

TRANSCENDENTAL FOLIATIONS WITH SLOPE LESS THAN 2

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Dedicated to the memory of Professor Gang Xiao

ABSTRACT. In this paper, we will construct a canonical resolution of double coverings of foliated surfaces and provide a concrete formula of c_1^2, c_2, χ of the double foliated surface. During the computation, we also derive a theorem about the Zariski decomposition of an adjoint divisor of type $K_{\mathcal{F}} + \Delta$, which generalizes McQuillan's theorem in the case that $\Delta = 0$. Then we prove the slope of double foliated surfaces of general type is at least 4 if the original foliation satisfies $c_1^2 \geq 4\chi$ and the ramification divisor avoids the set of saddle-nodes. In the last section, we will give some examples of transcendental double foliated surfaces with slope $\frac{12}{7}$.

1. INTRODUCTION

A foliated surface is a pair (X, \mathcal{F}) consisting of a smooth projective surface X and a (holomorphic) foliation \mathcal{F} . A holomorphic foliation \mathcal{F} on a smooth projective surface X can be defined by a differential equation of first order $\alpha = 0$, where α is a non-zero rational 1-form. We say \mathcal{F} is *algebraic* if the differential equation admits a rational first integral, otherwise, we say \mathcal{F} is *transcendental*. In 1891, Poincaré in [15, 16, 17], studied the following problem:

Problem 1.1 (Poincaré). *Is it possible to decide if a differential equation $\alpha = 0$ is algebraic?*

A similar problem was proposed by Painlevé [14], which is as follows:

Problem 1.2 (Painlevé). *Is it possible to decide if a differential equation $\alpha = 0$ has a rational first integral of a given genus g ?*

Lins-Neto [13] showed that the genus is not an invariant of differential equations, by constructing counterexamples.

On the other hand, several invariants of fibrations can be generalized to foliations. For example, the minimal models (Seidenberg [18], Brunella [4]), canonical divisor $K_{\mathcal{F}}$, pluri-canonical genera $p_n(\mathcal{F})$, Kodaira dimension $\kappa(\mathcal{F})$, and numerical Kodaira dimension $\nu(\mathcal{F})$ (McQuillan [11] and Mendes [9]), Chern numbers $c_1^2(\mathcal{F}), c_2(\mathcal{F})$ and $\chi(\mathcal{F}) = \frac{1}{12}(c_1^2(\mathcal{F}) + c_2(\mathcal{F}))$ (Tan [19]). For a foliation \mathcal{F} of general type, $c_1^2(\mathcal{F})$ and $\chi(\mathcal{F})$ are positive. Moreover, Tan [19] proved that \mathcal{F} is of general type iff $c_1^2(\mathcal{F}) > 0$. So we can define the slope of \mathcal{F} as

$$\lambda(\mathcal{F}) := \frac{c_1^2(\mathcal{F})}{\chi(\mathcal{F})}.$$

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From the Noether's equality, we have $0 < \lambda(\mathcal{F}) \leq 12$. For an algebraic foliation \mathcal{F} , i.e., a fibration, Xiao [20] proved

$$4 - \frac{4}{g} \leq \lambda(\mathcal{F}) \leq 12.$$

Here the lower bound is attained only for hyperelliptic fibrations, as proved by Konno [7]. Typically, the method of double coverings is useful for studying algebraic surfaces of general type. For example, Gang Xiao [21] studied the hyperelliptic fibration by considering the double covering of rational fibrations. Similarly, we will study the double covering of a foliated surface (X, \mathcal{F}) , which is called a *double foliated surface*, denoted by (Y, \mathcal{G}) .

In this paper, we will study the canonical resolution of the double covering (Y, \mathcal{G}) of a reduced foliated surface (X, \mathcal{F}) :

$$\begin{array}{ccccccccc} (\tilde{Y}, \tilde{\mathcal{G}}) & \xlongequal{\quad} & (Y_s, \mathcal{G}_s) & \longrightarrow & \cdots & \longrightarrow & (Y_1, \mathcal{G}_1) & \longrightarrow & (Y_0, \mathcal{G}_0) & \xlongequal{\quad} & (Y, \mathcal{G}) \\ \downarrow \tilde{\pi} & & \downarrow \pi_s & & & & \downarrow \pi_1 & & \downarrow \pi_0 & & \downarrow \pi \\ (\tilde{X}, \tilde{\mathcal{F}}) & \xlongequal{\quad} & (X_s, \mathcal{F}_s) & \xrightarrow{\sigma_s} & \cdots & \longrightarrow & (X_1, \mathcal{F}_1) & \xrightarrow{\sigma_1} & (X_0, \mathcal{F}_0) & \xlongequal{\quad} & (X, \mathcal{F}), \end{array}$$

where the branch locus B_i of π_i is a reduced even divisor, which can be written as

$$B_i = B_{i,h} + B_{i,v},$$

with $B_{i,v}$ consisting of all \mathcal{F}_i -invariant components of B_i . To simplify, we will assume (X, \mathcal{F}) is *minimal*, meaning that \mathcal{F} is reduced and there is no \mathcal{F} -exceptional curve on X . (Note that some other authors refer to such a foliated surface as *relatively minimal*.) We will also provide a construction of $\sigma := \sigma_1 \circ \cdots \circ \sigma_s$ such that $\tilde{\mathcal{G}}$ is a reduced foliation over the smooth surface \tilde{Y} .

For the case that $\nu(\mathcal{G}) \geq 0$, we will describe the Zariski decomposition of the canonical divisor $K_{\tilde{\mathcal{G}}}$ of the double foliation $\tilde{\mathcal{G}}$, specifically, $K_{\tilde{\mathcal{G}}} \equiv P(\tilde{\mathcal{G}}) + N(\tilde{\mathcal{G}})$. Note that

$$K_{\tilde{\mathcal{G}}} = \tilde{\pi}^* \left(K_{\tilde{\mathcal{F}}} + \frac{1}{2} \tilde{B}_h \right).$$

If we denote the Zariski decomposition of $K_{\tilde{\mathcal{F}}} + \frac{1}{2} \tilde{B}_h$ by

$$K_{\tilde{\mathcal{F}}} + \frac{1}{2} \tilde{B}_h \equiv P(\tilde{B}_h) + N(\tilde{B}_h),$$

then we have

$$P(\tilde{\mathcal{G}}) = \tilde{\pi}^* P(\tilde{B}_h), \quad N(\tilde{\mathcal{G}}) = \tilde{\pi}^* N(\tilde{B}_h).$$

Thus, it suffices to study the Zariski decomposition of $K_{\tilde{\mathcal{F}}} + \frac{1}{2} \tilde{B}_h$. More generally, we will study the Zariski decomposition of a pseudo-effective divisor of type $K_{\mathcal{F}} + \Delta$, where \mathcal{F} is a reduced foliation over a smooth surface X and either $\Delta = 0$ or $\Delta = \sum_{i=1}^l a_i C_i$ is an effective \mathbb{Q} -divisor such that C_i is not \mathcal{F} -invariant and $a_i \in [\frac{1}{2}, 1)$. For the case where $\Delta = 0$, this is just the famous McQuillan Theorem ([4], Theorem 8.1).

Our goal is to compute the Chern numbers $c_1^2(\mathcal{G}), c_2(\mathcal{G}), \chi(\mathcal{G})$ of the double foliation \mathcal{G} . More precisely, we state the following theorem (Theorem 4.19).

Theorem 1.3. *If (X, \mathcal{F}) is minimal with $\nu(\mathcal{F}) \geq 0$, then*

$$\begin{aligned} c_1^2(\mathcal{G}) &= 2c_1^2(\mathcal{F}) + \frac{3}{2}K_{\mathcal{F}}B_h + 2N^2 - 2N(B_h)^2 + \sum_{p \in S_{l,m}} T_1(p), \\ c_2(\mathcal{G}) &= 2c_2(\mathcal{F}) - 2N^2 + 2N(B_h)^2 - \frac{3}{2}s(B) + \sum_{p \in S_{l,m}} T_2(p) - \ell(\tilde{\mathcal{G}}), \\ \chi(\mathcal{G}) &= 2\chi(\mathcal{F}) + \frac{1}{8}K_{\mathcal{F}}B_h - \frac{1}{8}s(B) + \sum_{p \in S_{l,m}} \frac{1}{12}(T_1(p) + T_2(p)) - \frac{1}{12}\ell(\tilde{\mathcal{G}}). \end{aligned}$$

where $\ell(\tilde{\mathcal{G}})$ is the number of $\tilde{\mathcal{G}}$ -exceptional curves containing saddle-nodes.

As a consequence, we have the following claim (Theorem 4.31).

Theorem 1.4. *If $c_1^2(\mathcal{F}) \geq 4\chi(\mathcal{F})$ and the ramification divisor B of π misses the set of saddle-nodes, then*

$$c_1^2(\mathcal{G}) \geq 4\chi(\mathcal{G}).$$

In fact, the authors of [5] have proved a special version of the theorem above. In [5], they consider the case where (X, \mathcal{F}) is transcendental Riccati foliated surface ($c_1^2(\mathcal{F}) = \chi(\mathcal{F}) = 0$) and B is a normal-crossing divisor that contains no \mathcal{F} -invariant curves and disjoint from the singularities of \mathcal{F} .

2. PRELIMINARIES

2.1. Classification of foliated surfaces. By $\kappa(\mathcal{F})$ and $\nu(\mathcal{F})$, we can classify the foliations, which is due to the work of several authors: Miyaoka[12], McQuillan[10], Mendes[9], and Brunella[3]. See [4] for more details.

TABLE 1. Classification of foliations.

$\kappa(\mathcal{F})$	Algebraic foliations	Transcendental foliations
$-\infty$	$g = 0, (\nu(\mathcal{F}) = -\infty)$	Hilbert modular foliations ($\nu(\mathcal{F}) = 1$)
0	$g = 1$, isotrivial	Foliations induced by a holomorphic vector field (up to a group action)
1	$g = 1$, non-isotrivial; $g \geq 2$, isotrivial	Riccati foliations or Turbulent foliations
2	$g \geq 2$, non-isotrivial	Foliations of general type

Here note that if $\kappa(\mathcal{F}) \geq 0$, then $\kappa(\mathcal{F}) = \nu(\mathcal{F})$ and $\kappa(\mathcal{F}) < 2$ if and only if $c_1^2(\mathcal{F}) = 0$.

2.2. Chern numbers of foliations. For a given complex number a , we define

$$\beta(a) := \begin{cases} \frac{\gcd(m, n)^2}{mn}, & \text{if } a = \frac{m}{n} \in \mathbb{Q} - \{0\} \\ 0, & \text{others.} \end{cases}$$

For a reduced singularity p of \mathcal{F} , we define

$$\begin{aligned} \beta_p(\mathcal{F}) &:= \beta(-\lambda_p), \\ \chi_p(\mathcal{F}) &:= -\frac{1}{12}(\text{BB}(\mathcal{F}, p) + m_p(\mathcal{F}) - \beta_p(\mathcal{F})), \end{aligned}$$

where $\text{BB}_p(\mathcal{F})$ is the Baum-Bott index of \mathcal{F} at p (see [4], Ch.3, Sec.1). In particular, if $\lambda_p \neq 0$, then

$$\chi_p(\mathcal{F}) = \frac{1}{12} \left(\lambda_p + \frac{1}{\lambda_p} - \beta_p(\mathcal{F}) \right) - \frac{1}{4}.$$

Definition 2.1 (Tan). *Suppose (X, \mathcal{F}) is a minimal foliated surface. If $\nu(\mathcal{F}) \geq 0$, we define the following three Chern numbers:*

$$(2.1) \quad \begin{aligned} c_1^2(\mathcal{F}) &= K_{\mathcal{F}}^2 + \sum_{p \in N} \beta_p(\mathcal{F}), \\ c_2(\mathcal{F}) &= \sum_{p \notin N} \beta_p(\mathcal{F}), \\ \chi(\mathcal{F}) &= \chi(\mathcal{O}_X) + \frac{1}{4} K_{\mathcal{F}} \cdot N_{\mathcal{F}} + \sum_p \chi_p(\mathcal{F}), \end{aligned}$$

where N is the negative part of the pseudo-effective divisor $K_{\mathcal{F}} \stackrel{\text{num}}{=} P + N$. If $\nu(\mathcal{F}) = 0$, then we define $c_1^2(\mathcal{F}) = c_2(\mathcal{F}) = \chi(\mathcal{F}) = 0$.

Tan proved the following properties in [19]:

Proposition 2.2. *Suppose (X, \mathcal{F}) is a minimal foliated surface. Then we have*

- (1) $c_1^2(\mathcal{F}), c_2(\mathcal{F}), \chi(\mathcal{F})$ are non-negative rational numbers.
- (2) $c_1^2(\mathcal{F}), c_2(\mathcal{F}), \chi(\mathcal{F})$ are birational invariants.
- (3) If $K_{\mathcal{F}}$ has a Zariski decomposition with positive part P , then $c_1^2(\mathcal{F}) = P^2$.
- (4) (Noether's equality) $c_1^2(\mathcal{F}) + c_2(\mathcal{F}) = 12\chi(\mathcal{F})$.
- (5) If \mathcal{F} is birationally equivalent to a fibration $f : X \rightarrow C$, then

$$c_1^2(\mathcal{F}) = \kappa(f), \quad c_2(\mathcal{F}) = \delta(f), \quad \chi(\mathcal{F}) = \lambda(f),$$

where $\kappa(f), \delta(f), \lambda(f)$ are modular invariants of f .

By the formulas above, we can define the Chern numbers of any foliated surface (X, \mathcal{F}) by its minimal model. More precisely, if (X', \mathcal{F}') is the minimal model of (X, \mathcal{F}) , then we define the Chern numbers of \mathcal{F} as

$$c_1^2(\mathcal{F}) := c_1^2(\mathcal{F}'), \quad c_2(\mathcal{F}) := c_2(\mathcal{F}'), \quad \chi(\mathcal{F}) := \chi(\mathcal{F}').$$

Definition 2.3. *For any reduced foliated surface (X, \mathcal{F}) , we define*

$$(2.2) \quad \ell(\mathcal{F}) := K_{\mathcal{F}}^2 + \sum_{p \in X} \beta_p(\mathcal{F}) - 12\chi(\mathcal{F}).$$

Lemma 2.4. $\ell(\mathcal{F}) \geq 0$. *In particular, $\ell(\mathcal{F}) = 0$ if and only if there is no \mathcal{F} -exceptional curve which is the strong separatrix of a saddle-node.*

Proof. Let (X', \mathcal{F}') be the minimal model of (X, \mathcal{F}) with $\sigma : (X, \mathcal{F}) \rightarrow (X', \mathcal{F}')$. By the Noether formula, we have

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') = \frac{1}{12} \left[K_{\mathcal{F}'}^2 + \sum_{p \in X'} \beta_p(\mathcal{F}') \right].$$

Then we can see $\ell(\mathcal{F})$ denote the number of the exceptional curves in σ , which are the strong separatrix of saddle-nodes. Next is clear. \square

2.3. Let (X, \mathcal{F}) be a foliated surface. Consider a blowing-up $\sigma : X' \rightarrow X$ centered at a point $p \in X$ with an exceptional curve $E \subseteq X'$. One can get a foliation $\mathcal{F}' = \sigma^* \mathcal{F}$ on X' as a pulling-back of \mathcal{F} . It is well-known that

$$K_{\mathcal{F}'} = \sigma^* K_{\mathcal{F}} + (1 - l(p))E,$$

where $l(p)$ is the order of p .

Let R be a reduced curve in X containing no \mathcal{F} -invariant component, and let ν_p be the multiplicity of R at p . We have

$$\sigma^* R = R' + \nu_p E,$$

where R' is the strict transform of R under σ . Let p_1, \dots, p_s be the intersections of R' and E . By straightforward computation, one has

$$(2.3) \quad \text{tang}(\mathcal{F}, R, p) = \nu_p(\nu_p + l(p) - 1) + \sum_{i=1}^s \text{tang}(\mathcal{F}', R', p_i),$$

where $\text{tang}(\mathcal{F}, R, p)$ is abbreviated as t_p .

Obviously, if $\nu_p \geq 2$ or $\nu_p = l(p) = 1$, then $\text{tang}(\mathcal{F}, R, p) > \text{tang}(\mathcal{F}', R', p_i)$ for each p_i . In the case where $\nu_p = 1$ and $l(p) = 0$, we can see $s = 1$ and $t_p = t_{p_1}$. If $t_p > 0$, then $\nu_{p_1} = l(q_1) = 1$. Therefore, we can iterate this blowing-up procedure and stop at some moment when either

- i) $\nu_p = 0$ or
- ii) $\nu_p = 1$ and $l(p) = t_p = 0$,

for any point p . More precisely, we have the following lemma:

Lemma 2.5. *There is a bimeromorphic morphism $\sigma : \tilde{X} \rightarrow X$ satisfying that the pulling back foliation $\tilde{\mathcal{F}} = \rho^* \mathcal{F}$ of \mathcal{F} is reduced and the strict transform \tilde{R} of R is a smooth curve transverse to $\tilde{\mathcal{F}}$.*

3. THE ZARISKI DECOMPOSITION OF $K_{\mathcal{F}} + \Delta$

Let \mathcal{F} be a reduced foliation on a smooth surface X . Let $\Delta = 0$ or $\Delta = \sum_{i=1}^l a_i C_i$ be an effective \mathbb{Q} -divisor on X , where C_i 's are irreducible curves and $\frac{1}{2} \leq a_i \leq 1$ for each i . We assume that each C_i is not \mathcal{F} -invariant and $K_{\mathcal{F}} + \Delta$ is pseudo-effective with the Zariski decomposition

$$(3.1) \quad K_{\mathcal{F}} + \Delta \equiv P(\Delta) + N(\Delta).$$

In particular, if $\nu(\mathcal{F}) \geq 0$, i.e., $K_{\mathcal{F}}$ is pseudo-effective, then $K_{\mathcal{F}} + \Delta$ is also pseudo-effective obviously.

Our main goal in this section is to describe the construction of the negative part $N(\Delta)$ of the adjoint canonical divisor $K_{\mathcal{F}} + \Delta$.

3.1. (Δ, \mathcal{F}) -chains. Let Δ be as above. A (Δ, \mathcal{F}) -chain Θ is an \mathcal{F} -chain satisfying that $\Delta\Gamma < 1$ (resp., $\Delta\Gamma' = 0$) for its first component Γ (resp., any other component Γ'). For convenience, we say $\theta := \Delta\Gamma$ is the *multiplicity* of Θ w.r.t. Δ . Θ is said to be maximal if it cannot be contained in any other (Δ, \mathcal{F}) -chain. Any (Δ, \mathcal{F}) -chain is clearly contained in a maximal one and two maximal (Δ, \mathcal{F}) -chains are disjoint (see [4], p.97).

Let $\Theta = \Gamma_1 + \dots + \Gamma_r$ be a (Δ, \mathcal{F}) -chain with the first component Γ_1 and the multiplicity $\theta := \Delta\Gamma_1$ w.r.t. Δ .

One can find a unique effective \mathbb{Q} -divisor $M(\Theta) = \sum_{i=1}^r \gamma_i \Gamma_i$ supported on Θ such that

$$M(\Theta)\Gamma_1 = -1, \quad M(\Theta)\Gamma_i = 0 \text{ for } i > 1.$$

By a straightforward computation, one can find that $\gamma_i = \frac{\lambda_i}{n}$ for

$$n = [e_1, \dots, e_r] > \lambda_1 = [e_2, \dots, e_r] > \dots > \lambda_r = 1,$$

where $e_i = -\Gamma_i^2$ and $[e_1, \dots, e_r]$ denote the determinant of the intersection matrix $(-\Gamma_i \cdot \Gamma_j)_{1 \leq i, j \leq r}$. So $\gamma_r \leq \frac{1}{2}$ and $\gamma_r = \frac{1}{2}$ iff $i = 1$ and $\Gamma_1^2 = -2$. In particular, for any irreducible \mathcal{F} -invariant curve C meeting transversely with Θ ($C \neq \Gamma_i$ for all i 's), one has

$$(3.2) \quad M(\Theta)C = \gamma_r C\Gamma_r = \gamma_r \leq \frac{1}{2}.$$

Moreover, one has

Lemma 3.1 ([19]). $M(\Theta)^2 = -\sum_{p \in \Theta} \beta_p(\mathcal{F})$.

From a straightforward computation one has

Lemma 3.2. *For each irreducible component C_i of Δ , we have*

$$\theta M(\Theta)C_i = a_i M(\Theta)C_i \geq \frac{1}{2} M(\Theta)C_i.$$

In particular, $\theta = a_i$ whenever $C_i \Gamma_1 > 0$.

If $\nu(\mathcal{F}) \geq 0$ and $K_{\mathcal{F}} \equiv P + N$ is the Zariski decomposition of $K_{\mathcal{F}}$, then we write $N = M + Z$ where the support of M is contained in $\text{supp}(\Theta)$ and Z contains no component of Θ .

Lemma 3.3. *For any irreducible component of $M(\Theta)$, we have $M\Gamma \leq M(\Theta)\Gamma$. Therefore $M - M(\Theta) \geq 0$.*

Proof. Firstly, we claim that each component of Θ lies in the support of N . Since $K_{\mathcal{F}}\Gamma_1 = -1$, $\Gamma_1 N < 0$. So it is a component of N . Suppose that $\Gamma_1, \dots, \Gamma_{i-1}$ are the components of N and Γ_i is outside of the support of N . Since $\Gamma_{i-1}\Gamma_i > 0$, $N\Gamma_i > 0$. Hence $K_{\mathcal{F}}\Gamma_i > 0$, a contradiction. So $\text{supp}(\Theta) \subseteq \text{supp}(N)$.

For any i , one has

$$(M - M(\Theta))\Gamma_i \leq (N - M(\Theta))\Gamma_i = K_{\mathcal{F}}\Gamma_i - M(\Theta)\Gamma_i = 0.$$

Hence $M - M(\Theta) \geq 0$. □

3.2. The Zariski decomposition of $K_{\mathcal{F}} + \Delta$.

Definition 3.4. *We say E is (Δ, \mathcal{F}) -exceptional if it is an \mathcal{F} -exceptional curve satisfying $(K_{\mathcal{F}} + \Delta_{\text{red}})E \leq 0$. In particular, for the case that $\nu(\mathcal{F}) \geq 0$, we say E is (Δ, \mathcal{F}) -exceptional of type H-J if E is an (Δ, \mathcal{F}) -exceptional curve contained in N .*

Lemma 3.5. *Suppose $\Theta = \Gamma_1 + \dots + \Gamma_r$ is a (Δ, \mathcal{F}) -chain. Then*

$$N(\Delta) - (1 - \theta)M(\Theta) \geq 0.$$

Proof. Since $(K_{\mathcal{F}} + \Delta)\Gamma_1 = -1 + \theta < 0$, we can see $\Gamma_1 \in \text{Supp}(N(\Delta))$ clearly. Now we assume $\Gamma_1, \dots, \Gamma_k \in \text{Supp}(N(\Delta))$ ($k \leq r$). Let $\Theta' = \Gamma_1 + \dots + \Gamma_k$. So

$$[N(\Delta) - (1 - \theta)M(\Theta')]\Gamma_j = 0,$$

for any $j = 1, \dots, k$. Then we obtain $N(\Delta) - (1 - \theta)M(\Theta') \geq 0$.

If $k = r$, we are done. Next we assume $k < r$ and we will show $\Gamma_{k+1} \in \text{Supp}(N(\Delta))$. If $\Gamma_{k+1} \notin \text{Supp}(N(\Delta))$, then

$$(K_{\mathcal{F}} + \Delta)\Gamma_{k+1} = P(\Delta)\Gamma_{k+1} + N(\Delta)\Gamma_{r+1} \geq (1 - \theta)M(\Theta')\Gamma_{k+1} > 0.$$

This is a contradiction with the fact that $(K_{\mathcal{F}} + \Delta)\Gamma_{k+1} = 0$. So we are done. □

Theorem 3.6. *Suppose \mathcal{F} is a reduced foliation on a smooth surface X with $\nu(\mathcal{F}) \geq 0$. Let $\Delta = \sum_{i=1}^l a_i C_i$ be as above. Then $K_{\mathcal{F}} + \Delta$ is pseudo-effective with the Zariski decomposition*

$$K_{\mathcal{F}} + \Delta \equiv P(\Delta) + N(\Delta),$$

where $P(\Delta)$ (resp. $N(\Delta)$) denote the positive (negative) part. If there is no (Δ, \mathcal{F}) -exceptional curve of type H-J on X , then $\text{Supp}(N(\Delta))$ is a disjoint union of maximal (Δ, \mathcal{F}) -chains.

More precisely, if $\{\Theta_1, \dots, \Theta_s\}$ is the set of all maximal (Δ, \mathcal{F}) -chains over X and $\theta_1, \dots, \theta_s$ are their multiplicities respectively. Then

$$N(\Delta) = \sum_{i=1}^s (1 - \theta_i) M(\Theta_i).$$

Moreover, $N \geq \sum_{i=1}^s M(\Theta_i) \geq N(\Delta)$.

Proof. Let \mathfrak{S} denote the set of non- \mathcal{F} -invariant curves D contained in $N(\Delta)$, and we set

$$(3.3) \quad T = \sum_{D \in \mathfrak{S}} \alpha_D D, \quad \alpha_D = \begin{cases} 1, & \text{if } D \not\subset \Delta, \\ 1 - a_i, & \text{if } D = C_i \subset \Delta \end{cases}$$

Note that $a_i \in [\frac{1}{2}, 1]$, so $T \geq 0$. Let $\{\Theta'_1, \dots, \Theta'_t\}$ be the set of all maximal elements of (Δ, \mathcal{F}) -chains disjoint from \mathfrak{S} , where $t \leq s$ and we assume $\Theta'_i \subset \Theta_i$ for $i = 1, \dots, t$. Let

$$V = \sum_{i=1}^t (1 - \theta_i) M(\Theta'_i).$$

Note that

$$\text{Supp}(V) = \mathfrak{S} \setminus \mathfrak{S}_0, \quad \mathfrak{S}_0 = \{C_i \subset \Delta \mid a_i = 1\},$$

where \mathfrak{S}_0 is disjoint from any (Δ, \mathcal{F}) -chain clearly. This implies $\Theta \cap \mathfrak{S} = \emptyset$ iff $\Theta T = 0$, for any (Δ, \mathcal{F}) -chain Θ . Thus if $T = 0$, then $t = s$, $\Theta'_i = \Theta_i$, and so $V = \sum_{i=1}^s (1 - \theta_i) M(\Theta_i)$. By the lemma 3.5, $N(\Delta) - V \geq 0$. Then it suffices to show that $M := N(\Delta) - V + T \equiv 0$.

Firstly, we claim $MC \geq 0$ for any $C \in \mathfrak{S}$. Indeed, by definition of V and C , we see $VC = 0$ and $C^2 < 0$. So

$$MC = (K_{\mathcal{F}} + \Delta + T)C \geq K_{\mathcal{F}}C + \Delta C + \alpha_C C^2.$$

If $C \not\subset \Delta$, then $\alpha_C = 1$ and $\Delta C \geq 0$. If $C \subset \Delta$, say $C = C_1$, then $\alpha_C = 1 - a_1$ and $\Delta C \geq a_1 C^2$. They both imply $\Delta C + \alpha_C C^2 \geq C^2$, so

$$MC \geq K_{\mathcal{F}}C + C^2 = \text{tang}(\mathcal{F}, C) \geq 0.$$

Secondly, we claim that $MC = 0$ for any component C of V . Without loss of generality, we assume that C is contained in $\Theta := \Theta'_1 = \Gamma_1 + \dots + \Gamma_r$, with $\theta = \theta_1$. By a straightforward computation, we have

$$(VC, \Delta C, K_{\mathcal{F}}C) = \begin{cases} (\theta - 1, \theta, -1), & \text{if } C = \Gamma_1, \\ (0, 0, 0), & \text{others.} \end{cases}$$

Hence

$$MC = (N(\Delta)C - V)C = (K_{\mathcal{F}}C + \Delta C - V)C = 0.$$

Suppose that $M > 0$. One has $M^2 < 0$ by $\text{Supp}(M) \subset \text{Supp}(N(\Delta))$. One can find an irreducible component C of M satisfying $MC < 0$. By the above discusses, $C \notin \mathfrak{S} \cup \text{Supp}(V)$. Hence C is an \mathcal{F} -invariant component of $N(\Delta)$ which is not contained in V . We will show that such curve C does not exist and hence $M = 0$.

Without loss of generality, we assume that C meets transversely with (Δ, \mathcal{F}) -chains $\Theta'_1, \dots, \Theta'_k$ at p_1, \dots, p_k respectively, and contains h other singularities of \mathcal{F} , through which there is a separatrix not contained in $C + \Theta'_1 + \dots + \Theta'_k$, says

q_1, \dots, q_h . From the separatrix theorem ([4], Theorem 3.4), one can find that $h \geq 1$. By (3.2) and $Z(\mathcal{F}, C) \geq h + k$, we can find

$$VC = \sum_{i=1}^k (1 - \theta_i) M(\Theta'_i) C \leq \frac{k}{2}$$

and

$$K_{\mathcal{F}}C = Z(\mathcal{F}, C) - \chi(C) \geq h + k - 2 + 2p_a(C).$$

So

$$0 > MC = (K_{\mathcal{F}} + \Delta - V + T)C \geq h + \frac{k}{2} - 2 + 2p_a(C) + (\Delta + T)C.$$

If $TC > 0$, then $(\Delta + T)C \geq 1$ by (3.3). In this case, we get $MC \geq 0$, a contradiction. So $TC = 0$. Similarly, we have $p_a(C) = 0$, $h = 1$ and $k \leq 1$. Thus

$$\frac{k}{2} - \Delta C \geq VC - \Delta C > K_{\mathcal{F}}C \geq k - 1.$$

Therefore one of the following two cases occurs:

- i) $k = 0, K_{\mathcal{F}}C = -1$ and $\Delta C < 1$ (i.e., $\Delta_{\text{red}}C \leq 1$),
- ii) $k = 1, K_{\mathcal{F}}C = 0$ and $\Delta C < \frac{1}{2}$ (i.e., $\Delta_{\text{red}}C = 0$).

One can see that $(K_{\mathcal{F}} + \Delta_{\text{red}})C \leq 0$ and all singularities of \mathcal{F} on C are non-degenerated in both above cases. By our assumption, one has $C^2 < -1$, otherwise, C is a (Δ, \mathcal{F}) -exceptional curve, a contradiction. Therefore either C is a (Δ, \mathcal{F}) -chain disjoint from \mathfrak{S} or $\Theta'_i + C$ is a (Δ, \mathcal{F}) -chain disjoint from \mathfrak{S} for some i , a contraction with our assumptions.

Now we are done. \square

Remark 3.7. *If $\nu(\mathcal{F}) = -\infty$ and $K_{\mathcal{F}} + \Delta$ is pseudo-effective, we can get a similar result about $N(\Delta)$. In particular, if $\Delta = 0$, then Theorem 3.6 is just the McQuillan Theorem ([4], Theorem 8.1). By the way, in the paper [8], the author also studied the Zariski decomposition of such adjoint divisor, by using the $(K_{\mathcal{F}} + \Delta)$ -MMP method.*

Corollary 3.8. *Suppose $\nu(\mathcal{F}) \geq 0$ and there is no (Δ, \mathcal{F}) -exceptional curve of type H-J over X . Then for $\mu = \min\{a_i\}_{i=1}^l$,*

$$(3.4) \quad [(1 - \mu)N - N(\Delta)]\Delta \geq 0.$$

In particular,

$$(3.5) \quad \left[\frac{1}{2}N - N(\Delta)\right]\Delta \geq 0.$$

Proof. Using the notations in the proof of Theorem 3.6, for any irreducible component C of Δ , say $C = C_1$,

$$\begin{aligned} [(1 - \mu)N - N(\Delta)]C &= (1 - \mu)NC - (1 - a_1) \sum_{i=1}^s M(\Theta_i)C \quad (\text{Lemma 3.2}) \\ &\geq (1 - \mu) \left(N - \sum_{i=1}^s M(\Theta_i) \right) C \geq 0. \quad (\text{Lemma 3.3}) \end{aligned}$$

\square

Corollary 3.9. *If $\nu(\mathcal{F}) \geq 0$ and there is no (Δ, \mathcal{F}) -exceptional curve of type H-J over X , then*

$$(3.6) \quad (N + N(\Delta))\Delta + N^2 - N(\Delta)^2 = P(\Delta)N \geq 0.$$

Proof.

$$\begin{aligned} (N + N(\Delta))\Delta + N^2 - N(\Delta)^2 &= (\Delta + N - N(\Delta))(N + N(\Delta)) \\ &= P(\Delta)(N + N(\Delta)) = P(\Delta)N \geq 0. \end{aligned}$$

□

Corollary 3.10. *If $\nu(\mathcal{F}) \geq 0$ and there is no (Δ, \mathcal{F}) -exceptional curve of type H-J over X , then*

$$(3.7) \quad (2 - \mu)N\Delta + N^2 - N(\Delta)^2 \geq 0,$$

where $\mu = \min\{a_i\}_{i=1}^l \geq \frac{1}{2}$. In particular,

$$(3.8) \quad \frac{3}{2}N\Delta + N^2 - N(\Delta)^2 \geq 0.$$

Corollary 3.11. *Under the assumptions and notations in Theorem 3.6, one has*

$$\begin{aligned} N(\Delta)^2 &= - \sum_{i=1}^s \sum_{p \in \Theta_i} (1 - \theta_i)^2 \beta_p(\mathcal{F}), \\ N(\Delta)\Delta &= \sum_{i=1}^s \sum_{p \in \Theta_i} \theta_i(1 - \theta_i) \beta_p(\mathcal{F}). \end{aligned}$$

Definition 3.12 ([11], Definition I.1.5). *Let (X, \mathcal{F}, Δ) be a foliated triple and $f : X' \rightarrow X$ be a proper birational morphism. For any divisor E on X' , we define the discrepancy of (\mathcal{F}, Δ) along E to be*

$$a(E, \mathcal{F}, \Delta) = \text{ord}_E(K_{\mathcal{F}'} - f^*(K_{\mathcal{F}} + \Delta)).$$

We say (X, \mathcal{F}, Δ) is canonical if $a(E, \mathcal{F}, \Delta) \geq 0$ for every f -exceptional divisor E over X' .

Let \mathcal{F} be a reduced foliation over a smooth surface X and Δ be as above.

Lemma 3.13. *(X, \mathcal{F}, Δ) is canonical if and only if $\text{tang}(\mathcal{F}, \Delta_{\text{red}}) = 0$.*

3.3. Canonical resolution of (X, Δ, \mathcal{F}) . In this section, we assume (X, \mathcal{F}) is reduced with $\nu(\mathcal{F}) \geq 0$.

A point $p \in \Delta$ is called *regular* w.r.t. (Δ, \mathcal{F}) if Δ_{red} and \mathcal{F} are both non-singular at p and $t_p := \text{tang}(\mathcal{F}, \Delta_{\text{red}}, p) = 0$. A non-regular point $p \in \Delta$ is said to be a *wild* (resp., *tame*) *singularity* of (Δ, \mathcal{F}) if $\nu_p \geq 1 - l(p), \nu_p > 0$ (resp., $l(p) = 0, 0 < \nu_p < 1$). Note that the assumption $0 < \nu_p < 1$ is equal to say $m_p(\Delta_{\text{red}}) = 1$ and $p \in C_i$ for some component C_i of Δ with $a_i < 1$.

Consider the blowing-up $\sigma : (X', \mathcal{F}', E) \rightarrow (X, \mathcal{F}, p)$ over a non-regular point $p \in \Delta$. Recall

$$K_{\mathcal{F}'} + \Delta' = \sigma^*(K_{\mathcal{F}} + \Delta) + (1 - l(p) - \nu_p)E.$$

Definition 3.14. *Let Γ be an irreducible component of N passing through p .*

- (1) Γ is called the *first potential curve* of (Δ, \mathcal{F}) at p if $\Gamma \cap \text{Sing}(\mathcal{F}) = \{p\}$ and $0 < \Delta\Gamma = (\Delta\Gamma)_p < 1$.
- (2) Γ is called the *second potential curve* of (Δ, \mathcal{F}) at p if
 - (i) $\Gamma \cap \text{Sing}(\mathcal{F}) = \{p\}$ and $1 \leq \Delta\Gamma < (\Delta\Gamma)_p + 1$, or
 - (ii) $\Gamma \cap \text{Sing}(\mathcal{F}) = \{p, p'\}$, $\Delta\Gamma = (\Delta\Gamma)_p$, and Γ transversely intersects with some maximal (Δ, \mathcal{F}) -chain Θ at p' .

Moreover, a potential curve Γ is called *simple* if $(\Delta\Gamma)_p = m_p(\Delta)$.

For convenience, we let

$$(3.9) \quad \theta_p := \begin{cases} \Delta\Gamma - (\Delta\Gamma)_p, & \text{for (i),} \\ \text{the multiplicity of } \Theta \text{ w.r.t. } \Delta, & \text{for (ii).} \end{cases}$$

Lemma 3.15. *Suppose p is a wild singularity of (Δ, \mathcal{F}) , i.e., $\nu_p \geq 1 - l(p), \nu_p > 0$, and there is no (Δ, \mathcal{F}) -exceptional curve of type H-J.*

- (1) *There is no (Δ', \mathcal{F}') -exceptional curve of type H-J.*
- (2) *p is outside of $N(\Delta)$ unless there is a first potential curve of (Δ, \mathcal{F}) at p .*
- (3) *$N(\Delta') = \sigma^*N(\Delta)$ unless there is a simple potential curve of (Δ, \mathcal{F}) at p . Furthermore, there is at most one simple potential curve of \mathcal{F} at p .*
- (4) *Let Γ be a simple potential curve of (Δ, \mathcal{F}) at p with $\Gamma^2 = -e$ and Γ' be the strict transform of Γ under σ meeting transversely with E at q . Let Θ' be the (Δ', \mathcal{F}') -chain containing Γ' with the multiplicity θ . We have*

$$N(\Delta)^2 - N(\Delta')^2 = \begin{cases} \frac{1}{e+1} - \frac{(1-\nu_p)^2}{e}, & \text{if } \Gamma' = \Theta', \\ (1-\theta)^2\beta(-\lambda_q), & \text{otherwise.} \end{cases}$$

Here $q = \Gamma' \cap E$.

Proof. (1) Suppose that we have a (Δ', \mathcal{F}') -exceptional curve C' of type H-J. If $\nu_p + l(p) > 1$,

$$(K_{\mathcal{F}'} + \Delta'_{\text{red}})E \geq (K_{\mathcal{F}'} + \Delta')E > 0.$$

If $(\nu_p, l(p)) = (1, 0)$, then $K_{\mathcal{F}'}E = -1$ and $\Delta'E = 1$. Hence

$$(K_{\mathcal{F}'} + \Delta'_{\text{red}})E > (K_{\mathcal{F}'} + \Delta')E = 0.$$

In both cases, we have $C' \neq E$.

Let C be the image curve of C' under σ . Note that C lies in N . Hence $-1 = C'^2 \leq C^2 < 0$. It implies that C is a (-1) -curve not passing through p . Namely, C is a (Δ, \mathcal{F}) -exceptional curve of type H-J, a contradiction.

(2) Assume that $(\nu_p, l(p)) = (1, 0)$. In this case, $N(\Delta') = \sigma^*N(\Delta)$. Suppose that $p \in N(\Delta)$. It lies in some (Δ, \mathcal{F}) -chain Θ with multiplicity θ . Thus p lies in the first component of Θ and $1 \geq \theta \geq \nu_p = 1$. So $\theta = 1$, which is a contradiction with the fact that $\theta < 1$.

It's sufficient to consider the case of $\nu_p + l(p) > 1$. Suppose that some (Δ, \mathcal{F}) -chain passes through p . If $l(p) = 0$, then $\nu_p > 1$. So the first component Γ of this chain passes through p and $\Delta\Gamma \geq \nu_p > 1$, a contradiction. So $l(p) = 1$ and $\nu_p > 0$. It implies that Γ is a unique component of this (Δ, \mathcal{F}) -chain.

(3) It's enough to consider the case that $\nu_p + l(p) > 1$. If p occurs in the exceptional case in (2), then one gets

$$N(\Delta)^2 - N(\Delta')^2 = (1 - \nu_p)^2 M(\Gamma)^2 - M(\Gamma')^2 = \frac{1}{e+1} - \frac{(1-\nu_p)^2}{e}.$$

So $N(\Delta') \neq \sigma^*N(\Delta)$.

In what follows, we assume that there is no first potential curve passing through p . By (2), the pulling-backs of all (Δ, \mathcal{F}) -chains under σ are (Δ', \mathcal{F}') -chains.

Let Θ' be a maximal (Δ', \mathcal{F}') -chain which is not the pulling-back of some (Δ, \mathcal{F}) -chain. Thus $\Theta'E > 0$. From $Z(\mathcal{F}, E) \leq 2$, there is at most another one (Δ', \mathcal{F}') -chain Θ'' meeting with E . Suppose that Θ'' exists. From the separatrix theorem, $\Theta' + \Theta'' + E$ is not contractible. One can get an \mathcal{F} -invariant rational curve C with $C^2 = 0$ after contracting all contractible components of $\Theta' + \Theta'' + E$. Thus the corresponding foliation of \mathcal{F} from the contractions is generated by rational curves, a contradiction. Therefore there is at most one such (Δ', \mathcal{F}') -chain Θ' .

Let Γ' be the last component of Θ' . So $\Gamma'E = 1$, $(\Theta' - \Gamma')E = 0$. Obviously, either $\Theta' = \Gamma'$ or the image of $\Theta' - \Gamma'$ under σ is a (Δ, \mathcal{F}) -chain. Let Γ be the image curve of Γ' under σ . If Γ occurs in $N(\Delta)$, then Θ' is a pulling-back of a (Δ, \mathcal{F}) -chain, a contradiction. So $\Gamma' \notin N(\Delta')$.

(4) It's from a straightforward computation. \square

Lemma 3.16. *Suppose p is a tame singularity of (Δ, \mathcal{F}) and there is no (Δ, \mathcal{F}) -exceptional curve of type H-J.*

- (1) *E is the only (Δ', \mathcal{F}') -exceptional curve of type H-J and $p' := E \cap \Delta'$ is a wild singularity of (Δ', \mathcal{F}') with $l(p') = 1$ and $m_p(\Delta'_{\text{red}}) = 1$.*
- (2) *After a blow-up $\sigma' : (X'', \mathcal{F}'', E') \rightarrow (X', \mathcal{F}', p')$, there is no $(\Delta'', \mathcal{F}'')$ -exceptional curve of type H-J.*

Corollary 3.17. *For each non-regular point $p \in (X, \mathcal{F}, \Delta)$, we have*

$$N(\Delta)^2 - N(\Delta')^2 = \begin{cases} \frac{1}{e+1} - \frac{(1-\nu_p)^2}{e}, & \text{if } p \text{ lies on a first potential curve,} \\ (1-\theta)^2 \beta(-\lambda_q), & \text{if } p \text{ lies on a simple second potential curve with } l(p) = 1, \\ (1-\nu_p)^2, & \text{if } p \text{ is tame,} \\ 0, & \text{others.} \end{cases}$$

In particular, $0 \leq N(\Delta)^2 - N(\Delta')^2 < 1$.

In what follows, we assume that there is no (Δ, \mathcal{F}) -exceptional curve of type H-J on (X, \mathcal{F}) . Consider the following blow-ups

$$(\bar{X}, \bar{\mathcal{F}}) = (X_r, \mathcal{F}_r) \xrightarrow{\sigma_r} (X_{r-1}, \mathcal{F}_{r-1}) \xrightarrow{\sigma_{r-1}} \cdots \xrightarrow{\sigma_1} (X_0, \mathcal{F}_0) = (X, \mathcal{F})$$

where $\mathcal{F}_{i+1} = \sigma_{i+1}^* \mathcal{F}_i$ is the pulling-back foliation and Δ_{i+1} is the strict transform of Δ_i and q_i denotes the blow-up point of σ_i satisfying that q_i 's are all non-regular points over (Δ, \mathcal{F}) . From the above discussions, we can see

- (1) there is no $(\bar{\Delta}, \bar{\mathcal{F}})$ -exceptional curve of type H-J over \bar{X} ,
- (2) for any $q \in \bar{X}$, $t_p = 0$. Or say, $(\bar{X}, \bar{\mathcal{F}}, \bar{\Delta})$ is canonical.

For convenience, $\sigma := \sigma_r \circ \cdots \circ \sigma_1$ is said to be the *canonical resolution* of (X, \mathcal{F}, Δ) .

4. DOUBLE COVERS OVER FOLIATED SURFACES (X, \mathcal{F}) WITH $\nu(\mathcal{F}) \geq 0$

Let \mathcal{F} be a reduced foliation on a smooth surface X with $\nu(\mathcal{F}) \geq 0$. Let $K_{\mathcal{F}} \equiv P + N$ be the Zariski decomposition of the canonical divisor, where P (resp. N) is the positive (resp. negative) part. Let $\pi : Y \rightarrow X$ be a double cover over X with the branch locus B , which is a reduced even effective divisor. We set $\mathcal{G} := \pi^* \mathcal{F}$.

One can write

$$B = B_v + B_h,$$

where B_v consists of irreducible \mathcal{F} -invariant components of B and B_h consists of other components. For convenience, B_v (resp., B_h) is said to be the \mathcal{F} -invariant (resp., non- \mathcal{F} -invariant) part of B .

It is clear that $K_{\mathcal{F}} + \frac{1}{2}B_h$ is pseudo-effective with the Zariski decomposition

$$K_{\mathcal{F}} + \frac{1}{2}B_h \equiv P(B_h) + N(B_h),$$

where $P(B_h)$ (resp. $N(B_h)$) denotes the positive (resp. negative) part.

Remark 4.1. *Without causing confusions, the (B_h, \mathcal{F}) -chain means the $(\frac{1}{2}B_h, \mathcal{F})$ -chain. (B_h, \mathcal{F}) -exceptional curve is similar.*

4.1. Double cover with a smooth branch locus. In this section, we assume the branch locus B is smooth and reduced.

4.1.1. *Classification of the singularities of \mathcal{G} .* Let $q \in X$ be any point over X and $p = \pi(q) \in Y$.

- (I) Suppose $p \notin B$. Then the $\pi^{-1}(p)$ consists of two reduced singularities of \mathcal{G} which are exactly the copies of q .
 (II) Suppose $p \in B_h$.
 (1) If p is a regular point of \mathcal{F} , then B, \mathcal{F}, π can be locally defined by

$$B = (x + y^l = 0), \quad \omega = dx, \quad \begin{cases} x + y^l = u^2, \\ y = v, \end{cases}$$

where $p = (0, 0)$, $l = t_p + 1 \geq 1$ and $\tilde{B} := (\pi^*B)_{\text{red}} = (u = 0)$. Then around q , \mathcal{G} is locally defined by

$$\tilde{\omega} = \pi^*(\omega) = d(u^2 - v^l) = 2udu - lv^{l-1}dv.$$

This implies that q is reduced iff $t_p \leq 1$. In particular, q is regular if $t_p = 0$ and $\lambda_q = -1$ if $t_p = 1$.

- (2) If p is a singularity of \mathcal{F} , we can assume B, \mathcal{F}, π are locally defined by

$$B = (x + y^l = 0), \quad \omega = ye_1(x, y)dx + xe_2(x, y)dy, \quad \begin{cases} x + y^l = u^2, \\ y = v, \end{cases}$$

where $l \geq 1$, $e(0, 0) \neq 0$, $e_2(0, 0) \neq 0$ and $\tilde{B} = (u = 0)$. Then around q , \mathcal{G} is locally defined by

$$\tilde{\omega} = \pi^*\omega = 2uve_1du + (u^2e_2 - v^le_2 - lv^le_1)dv.$$

This implies that q is a singularity of \mathcal{G} , which is not reduced.

- (III) Suppose $p \in B_v$. By choosing a suitable local coordinate, one can assume that B_v (resp., π) is defined by $x = 0$ (resp., $z^2 = x$) and \mathcal{F} (resp., \mathcal{G}) has a 1-form ω (resp., $\tilde{\omega}$) nearby $p = (0, 0)$ (resp., $q = (0, 0)$) occurring one of the following cases:

- (1) p is a regular point of \mathcal{F} , $\omega = dx$ and $\tilde{\omega} = du$.
 (2) p is a non-degenerate singularity of \mathcal{F} , $\omega = \lambda y dx + x dy$ and $\tilde{\omega} = 2\lambda y dz + z dy$;
 (3) p is a saddle-node of \mathcal{F} with a strong separatrix B_v and

$$\begin{aligned} \omega &= (y(1 + \nu x^k) + xo(k))dx - x^{k+1}dy, \\ \tilde{\omega} &= 2(y(1 + \nu z^{2k}) + z^2o(k))dz - z^{2k+1}dy; \end{aligned}$$

- (4) p is a saddle-node of \mathcal{F} with a weak separatrix B_v and

$$\begin{aligned} \omega &= x(1 + \nu y^k + o(k))dy - y^{k+1}dx, \\ \tilde{\omega} &= z(1 + \nu y^k + o(k))dy - 2y^{k+1}dz. \end{aligned}$$

Therefore q is also a reduced singularity (resp. regular point) of \mathcal{G} in (2)-(4) (resp. (1)).

Thus we obtain the following proposition clearly.

Proposition 4.2. *Under the notations above. \mathcal{G} is a reduced foliation over a smooth surface Y if and only if*

- i) *the branch locus B is smooth and reduced;*
 ii) *for any point $p \in B_h$, $p \notin \text{Sing}\mathcal{F}$ and $t_p := \text{tang}(\mathcal{F}, B_h, p) \leq 1$.*

Proposition 4.3. *\mathcal{G} is a reduced foliation over a smooth surface Y if*

- i) *the branch locus B is smooth and reduced;*
 ii) *$\text{tang}(\mathcal{F}, B_h) = 0$.*

Under the assumption of Proposition 4.3, we can divide the (reduced) singularities q of \mathcal{G} into the following cases:

- (A) $p = \pi(q)$ is a singularity of \mathcal{F} outside of B ;
- (B) $p = \pi(q)$ is a singularity of \mathcal{F} over B_v .
- (\mathfrak{B}_1) p is non-degenerate with $\text{CS}(\mathcal{F}, B_v, p) = -\frac{n}{m}$ satisfying $\text{gcd}(m, n) = 1$ and n is odd.
- (\mathfrak{B}_2) Other cases.

Lemma 4.4. *For any point $p \in X$, we have*

$$\sum_{q \in \pi^{-1}(p)} \beta_q(\mathcal{G}) = \begin{cases} \frac{1}{2}\beta_p(\mathcal{F}), & \text{if } p \in \mathfrak{B}_1, \\ 2\beta_p(\mathcal{F}), & \text{others.} \end{cases}$$

Moreover,

$$\sum_{q \in \text{Sing}\mathcal{G}} \beta_q(\mathcal{G}) = \sum_{p \in \text{Sing}\mathcal{F}} 2\beta_p(\mathcal{F}) - \sum_{p \in \mathfrak{B}_1} \frac{3}{2}\beta_p(\mathcal{F}).$$

Theorem 4.5. *Under the assumption of Proposition 4.3. If $\nu(\mathcal{F}) \geq 0$, we have*

$$\begin{aligned} c_1^2(\mathcal{G}) &= 2c_1^2(\mathcal{F}) + \frac{3}{2}K_{\mathcal{F}}B_h + 2N^2 - 2N(B_h)^2, \\ \chi(\mathcal{G}) &= 2\chi(\mathcal{F}) + \frac{1}{8}K_{\mathcal{F}}B_h - \sum_{p \in \mathfrak{B}_1} \frac{1}{8}\beta_p(\mathcal{F}) + \frac{1}{6}\ell(\mathcal{F}) - \frac{1}{12}\ell(\mathcal{G}), \\ c_2(\mathcal{G}) &= 2c_2(\mathcal{F}) - 2N^2 + 2N(B_h)^2 - \sum_{p \in \mathfrak{B}_1} \frac{3}{2}\beta_p(\mathcal{F}) + 2\ell(\mathcal{F}) - \ell(\mathcal{G}). \end{aligned}$$

where $\ell(\mathcal{G})$ (resp., $\ell(\mathcal{F})$) is the number of \mathcal{G} (resp., \mathcal{F})-exceptional curves containing saddle-nodes.

Proof. It's easy to see that $K_{\mathcal{G}}$ has a Zariski decomposition $K_{\mathcal{G}} = P(\mathcal{G}) + N(\mathcal{G})$ where

$$P(\mathcal{G}) = \pi^*P(B_h), \quad N(\mathcal{G}) = \pi^*N(B_h).$$

Hence

$$c_1^2(\mathcal{G}) = P(\mathcal{G})^2 = 2P(B_h)^2 = 2(K_{\mathcal{F}} + \frac{1}{2}B_h)^2 - 2N(B_h)^2.$$

Since $\text{tang}(\mathcal{F}, B_h) = K_{\mathcal{F}}B_h + B_h^2 = 0$ and $P^2 = c_1^2(\mathcal{F})$, we have

$$c_1^2(\mathcal{G}) = 2c_1^2(\mathcal{F}) + \frac{3}{2}K_{\mathcal{F}}B_h + 2N^2 - 2N(B_h)^2.$$

Similarly,

$$K_{\mathcal{G}}^2 = 2K_{\mathcal{F}}^2 + \frac{3}{2}K_{\mathcal{F}}B_h.$$

By Lemma 4.4, we have

$$\sum_{q \in \text{Sing}\mathcal{G}} \beta_q(\mathcal{G}) = \sum_{p \in \text{Sing}\mathcal{F}} 2\beta_p(\mathcal{F}) - \sum_{p \in \mathfrak{B}_1} \frac{3}{2}\beta_p(\mathcal{F}).$$

From

$$12\chi(\mathcal{G}) = K_{\mathcal{G}}^2 + \sum_{q \in \text{Sing}\mathcal{G}} \beta_q(\mathcal{G}) - \ell(\mathcal{G}),$$

and the above discussions, one gets

$$\begin{aligned} 12\chi(\mathcal{G}) &= 2K_{\mathcal{F}}^2 + \frac{3}{2}K_{\mathcal{F}}B_h + 2 \sum_{p \in \text{Sing}\mathcal{F}} \beta_p(\mathcal{F}) - \frac{3}{2} \sum_{p \in \mathfrak{B}_1} \beta_p(\mathcal{F}) - \ell(\mathcal{G}) \\ &= 24\chi(\mathcal{F}) + \frac{3}{2}K_{\mathcal{F}}B_h - \frac{3}{2} \sum_{p \in \mathfrak{B}_1} \beta_p(\mathcal{F}) + 2\ell(\mathcal{F}) - \ell(\mathcal{G}). \end{aligned}$$

Finally, $c_2(\mathcal{G})$ is from the Noether formula $12\chi(\mathcal{G}) = c_1^2(\mathcal{G}) + c_2(\mathcal{G})$. \square

4.2. In this section, we firstly assume that there is no $(\frac{1}{2}B_h, \mathcal{F})$ -exceptional curve of type H-J on X . By Theorem 3.6, we have

$$N(B_h) = \sum_{i=1}^s (1 - \theta_i) M(\Theta_i),$$

where $\{\Theta_1, \dots, \Theta_s\}$ is the set of all maximal $(\frac{1}{2}B_h, \mathcal{F})$ -chain and $\theta_i = \frac{1}{2}B_h \Theta_i$ for all i .

Definition 4.6. For any $q \in \text{Sing}\mathcal{F}$, we define $\beta_q(B_v)$ as

$$(4.1) \quad \beta_q(B_v) := \begin{cases} \beta_q(\mathcal{F}), & \text{if } q \in \mathfrak{D}, \\ 4/3, & \text{if } q \in B_v \setminus B_h \text{ with } m_q(B_v) = 2, \lambda_q = 0, \\ 0, & \text{others,} \end{cases}$$

where \mathfrak{D} is set of singularities q of \mathcal{F} contained in $B_v \setminus B_h$, satisfying $\lambda_q = -\frac{n}{m} \in \mathbb{Q}^-$, $\gcd(m, n) = 1$ and

- (i) for $m_q(B_v) = 1$, $\text{CS}(\mathcal{F}, B_v, q) = -\frac{n}{m} \in \mathbb{Q}^-$, n is odd,
- (ii) for $m_q(B_v) = 2$, $m + n$ is odd.

Under the assumption of Proposition 4.3, it is clear that

$$(4.2) \quad s(B_v) := \sum_{p \in \text{Sing}\mathcal{F}} \beta_p(B_v) = \sum_{p \in \mathfrak{B}_1} \beta_p(\mathcal{F}).$$

Lemma 4.7. For a maximal $(\frac{1}{2}B_h, \mathcal{F})$ -chain $\Theta = \Gamma_1 + \dots + \Gamma_r$,

$$(4.3) \quad \sum_{q \in \Theta \cap \text{Sing}\mathcal{F}} \beta_q(B_v) = \begin{cases} \sum_{q \in \Theta} \beta_q(\mathcal{F}), & \text{if } B_h \Gamma_1 = 1 \text{ and } B_h \cap \Theta \cap \text{Sing}\mathcal{F} = \emptyset; \\ 0, & \text{others.} \end{cases}$$

Proof. If $B_h \cap \Gamma_1 \cap \text{Sing}\mathcal{F} \neq \emptyset$, then $r = 1$ and $\Theta \cap \text{Sing}\mathcal{F} = \{p_1\}$ for $p_1 = \Gamma_1 \cap B_h$. So

$$\sum_{q \in \Theta \cap \text{Sing}\mathcal{F}} \beta_q(B_v) = \beta_{p_1}(B_v) = 0.$$

Next we assume $B_h \cap \Gamma_1 \cap \text{Sing}\mathcal{F} = \emptyset$. Consider a sequence of blow-ups $\sigma : (X', \mathcal{F}') \rightarrow (X, \mathcal{F})$, whose blow-up points are the set of points $q \in \Theta \cap \text{Sing}\mathcal{F}$ with $m_q(B_v) = 2$. Let $\Theta' = (\sigma^* \Theta)_{\text{red}}$. We write $\Theta' = D_1 + \dots + D_l$, which has a similar construction as an \mathcal{F}' -chain but we permit $D_j^2 = -1$ for some j . We set $e_j = -D_j^2$. Note that $D_1 = \bar{\Gamma}_1$. Let $p_i = D_i \cap D_{i+1}$ for $i = 1, \dots, l-1$ and let p_l be the other singularities of \mathcal{F}' over D_l . Then it suffices to prove

$$\sum_{i=1}^l \beta_{p_i}(B'_v) = \begin{cases} \sum_{i=1}^l \beta(-\lambda_{p_i}), & \text{if } B'_h D_1 = 1, \\ 0, & \text{if } B'_h D_1 = 0. \end{cases}$$

- (1) If $B'_h D_1 = 0$ and $D_1 \not\subset B'_v$, then $D_1, \dots, D_l \not\subset B'_v$ and $B'_v D_l = 0$. So $\beta_{p_i}(B'_v) = 0$ for any $i = 1, \dots, l$.
- (2) If $B'_h D_1 = 0$ and $D_1 \subset B'_v$, then $D_1, D_3, \dots, D_{2[\frac{l-1}{2}]+1} \subset B'_v$ and $e_{2i+1} \equiv 0 \pmod{2}$. In this case, $p_1, \dots, p_l \in B'_v$ and $m_{p_i}(B'_v) = 1$ for all i . We set $\text{CS}(\mathcal{F}', B'_v, p_i) = -\frac{n_i}{m_i}$ for $\gcd(n_i, m_i) = 1$, $i = 1, \dots, l$. So $e_1 \equiv 0 \pmod{2}$, $n_1 \equiv 0 \pmod{2}$, and $m_1 = 1$. By the C-S formula for D_2 :

$$\text{CS}(\mathcal{F}', D_2, p_1) + \text{CS}(\mathcal{F}', D_2, p_2) = D_2^2 = -e_2,$$

we obtain

$$e_2 = \frac{m_1}{n_1} + \frac{m_2}{n_2}.$$

So $n_2 m_1 = n_1(n_2 e_2 - m_2)$. Since n_1 is even and m_1 is odd, we see that n_2 is even. Similarly, by the C-S formula for D_3 , we get

$$e_3 = \frac{n_2}{m_2} + \frac{n_3}{m_3}.$$

So $n_3 m_2 = e_3 m_2 m_3 - m_3 n_2$. Since e_3, n_2 are even and m_2 is odd, we see n_3 is even. By induction, we see that n_1, \dots, n_l are even, so $\beta_{p_i}(B'_v) = 0$ for $i = 1, \dots, l$.

(3) Suppose $B'_h D_1 = 1$.

(i) If $D_1 \not\subset B'_v$, then $D_2, D_4, \dots, D_{2[\frac{l}{2}]} \subset B'_v$ and $e_{2i} \equiv 0 \pmod{2}$. In this case, $p_1, \dots, p_l \in B'_v$ and $m_{p_i}(B'_v) = 1$ for all i . Similar to the trick on (2), we see that n_1, n_2, \dots, n_l are odd. So $\beta_{p_i}(B'_v) = \beta(-\lambda_{p_i})$ for $i = 1, \dots, l$.

(ii) If $D_1 \subset B'_v$, then $D_1, D_3, \dots, D_{2[\frac{l-1}{2}]+1} \subset B'_v$ and $e_1 \equiv 1 \pmod{2}$, $e_{2i+1} \equiv 0 \pmod{2}$. In this case, $p_1, \dots, p_l \in B'_v$ and $m_{p_i}(B'_v) = 1$. Similar to the trick on (2), we see that n_1, n_2, \dots, n_l are odd. So $\beta_{p_i}(B'_v) = \beta(-\lambda_{p_i})$ for $i = 1, \dots, l$.

□

Corollary 4.8. For a maximal $(\frac{1}{2}B_h, \mathcal{F})$ -chain $\Theta = \Gamma_1 + \dots + \Gamma_r$,

$$\sum_{q \in \Theta \cap \text{Sing} \mathcal{F}} \beta_q(B_v) = \begin{cases} 0, & \text{if } B_h \cap \Theta \cap \text{Sing} \mathcal{F} \neq \emptyset \\ 2(1 - \theta)M(\Theta)B_h, & \text{others.} \end{cases}$$

Proposition 4.9. If $B_h \cap N(B_h) \cap \text{Sing} \mathcal{F} = \emptyset$, then $s(B_v) \geq 2N(B_h)B_h$.

4.3. Singularities of (B_h, \mathcal{F}) . In this section, we assume that there is no (B_h, \mathcal{F}) -exceptional curve of type H-J over (X, \mathcal{F}) .

Let $p \in B_h$ with $t_p \geq 1$. Consider a sequence of blowing-ups $\sigma = \sigma_1 \circ \dots \circ \sigma_r : X' \rightarrow X$:

$$(X', \mathcal{F}', B'_h) = (X_r, \mathcal{F}_r, B_{r,h}) \xrightarrow{\sigma_r} X_{r-1} \xrightarrow{\sigma_{r-1}} \dots \xrightarrow{\sigma_1} (X_0, \mathcal{F}_0, B_{0,h}) = (X, \mathcal{F}, B_h),$$

satisfying the following conditions:

- (i) $B_{i,h}$ is the strict transform of B_h , i.e., $B_{i,h} = (\sigma_1 \circ \dots \circ \sigma_i)_*^{-1} B_h$.
- (ii) Let q_i (resp. E_i) denote the blow-up point (resp. exceptional curve) of σ_i , then $q_1 = p$ and $q_i \in B_{i-1,h} \cap \text{Sing} \mathcal{F}_{i-1} \cap (\sigma_1 \circ \dots \circ \sigma_{r-1})^{-1}(p)$ for $i \geq 2$, where
- (iii) For any $q \in B'_h \cap \sigma^{-1}(p)$, q is a regular point of \mathcal{F}' , i.e., $l(q) = 0$.

Lemma 4.10. If there is no (B_h, \mathcal{F}) -exceptional curve of type H-J over (X, \mathcal{F}) , then there is no (B'_h, \mathcal{F}') -exceptional curve of type H-J over (X', \mathcal{F}') .

Proof. By Lemma 3.15 and Lemma 3.16. □

Definition 4.11. We call $(p = q_1 \rightarrow q_2 \rightarrow \dots \rightarrow q_r)$ a singularity of type $S_{l,m}$ for $l = l(p)$ and $m = m_p(B_h)$. We call the above sequence of blow-ups σ a bunch of blow-ups over a singularity of type $S_{l,m}$, for $l = l(p)$, $m = m_p(B_h)$. Note that $l \in \{0, 1\}$, $m \geq 1$.

We denote by $r = r_p$ the number of the blow-ups, which is called the rank of p .

Lemma 4.12. $K_{\mathcal{F}'} B'_h = K_{\mathcal{F}} B_h + 2(1 - l(p))m_p(B_h)$.

Proposition 4.13. *Suppose $p \in (S_{0,m})$. Then $N(B'_h) = \sigma^*N(B_h)$, unless*

$$(4.4) \quad \bar{E}_1 B'_h = E_1 B_{1,h} - (E_1 B_{1,h})_{q_2} \leq 1.$$

In particular, if (4.4) holds, then we obtain a maximal (B'_h, \mathcal{F}') -chain Θ_p contained in $\sigma^{-1}(p)$, with the first curve $\Gamma_1 = \bar{E}_1$ and $\theta_p = (B_{1,h}E_1 - (B_{1,h}E_1)_{q_2})/2$.

Proof. By Lemma 3.15 and Lemma 3.16. \square

For convenience, we divide $S_{0,m}$ -singularities into the following 2 cases:

$$(S_{0,m}^\theta) \quad p \in (S_{0,m}) \text{ with } \theta := \theta_p \in \{0, \frac{1}{2}\}.$$

$$(S_{0,m}^*) \quad p \in (S_{0,m}) \text{ with } \theta_p \geq 1/2.$$

Corollary 4.14. *Suppose $p \in S_{0,m}$. We can write $N(B'_h)$ as*

$$(4.5) \quad N(B'_h) = \sigma^*N(B_h) + (1 - \theta_p)M(\Theta_p),$$

where we set $M(\Theta_p) = 0$ if Θ_p does not exist. Moreover,

$$(4.6) \quad N(B_h)^2 - N(B'_h)^2 = \begin{cases} (1 - \theta_p)^2 \beta_p^-, & \text{if } p \in (S_{0,m}^\theta), \\ 0, & \text{if } p \in (S_{0,m}^*). \end{cases}$$

$$\text{Here } \beta_p^- := \sum_{q \in \sigma^{-1}(p) \cap \Theta_p} \beta_q(\mathcal{F}').$$

Proof. Clear. \square

Proposition 4.15. *Suppose $p \in (S_{1,m})$. Then $N(B'_h) = \sigma^*N(B_h)$ unless there is a potential curve Γ of (B_h, \mathcal{F}) passing through p . Moreover, if $N(B'_h) \neq \sigma^*N(B_h)$, then there is a unique maximal (B'_h, \mathcal{F}') -chain, say Θ_p (with multiplicity θ_p), which is not the pullback of a maximal (B_h, \mathcal{F}) -chain.*

Proof. By Lemma 3.15, (3). \square

For convenience, we divide $S_{1,m}$ -singularities into the following 3 cases:

$$(S_{1,m}^{I,e}) \quad p \text{ is contained in a first potential curve } \Gamma \text{ with } l(p) = 1 \text{ and } e = -\Gamma^2.$$

$$(S_{1,m}^{II,\theta}) \quad p \text{ is contained in a second potential curve } \Gamma \text{ with } l(p) = 1 \text{ and } \theta := \theta_p \in \{0, \frac{1}{2}\}, \text{ where } \theta_p \text{ is as in Definition 3.14.}$$

$$(S_{1,m}^*) \quad \text{Other cases in } (S_{1,m}).$$

4.3.1. *Zariski index $\alpha(p)$.*

Definition 4.16. *We define the Zariski index $\alpha(p)$ of p w.r.t. σ as*

$$(4.7) \quad \alpha(p) := \begin{cases} \frac{3e-1}{4e(e+1)} (> 0), & \text{if } p \in (S_{1,m}^{I,e}), \\ (1 - \theta_p)^2 \beta_p^-, & \text{if } p \in (S_{1,m}^{II,\theta}) \cup (S_{0,m}^\theta), \\ 0, & \text{if } p \in (S_{1,m}^*) \cup (S_{0,m}^*), \end{cases}$$

$$\text{where } \beta_p^- := \sum_{q \in \sigma^{-1}(p) \cap \Theta_p} \beta_q(\mathcal{F}').$$

Proposition 4.17. *For $p \in (S_{1,m})$, we have $0 \leq \alpha(p) = N(B_h)^2 - N(B'_h)^2 < 1$.*

Proof. By Corollary 3.17 and Corollary 4.14. \square

4.3.2. *Other indexes:* $s_0(p), s(p)$.

Definition 4.18. For any $p \in (S_{l,m})$, we define

$$(4.8) \quad s_0(p) := \#\{\text{blow-up point in } \sigma_i, \text{ which is a saddle-node.}\},$$

$$(4.9) \quad s(p) := \sum_{q \in \sigma^{-1}(p)} \beta_q(B'_v).$$

4.4. **Canonical resolution.** Recall the double cover over a foliated surface

$$\pi : (Y, \mathcal{G}) \longrightarrow (X, \mathcal{F}),$$

where we assume \mathcal{F} is a minimal foliation with $\nu(\mathcal{F}) \geq 0$ and the (reduced) ramification divisor is $B = B_h + B_v$. Consider the following canonical resolution

$$(4.10) \quad \begin{array}{ccccccc} (\tilde{Y}, \tilde{\mathcal{G}}) & \xlongequal{\quad} & (Y_s, \mathcal{G}_s) & \longrightarrow & (Y_{s-1}, \mathcal{G}_{s-1}) & \longrightarrow & \cdots \longrightarrow & (Y, \mathcal{G}) \\ & & \downarrow \tilde{\pi} & & \downarrow \pi_s & & \downarrow \pi_{s-1} & & \downarrow \pi \\ (\tilde{X}, \tilde{\mathcal{F}}) & \xlongequal{\quad} & (X_s, \mathcal{F}_s) & \xrightarrow{\sigma_s} & (X_{s-1}, \mathcal{F}_{s-1}) & \xrightarrow{\sigma_{s-1}} & \cdots \xrightarrow{\sigma_1} & (X, \mathcal{F}) \end{array}$$

where

- (1) q_i (resp. E_i) denotes the blow-up point (resp. exceptional curve) of σ_i ,
- (2) $\sigma = \sigma_1 \circ \cdots \circ \sigma_s$ is a minimal resolution such that
 - i) the branch locus \tilde{B} of $\tilde{\pi}$ is smooth and reduced,
 - ii) $\text{tang}(\tilde{\mathcal{F}}, \tilde{B}_h) = 0$.

Using the notations of $S_{l,m}$ -singularities of (B_h, \mathcal{F}) , the process of σ can be divided into the following 3 steps:

- (Step 1) During $\bar{\sigma} := \sigma_1 \circ \cdots \circ \sigma_t$, we blow up the set of $S_{l,m}$ -singularities over (B_h, \mathcal{F}) . In fact, $\bar{\sigma}$ is a canonical resolution of $(X, \mathcal{F}, \frac{1}{2}B_h)$ (see Sec. 3.3).
- (Step 2) For $t+1 \leq i \leq s$, σ_i is a blow-up over q_i and either $q_i \in \bar{B}_v \cap \bar{B}_h$ or $q_i \in \bar{B}_v \setminus \bar{B}_h$ with $m_{q_i}(\bar{B}_v) = 2$.

In fact, to compute Chern numbers of the double foliation \mathcal{F} , we just need to consider the step 1. More precisely, we have the following theorem.

Theorem 4.19. Under the notations above, we have

$$(4.11) \quad \begin{cases} c_1^2(\mathcal{G}) = 2c_1^2(\mathcal{F}) + \frac{3}{2}K_{\mathcal{F}}B_h + 2N^2 - 2N(B_h)^2 + \sum_{p \in S_{l,m}} T_1(p), \\ c_2(\mathcal{G}) = 2c_2(\mathcal{F}) - 2N^2 + 2N(B_h)^2 - \frac{3}{2}s(B_v) + \sum_{p \in S_{l,m}} T_2(p) - \ell(\tilde{\mathcal{G}}), \\ \chi(\mathcal{G}) = 2\chi(\mathcal{F}) + \frac{1}{8}K_{\mathcal{F}}B_h - \frac{1}{8}s(B_v) + \sum_{p \in S_{l,m}} \frac{1}{12}(T_1(p) + T_2(p)) - \frac{1}{12}\ell(\tilde{\mathcal{G}}). \end{cases}$$

Here

$$T_1(p) = (1 - l(p)) \frac{3m(p) - 4}{2} + 2\alpha(p),$$

$$T_2(p) = 2(1 - l(p))^2 - 2\alpha(p) + 2s_0(p) - \frac{3}{2}s(p).$$

Proof. Note that

$$\begin{aligned} \sum_{p \in (S_{l,m})} T_1(p) &= \left[\frac{3}{2} K_{\mathcal{F}} B_h + 2N^2 - 2N(B_h)^2 \right] - \left[\frac{3}{2} K_{\bar{\mathcal{F}}} \bar{B}_h + 2\bar{N}^2 - 2N(\bar{B}_h)^2 \right], \\ \sum_{p \in (S_{l,m})} T_2(p) &= \left[-2\bar{N}^2 + 2N(\bar{B}_h)^2 - \frac{3}{2} s(\bar{B}_v) + 2\ell(\mathcal{F}_i) \right] - \left[-2N^2 + 2N(B_h)^2 - \frac{3}{2} s(B_v) + 2\ell(\mathcal{F}) \right]. \end{aligned}$$

It suffices to prove

$$\begin{aligned} c_1^2(\mathcal{G}) &= 2c_1^2(\mathcal{F}) + \frac{3}{2} K_{\bar{\mathcal{F}}} \bar{B}_h + 2\bar{N}^2 - 2N(\bar{B}_h)^2, \\ c_2(\mathcal{G}) &= 2c_2(\mathcal{F}) - 2\bar{N}^2 + 2N(\bar{B}_h)^2 - \frac{3}{2} s(\bar{B}_v) + 2\ell(\bar{\mathcal{F}}) - \ell(\tilde{\mathcal{G}}). \end{aligned}$$

Let $T_1(\bar{\mathcal{F}})$ (resp. $T_2(\bar{\mathcal{F}})$) denote the right side of the first (resp. second) equation above. Recall for $s+1 \leq i \leq n$, either $q_i \in \bar{B}_v \cap \bar{B}_h$ or $q_i \in \bar{B}_v \setminus \bar{B}_h$ with $m_{q_i}(\bar{B}_v) = 2$.

(1) If $q_i \in \bar{B}_v \cap \bar{B}_h$, then $l(q_i) = 0$ and

$$\begin{cases} K_{\mathcal{F}_i} B_{i,h} = K_{\mathcal{F}_{i-1}} B_{i-1,h} + 1, & N_i^2 = N_{i-1}^2 - 1, & N(B_{i,h})^2 = N(B_{i-1,h})^2 - \frac{1}{4}, \\ s(B_{i,v}) = s(B_{i-1,v}) + 1, & \ell(\mathcal{F}_i) = \ell(\mathcal{F}_{i-1}). \end{cases}$$

(2) If $q_i \in \bar{B}_v \setminus \bar{B}_h$ with $m_{q_i}(\bar{B}_v) = 2$ and $\lambda_p \neq 0$, then $l(q_i) = 1$ and

$$\begin{cases} K_{\mathcal{F}_i} B_{i,h} = K_{\mathcal{F}_{i-1}} B_{i-1,h}, & N_i^2 = N_{i-1}^2, & N(B_{i,h})^2 = N(B_{i-1,h})^2, \\ s(B_{i,v}) = s(B_{i-1,v}), & \ell(\mathcal{F}_i) = \ell(\mathcal{F}_{i-1}). \end{cases}$$

(3) If $q_i \in \bar{B}_v \setminus \bar{B}_h$ with $m_{q_i}(\bar{B}_v) = 2$ and $\lambda_p = 0$, then $l(q_i) = 1$ and

$$\begin{cases} K_{\mathcal{F}_i} B_{i,h} = K_{\mathcal{F}_{i-1}} B_{i-1,h}, & N_i^2 = N_{i-1}^2, & N(B_{i,h})^2 = N(B_{i-1,h})^2, \\ s(B_{i,v}) = s(B_{i-1,v}) + \frac{4}{3}, & \ell(\mathcal{F}_i) = \ell(\mathcal{F}_{i-1}) + 1. \end{cases}$$

(1), (2) and (3) all imply $T_1(\mathcal{F}_i) = T_1(\mathcal{F}_{i-1}), T_2(\mathcal{F}_i) = T_2(\mathcal{F}_{i-1})$. So

$$T_1(\bar{\mathcal{F}}) = T_1(\tilde{\mathcal{F}}), \quad T_2(\bar{\mathcal{F}}) = T_2(\tilde{\mathcal{F}}).$$

By Theorem 4.5, we have seen $c_1^2(\mathcal{G}) = T_1(\tilde{\mathcal{F}}), c_2(\mathcal{G}) = T_2(\tilde{\mathcal{F}})$. Thus

$$c_1^2(\mathcal{G}) = T_1(\bar{\mathcal{F}}), \quad c_2(\mathcal{G}) = T_2(\bar{\mathcal{F}}).$$

Finally, the computation of $\chi(\mathcal{G})$ is from the Noether formula $12\chi(\mathcal{G}) = c_1^2(\mathcal{G}) + c_2(\mathcal{G})$. \square

4.5. Computation of initial invariants. In this section, we assume \mathcal{F} is reduced with $\nu(\mathcal{F}) \geq 0$ and there is no (B_h, \mathcal{F}) -exceptional curve of type H-J over X . We will discuss the positivity of the following two invariants:

$$(4.12) \quad \begin{aligned} T_1(B, \mathcal{F}) &:= 2c_1^2(\mathcal{F}) + \frac{3}{2} K_{\mathcal{F}} B_h + 2N^2 - 2N(B_h)^2, \\ T_2(B, \mathcal{F}) &:= 2c_2(\mathcal{F}) - 2N^2 + 2N(B_h)^2 - \frac{3}{2} s(B_v) + 2\ell(\mathcal{F}). \end{aligned}$$

Proposition 4.20. $T_1(B, \mathcal{F}) \geq 0$.

Proof. By Corollary 3.10,

$$T_1(B, \mathcal{F}) \geq \frac{3}{2} N B_h + 2N^2 - 2N(B_h)^2 \geq 0.$$

\square

Proposition 4.21. $T_2(B, \mathcal{F}) \geq 0$, if for any saddle-node $q \in B_v \setminus B_h$, $m_q(B_v) = 1$.

Proof. We write $s(B_v) = s'(B_v) + s''(B_v) + s'''(B_v)$, where

$$s'(B) = \sum_{q \notin N} \beta_q(B_v), \quad s''(B) = \sum_{q \in N \setminus N(B_h)} \beta_q(B_v), \quad s'''(B) = \sum_{q \in N(B_h)} \beta_q(B_v).$$

By assumption and the definition of $\beta_q(B_v)$, we see $\beta_q(B_v) \leq \beta_q(\mathcal{F})$. So

$$(I) \quad 2c_2(\mathcal{F}) + 2\ell(\mathcal{F}) - \frac{3}{2}s'(B) \geq 2 \sum_{q \notin N} \beta_q(\mathcal{F}) - \frac{3}{2} \sum_{q \notin N} \beta_q(\mathcal{F}) = \sum_{q \notin N} \frac{1}{2} \beta_q(\mathcal{F}) \geq 0,$$

and

$$(II) \quad \sum_{q \in N \setminus N(B_h)} 2\beta_q(\mathcal{F}) - \frac{3}{2}s''(B) \geq \sum_{q \in N \setminus N(B_h)} \frac{1}{2} \beta_q(\mathcal{F}) \geq 0.$$

Recall $N(B_h)^2 = \sum_{i=1}^s (1-\theta_i)^2 M(\Theta_i)^2$, where $\Theta_1, \dots, \Theta_s$ are maximal $(\frac{1}{2}B_h, \mathcal{F})$ -chains with $\theta_i = B_h \Theta_i / 2$. Next we will show

$$(III) \quad 2 \sum_{q \in \Theta_i} \beta_q(\mathcal{F}) - 2(1-\theta_i)^2 \sum_{q \in \Theta_i} \beta_q(\mathcal{F}) - \frac{3}{2} \sum_{q \in \Theta_i} \beta_q(B_v) \geq 0.$$

This is true, just by Lemma 4.7:

$$\sum_{q \in \Theta_i} \beta_q(B_v) \begin{cases} = 0, & \text{if } \theta_i = 0; \\ \leq \sum_{q \in \Theta_i} \beta_q(\mathcal{F}), & \text{if } \theta_i = \frac{1}{2}. \end{cases}$$

Now from (I), (II) and (III), $T_2(B, \mathcal{F}) \geq 0$ is clear. \square

Proposition 4.22. *If $2c_1^2(\mathcal{F}) \geq c_2(\mathcal{F})$, $\ell(\mathcal{F}) = 0$ and $B_h \cap N(B_h) \cap \text{Sing}\mathcal{F} = \emptyset$, then $2T_1(B, \mathcal{F}) \geq T_2(B, \mathcal{F})$.*

Proof. In this case,

$$\begin{aligned} & 2T_1(B, \mathcal{F}) - T_2(B, \mathcal{F}) \\ &= (2c_1^2(\mathcal{F}) - c_2(\mathcal{F})) + 3K_{\mathcal{F}}B_h + 6N^2 - 6N(B_h)^2 + \frac{3}{2}s(B) \\ &\geq 3NB_h + 6N^2 - 6N(B_h)^2 + \frac{3}{2} \cdot 2N(B_h)B_h, \quad (\text{Proposition 4.9}) \\ &\geq 3[(N + N(B_h))B_h + 2N^2 - 2N(B_h)^2] \\ &\geq 0 \quad (\text{Corollary 3.9}). \end{aligned}$$

\square

4.6. Computation of local invariants. In this section, we assume \mathcal{F} is reduced and there is no (B_h, \mathcal{F}) -exceptional curve of type H-J over X . We will compute the contribution of the S^j -singularity $p \in B_h$ to $\alpha(p)$, $s_0(p)$, $s(p)$, $T_1(p)$, $T_2(p)$, where

$$(4.13) \quad \begin{aligned} T_1(p) &= (1-l(p)) \frac{3m(p)-4}{2} + 2\alpha(p), \\ T_2(p) &= 2(1-l(p))^2 - 2\alpha(p) + 2s_0(p) - \frac{3}{2}s(p). \end{aligned}$$

Definition 4.23. *We define*

$$(4.14) \quad \Delta(t_p) := \text{tang}(\mathcal{F}, B_h) - \text{tang}(\mathcal{F}', B_h') = m_1(m_1 - 1 + l(p)) + \sum_{i=2}^s m_i^2.$$

It is clear that $\Delta(t_p) \leq t_p$. If $\Delta(t_p) = t_p$, then there is no more $S_{l,m}$ -singularity after p . So the blow-up process of $S_{l,m}$ -singularities will continue until the equation $\Delta(t_p) = t_p$ holds.

Lemma 4.24. *If $p \in S_{l,m}$ and p is not a saddle-node, then $2T_1(p) \geq T_2(p)$.*

Proof. In this case, $s_0(p) = 0$. So

$$\begin{aligned} 2T_1(p) - T_2(p) &= 2 \left((1 - l(p)) \frac{3m - 4}{2} + 2\alpha(p) \right) - \left(2(1 - l(p))^2 - 2\alpha(p) - \frac{3}{2}s(p) \right) \\ &= (1 - l(p))(3m - 6 + 2l(p)) + 6l(p) + \frac{3}{2}s(p). \end{aligned}$$

If $l(p) = 1$ or $l(p) = 0$ and $m \geq 2$, $2T_1(p) \geq T_2(p)$ is clear. Next we assume $l(p) = 0$ and $m = 1$. By Lemma 4.28,

$$T_1(p) = \frac{3t_p - 1}{2(t_p + 1)}, \quad T_2(p) = \frac{3 \left(t_p - 2 \left\lfloor \frac{t_p}{2} \right\rfloor \right) + 1}{2(t_p + 1)}.$$

So

$$2T_1(p) - T_2(p) = \frac{3(t_p - 1) + 6 \left\lfloor \frac{t_p}{2} \right\rfloor}{2(t_p + 1)} \geq 0,$$

where $t_p \geq 1$. Then we are done. \square

4.6.1. $S_{1,m}$.

Proposition 4.25. *Suppose $p \in S_m^1$, Then $l(p) = 1$ and $0 \leq \alpha(p) < 1$. Hence*

$$(4.15) \quad T_1(p) = 2\alpha(p), \quad T_2(p) = -2\alpha(p) + 2s_0(p) - \frac{3}{2}s(p).$$

Here

- (1) $T_1(p) \geq 0$ and $T_1(p) = 0$ unless p belongs to (e1) or (e2) or (e3).
- (2) If $\lambda_p = 0$, $T_2(p) \geq \frac{1}{2}s_0(p) \geq \frac{1}{2}$, and if $\lambda_p \neq 0$, $0 \geq T_2(p) > -\frac{7}{2}\beta_p(\mathcal{F}) \geq -\frac{7}{2}$.

Proof. Clear. \square

4.6.2. $S_{0,m}$.

Proposition 4.26. *Suppose $p \in S_{0,m}$. Then $l(p) = 0$, $0 \leq \alpha(p) < 1$ and*

$$(4.16) \quad T_1(p) = \frac{3}{2}m(p) - 2 + 2\alpha(p) \geq 0, \quad T_2(p) = 2 - 2\alpha(p) - \frac{3}{2}s(p) \geq 0.$$

Proof. The proof of $T_2(p) \geq 0$ is similar to the proof of $T_2(B, \mathcal{F}) \geq 0$, see Proposition 4.21. Next we will show $T_1(p) > 0$. If $m(p) \geq 2$, then

$$T_1(p) \geq \frac{3}{2} \cdot 2 - 2 = 1 > 0.$$

If $m = 1$, by Lemma 4.28,

$$T_1(p) = \frac{3t_p - 1}{2(t_p + 1)} > 0,$$

where $t_p \geq 1$. So we are done. \square

Definition 4.27. *Suppose $p \in B_h$ is a regular point of \mathcal{F} , i.e., $l(p) = 0$. Let F denote the separatrix through p . If F (resp. B_h) is locally defined by $g = 0$ (resp. $f = 0$), then we define*

$$\eta_p := I_p < f, g >.$$

Note that the definition of η_p does depend on the choice of f and g .

Lemma 4.28 ($S_{0,1}$). *Suppose $p \in S_{0,1}$. Then $\Delta(t_p) = t_p \geq 1$, $r = \eta_p = t_p + 1$ and*

$$\beta_p^- = \frac{t_p}{t_p + 1}, \quad \theta_p = 0, \quad s(p) = \frac{2 \left\lfloor \frac{t_p}{2} \right\rfloor - t_p + 1}{t_p + 1}.$$

Moreover, $\alpha(p) = \frac{t_p}{t_p + 1}$ and

$$T_1(p) = \frac{3t_p - 1}{2(t_p + 1)}, \quad T_2(p) = \frac{3 \left(t_p - 2 \left\lfloor \frac{t_p}{2} \right\rfloor \right) + 1}{2(t_p + 1)}.$$

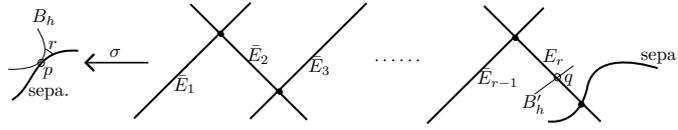


FIGURE 1. $p \in S_m^0$, $m = 1$, $E_r^2 = -1$, $\bar{E}_i^2 = -2$ ($i < r$).

Proof. In this case, $r = t_p + 1$, $q_1 = p$ and $q_i = E_{i-1} \cap B_{i-1,h}$ for $i = 2, \dots, r$, see Figure 1. It is clear that $\Theta_p = \bar{E}_1 + \dots + \bar{E}_{r-1}$ is a maximal $(\frac{1}{2}B_h, \mathcal{F}')$ -chain with $\theta_p := \frac{1}{2}\Theta_p B_h = 0$. So

$$\beta_p^- = \sum_{i=1}^{r-1} \beta(-\lambda_{p^i}) = \sum_{i=1}^{r-1} \frac{1}{i \cdot (i+1)} = 1 - \frac{1}{r} = \frac{r-1}{r} = \frac{t_p}{t_p + 1},$$

where $p^i = \bar{E}_i \cap \bar{E}_{i+1}$ for $i = 1, \dots, r-1$. Let p^r be another singularity of \mathcal{F}' over E_r . Next we compute $s(p)$.

By Lemma 4.7, $\beta_{p^i}(B'_v) = 0$ for $i = 1, \dots, r-1$. So it suffices to compute $\beta_{p^r}(B'_v)$.

i) If $p \in B_v$, then $E_r \not\subset B'_v$. So $p^r \in B'_v$ with $m_{p^r}(B'_v) = 1$ and

$$\text{CS}(\mathcal{F}', B'_v, p^r) = -r.$$

Thus $\beta_{p^r}(B'_v) = 0$ if r is even and $\beta_{p^r}(B'_v) = \frac{1}{r}$ if r is odd.

ii) If $p \notin B_v$ and r is even, then $E_r \not\subset B'_v$, which implies $p^r \notin B'_v$. So $\beta_{p^r}(B'_v) = 0$.

iii) If $p \notin B_v$ and r is odd, then $E_r \subset B'_v$, which implies $p^r \in B'_v$ with $m_{p^r}(B'_v) = 1$ and

$$\text{CS}(\mathcal{F}', B'_v, p^r) = -\frac{1}{r}.$$

So $\beta_{p^r}(B'_v) = \frac{1}{r}$.

Therefore,

$$s(p) = \beta_{p^r}(B'_v) = \left(2 \left\lfloor \frac{r-1}{2} \right\rfloor + 1 - (r-1) \right) \frac{1}{r} = \frac{2 \left\lfloor \frac{t_p}{2} \right\rfloor - t_p + 1}{t_p + 1}.$$

□

For any $p \in S_{0,m}^*$, it is clear that $m \geq 2$. Next we consider a special case $S_{0,m}^{**} := \{p \in S_{0,m}^* \mid m(p) = \eta_p\}$.

Lemma 4.29 ($S_{0,m}^{**}$). *Suppose $p \in S_{0,m}^{**}$. Then $\Delta(t_p) = m(m-1)$, $r = 1$ and*

$$\beta_p^- = 0, \quad s(p) = m - 2 \left\lfloor \frac{m}{2} \right\rfloor.$$

Moreover, $\alpha(p) = 0$ and

$$T_1(p) = \frac{3}{2}m - 2, \quad T_2(p) = 2 - \frac{3}{2}m + 3 \left\lfloor \frac{m}{2} \right\rfloor.$$

Proof. $r = 1$ and $\beta_p^- = 0$ are clear. Next we compute $s(p)$.

- (i) Suppose m is even. If $p \in B_v$, then $m_p(B) = m+1$ is odd and $E_1 \subset B'_v$. So $m_q(B'_v) = 2$ and $\lambda_q = -1$ for $q = E_1 \cap \text{Sing}\mathcal{F}$, which implies $\beta_q(B'_v) = 0$. If $p \notin B_v$, then $E_1 \not\subset B'_v$. So $q \notin B'_v$ and $\beta_q(B'_v) = 0$. Thus in this case that m is even, $s(p) = \beta_q(B'_v) = 0$.
- (ii) Suppose m is odd. If $p \in B_v$ (resp. $p \notin B_v$), then $E_1 \not\subset B'_v$ (resp. $E_1 \subset B'_v$). They both imply $m_q(B'_v) = 1$ and $\lambda_q = -1$. So $\beta_q(B'_v) = 1$. Thus in this case that m is odd, $s(p) = \beta_q(B'_v) = 1$.

□

4.7. Slope inequality. Using the notations in Section 4.4, by blowing up all $S_{1,m}^I$ -singularities, it suffices to assume (X, \mathcal{F}) is reduced satisfying

- there is no (B_h, \mathcal{F}) -exceptional curves of type H-J,
- there is no $S_{1,m}^I$ -singularities,
- $\ell(\mathcal{F}) = 0$.

So $N(B_h) \cap B_h \cap \text{Sing}\mathcal{F} = \emptyset$. We set

$$F_\lambda(\cdot) := \frac{12-\lambda}{12}T_1(\cdot) - \frac{\lambda}{12}T_2(\cdot).$$

By Proposition 4.22 and Lemma 4.24, we have the following claims.

Lemma 4.30. *Let λ be a positive rational number with $\lambda \leq 4$.*

- (i) *If $c_1^2(\mathcal{F}) \geq \lambda\chi(\mathcal{F})$, then*

$$F_{B,\lambda}(\mathcal{F}) = \frac{12-\lambda}{12}T_1(B, \mathcal{F}) - \frac{\lambda}{12}T_2(B, \mathcal{F}) \geq 0.$$

- (ii) *For any $p \in S_{l,m}$, if p is not a saddle-node, then*

$$F_\lambda(p) = \frac{12-\lambda}{12}T_1(p) - \frac{\lambda}{12}T_2(p) \geq 0.$$

Thus

$$\begin{aligned} c_1^2(\mathcal{G}) - \lambda\chi(\mathcal{G}) &= 2(c_1^2(\mathcal{F}) - \lambda\chi(\mathcal{F})) + \frac{12-\lambda}{8}K_{\mathcal{F}}B_h + 2N^2 - 2N(B_h)^2 + \frac{1}{8}s(B_v) \\ &\quad + \sum_{p \in S_{l,m}} \frac{(12-\lambda)T_1(p) - \lambda T_2(p)}{12} + \frac{\lambda}{12}\ell(\tilde{\mathcal{G}}) \\ &= F_\lambda(B, \mathcal{F}) + \sum_{p \in S_{l,m}} F_\lambda(p) + \frac{\lambda}{12}\ell(\tilde{\mathcal{G}}). \end{aligned}$$

Theorem 4.31. *Under the notations above. If $c_1^2(\mathcal{G}) \geq 4\chi(\mathcal{G})$ and the branch locus B of π misses the saddle-nodes, then $c_1^2(\mathcal{G}) \geq 4\chi(\mathcal{G})$.*

Proof. Under the assumption above, by Lemma 4.30,

$$F_4(B, \mathcal{F}) \geq 0, \quad F_4(p) \geq 0, \text{ for any } p \in S_{l,m}.$$

So

$$c_1^2(\mathcal{G}) - 4\chi(\mathcal{G}) = F_4(B, \mathcal{F}) + \sum_{p \in S_{l,m}} F_4(p) + \frac{1}{3}\ell(\tilde{\mathcal{G}}) \geq 0.$$

□

Finally, we consider the case that (X, \mathcal{F}) is a relatively minimal elliptic fibration $f : X \rightarrow C$. In this case, $c_1^2(\mathcal{F}) = 0$ but $\chi(\mathcal{F}) > 0$. Since $c_1^2(\mathcal{F}) = \kappa(f)$ and $\chi(\mathcal{F}) = \lambda(f)$ are modular invariants of f , it suffices to assume f is semi-stable. So by ([4], p.22), we have

$$K_{\mathcal{F}} = f^*[(f_{*1}\mathcal{O}_X)^\vee] = K_f$$

where $\deg(f_{*1}\mathcal{O}_X)^\vee = \chi_f = \chi(\mathcal{F})$. So $K_{\mathcal{F}} \cdot B_h = \chi_f \cdot (B_h \cdot F)$, where F is the general fibre of f . Note that, in this case, $\ell(\mathcal{F}) = \ell(\mathcal{G}) = 0$ and $K_{\mathcal{F}} = K_f$ is nef, which implies $N = N(B_h) = 0$. So by Lemma 4.30,

$$c_1^2(\mathcal{G}) - 4\chi(\mathcal{G}) = -8\chi(\mathcal{F}) + K_{\mathcal{F}}B_h + \sum_{p \in S_{l,m}} \frac{2T_1(p) - T_2(p)}{3} \geq -8\chi(\mathcal{F}) + K_{\mathcal{F}}B_h.$$

In fact, (Y, \mathcal{G}) is a fibration f' of genus g , where $g = g(F')$ for the general fibre F' of f' . (We always call (Y, \mathcal{G}) is a bielliptic fibration.) Consider the fibers of the two fibrations, and we get a double cover of an elliptic curve with the ramification divisor B . Here we can easily see $\deg B = B_h \cdot F$. So by the Hurwitz's Theorem, we have

$$B_h \cdot F = \deg B = 2g - 2 - 2 \cdot (2 \cdot 1 - 2) = 2g - 2.$$

So

$$K_{\mathcal{F}}B_h = \chi_f(2g - 2).$$

Since f is semi-stable, which implies $\chi(\mathcal{G}) = \chi_f$, we have

$$c_1^2(\mathcal{G}) - 4\chi(\mathcal{G}) \geq -8\chi(\mathcal{F}) + (2g - 2)\chi(\mathcal{F}) = (2g - 10)\chi(\mathcal{F}).$$

In particular, if $g \geq 5$, then $c_1^2(\mathcal{G}) - 4\chi(\mathcal{G}) \geq 0$, or say $\lambda(\mathcal{G}) \geq 4$.

Proposition 4.32. *If (Y, \mathcal{G}) is a bielliptic fibration with $g \geq 5$, then $\lambda(\mathcal{G}) \geq 4$.*

Note that this result above have proved in [2], where the author considered the slope of bielliptic fibrations, in the sense of relative invariants.

5. EXAMPLE OF FOLIATIONS WITH SLOPE $\frac{12}{7}$

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, let $f : X \rightarrow \mathbb{P}^1$ be one of the rulings with a fiber F_0 , and let C_0 be a section. Choose a proper coordinate (x, y) nearby $p = (0, 0) \in C_0 \cap F_0$ such that C_0 (resp., F_0) is defined by $y = 0$ (resp., $x = 0$).

Example 5.1. *Let \mathcal{F} be a foliation on X locally generated by*

$$\omega = x^2 dy - y dx.$$

Let $\pi : (Y, \mathcal{G}) \rightarrow (X, \mathcal{F})$ be the double cover locally defined by

$$z^2 = y(y + x^{2k}(1 + y^2)), \quad (k \geq 1).$$

Then we have

$$c_1^2(\mathcal{G}) = 2k, \quad c_2(\mathcal{G}) = 12k, \quad \chi(\mathcal{G}) = \frac{7k}{6}, \quad \lambda(\mathcal{G}) = \frac{12}{7}.$$

APPENDIX A. CLASSIFICATION OF $S_{0,m}$ -SINGULARITIES FOR $m \leq 3$

A.1. $S_{0,1}$. We denote by A_0^η the set of points $p \in S_{0,1}$ with $\eta_p = \eta$. By Lemma 4.28, we see $\eta \geq 2$ and we have the following table.

TABLE 2. $S_{0,1}$.

	$\Delta(t_p)$	$\alpha(p)$	$s(p)$	$T_1(p)$	$T_2(p)$	$t_p = \Delta(t_p)?$
A_0^η	$\eta - 1$	$\frac{\eta - 1}{\eta}$	$\frac{2\{\eta/2\}}{\eta}$	$\frac{3\eta - 4}{2\eta}$	$\frac{2 - 3\{\eta/2\}}{\eta}$	Yes

Here we set $\{x\} := x - [x]$.

Remark A.1. *It is clear that $\Delta(t_p) \leq t_p$. That the equation $\Delta(t_p) = t_p$ holds means there is no more $S_{l,m}$ -singularities after p .*

A.2. $S_{0,2}$. In this case, we divide it into the following 4 cases:

- A_1^η . p is a node of B_h with $\eta_p = \eta$.
- $A_n^{\eta,I}$. p is a singularity of B_h of type A_n ($n \geq 2$) with $\eta = \eta_p \leq n - 1$.
- A_{2k}^{II} . p is a singularity of B_h of type A_{2k} ($k \geq 1$) with $\eta = 2k + 1$.
- $A_{2k-1}^{\eta,II}$. p is a singularity of B_h of type A_{2k-1} ($k \geq 2$) with $\eta \leq 2k$.

TABLE 3. $S_{0,2}$.

	$\Delta(t_p)$	$\alpha(p)$	$s(p)$	$T_1(p)$	$T_2(p)$	$t_p = \Delta(t_p)?$
A_1^η	η	$\frac{\eta - 2}{4\eta - 4}$	$\frac{\eta - 2 + 2\{\eta/2\}}{\eta - 1}$	$3 - \frac{4}{\eta}$	$\frac{2 - 3\{\eta/2\}}{\eta - 1}$	Yes
$A_n^{\eta,I}$	$2\eta - 2$	$1 - \frac{2}{\eta}$	0	$3 - \frac{4}{\eta}$	$\frac{4}{\eta}$	No
A_{2k}^{II}	$4k$	$\frac{2k - 1}{2k + 1}$	$\frac{2}{2k + 1}$	$\frac{6k - 1}{2k + 1}$	$\frac{1}{2k + 1}$	Yes
$A_{4k-1}^{\eta,II}$	$\eta + 4k - 2$	$\frac{2k - 1}{2k}$	$\frac{2\{\eta/2\}}{\eta - 2k}$	$\frac{3k - 1}{k}$	$\frac{1}{k} - \frac{3\{\eta/2\}}{\eta - 2k}$	
$A_{4k+1}^{\eta,II}$	$p \notin B_v$	$\eta + 4k$	$\frac{2k}{2k + 1}$	$\frac{1}{2k+1} - \frac{1-2\{\eta/2\}}{\eta-2k-1}$	$\frac{6k+1}{4k+2} + \frac{3-6\{\eta/2\}}{2(\eta-2k-1)}$	
	$p \in B_v$		$\frac{1}{2k+1}$	$\frac{1}{4k+2}$		

Proposition A.2. *Suppose $p \in A_n^{\eta,I}$. Then $\eta = 2r$ for some $r \geq 2$.*

- (i) *If $n = 2k - 1$, then after p , there are $k - r$ $S_{0,2}^{**}$ -singularities.*
- (ii) *If $n = 2k$, then after p , there are $k - r - 1$ $S_{0,2}^{**}$ -singularities and one A_0^2 -singularity.*

A.3. $S_{0,3}$. In this case, we divide it into the following cases:

- (1) p is a singularity of B_h of type D_n ($n \geq 4$).
 - $D_n^{\eta,I}$. the separatrix through p is not tangent to the component of B_h of type A_{n-3} at p .
 - $D_n^{\eta,II}$. the separatrix through p is tangent to the component of B_h of type A_{n-3} at p with $\eta \leq n - 3$.

- D_{2k+3}^{III} . the separatrix through p is tangent to the component of B_h of type A_{n-3} at p with $\eta = 2k + 2$.
 $D_{2k+2}^{\eta, III}$. the separatrix through p is tangent to the component of B_h of type A_{n-3} at p with $\eta \geq 2k + 1$.
 (2) p is a singularity of B_h of type E_6 .
 E_6^I . The separatrix through p is not tangent to B_h at p with $\eta = 3$.
 E_6^{II} . The separatrix through p is tangent to B_h at p with $\eta = 4$.
 (3) p is a singularity of B_h of type E_7 .
 E_7^I . The separatrix through p is not tangent to B_h at p with $\eta = 3$.
 $E_7^{\eta, II}$. The separatrix through p is tangent to B_h at p with $\eta \geq 5$.
 (4) p is a singularity of B_h of type E_8 .
 E_8^I . The separatrix through p is not tangent to B_h at p with $\eta = 3$.
 E_8^{II} . The separatrix through p is tangent to B_h at p with $\eta = 5$.

 TABLE 4. $S_{0,3}$.

	$\Delta(t_p)$	$\alpha(p)$	$s(p)$	$T_1(p)$	$T_2(p)$	$t_p = \Delta(t_p)?$
$D_n^{\eta, I}$	$\eta + 3$	0	$\frac{2\{\eta/2\}}{\eta - 2}$	$\frac{5}{2}$	$2 - \frac{3\{\eta/2\}}{\eta - 2}$	No
$D_n^{\eta, II}$	2η	$\frac{\eta - 3}{4\eta - 4}$	1	$\frac{3\eta - 4}{\eta - 1}$	$\frac{1}{\eta - 1}$	No
D_{2k+3}^{III}	$4k + 4$	$\frac{2k - 1}{4(2k + 1)}$	$\frac{2k - 1}{2k + 1}$	$\frac{6k + 2}{2k + 1}$	$\frac{4}{2k + 1}$	Yes
$D_{4k+2}^{\eta, III}$	$\eta + 4k + 1$	$\frac{2k - 1}{8k}$	$1 - \frac{1 - 2\{\eta/2\}}{\eta - 2k - 1}$	$3 - \frac{1}{4k}$	$\frac{1}{4k} + \frac{3(1 - 2\{\eta/2\})}{2(\eta - 2k - 1)}$	Yes
$D_{4k}^{\eta, III}$	$\eta + 4k - 1$	$\frac{k - 1}{2(2k - 1)}$	$\frac{2k - 2}{2k - 1} + \frac{2\{\eta/2\}}{\eta - 2k}$	$\frac{6k + 2}{2k + 1}$	$\frac{2}{2k - 1} - \frac{3\{\eta/2\}}{\eta - 2k}$	Yes
E_6^I, E_7^I, E_8^I	6	0	1	$\frac{5}{2}$	$\frac{1}{2}$	No
E_6^{II}	9	$\frac{1}{4}$	0	3	$\frac{3}{2}$	Yes
$E_7^{\eta, II}$	$\eta + 6$	$\frac{1}{3}$	$\frac{2}{3} - \frac{1 - 2\{\eta/2\}}{\eta - 1}$	$\frac{19}{6}$	$\frac{1}{3} + \frac{3(1 - 2\{\eta/2\})}{2(\eta - 1)}$	Yes
E_8^{II}	12	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{33}{10}$	$\frac{3}{10}$	Yes

Proposition A.3. Suppose $p \in D_n^{\eta, I}$.

- (i) If $n = 2k + 2$, then after p , there are $k - 1$ $S_{0,2}^{**}$ -singularities.
 (ii) If $n = 2k + 3$, then after p , there are $k - 1$ $S_{0,2}^{**}$ -singularities and one A_0^2 -singularity.

Proposition A.4. Suppose $p \in D_n^{\eta, II}$. Then $\eta = 2r + 1 \leq n - 3$.

- (i) If $n = 2k + 2$, then after p , there are $k - r$ $S_{0,2}^{**}$ -singularities.
 (ii) If $n = 2k + 3$, then after p , there are $k - r$ $S_{0,2}^{**}$ -singularities and one A_0^2 -singularity.

Proposition A.5. For $p \in E_i^I$ ($i = 6, 7, 8$), p is just one $S_{0,3}^{**}$ -singularity. If $p \in E_6^I$ (resp. E_7^I, E_8^I), then after p , there is a singularity of type A_0^3 (resp. A_1^3, A_2^{II}).

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