. The branch B defined by $f = 0$ is $\mathcal{F}$-invariant. In this case, p is an A-D-E singularity of $\mathcal{F}$ iff it is a simple singularity of the curve B (see* [BHPV04, Ch. II, Sec. 8]*).*

Example 3.4. *If p is a reduced singularity or a singularity whose eigenvalues are positive rational numbers, then it is an A-D-E singularity (see* [Bru15, pp. 7-8]*).*

Lemma 3.5. *Let (X, F) be a reduced foliation.*

(1) *There is a bimeromorphic morphism $\rho : (X, \mathcal{F}) \to (X_0, \mathcal{F}_0)$ onto a relatively minimal A-D-E foliation $(X_0, \mathcal{F}_0)$. Therefore each foliation has a relatively minimal A-D-E model.*
(2) *$K_X + K_{\mathcal{F}} = \rho^*(K_{X_0} + K_{\mathcal{F}_0}) + V$ for some $\mathbb{Q}^+$-divisor V supported on the exceptional set of ρ.*
(3) *$K_X + K_{\mathcal{F}}$ is pseudoeffective iff $K_{X_0} + K_{\mathcal{F}_0}$ is pseudoeffective.*
(4) *For any (-1)-curve $E \subset X_0$, one has $K_{\mathcal{F}_0}E \geq 2$.*

Proof. (1) It's obvious that $(X, \mathcal{F})$ is an A-D-E foliation from Example 3.2. If it is not a relatively minimal A-D-E foliation, then we can find a (-1)-curve whose contraction produces a new A-D-E foliation. One can iterate the contraction procedure and must stop it after finite steps because the rank of the Néron-Severi group of the surface is strictly monotonic decreasing. Thus we get a relatively minimal A-D-E foliation $(X_0, \mathcal{F}_0)$ with a bimeromorphic morphism $\rho : (X, \mathcal{F}) \to (X_0, \mathcal{F}_0)$.

(2) By the above discussion, ρ factorizes through some blowing-ups :

$$(X, \mathcal{F}) := (X_r, \mathcal{F}_r) \xrightarrow{\sigma_r} (X_{r-1}, \mathcal{F}_{r-1}) \xrightarrow{\sigma_{r-1}} \cdots (X_1, \mathcal{F}_1) \xrightarrow{\sigma_1} (X_0, \mathcal{F}_0).$$

Let $E_i \subset X_i$ be the exceptional curve of the blowing-up σ_i centred at a point $p_{i-1} \in X_{i-1}$ and $\mathcal{E}_i$ be the total transform of E_i in X ($i = 1, \ldots, r$). By (2.1), one gets $K_X + K_{\mathcal{F}} = \rho^*(K_{X_0} + K_{\mathcal{F}_0}) + V$ where

$$V = \sum_{i=1}^{r} (2 - l(p_{i-1}))\, \mathcal{E}_i. \tag{3.1}$$

Note that each p_{i-1} is an A-D-E singularity and hence $l(p_{i-1}) \leq 2$. So V is a $\mathbb{Q}^+$-divisor.

(3) ($\Longrightarrow$) Assume that $K_X + K_{\mathcal{F}}$ is pseudoeffective. For any ample divisor H_0 in X_0, ρ^*H_0 is nef. So one has

$$(K_{X_0} + K_{\mathcal{F}_0})H_0 = (K_X + K_{\mathcal{F}})\rho^*H_0 \geq 0.$$

($\Longleftarrow$) Assume that $K_{X_0} + K_{\mathcal{F}_0}$ is pseudoeffective. Consider the Zariski decomposition (1.2) of $K_{X_0} + K_{\mathcal{F}_0}$. Thus we have

$$K_X + K_{\mathcal{F}} = \rho^*(P_0) + (\rho^*N_0 + V).$$

For any ample divisor $H \subset X$, one can see that

$$(K_X + K_{\mathcal{F}})H \geq \rho^*P_0 \cdot H \geq 0$$

from ρ^*P_0 is nef.

(4) Let $E \subset X_0$ be a (-1)-curve. Consider a contraction $\sigma : (X_0, \mathcal{F}_0, E) \to (Y, \mathcal{G}, p)$ sending E to a point $p = \sigma(E)$. It produces a new foliation $\mathcal{G}$ with a singularity p with order l. The minimality of the A-D-E foliation $\mathcal{F}_0$ implies that $l \geq 3$. By (2.1), one has $K_{\mathcal{F}_0}E = l - 1 \geq 2$. □

Remark 3.6. *However the relatively minimal A-D-E model of a foliation is not necessarily unique. For example, we consider a Riccati foliation $\mathcal{F}_\lambda$ on $X = \mathbb{P}^1 \times \mathbb{P}^1$, defined by $x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$ ($\lambda \in \mathbb{C}$ and $\lambda \neq 0$), with respect to a ruling map*

$$pr_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1, \quad (x, y) \to y.$$

$\mathcal{F}_\lambda$ is a relatively minimal A-D-E foliation.

We have a bimeromorphic map $\sigma : (X, \mathcal{F}_{\lambda-1}) \dashrightarrow (X, \mathcal{F}_\lambda)$ sending (x, y) to (x, xy) for each $\lambda \neq 0, 1$.

Corollary 3.7. *Given a bimeromorphic map $\sigma : (Y, \mathcal{G}) \dashrightarrow (X_0, \mathcal{F}_0)$ from a relatively minimal A-D-E foliation $(Y, \mathcal{G})$. $K_{X_0} + K_{\mathcal{F}_0}$ is pseudoeffective iff $K_Y + K_{\mathcal{G}}$ is pseudoeffective.*

Proof. We may find a reduced foliation $(Y_0, \mathcal{G}_0)$ associated with two bimeromorphic morphisms $\rho : (Y_0, \mathcal{G}_0) \to (X_0, \mathcal{F}_0)$ and $\tau : (Y_0, \mathcal{G}_0) \to (Y, \mathcal{G})$ satisfying $\rho = \sigma\tau$. By Lemma 3.5, $K_{X_0} + K_{\mathcal{F}_0}$ (resp., $K_Y + K_{\mathcal{G}}$) is pseudoeffective iff $K_{Y_0} + K_{\mathcal{G}_0}$ is pseudoeffective. □

Lemma 3.8. *Let $(X, \mathcal{F})$ be a relatively minimal foliation. If $K_X + K_{\mathcal{F}}$ is pseudoeffective, then $\mathcal{F}$ is minimal.*

Proof. Suppose that $\mathcal{F}$ be not minimal. We will get a contradiction.

From [Bru15, Theorem 5.1], $\mathcal{F}$ is biholomorphic to one of the following foliations:

(1) rational fibrations;
(2) nontrivial Riccati foliations;
(3) the very special foliation.

In case (1), we have $(K_X + K_{\mathcal{F}})F = -4$ for a general fiber F of the rational fibration generating $\mathcal{F}$. Hence $K_X + K_{\mathcal{F}}$ is not pseudoeffective, a contradiction.

In case (2), we have $(K_X + K_{\mathcal{F}})F = -2$ for a general fiber F of the rational fibration adapted to the Riccati foliation $\mathcal{F}$. We get a contradiction again.

In case (3), from [Per05, Sec. 5], $(X, \mathcal{F})$ has a relatively minimal A-D-E model $(\mathbb{P}^2, \mathcal{F}_0)$ induced by a homogeneous one-form on $\mathbb{P}^2$

$$\Omega := Z(-Y^2 - XZ + 2XY)dX + 3XZ(Y - X)dY + X(XZ - 2Y^2 + XY)dZ,$$

One has $K_{\mathbb{P}^2} + K_{\mathcal{F}_0} = -2L$ for a general line L in $\mathbb{P}^2$, a contradiction. □

3.2. **Classification of A-D-E singularities of foliations.** For convenience, we assume that $(X, \mathcal{F})$ is relatively minimal. Let $(X_0, \mathcal{F}_0)$ be the relatively minimal A-D-E model of $(X, \mathcal{F})$ with a bimeromorphic morphism $\rho : (X, \mathcal{F}) \to (X_0, \mathcal{F}_0)$.

The morphism ρ can factorize through a bimeromorphic morphism $\rho' : (X, \mathcal{F}) \to (X', \mathcal{F}')$ onto a foliation $(X', \mathcal{F}')$ satisfying

(1) each singularity of $(X', \mathcal{F}')$ has an eigenvalue, namely, it is either a reduced singularity or a singularity whose eigenvalues are positive rational numbers;
(2) $\sigma : (X', \mathcal{F}') \to (X_0, \mathcal{F}_0)$ consists of blowing-ups and satisfies $\rho = \sigma\rho'$;

$$\begin{array}{ccc} (X, \mathcal{F}) & & \\ {\scriptstyle \rho'}\downarrow & \overset{\rho}{\searrow} & \\ (X', \mathcal{F}') & \xrightarrow{\ \sigma\ } & (X_0, \mathcal{F}_0) \end{array}$$

(3) for any (-1)-curve $E \subset X'$ in the exceptional set of σ, the contraction of E to a point p produces a new foliation $(Y, \mathcal{G})$ which has at p a singularity without eigenvalue.

Let p_0 be an A-D-E singularity of $\mathcal{F}_0$ without eigenvalue in a neighbourhood U_0. From the above discussion, σ gives a partial resolution of p_0:

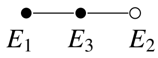
$$(X', \mathcal{F}') \xrightarrow{\sigma_r} (U_{r-1}, \mathcal{F}_{r-1}, p_{r-1}) \xrightarrow{\sigma_{r-1}} \cdots (U_1, \mathcal{F}_1, p_1) \xrightarrow{\sigma_1} (U_0, \mathcal{F}_0, p_0)$$

where σ_{i+1} is a blowing-up of a neighborhood U_i at the A-D-E singularity p_i of $\mathcal{F}_i$ without eigenvalue, $\mathcal{F}_{i+1} = \sigma_{i+1}^* \mathcal{F}_i$ is the pulling-back of the foliation $\mathcal{F}_i$.

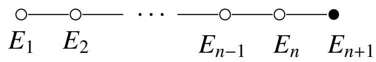
Let E_i be the exceptional curve of σ_i and $\mathcal{E}_i$ be the total transform of E_i in X'. For convenience, we also denote the strict transform of E_i in X' by E_i. One can see that each $E_i \subset X'$ is $\mathcal{F}'$-invariant. If not, p_{i-1} is a singularity with an eigenvalue 1, a contradiction.

Theorem 3.9. *The singularity p_0 is an A-D-E one without eigenvalue iff each irreducible components of $\mathcal{E}_1$ is an $\widetilde{\mathcal{F}}$-invariant rational curve and one of the following cases occurs (we denote a (-2)-curve by $\circ$ and the other curve by $\bullet$):*

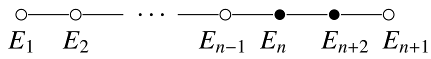
(A_2) $\mathcal{E}_1 = E_1 + E_2 + 2E_3$ *($r = 3$) where E_1 (resp., E_2, E_3) is a (-3)-curve (resp., (-2)-curve, (-1)-curve) with $Z(\mathcal{F}', E_1) = 1$ (resp., $Z(\mathcal{F}', E_2) = 1$, $Z(\mathcal{F}', E_3) = 3$).*



(A_{2n+1}) $\mathcal{E}_1 = E_1 + \cdots + E_n + E_{n+1}$ *($r = n + 1 \geq 2$) where $E_1 + \cdots + E_n$ is a maximal simple $\mathcal{F}'$-chain with $Z(\mathcal{F}', E_1) = 1$ and E_{n+1} is a (-1)-curve with $Z(\mathcal{F}', E_{n+1}) = 3$.*

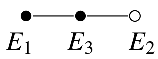


(A_{2n}) $\mathcal{E}_1 = E_1 + \cdots + E_n + E_{n+1} + 2E_{n+2}$ *($r = n + 2 \geq 4$) where $E_1 + \cdots + E_{n-1}$ (resp., E_{n+1}) is a maximal simple $\mathcal{F}'$-chain with $Z(\mathcal{F}', E_1) = 1$ (resp., $Z(\mathcal{F}', E_{n+1}) = 1$) and E_n (resp., E_{n+2}) is a (-3)-curve (resp., (-1)-curve) with $Z(\mathcal{F}', E_n) = 2$ (resp., $Z(\mathcal{F}', E_{n+2}) = 3$).*

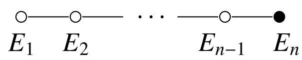


(D_4) $\mathcal{E}_1 = E_1$ *($r = 1$) where E_1 is a (-1)-curve with $Z(\mathcal{F}', E_1) = 3$.*

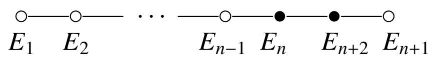
(D_5) $\mathcal{E}_1 = E_1 + E_2 + 2E_3$ *($r = 3$) where E_1 (resp., E_2, E_3) is a (-3)-curve (resp., (-2)-curve, (-1)-curve) with $Z(\mathcal{F}', E_1) = 2$ (resp., $Z(\mathcal{F}', E_2) = 1$, $Z(\mathcal{F}', E_3) = 3$).*



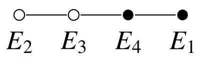
(D_{2n+2}) $\mathcal{E}_1 = E_1 + \cdots + E_{n-1} + E_n$ *($r = n \geq 2$) where E_i is a (-2)-curve with $Z(\mathcal{F}', E_i) = 2$ ($i = 1, \ldots, n-1$) and E_n is a (-1)-curve with $Z(\mathcal{F}', E_n) = 3$.*



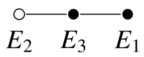
(D_{2n+3}) $\mathcal{E}_1 = E_1 + \cdots + E_n + E_{n+1} + 2E_{n+2}$ *($r = n + 2 \geq 4$) where E_i is a (-2)-curve with $Z(\mathcal{F}', E_i) = 2$ ($i = 1, \ldots, n-1$) and E_n (resp., E_{n+1}, E_{n+2}) is a (-3)-curve (resp., (-2)-curve, (-1)-curve) with $Z(\mathcal{F}', E_n) = 2$ (resp., $Z(\mathcal{F}', E_{n+1}) = 1$, $Z(\mathcal{F}', E_{n+2}) = 3$).*



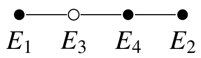
(E_6) $\mathcal{E}_1 = E_1 + E_2 + 2E_3 + 3E_4$ *($r = 4$) where $E_2 + E_3$ is a maximal simple $\mathcal{F}'$-chain with $Z(\mathcal{F}', E_2) = 1$ and E_1 (resp., E_4) is a (-4)-curve (resp., (-1)-curve) with $Z(\mathcal{F}', E_1) = 1$ (resp., $Z(\mathcal{F}', E_4) = 3$).*



(E_7) *$\mathcal{E}_1 = E_1 + E_2 + 2E_3$ ($r = 3$) where E_1 (resp., E_2, E_3) is a (-3)-curve (resp., (-2)-curve, (-1)-curve) with $Z(\mathcal{F}', E_1) = 1$ (resp., $Z(\mathcal{F}', E_2) = 2$, $Z(\mathcal{F}', E_3) = 3$).*



(E_8) *$\mathcal{E}_1 = E_1 + E_2 + 2E_3 + 3E_4$ ($r = 4$) where E_i is a (-3)-curves with $Z(\mathcal{F}', E_i) = 1$ ($i = 1, 2$), and E_3 (resp., E_4) is a (-2)-curve (resp., (-1)-curve) with $Z(\mathcal{F}', E_3) = 2$ (resp., $Z(\mathcal{F}', E_4) = 3$).*



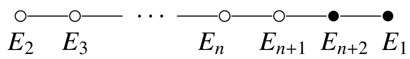
Proof. ($\Longleftarrow$) It's from Lemma 2.4.

($\Longrightarrow$) Consider the blowing-up $\sigma_1 : (U_1, \mathcal{F}_1, p_1) \to (U_0, \mathcal{F}_0, p_0)$. By (2.7), one has $Z(\mathcal{F}_1, E_1) = 1 + l(p_0) \leq 3$. Therefore one the following cases occurs.

(1) There are exactly three singularities of $\mathcal{F}_1$ on E_1.
(2) There are two singularities of $\mathcal{F}_1$, says q and p_1, satisfying $\mu_q(\mathcal{F}_1, E_1) = 1$ and $\mu_{p_1}(\mathcal{F}_1, E_1) = 2$.
(3) There are a unique singularity p_1 of $\mathcal{F}_1$ with $\mu_{p_1}(\mathcal{F}_1, E_1) \leq 3$.

It's easy to see that p_0 is of type D_4 in case (1).

In case (2), q has an eigenvalue by Lemma 2.3. We apply induction on the number r of the blowing-ups and assume that p_1 is of type A-D-E. If p_1 has an eigenvalue, then p_1 is a saddle-node with a weak separatrix E_1 ans so p_0 is D_4 again. If p_1 of type A_{2n+1}, then the exceptional set of p_0 in X' is as follows.



By (2.7) again, one has

$$3 \geq 1 + l(p_0) \geq (n+1)(Z(E_{n+2}) - 2) + (Z(\mathcal{F}', E_1) - 1) \geq n + 2.$$

So $n = 1$. Thus p_0 is of type D_5 . Similarly, p_0 may also be of type D_{n+2} if p_1 is of type D_n. However, p_1 cannot be one of other types. If not, one can find that $E_1 \subset X_1$ is not smooth, a contradiction.

By a similar discussion, in case (3), one can find that p_0 may be of type A_n ($n \geq 2$), E_6, E_7 and E_8. □

4. The proves of our Theorems and Corollaries

Let $(X, \mathcal{F})$ be a a relatively minimal foliation and $(X_0, \mathcal{F}_0)$ be the relatively minimal A-D-E model of $(X, \mathcal{F})$ with a bimeromorphic morphism $\rho : (X, \mathcal{F}) \to (X_0, \mathcal{F}_0)$. One has

$$K_X + K_{\mathcal{F}} = \rho^*(K_{X_0} + K_{\mathcal{F}_0}) + V$$

where V is a $\mathbb{Q}^+$-divisor supported on the exceptional set of ρ.

In what follows, we assume that both $K_{\mathcal{F}_0}$ and $K_{X_0} + K_{\mathcal{F}_0}$ are pseudo-effective. Consider the Zariski decomposition of $K_{X_0} + K_{\mathcal{F}_0}$ as in (1.2).

Lemma 4.1. *For an irreducible curve C in X_0, we have $N_0C < 0$ iff C is a (-2)-curve as the first component of some $\mathcal{F}_0$-chain.*

Proof. ($\Longrightarrow$) Since $N_0C < 0$ and $N_0 \geq 0$, C is a component of N_0 with $C^2 < 0$. If C is not $\mathcal{F}_0$-invariant, then

$$N_0C = (K_{X_0} + K_{\mathcal{F}_0})C = 2(p_a(C) - 1 - C^2) + tang(\mathcal{F}_0, C) \geq tang(\mathcal{F}_0, C) \geq 0,$$

a contradiction. Therefore C is $\mathcal{F}_0$-invariant.

Since $(K_{X_0} + K_{\mathcal{F}_0})C < 0$, one has

$$-1 \leq K_{\mathcal{F}_0}C \leq -1 - K_{X_0}C = -2p_a(C) + 1 + C^2.$$

Thus

$$0 \leq p_a(C) \leq 1 + \frac{1}{2}C^2.$$

It implies that C is a (-2)-curve or (-1)-curve. If C is a (-1)-curve, then $K_{\mathcal{F}_0}C \geq 2$ by Lemma 3.5, a contradiction. Thus C is a (-2)-curve and hence $K_{\mathcal{F}_0}C = -1$. By Corollary 2.7, C is the first component of an $\mathcal{F}_0$-chain.

($\Longleftarrow$) Since $K_{X_0}C = 0$ and $K_{\mathcal{F}_0}C = -1$,

$$N_0C \leq P_0C + N_0C = (K_{X_0} + K_{\mathcal{F}_0})C = -1.$$

Up to now, we complete this proof. □

Lemma 4.2. *These maximal simple $\mathcal{F}_0$-chains are disjoint. Furthermore, they are contained in the support of N_0. In particular, There are finite maximal A-chains.*

Proof. The first part is from separatrix Theorem.

Let $\Theta = \Gamma_1 + \cdots + \Gamma_l$ be a maximal simple $\mathcal{F}_0$-chain with the first component Γ_1 and $\Gamma_i\Gamma_{i+1} = 1$ ($i = 1, \ldots, l-1$). By Lemma 4.1, Γ_1 is in N_0. Suppose that Γ_k be not in N_0 for some k. Without loss of generality, we assume Γ_{k-1} is in N_0. So $N_0\Gamma_k > 0$. However one has

$$0 \geq K_{\mathcal{F}_0}\Gamma_k = (K_{X_0} + K_{\mathcal{F}_0})\Gamma_k \geq N_0\Gamma_k > 0,$$

a contradiction. □

Let T be the sum of all curves in N_0 which are not $\mathcal{F}_0$-invariant. Consider a maximal simple $\mathcal{F}_0$-chain $\Theta = \sum\limits_{i=1}^{l} \Gamma_i$ as above. Let r be the minimal number such that Γ_{r+1} meets with T (if C and T are disjoint, then we take $r = l$). We define

$$M(\Theta) := \begin{cases} \frac{1}{r+1}\sum\limits_{i=1}^{r}(r+1-i)\Gamma_i, & \text{if } r > 0, \\ 0, & \text{if } r = 0. \end{cases}$$

It's easy to see that

$$M(\Theta)\Gamma_i = \begin{cases} -1, & \text{if } i = 1, \\ \frac{1}{r+1}, & \text{if } i = r+1, \\ 0, & \text{if } i \neq 1, r+1 \end{cases} \tag{4.1}$$

whenever $r > 0$. Thus one has

$$(N_0 - M(\Theta))\Gamma_i = \begin{cases} 0, & \text{if } i \neq r+1, \\ -\frac{1}{r+1}, & \text{if } i = r+1. \end{cases} \tag{4.2}$$

Note that the above equalities hold also in the case that $r = 0$.

For any irreducible $\mathcal{F}_0$-invariant C_0 outside of Θ, either $C_0\Theta = 0$ or C_0 meets transversely with the last component Γ_l of Θ. Hence

$$M(\Theta)C_0 = \begin{cases} \frac{1}{l+1}, & \text{if } r = l \text{ and } C_0\Gamma_l > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

In particular, $M(\Theta)C_0 \leq \frac{1}{2}$.

Let $\Theta_1, \ldots, \Theta_s$ be all maximal simple $\mathcal{F}_0$-chainx. Take

$$\overline{N}_0 = N_0 - \sum_{i=1}^{s} M(\Theta_i).$$

Lemma 4.3. $\overline{N}_0 \geq 0$.

Proof. Write

$$N_0 = D + \sum_{i=1}^{s} Z_i$$

where the support of Z_i is contained in Θ_i ($i = 1, \ldots, s$) and the support of D contains no irreducible component in Θ_i's. Since $N_0 \geq 0$, we have $D \geq 0$ and $Z_i \geq 0$ ($i = 1, \ldots, s$).

It's enough to prove $Z_i \geq M(\Theta_i)$. Take $\Theta = \Theta_i$ and adopt all notations as above. If $r = 0$, then $M(\Theta) = 0$ and hence $Z_i \geq M(\Theta)$. We assume that $r > 0$. By (4.2), one has

$$(Z_i - M(\Theta))\Gamma \leq (N_0 - M(\Theta))\Gamma \leq 0$$

for each irreducible component Γ of Θ. It implies that $Z_i - M(\Theta_i) \geq 0$. □

Lemma 4.4. *We have $(\overline{N}_0 + T)C \geq 0$ if C occurs in one of the following cases:*

(1) *C is a component of T.*

(2) *C is an irreducible component of a maximal simple $\mathcal{F}_0$-chain.*

Proof. (1) Let C be a component of T. One has

$$(\overline{N}_0 + T)C \geq (\overline{N}_0 + C)C = (N_0 + C)C = K_{X_0}C + K_{\mathcal{F}_0}C + C^2 = tang(\mathcal{F}_0, C) + K_{X_0}C.$$

Suppose that $(\overline{N}_0 + T)C < 0$. Note C is in N_0. So $C^2 < 0$. If $K_{X_0}C < 0$, then C is a (-1)-curve and so $K_{X_0}C = -1$. Hence the above inequality implies that $tang(\mathcal{F}_0, C) = 0$. Thus $K_{\mathcal{F}_0}C = 1$. However $K_{\mathcal{F}_0}C \geq 2$ by Lemma 3.5, a contradiction.

(2) Without loss of generality, we assume C is a component of $\Theta_1 = \Gamma_1 + \cdots + \Gamma_l$, says $C = \Gamma_i$. Let r be the minimal subscript such that Γ_{r+1} meets with T. By (4.2), one has

$$\overline{N}_0\Gamma_i = \begin{cases} 0, & \text{if } i \neq r+1, \\ -\frac{1}{r+1}, & \text{if } i = r+1. \end{cases}$$

Note that $T\Gamma_{r+1} \geq 1$ and $T\Gamma_i \geq 0$ ($i \neq r+1$). Thus one has $(\overline{N}_0 + T)C \geq 0$. □

Lemma 4.5. $\overline{N}_0 + T = 0$.

Proof. By Lemma 4.4 and the negativity of $\overline{N}_0 + T$, we can find an $\mathcal{F}_0$-invariant curve C_0 in N_0 such that C_0 is outside of Θ_i's and

$$(\overline{N}_0 + T)C_0 < 0 \tag{4.4}$$

whenever $\overline{N}_0 + T \neq 0$.

Let k be the number of the intersections of Θ_i's and C_0. Let h be the number of else singularities of $\mathcal{F}_0$ on C_0. By (4.3), one gets

$$(\overline{N}_0 + T)C_0 = (N_0 + T)C_0 - \sum_{i=1}^{s} M(\Theta_i)C_0 \geq K_{X_0}C_0 + K_{\mathcal{F}_0}C_0 + TC_0 - \frac{k}{2}. \tag{4.5}$$

From Cerveau-Lins Neto formula, we have

$$K_{X_0}C_0 + K_{\mathcal{F}_0}C_0 = -C_0^2 + 2(p_a(C_0) + g(C_0) - 2) + \sum_{p \in C_0} \sum_{B \in C_0(p)} \mu(\mathcal{F}_0, B, p). \tag{4.6}$$

Combing (4.4), (4.5) and (4.6), one gets

$$-C_0^2 + 2(p_a(C_0) + g(C_0) - 2) + TC_0 + \sum_{p \in C_0} \sum_{B \in C_0(p)} \mu(\mathcal{F}_0, B, p) - \frac{k}{2} < 0. \tag{4.7}$$

Claim 1. $C_0 \cong \mathbb{P}^1$.

Firstly, we claim that C_0 is smooth. Suppose that C_0 have a singularity p. If $p \in \Theta_i$, then p is not a reduced singularity of $\mathcal{F}_0$ on Θ_i, a contradiction. So any simple $\mathcal{F}_0$-chain doesn't pass through p. Thus we have

$$-C_0^2 + 2(p_a(C_0) + g(C_0) - 2) + TC_0 + (1 + k) - \frac{k}{2} < 0. \tag{4.8}$$

From (4.8) and $-C_0^2 \geq 1$, we get $p_a(C_0) = 0$ (i.e., $C_0 \cong \mathbb{P}^1$), a contradiction. Hence C_0 is smooth.

Therefore, by (4.7), we have

$$-C_0^2 + 4(p_a(C_0) - 1) + TC_0 + \frac{k}{2} < 0.$$

It implies that $C_0 \cong \mathbb{P}^1$.

Let $p_1, \ldots, p_h$ be the singularities of $\mathcal{F}_0$ on C_0 outside Θ_i's.

Calim 2. $\mu_{p_i}(\mathcal{F}_0, C_0) = 1$ for each p_i and $h \leq 2$.

(4.7) implies that

$$-C_0^2 - 4 + TC_0 + \frac{k}{2} + \sum_{i=1}^{h} \mu_{p_i}(\mathcal{F}_0, C_0) < 0. \tag{4.9}$$

If $\mu_{p_i}(\mathcal{F}_0, C_0) \geq 2$ for some i, then C_0 is a (-1)-curve, $h = 1$, $k \leq 1$ and $\mu_{p_1}(\mathcal{F}_0, C_0) = 2$ from (4.9). By Cerveau-Lins Neto formula,

$$K_{\mathcal{F}_0} C_0 = -2 + k + \mu_{p_1}(\mathcal{F}_0, C_0) \leq 1.$$

However, $K_{\mathcal{F}_0} C_0 \geq 2$ by Lemma 3.5, a contradiction. Hence $\mu_{p_i}(\mathcal{F}_0, C_0) = 1$ for each p_i and $h \leq 2$.

Therefore we get

$$-C_0^2 - 4 + TC_0 + \frac{k}{2} + h < 0. \tag{4.10}$$

Claim 3. $h = k = 1$ and $C_0^2 = -2$.

From separatrix Theorem (see [Bru15, Theorem 3.4] or [Cam88]), one can find that $h > 0$. So one can find that $-C_0^2 \leq 2$ by (4.10).

If $k \geq 2$, then one can find two $\mathcal{F}_0$-invariant (-2)-curves, says Γ_1, Γ_2, meeting with C_0 transversely. Since $\Gamma_1 + \Gamma_2 + C_0$ is negative, $-C_0^2 \geq 2$. Thus

$$-C_0^2 - 4 + TC_0 + \frac{k}{2} + h \geq 0,$$

a contradiction. Hence $k \leq 1$. By Cerveau-Lins Neto formula, $K_{\mathcal{F}_0} C_0 = -2 + k + h \leq 1$. From Lemma 3.5 and $-C_0^2 \leq 2$, one gets $C_0^2 = -2$. So $h = 1$ and $k \leq 1$.

If $k = 0$, then $N_0 C_0 = \overline{N}_0 C_0 \leq (\overline{N}_0 + T) C_0 < 0$. By Lemma 4.1, C_0 is contained in some simple $\mathcal{F}_0$-chain, a contradiction. So $k = 1$.

Claim 4. $C_0 + \Theta_1$ is a simple $\mathcal{F}_0$-chain.

By the above discussion, C_0 has two singularities of $\mathcal{F}_0$: p_1 and $q_1 = \Theta_1 \cap C_0$. Let λ_{p_1} (resp., λ_{q_1}) be the eigenvalue of p_1 (resp., q_1) along C_0. More precisely, $\lambda_{q_1} = -\frac{l+1}{l}$ by Camacho-Sad formula where l is the number of irreducible components of Θ_1. Note that $C_0^2 = -2$. By Camacho-Sad formula again, one has $\lambda_{p_1} = -\frac{l+1}{l+2}$. Thus $C_0 + \Theta_1$ is a simple $\mathcal{F}_0$-chain. However, Θ_1 is a maximal simple $\mathcal{F}_0$-chain, a contradiction.

Up to now, this proof is completed. □

Proof of Theorem 1.2. From Lemma 3.8, $\mathcal{F}$ is minimal. Lemma 4.5 implies that $N_0 = \sum_{i=1}^{s} M(\Theta_i)$. So $\lfloor N_0 \rfloor = 0$.

If $\rho^* N_0$ meets with the exceptional set E of ρ, then ρ contracts some exceptional curves to a point, says p, on a maximal simple $\mathcal{F}_0$-chain. Thus p is either smooth or reduced. However, $\mathcal{F}$ is relatively minimal, a contradiction. Hence $\rho^* N_0$ is disjoint from E. □

Proof of Theorem 1.4. Since $K_{\mathcal{F}}$ is pseudoeffective, $h^2(K_X + K_{\mathcal{F}}) = h^0(-K_{\mathcal{F}}) = 0$. From Riemann-Roch formula, one has

$$h^0(K_X + K_{\mathcal{F}}) = h^1(K_X + K_{\mathcal{F}}) + \chi(\mathcal{O}_X) + \rho(X) \geq \rho(X) + \chi(\mathcal{O}_X). \tag{4.11}$$

If P is big, then $h^1(K_X + K_{\mathcal{F}}) = 0$ by Kawamata-Viehweg vanishing theorem and the fact that $\lfloor N \rfloor = 0$ where P is as in (1.1). Thus one gets

$$h^0(K_X + K_{\mathcal{F}}) = \chi(\mathcal{O}_X) + \rho(X). \tag{4.12}$$

In the case that $\mathrm{kod}(X) \geq 0$, one can find that K_X is pseudoeffective. If not, $h^0(nK_X) = 0$ for all $n \geq 1$, namely, $\mathrm{kod}(X) = -\infty$, a contradiction. So $K_X + K_{\mathcal{F}}$ is also pseudoeffective.

In what follows, we assume that $\mathrm{kod}(X) = -\infty$. Note that $p_g(X) = 0$. One has

$$h^0(K_X + K_{\mathcal{F}}) \geq \rho(X) + 1 - q(X)$$

from (4.11). So $K_X + K_{\mathcal{F}}$ is pesudoeffective whenever $\rho(X) \geq q(X)$. □

Proof of Corollary 1.5. Since $h^0(K_{\mathcal{F}}) > 0$, we have

$$h^0(K_X + K_{\mathcal{F}}) \geq h^0(K_X) = p_g(X).$$

From (4.12), we get $q(X) \leq 1 + \rho(X)$. □

Proof of Corollary 1.7. In this case, $K_{\mathcal{F}} = K_f$ (see [Bru15, Ch.2, Sec.3, Example (5)]). It's well-known, K_f is a nef and big divisor. By (4.12) and a straightforwards computation, one gets (1.4) and (1.5).

If $b \geq 1$, then (1.4) implies $h^0(K_X + K_{\mathcal{F}}) > 0$. If $b = 0$, one gets again

$$h^0(K_X + K_{\mathcal{F}}) \geq \chi_f + K_f^2 - 3(g-1) \geq g - 1 > 0$$

from (1.4) and the equality $K_f^2 \geq 4g - 4$ in [TTZ05, Theorem 2.1]. So $K_X + K_{\mathcal{F}}$ is pseudo-effective.

Now we will claim $K_X + K_{\mathcal{F}}$ is nef, i.e., the negative part $\overline{N} = 0$. We adopt all notations and assumptions in Sec. 3.2.

Note that each singularities of $\mathcal{F}$ has an eigenvalue -1 from f is semistable. The key fact implies that

(1) $N_0 = 0$;
(2) the eigenvalue of each non-reduced singularity of $(X', \mathcal{F}')$ is 1;
(3) the singularities of $(X_0, \mathcal{F}_0)$ is at worst of type D_{2n+2} from Theorem 3.9.

By (3.1), for a singularity p_0 of type D_{2n+2}, the contribution of p_0 to V is exactly zero. Hence $\overline{N} = 0$.

Since $(K_X + K_{\mathcal{F}})F = 4g - 4 > 0$ for ageneral fiber F, $K_X + K_{\mathcal{F}} \not\equiv_{\mathrm{num}} 0$, that is, $\bar{\nu}(\mathcal{F}) \geq 1$. □

5. An example for a foliation $\mathcal{F}$ with $\bar{\nu}(\mathcal{F}) = 0$

Let $X_0 = \mathbb{P}^2$. Consider a family of curves as follows:

$$C_t : (X^4 + Y^4 + Z^4) + t(X^2Y^2 + Y^2Z^2 + Z^2X^2) = 0, \quad t \in \mathbb{C}^1$$

and C_∞ is defined by $X^2Y^2 + Y^2Z^2 + Z^2X^2 = 0$. The family of curves induces a foliation $\mathcal{F}_0$. More precisely, in the neighbourhood $U_0 = \{[x, y, 1] \mid x, y \in \mathbb{C}\}$, the generator of $\mathcal{F}_0$ is

$$v = y(x^2y^2 + y^2 - x^4 - 1)\frac{\partial}{\partial x} - x(x^2y^2 + x^2 - y^4 - 1)\frac{\partial}{\partial y}.$$

The foliation $\mathcal{F}_0$ is an A-D-E foliation. All non-reduced singularities are as follows:

$$p_1 = [\omega, \omega^2, 1], \quad p_2 = [-\omega, \omega^2, 1], \quad p_3 = [\omega, -\omega^2, 1], \quad p_4 = [-\omega, -\omega^2, 1],$$
$$p_5 = [\omega^2, \omega, 1], \quad p_6 = [-\omega^2, \omega, 1], \quad p_7 = [\omega^2, -\omega, 1], \quad p_8 = [-\omega^2, -\omega, 1].$$

Each p_i has an eigenvalue $\frac{1}{2}$ and lies in C_2. Each reduced singularity of $\mathcal{F}_0$ has an eigenvalue -1.

Consider a minimal resolution $\rho : (X, \mathcal{F}) \to (X_0, \mathcal{F}_0)$ of all p_i's such that the exceptional set of p_i is $E_{2i-1} + E_{2i}$ where E_{2i-1} (resp., E_{2i}) is a (-2)-curve (resp., (-1)-curve) and $E_{2i-1}E_{2i} = 1$. The pulling-back foliation $\mathcal{F} = \rho^*\mathcal{F}_0$ is relatively minimal.

In fact, $\mathcal{F}$ gives a minimal normal-crossing fibration $f : X \to \mathbb{P}^1$ of genus $g = 3$ with four singular fibers $F_t = \rho^*C_t - \sum_{i=1}^{16} E_i$ $(t = -2, -1, 2, \infty)$:

(1) F_{-2} is a reduce nodal curve consisting of four (-3)-curves;
(2) F_{-1} is reduce nodal curve consisting of two (-4)-curves;
(3) $F_2 = 2\Gamma + \sum_{i=1}^{8} E_{2i-1}$ where Γ is a (-4)-curve meeting transversely with each E_{2i-1};
(4) F_∞ is a irreducible nodal curve with three nodes.

We have $K_{\mathcal{F}_0} = 3L$, $K_{X_0} = -3L$ where L is a line in $\mathbb{P}^2$. Hence

$$K_X + K_{\mathcal{F}} = \rho^*(K_{X_0} + K_{\mathcal{F}_0}) + \sum_{i=1}^{16} E_i = \sum_{i=1}^{16} E_i.$$

So $\bar{\nu}(\mathcal{F}) = 0$ and $h^0(K_X + K_{\mathcal{F}}) = 1$.

Note that $C_2 = 2\Gamma_0$ where Γ_0 is a conic curve. One has

$$2\rho^*L \equiv \rho^*\Gamma_0 \equiv \Gamma + \sum_{i=1}^{16} E_i.$$

Therefore

$$K_{\mathcal{F}} = K_f - \Gamma \equiv \rho^*L + \Gamma + \sum_{i=1}^{8} E_{2i-1}.$$

The positive and negative parts of a Zariski decomposition of $K_{\mathcal{F}}$ are

$$P = \rho^*L + \Gamma + \frac{1}{2}\sum_{i=1}^{8} E_{2i-1}, \quad N = \frac{1}{2}\sum_{i=1}^{8} E_{2i-1}$$

respectively. Moreover, we have $c_1^2(\mathcal{F}) = 5$, $K_f^2 = 9$ and $\chi_f = 3$.

Acknowledgements The authors would like to give their thanks to professor Sheng-Li Tan, professor Xiaolei Liu and professor Hao Lin for many useful discussions on foliations and fibrations.

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