ON THE POINCARE PROBLEM FOR RICCATI FOLIATIONS

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Dedicated to the memory of Professor Gang Xiao

Abstract. In this paper, we will give some criteria on the algebraicity of a Riccati foliation.

1. Introduction

A holomorphic foliation on a smooth projective algebraic surface is said to be algebraic if it admits a rational first integral. In [Poi91], Poincaré studied the following problem which can be rephrased in modern terminology.

Question 1.1. *Is it possible to decided if a holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^2$ (alternatively, a rational ruled surface) is algebraic?*

Some research on holomorphic foliations is motivated by this problem (see [CN91], [Per02], [LN02], [Zam97], [Zam00], [Zam06] etc.). Painlevé [Pai74] asked the following question:

Question 1.2. *Can we recognize the genus g of an algebraic foliation from its defining differential equation?*

Lins-Neto [LN02] constructed counter-examples to show that the genus is not an invariant of differential equations. Therefore, one cannot define the genus for non-algebraic foliations.

In this paper we will answer the above questions in the case of Riccati foliations. Let $\mathcal{F}$ be a foliation on an algebraic surface X with a regular ruling map $\varphi : X \to B$. We say $\mathcal{F}$ is a Riccati foliation with respect to φ if $K_{\mathcal{F}}F = 0$ for a general fiber F of φ, i.e., F is transverse to $\mathcal{F}$ ([Bru15, Ch. 4]). Let x (resp., y) be the local coordinate of B (resp., F). A Riccati foliation can always be written locally as

$$\omega = (q_0(x)y^2 + q_1(x)y + q_2(x))dx - p(x)dy, \tag{1.1}$$

where q_i's and p are holomorphic functions. For convenience, we usually rewrite ω as in the following form:

$$\omega = (g_0(x)y^2 + g_1(x)y + g_2(x))dx - dy \tag{1.2}$$

where $g_i(x) := \frac{q_i}{p}$ for $i = 0, 1, 2$.

Up to a birational map, an algebraic Riccati foliation gives a fibration of genus g, i.e., a holomorphic map from a smooth algebraic surface to a smooth curve such that the general fiber is a smooth curve of genus g. Such a fibration is said to be a *Riccati fibration*.

First of all, a Riccati foliation $\mathcal{F}$ with Kodaira dimension $kod(\mathcal{F}) = -\infty$ is algebraic by Miyaoka Theorem [Miy85] (also see [Bru15, Theorem 7.1]). More precisely, such a Riccati fibration is a family of rational curves. We can classify all such Riccati foliations as follows.

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[1] This work is supported by NSFC, Science and Technology Commission of Shanghai Municipality (No. 18dz2271000).

[2] *2020 Mathematics Subject Classification.* 14C21, 14D06, 14H10, 32S65, 37F75

[3] *Key words and phrases.* Riccati foliation, isotrivial fibration, singular fiber.

Theorem 1.3. *A Riccati foliation $\mathcal{F}$ has Kodaira dimension $kod(\mathcal{F}) = -\infty$ if and only if $\mathcal{F}$ has a standard form (see Sec. 2.1) on $\mathbb{P}^1 \times \mathbb{P}^1$ which occurs in one of the following cases by choosing a suitable coordinate:*

(1) $\omega = dy$;
(2) $\omega = \lambda y dx - x dy$ ($\lambda \in \mathbb{Q}^+$ and $\lambda \leq \frac{1}{2}$);
(3) $\omega = (xy^2 + y - \lambda^2(x-1))dx - 2x(x-1)dy$ ($\lambda \in \mathbb{Q}^+$ and $\lambda \leq \frac{1}{2}$);
(4) $\omega = (xy^2 - 2(x-3)y - 3(x-1))dx - 12x(x-1)dy$;
(5) $\omega = (xy^2 - 4(x-3)y - 5(x-1))dx - 24x(x-1)dy$;
(6) $\omega = (xy^2 - 10(x-3)y - 11(x-1))dx - 60x(x-1)dy$;
(7) $\omega = (xy^2 - 10(x-3)y - 119(x-1))dx - 60x(x-1)dy$.

For an algebraic Riccati foliation with $kod(\mathcal{F}) \geq 0$, the corresponding Riccati fibration has a genus $g > 0$. For convenience, in what follows, we assume that such an fibration is *minimal normal-crossing*, i.e., each singular fiber is normal-crossing and each (-1)-curve in these fibers passes through at least 3 intersections. We can figure out the structure of the Riccati fibration firstly.

Theorem 1.4. *Let $f : X \to C$ be a minimal normal-crossing fibration of genus $g > 0$ with singular fibers $F_1, \ldots, F_s$. If f is a Riccati fibration, then f is an isotrivial fibration over $C \cong \mathbb{P}^1$ and occurs in one of the following cases:*

(A_0) $s = 0$ (i.e., f is trivial);
(A_{n-1}) $s = 2$ and $(\gamma_1, \gamma_2, d) = (n, n, n)$ ($n \geq 2$);
(D_{n+2}) $s = 3$ and $(\gamma_1, \gamma_2, \gamma_3, d) = (2, 2, n, 2n)$ ($n \geq 2$);
(E_6) $s = 3$ and $(\gamma_1, \gamma_2, \gamma_3, d) = (2, 3, 3, 12)$;
(E_7) $s = 3$ and $(\gamma_1, \gamma_2, \gamma_3, d) = (2, 3, 4, 24)$;
(E_8) $s = 3$ and $(\gamma_1, \gamma_2, \gamma_3, d) = (2, 3, 5, 60)$

where $\gamma_1 \leq \cdots \leq \gamma_s$ be the orders of periodic topology monodromies of F_i's respectively and d is an integer satisfying $\sum_{i=1}^{s}(1 - 1/\gamma_i) = 2 - 2/d$.

Conversely, each isotrivial fibration $f : X \to C(\cong \mathbb{P}^1)$ of genus $g > 1$ occurring in one of the above cases is a Riccati fibration.

Remark 1.5. Theorem 1.4 can also be rephrased as follows: f is a Riccati fibration iff f can become a trivial fibration after a base change $\pi : \mathbb{P}^1 \to C(\cong \mathbb{P}^1)$ of degree d uniformly ramified over s critical points of f with ramification index $\gamma_1, \ldots, \gamma_s$ respectively. It's well-known that such a uniformly ramified cover over $\mathbb{P}^1$ is given exactly by a finite subgroup of $\mathrm{Aut}(\mathbb{P}^1)$ which also corresponds with one kind of *A*-*D*-*E* surface singularities (see [Xia92, Theoreom A 3.6] for instance). An algebraic Riccati foliation is said to be of *type A_{n-1} (resp., D_n, E_k)* if the corresponding Riccati fibration is of type A_{n-1} (resp., D_n, E_k).

Remark 1.6. In what follows, we take $\gamma_1 = \gamma_2 = \gamma_3 = d = 1$ (if $s = 0$) or $\gamma_3 = 1$ (if $s = 2$) for convenience. The equality $\sum_{i=1}^{s}(1 - 1/\gamma_i) = 2 - 2/d$ still holds.

The genus of the fibration induced by an algebraic Riccati foliation can be determined by the following formula (see Lemma 4.2).

Corollary 1.7. *Let $\mathcal{F}$ be a standard form of an algebraic Riccati foliation w.r.t. a regular ruling map $\varphi : X \to B$ and $f : X \to \mathbb{P}^1$ be the fibration of genus g induced by $\mathcal{F}$. Let $F_1, \cdots, F_l$ be the $\mathcal{F}$-invariant fibers of φ and F' be a general fiber of f. Assume that F_i is of type $I_{\frac{m_i}{n_i}}$ (see Sec. 2.1) where $n_i > 1$ and $\gcd(m_i, n_i) = 1$ for $i = 1, \cdots, l$. We have*

$$\frac{2g-2}{d} = 2g(B) - 2 + \sum_{i=1}^{l}\left(1 - \frac{1}{n_i}\right)$$

where $d := FF'$.

From the above results, one can classify precisely all Riccati fibrations of $g = 1$ as well as their Riccati foliations.

Theorem 1.8. *A Riccati foliation $\mathcal{F}$ with $kod(\mathcal{F}) = 0$ is algebraic iff $\mathcal{F}$ is induced by an isotrivial elliptic fibration $f : X \to C$, up to a suitable coordinate, occurring in one of the following cases:*

(1) *f is the second projection $pr_2 : X = E \times \mathbb{P}^1 \to \mathbb{P}^1$ for some smooth elliptic curve E and hence $\mathcal{F}$ is a Riccati foliation of type A_0 w.r.t $pr_1 : X \to E$;*
(2) *f is an elliptic fibration over $\mathbb{P}^1$ with two singular fibers of nI_0 (see Lemma 2.12) and hence $\mathcal{F}$ is a suspension of the corresponding monodromy $\rho : \pi_1(Alb(X)) \to Aut(\mathbb{P}^1)$ w.r.t. the Albanese morphism $Alb : X \to E$ where E is a smooth elliptic curve (see* [Bru15, Ch. 7, Proposition 6]*);*
(3) *f is one of the following families from the Riccati foliation $\mathcal{F}$ w.r.t. the projection $pr_1 : X = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.*

Type	Riccati foliations	Families	Singular fibers
A_1	$(3x^2+1)ydx - 2(x^3+x+c)dy$	$y^2 = t(x^3+x+c)$	$\mathrm{I}_0^*, \mathrm{I}_0^*$
	$3x^2ydx - 2(x^3+1)dy$	$y^2 = t(x^3+1)$	
A_2	$(2x-1)ydx - 3x(x-1)dy$	$y^3 = tx(x-1)$	$\mathrm{IV}, \mathrm{IV}^*$
A_3	$(2x-1)ydx - 4x(x-1)dy$	$y^4 = tx(x-1)$	$\mathrm{III}, \mathrm{III}^*$
A_5	$(3x-2)ydx - 6x(x-1)dy$	$y^6 = tx^2(x-1)$	$\mathrm{II}, \mathrm{II}^*$
E_6	$(3y^2-2xy-1)dx - 6(x^2-1)dy$	$z^3 = t(x^2-1)$	$\mathrm{IV}, \mathrm{IV}^*, 2\mathrm{I}_0$
D_{n+2}	$\frac{\psi'}{\psi(\psi-1)}(y^2+n(\psi-1)y-\psi)dx - 2ndy$	$\left(\frac{y+\sqrt{\psi}}{y-\sqrt{\psi}}\right)^n = t\left(\frac{\sqrt{\psi}+1}{\sqrt{\psi}-1}\right)$	$\mathrm{I}_0^*, \mathrm{I}_0^*, \mathrm{nI}_0$

where $c \in \mathbb{C}$ satisfies $4 + 27c^3 \neq 0$,

$$z := \frac{(4x^2-3)y^4 - 4xy^3 + 6y^2 - 4xy + 1}{3y^4 - 8xy^3 + 6y^2 - 1}$$

and $\psi = \frac{xf^2}{(x-1)(x-\lambda)g^2}$ ($f, g \in \mathbb{C}[x]$) satisfies

$$xf^2 - (x-1)(x-\lambda)g^2 = h^n \tag{1.3}$$

for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$ and $h \in \mathbb{C}[x]$ (see Example 6.2).

In this paper, we consider the case that $\mathcal{F}$ is a Riccati foliation with respect to a Hirzebruch surface $\varphi : \mathbb{F}_e \to \mathbb{P}^1$ of degree e. In this case, g_i's in (1.2) are rational functions in $\mathbb{C}[x]$ (see Lemma 3.4). For convenience, the tautological section Γ_∞ of φ with $\Gamma_\infty^2 = -e$ is defined by $y = \infty$ in what follows.

One can define the *discriminant* of ω as follows:

$$\Delta(\omega) = \frac{1}{2}\left(g_1 + \frac{g_0'}{g_0}\right)' - \frac{1}{4}\left(g_1 + \frac{g_0'}{g_0}\right)^2 + g_0 g_2. \tag{1.4}$$

whenever $g_0 \neq 0$. $\Delta(\omega)$ is an invariant of $\mathcal{F}$ under any affine transformation

$$y = a(x)\bar{y} + b(x) \tag{1.5}$$

where $a, b \in \mathbb{C}(x)$ and $a \neq 0$ (Lemma 3.5).

Now our main results can be stated as follows.

Theorem 1.9. *Assume that $g_0 \neq 0$. The following conditions are equivalent:*

(1) *$\mathcal{F}$ is algebraic;*
(2) *by choosing a proper affine transformation* (1.5)*, g_i's in* (1.2) *can be taken as*

$$g_0 = \frac{1}{d} \cdot \frac{\psi'}{(\psi-1)}, \quad g_2 = \frac{1}{\gamma_2} \cdot \frac{\psi'}{(\psi-1)} - \left(1 - \frac{1}{\gamma_1}\right)\frac{\psi'}{\psi}, \quad g_2 = \left(\frac{1}{d} - \frac{1}{\gamma_3}\right)\frac{\psi'}{\psi}$$

where $\psi \in \mathbb{C}(x)$, γ_i's and d are as in Theorem 1.4 and Remark 1.6;

(3) *there is a rational function* $\psi \in \mathbb{C}(x)$ *satisfying*

$$\Delta(\omega) = \frac{1}{2}\left(\frac{\psi''}{\psi'}\right)' - \frac{1}{4}\left(\frac{\psi''}{\psi'}\right)^2 + \frac{1}{4}\left(1-\frac{1}{\gamma_1^2}\right)\left(\frac{\psi'}{\psi}\right)^2 + \frac{1}{4}\left(1-\frac{1}{\gamma_2^2}\right)\left(\frac{\psi'}{\psi-1}\right)^2 + \frac{1}{4}\left(\frac{1}{\gamma_1^2}+\frac{1}{\gamma_2^2}-\frac{1}{\gamma_3^2}-1\right)\left(\frac{\psi'}{\psi}\right)\left(\frac{\psi'}{\psi-1}\right);$$

(4) *there is a Riccati foliation* $\mathcal{F}_0$ *with* $kod(\mathcal{F}_0) = -\infty$ *w.r.t. a rational ruled surface* $\varphi_0 : X_0 \to \mathbb{P}^1$ *such that* $\mathcal{F}$ *is the pulling-back foliation of* $\mathcal{F}_0$ *after a base change* $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ *and a birational map* $\sigma : X \dashrightarrow X_1$ *as in the following commutative diagram,*

$$\begin{array}{ccccc} (\mathcal{F}, X) & \overset{\sigma}{\dashrightarrow} & (\psi^*\mathcal{F}_0, X_1) & \longrightarrow & (\mathcal{F}_0, X_0) \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} \\ & & \mathbb{P}^1 & \xrightarrow{\psi} & \mathbb{P}^1 \end{array}$$

where $\varphi : X \to \mathbb{P}^1$ *(resp.,* $\varphi_1 : X_1 \to \mathbb{P}^1$*) is the ruling map adapted to* $\mathcal{F}$ *(resp.,* $\psi^*\mathcal{F}_0$*).*

Theorem 1.10. *Assume that* $g_0 = 0$*. The following conditions are equivalent:*

(1) $\mathcal{F}$ *is algebraic;*
(2) $\mathcal{F}$ *is of type* A_{n-1} $(n \geq 1)$*;*
(3) *there is an* $\mathcal{F}$*-invariant section* Γ *of* φ *except the tautological section* $y = \infty$*;*
(4) *by choosing a proper affine transformation* (1.5)*, we can take*

$$g_1 = \frac{\psi'}{n\psi}, \quad g_2 = 0 \ (n \geq 1)$$

for some $\psi \in \mathbb{C}(x)$*;*

(5) $\mathcal{F}$ *is the pulling-back foliation of* $\mathcal{F}_0$ *defined by* $\omega_0 = ydx - nxdx$ *after a base change and a birational map as in the commutative diagram in Theorem 1.9 (4).*

Remark 1.11. $g_0 = 0$ iff the tautological section Γ_∞ is $\mathcal{G}$-invariant (see Lemma 3.4).

Remark 1.12. Theorem 1.9 and Theorem 1.10 also hold for the Riccati foliations with Kodaira dimension $-\infty$. So we can also classify them according to *ADE* types (see Example 6.1).

Based on the above theorems, we can get some criteria for the algebraicity or transcendency of a Riccati foliation $\mathcal{F}$ w.r.t. a rational fibration. For convenience, in what follows, we assume $\mathcal{F}$ is a standard form w.r.t. $\varphi : X(= \mathbb{F}_e) \to B(= \mathbb{P}^1)$ and each singularity of $\mathcal{F}$ is a non-degenerated one with a rational eigenvalue (see Sec. 2.1) .

Corollary 1.13. *Under our assumptions, the following conditions are equivalent:*

(1) $\mathcal{F}$ *is an algebraic foliation of type* A_{n-1}*;*
(2) $\mathcal{F}$ *has two disjoint* $\mathcal{F}$*-invariant sections of* φ*.*
(3) $\mathcal{F}$ *occurs in one of the following cases:*
   (i) $g_0 = 0$ *and* $g_1 f + g_2 = f'$ *for some* $f(x) \in \mathbb{C}[x]$ *with* $\deg f \leq e$*;*
   (ii) $g_0 \neq 0$*,* $e = 0$*,* $g_1 = c_1 g_0$ *and* $g_2 = c_2 g_0$ *for some* $c_1, c_2 \in \mathbb{C}$ *satisfying* $c_1^2 - 4c_2 \neq 0$*.*

Theorem 1.4 and Corollary 1.13 provide a new viewpoint for a fibration $f : X \to \mathbb{P}^1$ with two singular fibers.

Corollary 1.14. *A fibration* $f : X \to \mathbb{P}^1$ *with two singular fibers is a Riccati fibration of type* A_{n-1}*. Furthermore, if* X *is a rational surface, then* f *can be obtained by a pencil as* *follows:*

$$y^n = t\prod_{i=1}^{\ell}(x-a_i)^{m_i}, \quad \forall t \in \mathbb{P}^1$$

where n and m_i's are positive integers.

Remark 1.15. It is well-known that if $f : S \to \mathbb{P}^1$ is non-trivial (resp. non-isotrivial), then $s \geq 2$ (resp. 3, see [Bea81]). For a fibration over $\mathbb{P}^1$ with two singular fibers, each singular fiber is dual to each other and hence they have the same order of periodic topology monodromy (see [GLT16, Theorem 1.1]). Furthermore, the authors in [GLT16] classify all such fibrations of genus 2 .

Corollary 1.16. *Under our assumptions, $\mathcal{F}$ is an algebraic foliation of type D_{n+2} iff it satisfies the following conditions:*

1. *there is a horizontal irreducible $\mathcal{F}$-invariant curve Γ defined by*
$$(y+a)^2 - \mu = 0, \quad \textit{for some } a, \mu \in \mathbb{C}(x);$$
2. $g_0 \neq 0$ *and* $\tilde{\omega} := \left(ng_0y^2 + \frac{\mu'}{2\mu}y - ng_0\mu\right)dx - dy$ *gives an algebraic Riccati foliation of type A_{n-1}.*

Corollary 1.17. *If there is a singularity p of $\mathcal{F}$ with eigenvalue $\lambda_p = \frac{m}{n}$ ($n > 1$ and* $\gcd(m,n) = 1$*) satisfying $n \geq 6$, then $\mathcal{F}$ occurs in one of the following cases:*

1. *$\mathcal{F}$ is of type A or D;*
2. *$\mathcal{F}$ is not an algebraic Riccati foliation.*

2. Preliminaries

2.1. **Riccati foliations.** Let $(X, \mathcal{F})$ be a Riccati foliation w.r.t. a minimal rational fibration $\varphi_0 : X \to B$. A fiber of φ_0 is $\mathcal{F}$-invariant if and only if it contains the singularities of $\mathcal{F}$. Note that $K_{\mathcal{F}} \sim rF$, where F is a fiber of φ_0. We call r the *degree* of $\mathcal{F}$, and denote it by $\deg \mathcal{F} = r$.

By choosing proper flipping maps, one can get a standard form $(Y, \mathcal{G})$ of $(X, \mathcal{F})$ where Y admits a minimal rational fibration $\varphi : Y \to B$ (see [Bru15, Ch. 4, Prop. 4.2]) and each $\mathcal{G}$-invariant fiber F is of the following form:

- (I_a) F admits two singular points with nonzero eigenvalues $\pm a$ along F, where $0 \leq \operatorname{Re} a \leq \frac{1}{2}$.
- (II) F admits a saddle-node of multiplicity two, whose weak separatrix is contained in F.
- (III) F admits two saddle-nodes of the same multiplicity, whose strong separatrices are contained in F.
- (IV) F admits only one nilpotent singularity.

and that its reduced standard form $\rho : (\widetilde{Y}, \widetilde{\mathcal{G}}) \to (Y, \mathcal{G})$ is relatively minimal.

An algebraic Riccati foliation has at most singularities of type I_a ($a \in \mathbb{Q}^+$ and $a \leq \frac{1}{2}$). In this paper, our main goal is to answer Poincare problem on the algebraicity of the Riccati foliation. So we impose that following condition on a Riccati foliation to simplify our discussion in what follows.

Assumption. *All $\mathcal{G}$-invariant fibers of $\mathcal{G}$ are type I_a ($a \in \mathbb{Q}^+$ and $a \leq \frac{1}{2}$).*

In this case, ρ restricted on a fiber F is exactly a resolution of the singularity with positive eigenvalue in F.

For a given $\mathcal{G}$-invariant fiber F of type I_a ($a = \frac{m}{n}$, $(m,n) = 1$) , we denote by $n_F = n$. We have the following facts for such a Riccati foliation (see [HLT20]). The total transform of F under ρ is

$$\rho^*F = n_F(\Theta_F + N_F + N_F'), \tag{2.1}$$

where Θ_F is a (-1) curve, N_F and N'_F are $\mathbb{Q}^+$-divisors. There is a Zariski decomposition

$$K_{\widetilde{\mathcal{G}}} = \rho^* K_{\mathcal{G}} - \sum_F \Theta_F \sim \Big(\deg \mathcal{G} - \sum_F \frac{1}{n_F}\Big)\rho^* F_0 + \sum_F (N_F + N'_F) \tag{2.2}$$

whenever $\deg \mathcal{G} \geq \sum\limits_F \frac{1}{n_F}$, where F runs over all $\mathcal{G}$-invariant fibers and F_0 is a general fiber of φ.

Remark 2.1. From (2.2), the Kodaira dimension $\mathrm{Kod}(\widetilde{\mathcal{G}}) \leq 1$. Furthermore, for any relatively minimal Riccati foliation $\mathcal{F}$, its Kodaira dimension $\mathrm{Kod}(\mathcal{F})$ is consistent with the numerical Kodaira dimension $\nu(\mathcal{F})$ (cf. [Bru15, Ch.9, Sec. 5]). So $\mathrm{Kod}(\mathcal{F}) = -\infty$ iff $K_{\mathcal{F}}$ is not pseudo-effective.

Remark 2.2. The support of N_F (resp., N'_F) in (2.1) is a $\widetilde{\mathcal{G}}$-chains, i,e., a Hirzebruch-Jung string $C = C_1 + \cdots + C_r$ consisting of $\widetilde{\mathcal{G}}$-curves C_i's satisfying that

(1) all singularities of $\widetilde{\mathcal{G}}$ on C are reduced and non-degenerated;
(2) there is only one singularity of $\widetilde{\mathcal{G}}$, says $p_r (\in C_r)$, on $C - \{p_1, \ldots, p_{r-1}\}$ where $p_i = C_i \cap C_{i+1}$ $(i = 1, \ldots, r-1)$.

In particular, there is at most one $\widetilde{\mathcal{G}}$-curve meeting transversely with C.

One can write $N_F = \sum\limits_{i=1}^{r} \frac{\mu_i}{n_F} C_i$ where $1 = \mu_r < \mu_{r-1} < \cdots < \mu_1 < n_F$. N_F satisfies that $N_F C_1 = -1$ and $NC_i = 0$ for else i. All μ_i's can be determined uniquely by these equalities. More details can be found in [Bru15, Ch.8, Sec.2].

The following Lemmas are useful.

Lemma 2.3. *Let Γ be a section of $\tilde{\varphi} = \varphi\rho : \widetilde{Y} \to B$. Then $\Theta_F \Gamma = 0$. Moreover, Γ meets transversely with one of N_F, N'_F at some singularity and disjoints from another.*

In particular, there are at most two $\widetilde{\mathcal{G}}$-invariant sections of $\tilde{\varphi}$ whenever there is a $\mathcal{G}$-invariant F.

Proof. If $\Theta_F \Gamma > 0$, then (2.1) implies that $\rho^* F \cdot \Gamma \geq n_F > 1$, a contradiction. So $\Theta_F \Gamma = 0$. Thus one has $\Gamma N_F > 0$ or $\Gamma N'_F > 0$.

Without loss of generality, we assume $\Gamma N_F > 0$. Note that $n_F N_F$ and $n_F N'_F$ are $\mathbb{Z}$-divisor (Remark 2.2). Therefore we have $n_F N_F \Gamma = 1$ and $N'_F \Gamma = 0$ from $\rho^* F \cdot \Gamma = 1$. Namely, Γ meets transversely with an irreducible component of N_F at some singularity and disjoints from N'_F.

The latter part is from Remark 2.2. □

Corollary 2.4. *Let D_1, D_2 are the $\mathcal{G}$-invariant sections of $\varphi : Y \to B$. Then D_1, D_2 are disjoint. In particular, if $\varphi : Y (= \mathbb{F}_e) \to B (= \mathbb{P}^1)$ is a Hirzebruch surface of degree $e > 0$, then one of D_i's is a tautological section (i.e., a section with self-intersection number $(-e)$).*

Proof. Suppose that D_1, D_2 have an intersection p. Let F be the fiber passing through p. Since D_1, D_2 and F are $\mathcal{G}$-invariant, p has an eigenvalue $\lambda_p > 0$.

Let q be another singularity in F' with eigenvalue $\lambda_q < 0$. Since $D_1 F' = D_2 F' = 1$, q is a reduced non-degenerated singularity outside of D_1, D_2.

Let Γ_i be the inverse image of D_i under $\rho : \widetilde{Y} \to Y$ $(i = 1, 2)$. From Lemma 2.3, we can assume that Γ_1 (resp., Γ_2) meets transversely with N_F (resp., N'_F) at some singularity $\tilde{p}_1$ (resp., $\tilde{p}_2$) and disjoints from N'_F (resp., N_F).

Note that only one of $\tilde{p}_i$'s is exactly the inverse image of q. Thus only one of D_i's passes through q, a contradiction.

The latter part is from the well-known facts of a rational ruled surface. □

Let $F_1, \ldots, F_l$ be the $\mathcal{G}$-invariant fibers of φ with $n_1 \leq \cdots \leq n_l$ respectively where $n_i := n_{F_i}$ $(i = 1, \ldots, l)$.

Lemma 2.5. *We have* $\deg \mathcal{G} = 2g(B) - 2 + l$. *Furthermore,* $\mathrm{Kod}(\widetilde{\mathcal{G}}) = -\infty$ *iff* $B \cong \mathbb{P}^1$ *and* $\sum_{1\le i\le l}\left(1 - \frac{1}{n_i}\right) < 2$. *In this case,* $\mathcal{G}$ *is algebraic and* n_i*'s satisfy one of the following conditions:*

(1) $l \le 2$;
(2) $l = 3$, $n_1 = n_2 = 2$;
(3) $l = 3$, $n_1 = 2$, $n_2 = 3$, $n_3 \le 5$.

Proof. Let $m(\mathcal{G})$ be the sum of the multiplicities of the singularities of $\mathcal{G}$. From [Bru15, Proposition 2.1], one has

$$m(\mathcal{G}) = K_{\mathcal{G}}^2 - K_{\mathcal{G}}K_Y + c_2(Y) = 2\deg \mathcal{G} + 4 - 4g(B).$$

Under our assumption, we have also $m(\mathcal{G}) = 2l$. Thus

$$\deg \mathcal{G} = 2g(B) - 2 + l.$$

From (2.2), $K_{\widetilde{\mathcal{G}}}$ is not pseudo-effective iff $\deg \mathcal{G} < \sum_{1\le i\le l} \frac{1}{n_i}$, that is,

$$2g(B) - 2 + \sum_{1\le i\le l}\left(1 - \frac{1}{n_i}\right) < 0.$$

The above inequality holds iff $g(B) = 0$ and $\sum_{1\le i\le l}\left(1 - \frac{1}{n_i}\right) < 2$. In this case, it's algebraic from Miyaoka Theorem [Miy85].

The latter consequence is from a straightforwards computation. □

Similarly, one can get the following result.

Lemma 2.6. *The Kodaira dimension* $kod(\widetilde{\mathcal{G}}) = 0$ *iff either*

(I) *B is a smooth elliptic curve and $\mathcal{G}$ is a suspension of a representation* $\mu : \pi_1(B) \to \mathrm{Aut}(\mathbb{P}^1)$ *(see* [Bru15, Proposition 6.6]*) or*
(II) $B \cong \mathbb{P}^1$ *and one of the following cases occurs:*
  (1) $l = 3$ *and* $(n_1, n_2, n_3) = (3, 3, 3)$;
  (2) $l = 3$ *and* $(n_1, n_2, n_3) = (2, 4, 4)$;
  (3) $l = 3$ *and* $(n_1, n_2, n_3) = (2, 3, 6)$;
  (4) $l = 4$ *and* $(n_1, n_2, n_3, n_4) = (2, 2, 2, 2)$.

2.2. **Foliations induced by fibrations.** Let $f : X \to C$ be a minimal normal-crossing fibration of genus $g \ge 1$ with singular fibers $F_1, \dots, F_s$. From [Bru15, p.21, p.62], f gives a relative minimal foliation $\mathcal{F}$ with a canonical divisor

$$K_{\mathcal{F}} = K_{X/C} - \sum_{i=1}^{s} (F_i - F_{i,\mathrm{red}}) \tag{2.3}$$

where $F_{i,\mathrm{red}}$ is the reduce part of F_i. Since $g \ge 1$, $K_{\mathcal{F}}$ is pseudoeffective (see [Bru15, Theorem 7.1]). $K_{\mathcal{F}}$ gives a Zariski decomposition $K_{\mathcal{F}} = P + N$ where N consists of some Hirzebruch-Jung branches lying the fibers of f ([Ser92, Theorem 3.4]).

The fibration f is said to be *isotrivial* if all smooth fibers are isomorphic to a fixed smooth curve. By [Ser92] or [Bru15, § 9.2], one has

Lemma 2.7. *Let $f, \mathcal{F}$ be as above and* $\mathrm{kod}(\mathcal{F})$ *be the Kodaira dimension.*

(1) $\mathrm{kod}(\mathcal{F}) = 0$ *iff f is an isotrivial elliptic fibration;*
(2) $\mathrm{kod}(\mathcal{F}) = 1$ *iff f is either non-isotrivial ($g = 1$) or isotrivial ($g > 1$).*
(3) $\mathrm{kod}(\mathcal{F}) = 2$ *iff f is a non-isotrivial fibration of genus $g > 1$.*

Corollary 2.8. *If $kod(\mathcal{F}) = 1$, then $|mP|$ (for $m \gg 0$) as a base point free linear system gives a fibration $\varphi : X \to B$ with $P \sim \gamma F'$ ($\gamma \in \mathbb{Q}^+$) for a general fiber F' of φ.*

Furthermore, f coincides with φ if and only if $g = 1$.

Proof. The first part of this corollary is trivial.

From [Ser92, Theorem 3.4], N consists of Hirzebruch-Jung branches in all singular fibers of f. So $NF = 0$ for a general fiber F of f. By (2.3), one gets

$$PF = K_{\mathcal{F}}F = 2g - 2.$$

If $g > 1$, then $PF > 0$. So F is a horizontal curve in the fibration $\varphi : X \to B$. If $g = 1$, then $PF = 0$ implies that $F'F = 0$, i.e., $\varphi = f$. □

For an isotrivial fibration f, each singular fiber F can be written as follows

$$F = \gamma\left(\Gamma + \sum_{i=1}^{b} \Theta_i\right) \tag{2.4}$$

where Θ_i's are disjoint Hirzebruch-Jung branches, Γ is a smooth curve of genus g' meeting transversely with each Θ_i at one point, γ (> 1) is the order of the topology monodromy of the fiber germ (f, F) (see [GLT16, p. 88]). The component Γ is said to be *principal* (see [Xia90, p. 383]).

Let $F_1, \ldots, F_s$ be the singular fibers of f with principal components $\Gamma_1, \ldots, \Gamma_s$ and the orders of topology monodromy $\gamma_1, \cdots \gamma_s$ respectively. Set $d = FF'$.

Corollary 2.9. *Under the notations and assumptions in Corollary 2.8, one has*

$$\frac{2g(F') - 2}{d} = 2g(C) - 2 + \sum_{i=1}^{s}\left(1 - \frac{1}{\gamma_i}\right)$$

whenever $g > 1$.

Proof. From Corollary 2.8, $P \sim \gamma F'$ ($\gamma \in \mathbb{Q}^+$) and hence $PF' = 0$. Since $PN = 0$, one has $F'N = 0$. So $F'K_{\mathcal{F}} = PF' + NF' = 0$.

By (2.4), the support of $F_i - \gamma_i\Gamma_i$ consists of some Hirzebruch-Jung branches of F_i. Since all Hirzebruch-Jung branches lie in N and $NF' = 0$, one gets $(F_i - \gamma_i\Gamma_i)F' = 0$, i.e., $F_iF' = \gamma_i\Gamma_iF'$. Similarly, one has also $F_{i,\mathrm{red}}F' = \Gamma_iF'$.

Thus we obtain

$$\sum_{i=1}^{s} (F_i - F_{i,\mathrm{red}})\, F' = d\sum_{i=1}^{s}\left(1 - \frac{1}{\gamma_i}\right). \tag{2.5}$$

Since $K_XF' = 2g(F') - 2$, one has

$$K_{X/C}F' = 2g(F') - 2 - (2g(C) - 2)FF'. \tag{2.6}$$

Combining (2.3), (2.5), (2.6) and $K_{\mathcal{F}}F' = 0$, one gets (4.1). □

Corollary 2.10. *The isotrivial fibration* $f : X \to \mathbb{P}^1$ *of genus* $g > 1$ *satisfying* $\sum_{i=1}^{s}\left(1 - \frac{1}{\gamma_i}\right) < 2$ *(i.e., the conditions in Lemma 2.5) is a Riccati fibration.*

Proof. By Corollary 2.9, $F' \cong \mathbb{P}^1$, i.e., $\varphi : X \to B$ in Corollary 2.8 is a ruled surface. So $K_{\mathcal{F}}F' = 0$, namely, $\mathcal{F}$ is a Riccati foliation. □

For an elliptic fibration on a birationally ruled surface, we have the following well-known result (see [Xia92, Theorem 3.2.4] or [FM94, Proposition 3.23]).

Lemma 2.11. *Let* $f : X \to C$ *be an elliptic fibration with* $kod(X) = -\infty$ *and* $F_1, \ldots, F_k$ *be the multiple fibers with the multiplicities* $m_1 \le \cdots \le m_k$ *respectively. Then* $C \cong \mathbb{P}^1$ *and one of the following cases holds:*

(1) $\chi(O_X) = 0$ *(i.e., all singular fibers of f are multiple fibers),* $k = 0$ *or* $k = 2$ *and* $m_1 = m_2$*. In this case, X is a minimal elliptic ruled surface.*
(2) $\chi(O_X) = 1$*,* $k \le 1$*. In this case, X is a rational surface. In particular, if* $k = 1$ *and f is relatively minimal, then* $F_1 \equiv_{\mathrm{linear}} -m_1K_X$*.*

A fibration $f : X \to \mathbb{P}^1$ with 2 singular fibers is isotrivial from [Bea81] (also see [GLT16]). In particular, such an elliptic fibration can be classified as follows (see [Tan10, Theorem 3.2], [Hir85] or [MP86]).

Lemma 2.12. *Let $f : X \to \mathbb{P}^1$ be an elliptic fibration with 2 singular fibers. Then f is isomorphic to one of the following families.*

(I) *$X = (E \times \mathbb{P}^1)/\mathbb{Z}_n$ where E is an elliptic curve and the n-cyclic group $\mathbb{Z}_n = \{\sigma^k\}$ acts on $E \times \mathbb{P}^1$ by $\sigma^k(p, [x, y]) = (p + k\delta, [x, \xi^k y])$;*

(I*) *$y^2 = \lambda(x^3 + x + c)$ ($4 + 27c^2 \neq 0$) or $y^2 = \lambda(x^3 + 1)$;*

(II) *$y^2 = x^3 + \lambda$;*

(III) *$y^2 = x^3 + \lambda x$;*

(IV) *$y^3 = x^3 + \lambda x$.*

The types of the singular fibers are respectively (nI_0, nI_0), (I_0^*, I_0^*), (II, II^*), (III, III^*), (IV, IV^*).

Remark 2.13. In case (I) of Lemma 2.12 , X is a minimal elliptic ruled surface by Lemma 2.11. So the foliation $\mathcal{F}$ induced by f is a Riccati foliation w.r.t. the ruling map. It has no singularity and is a non-trivial holomorphic vector field with $\mathrm{kod}(\mathcal{F}) = 0$. By [Bru15, Theorem 6.6], $\mathcal{F}$ is a suspension of a representation $\rho : \pi_1(Alb(X)) \to Aut(\mathbb{P}^1)$.

3. Riccati foliations on a rational surface

All notations and assumptions in Sec. 2.1 are adopted. In this section, we consider the case that X is a rational surface, i.e., $\varphi : Y(= \mathbb{F}_e) \to B(= \mathbb{P}^1)$ is a Hirzebruch surface of degree e. In this case, $\deg \mathcal{G} = l - 2$ by Lemma 2.5 .

Let Γ_∞ be a tautological section with $\Gamma_\infty^2 = -e$ and F be a general fiber of φ. Let x (resp., y) be the coordinate of B (resp., F). We assume that Γ_∞ is defined by $y = \infty$. Let $F_1, \ldots, F_l$ be the $\mathcal{G}$-invariant fiber of φ. Without loss of generality, we assume $F_l = \varphi^{-1}(\infty)$ whenever $l > 0$.

Remark 3.1. The birational map $\sigma : (X, \mathcal{F}) \dashrightarrow (Y, \mathcal{G})$ can be realized as a Möbius transformation

$$y = \frac{a\bar{y} + b}{c\bar{y} + d}, \quad a, b, c, d \in \mathbb{C}(x),\ ad - bc \neq 0,$$

where $\bar{y}$ is the coordinate of a general fiber of $\varphi_0 : X \to B$. Moreover, it can be decomposed into more simple transformations: $y = (x - r)^{\pm 1} \cdot \bar{y}$ (i.e., flipping map), $y = s\bar{y} + r$ and $y = \frac{1}{\bar{y}}$ ($r, s \in \mathbb{C}$, $s \neq 0$).

3.1. Discriminant of a Riccati foliation.

Lemma 3.2. *Under our assumptions, we have*

(1) *if Γ_∞ is not $\mathcal{G}$-invariant, then $l \geq 2 + e$ and the equality holds iff Γ_∞ transverses to $\mathcal{G}$;*

(2) *if Γ_∞ is $\mathcal{G}$-invariant, then $l \geq 2e$ and the equality holds iff either $l = e = 0$ or each singularity $p_i = \Gamma_i \cap F_i$ has an eigenvalue $-\frac{1}{2}$ ($i = 1, \ldots, l$).*

Moreover, we have always $l \neq 1$.

Proof. (1) By Lemma 2.5, $K_{\mathcal{G}}\Gamma_\infty = \deg \mathcal{G} = l - 2$. If Γ_∞ is not $\mathcal{G}$-invariant, then

$$K_{\mathcal{G}}\Gamma_\infty = \mathrm{tang}(\mathcal{G}, \Gamma_\infty) + e \geq e, \tag{3.1}$$

i.e., $l \geq 2 + e \geq 2$, and the first equality holds iff Γ_∞ transverse to $\mathcal{G}$ from [Bru15, Proposition 2.2].

(2) Assume that $l > 0$. If Γ_∞ is $\mathcal{G}$-invariant, then

$$-e = \sum_{1 \leq i \leq l} \frac{m_i}{n_i} \tag{3.2}$$

where $\frac{m_i}{n_i}$ is the eigenvalue of the singularity $p_i = F_i \cap \Gamma_\infty$ ($i = 1, \ldots, l$) from Camacho-Sad formula ([CS82, Suw98]). Note that $|\frac{m_i}{n_i}| \le \frac{1}{2}$. Thus $e \le \frac{l}{2}$ and the equality holds iff each $\frac{m_i}{n_i} = -\frac{1}{2}$. If $l = 1$, $e = -\frac{m_1}{n_1}$ is not an integer, a contradiction. So $l \ne 1$.

Similarly, in case of $l = 0$, the Camacho-Sad formula implies $e = 0$. □

Corollary 3.3. *If $l = 0$, then $\mathcal{G}$ is defined by $\omega = dy$ w.r.t. the first projection*

$$\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1, \quad (x, y) \to x.$$

Proof. From Lemma 3.2, $e = 0$ and Γ_∞ is $\mathcal{G}$-invariant. Since $l = 0$, $\mathcal{G}$ is algebraic (Lemma 2.5). In this case, the Riccati fibration $f : Y \to C$ induced by $\mathcal{G}$ is smooth.

For any irreducible $\mathcal{G}$-invariant component $\Gamma(\ne \Gamma_\infty)$, one can claim that $\Gamma_\infty\Gamma = 0$. If not, their intersections give at least one singularity of $\mathcal{G}$ and hence there is a $\mathcal{G}$-invariant fiber of φ passing through it, a contradiction. So Γ is defined by $y = c$ for some $c \in \mathbb{C}$.

Therefore f is exactly the second projection

$$f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1, \quad (x, y) \to y.$$

Namely, $\mathcal{G}$ can be defined by $\omega = dy$. □

In what follows, we assume that $l \ge 2$. From Lemma 3.2, we have always $l \ge e + 1$.

Lemma 3.4. *Each Riccati foliation $\mathcal{F}$ has an expression* (1.1) *or* (1.2) *satisfying*

(1) *$p, q_i \in \mathbb{C}[x]$ (i.e., $g_i \in \mathbb{C}(x)$) for $i = 0, 1, 2$;*
(2) *Γ_∞ is $\mathcal{G}$-invariant iff $q_0 = 0$ (i.e., $g_0 = 0$);*
(3) *if $\mathcal{F}$ is a standard form, then p has no multiple root (i.e., the order of each pole of g_i's on $\mathbb{P}^1 - \{\infty\}$ is 1) and $\deg q_i < \deg p + (i-1)e$ (i.e., $\deg g_i := \deg q_i - \deg p < (i-1)e$ for $i = 0, 1, 2$).*

Proof. From Remark 3.1, it's enough to consider the standard form $\mathcal{G}$.

It's well known that ω is a section of $V := H^0(Y, \Omega_Y \otimes \mathcal{O}_Y(N_\mathcal{G}))$ where

$$N_\mathcal{G} := K_\mathcal{G} - K_Y = 2\Gamma_\infty + (l + e)F$$

is the normal bundle of $\mathcal{G}$ (see [Bru15]). One can constructs a subspace V' of V consisting of the following differential forms

$$\omega = \sum_{i=0}^{2} q_i(x) y^{2-i} dx - p(x) dy + c x^{l-1}(e y dx - x dy), \quad q_i, p \in \mathbb{C}[x],\ c \in \mathbb{C},$$

where $\deg q_i \le l - 2 + (i-1)e$ ($i = 0, 1, 2$) and $\deg p \le l - 1$. It's easy to see that $\dim V' = 4l - 2$.

We will claim $V = V'$. For this purpose , we need compute $\dim V$. Consider the exact sequence

$$0 \longrightarrow \varphi^*\Omega_B \otimes \mathcal{O}_Y(N_\mathcal{G}) \longrightarrow \Omega_Y \otimes \mathcal{O}_Y(N_\mathcal{G}) \longrightarrow \Omega_{Y/B} \otimes \mathcal{O}_Y(N_\mathcal{G}) \longrightarrow 0$$

where $\Omega_{Y/B} = \mathcal{O}_Y(-2\Gamma_\infty - eF)$ be the relatively canonical sheaf of φ.

By Leray spectral sequence and $R^1\varphi_*\mathcal{O}_Y(2\Gamma_\infty) = 0$, one has

$$h^k(Y, \varphi^*\Omega_B \otimes \mathcal{O}_Y(N_\mathcal{G})) = h^k(B, \varphi_*\mathcal{O}_Y(2\Gamma_\infty) \otimes \mathcal{O}_B(l + e - 2)),\ k = 0, 1.$$

Since $\varphi_*\mathcal{O}_Y(2\Gamma_\infty) = \mathcal{O}_B \oplus \mathcal{O}_B(-e) \oplus \mathcal{O}_B(-2e)$ and $l \ge e + 1$, we get

$$h^1(Y, \varphi^*\Omega_B \otimes \mathcal{O}_Y(N_\mathcal{G})) = 0, \quad h^0(Y, \varphi^*\Omega_B \otimes \mathcal{O}_Y(N_\mathcal{G})) = 3l - 3.$$

Note $h^0(\Omega_{Y/B} \otimes \mathcal{O}_Y(N_\mathcal{G})) = h^0(Y, \mathcal{O}_Y(lF)) = l + 1$, we obtain

$$\dim V = h^0(Y, \varphi^*\Omega_B \otimes \mathcal{O}_Y(N_\mathcal{G})) + h^0(\Omega_{Y/B} \otimes \mathcal{O}_Y(N_\mathcal{G})) = 4l - 2.$$

Now we investigate the neighbourhood near by $F_l = \varphi^{-1}(\infty)$. Take a coordinate transformation $(x, y) = (\frac{1}{t}, \frac{u}{t^e})$. We get the expression of ω in the neighbourhood as follows:

$$\tilde{\omega} = -\sum_{i=0}^{2} \tilde{q}_i y^{2-i} dt + \tilde{p}(eudt - tdu) - cdu,$$

where $\tilde{q}_i := q_i t^{l-2+(i-1)e}$ $(i = 0, 1, 2)$ and $\tilde{p} := pt^{l-1}$ are still polynomials in $\mathbb{C}[x]$. Note that $\mathcal{G}$-invariant fiber F_l is defined by $t = 0$. So $c = 0$. Thus we get the expression (1.1) with coefficients $q_i, p \in \mathbb{C}[x]$ (i.e., g_i's are in $\mathbb{C}(x)$).

Let $x = a_i$ be the equation of F_i $(i = 1, \ldots, l-1)$. Since F_i's are $\mathcal{G}$-invariant, $x = a_1, \ldots, a_{l-1}$ are the roots of p. Note that $\deg p \le l-1$. So $\deg p = l-1$ and p has no multiple root.

Take $y = \frac{1}{v}$. One get the differential form

$$\tilde{\omega} = \sum_{i=0}^{2} q_i v^i dt + pdv.$$

Note that Γ_∞ is defined by $v = 0$. Thus Γ_∞ is $\mathcal{G}$-invariant iff $v \mid q_0(x)$ (i.e., $q_0 = 0$). □

For convenience, we usually replace the expression (1.1) by (1.2). We define the discriminant of ω as in (1.4). Let $(\overline{X}, \overline{\mathcal{F}})$ be a Riccati foliation w.r.t. $\bar{\varphi} : \overline{X} \to \mathbb{P}^1$ and

$$\bar{\omega} = (\bar{g}_0 \bar{y}^2 + \bar{g}_1 \bar{y} + \bar{g}_0)dx - d\bar{y}$$

be the differential form of $\overline{\mathcal{F}}$.

Lemma 3.5. *Assume that* $g_0 \bar{g}_0 \neq 0$. *Then* $\Delta(\omega) = \Delta(\bar{\omega})$ *iff there is a birational map* $\sigma : (\overline{X}, \overline{\mathcal{F}}) \dashrightarrow (X, \mathcal{F})$ *defined by an affine transformation as in* (1.5).

Proof. (⇒) By a transform

$$y = \frac{z}{g_0} - \frac{1}{2g_0}\left(g_1 + \frac{g_0'}{g_0}\right) \quad \left(\text{resp., } \bar{y} = \frac{z}{\bar{g}_0} - \frac{1}{2\bar{g}_0}\left(\bar{g}_1 + \frac{\bar{g}_0'}{\bar{g}_0}\right)\right),$$

one gets a Riccati foliation defined by

$$\omega' = (z^2 + \Delta)dx - dz$$

where $\Delta := \Delta(\omega) = \Delta(\bar{\omega})$. Hence a birational map $\sigma : (\overline{X}, \overline{\mathcal{F}}) \dashrightarrow (X, \mathcal{F})$ can be obtained by the transformation

$$y = \frac{\bar{g}_0}{g_0}\bar{y} - \frac{1}{2g_0}\left(g_1 - \bar{g}_1 + \frac{g_0'}{g_0} - \frac{\bar{g}_0'}{\bar{g}_0}\right).$$

(⇐) By Remark 3.1, it's enough to consider the transformations: $y = (x-r)^{\pm 1}\bar{y}$ and $y = s\bar{y} + r$ $(s, r \in \mathbb{C},\ s \neq 0)$.

Take a transformation $y = (x-r)\bar{y}$. One has

$$\bar{g}_0 = g_0(x-r), \quad \bar{g}_1 = g_1 - \frac{1}{x-r}, \quad \bar{g}_2 = \frac{g_2}{x-r}.$$

From a straightforwards computation, we get $\Delta(\bar{\omega}) = \Delta(\omega)$. The other cases can also be checked similarly. □

Example 3.6. Consider a standard form $\mathcal{F}$ w.r.t. $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ with three $\mathcal{F}$-invariant fibers. By choosing a proper coordinate, we can assume that

(1) F_1, F_2 and F_3 are defined by $x = 0$, $x = 1$ and $x = \infty$ respectively;
(2) $p_1 = (0, \infty)$, $p_2 = (1, 0)$ are singularities of $\mathcal{G}$ with eigenvalues λ_1, λ_2 respectively.

Firstly, we consider the case that the sections $\Gamma_\infty : y = \infty$ and $\Gamma_0 : y = 0$ passe through both singularities on F_3. In this case, both sections are $\mathcal{G}$-invariant. If not, $1 = K_{\mathcal{G}}\Gamma = \operatorname{tang}(\mathcal{G}, \Gamma) \geq 2$ for $\Gamma = \Gamma_\infty$ or Γ_0, a contradiction. So

$$\omega = \left(\frac{-\lambda_1}{x} + \frac{\lambda_2}{x-1}\right)ydx - dy \tag{3.3}$$

and the eigenvalue λ_3 of the singularity $p_3 = \Gamma_\infty \cap F_3$ satisfies $\lambda_1 - \lambda_2 + \lambda_3 = 0$ by Camacho-Sad formula. In particular, the foliation is an algebraic one of type A.

In what follows, we assume there is a singularity on F_3, says p_3, outside of Γ_∞ and Γ_0. By choosing a proper coordinate, we can take $p_3 = (\infty, -1)$ with eigenvalue λ_3. From Lemma 3.4, we get

$$\omega = \left(\frac{\lambda_2 - \lambda_1 + \lambda_3}{2(x-1)} y^2 + \left(\frac{-\lambda_1}{x} + \frac{\lambda_2}{x-1}\right) y + \frac{\lambda_2 - \lambda_1 - \lambda_3}{2x}\right) dx - dy. \tag{3.4}$$

Hence

$$\Delta(\omega) = \frac{1}{4}\left(\frac{1-(\lambda_1-1)^2}{x^2} + \frac{1-\lambda_2^2}{(x-1)^2} + \frac{(\lambda_1-1)^2 + \lambda_2^2 - \lambda_3^2 - 1}{x(x-1)}\right).$$

3.2. **Riccati foliations with Kodaira dimension** $-\infty$**.** Let $\mathcal{F}$ be a Riccati foliation with $\mathrm{kod}(\mathcal{F}) = -\infty$. In this case, it's algebraic from Miyaoka Theorem [Miy85]. By Lemma 2.5, the rational fibration $\varphi : Y(= \mathbb{F}_e) \to B(= \mathbb{P}^1)$ adapted to the standard form $\mathcal{G}$ is a Hirzebruch surface of degree e.

Lemma 3.7. *We have $e \le 1$. Furthermore, up to a flipping map, we can assume always that $e = 0$.*

Proof. It's easy to see that $e \le 1$ and $l \le 3$ from Lemma 2.5 and Lemma 3.2.

Now we consider the case of $e = 1$. We hope to find a singularity with eigenvalue $\frac{1}{2}$ outside Γ_∞. Then we can make a flipping map by blowing-up the singularity with eigenvalue $\frac{1}{2}$ and get a new standard form of $\mathcal{F}$ w.r.t. the projection $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. For this purpose, we consider the following two cases.

Case 1. Γ_∞ is not $\mathcal{G}$-invariant.

From Lemma 3.2, $l = 3$ and Γ_∞ transverses to $\mathcal{G}$. In this case, there is a $\mathcal{G}$-invariant fiber of type $I_{\frac{1}{2}}$ by Lemma 2.5. Hence Γ_∞ doesn't pass through both singularities in such an fiber.

Case 2. Γ_∞ is $\mathcal{G}$-invariant.

Lemma 3.2 implies $l = 2$ and Γ_∞ passes through two singularities with eigenvalues $-\frac{1}{2}$ precisely. Namely, the singularities with eigenvalues $\frac{1}{2}$ are outside Γ_∞. □

In what follows, we assume that $e = 0$, i.e., $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is a projection satisfying $\varphi(x, y) = x$ where (x, y) is the coordinate of $\mathbb{P}^1 \times \mathbb{P}^1$. Let Γ_∞ (resp., Γ_0) be the section of φ defined by $y = \infty$ (resp., $y = 0$).

Corollary 3.8. *If $l = 2$, then $\mathcal{G}$ can be defined by $\omega = \lambda y dx - dy$ up to a suitable coordinate.*

Proof. One can choose a suitable coordinate such that the $\mathcal{G}$-invariant fibers of φ, says F_0, F_∞, are defined by $x = 0, \infty$ respectively. Furthermore, we can also assume that Γ_∞ and Γ_0 passes through the singularities in F_0 respectively.

We claim that Γ_∞ is $\mathcal{G}$-invariant. If not, one has $\mathrm{tang}(\mathcal{G}, \Gamma_\infty) \ge 1$ by our assumption. So (3.1) implies $l \ge 3$, a contradiction. Similarly, Γ_0 is also $\mathcal{G}$-invariant. Thus we get $\omega = \lambda y dx - x dy$ from Lemma 3.4. □

*Proof of Theorem 1.3.*The case for $l \le 2$ is from Corollary 3.3 and Corollary 3.8.

In what follows, we assume that $l = 3$. In this case, (n_1, n_2, n_3) satisfies Lemma 2.5 and so $n_1 = 2$. One can find that Γ_∞ (resp., Γ_0) is not $\mathcal{G}$-invariant and passes at most one singularity of $\mathcal{G}$ from (3.1) and (3.2).

By choosing a suitable coordinate, one can assume that $\mathcal{F}$ has a differential form ω as in Example 3.6 with $(\lambda_1, \lambda_2, \lambda_3) = (\frac{1}{2}, \frac{1}{n_2}, \frac{m}{n_3})$ where $0 < m \le \frac{n_3}{2}$ and $(m, n_3) = 1$.

If $n_2 = 2$, then

$$\omega = (\lambda_3 x y^2 + y - \lambda_3(x-1))dx - 2x(x-1)dy.$$

By replaceing y by $\frac{y}{\lambda_3}$, one gets an expression as in Theorem 1.3 (3). If $n_2 = 3$, then ω is as in Theorem 1.3 (4)–(7). □

4. Singular fibers of a Riccati fibration of genus $g \geq 1$

Let $f : X \to C$ be a Riccati fibration of genus $g \geq 1$ and $\mathcal{F}$ be the Riccati foliation induced by f with respect to a rational fibration. Without loss of generality, we assume that f is a minimal normal-crossing fibration whose singular fibers are $F_1, \ldots, F_s$ with principal components $\Gamma_1, \ldots, \Gamma_s$ respectively.

Let $(Y, \mathcal{G})$ be the standard form of $(X, \mathcal{F})$ w.r.t. a minimal rational fibration $\varphi : Y \to B$ and $\rho : (\widetilde{Y}, \widetilde{\mathcal{G}}) \to (Y, \mathcal{G})$ be the relatively minimal standard form w.r.t. a rational fibration $\tilde{\varphi} = \varphi\rho : \widetilde{Y} \to B$ as in Sec. 2.1. Under our assumption, $(\widetilde{Y}, \widetilde{\mathcal{G}}) = (X, \mathcal{F})$. Since $g \geq 1$, there is a Zariski decomposition $K_{\widetilde{\mathcal{G}}} = P + N$.

Let $F'_1, \ldots, F'_l$ be the $\mathcal{G}$-invariant fibers of $\mathcal{G}$ and take $n_i = n_{F'_i}$ $(i = 1, \ldots, l)$ where $n_{F'_i}$ is defined as in Sec 2.1. We set $d = FF'$.

4.1. **Proof of Theorem 1.4.**

Lemma 4.1. *Any Riccati fibration is isotrivial. Furthermore, the rational fibration $\tilde{\varphi}$ coincides with the fibration given by $|mP|$ as in Corollary 2.8 whenever $g > 1$.*

Proof. It's enough to consider the case of $\mathrm{kod}(\widetilde{\mathcal{G}}) = 1$ by Lemma 2.7 and $\mathrm{kod}(\widetilde{\mathcal{G}}) \leq 1$.

Let $\varphi' : \widetilde{Y} \to B$ be the fibration given by $|mP|$ and F' be a general fiber of φ' . Take a general fiber $\widetilde{F}$ of $\tilde{\varphi}$. One has $K_{\widetilde{\mathcal{G}}}\widetilde{F} = 0$ since $\widetilde{\mathcal{G}}$ is a Riccati foliation. Noting that both P and $\widetilde{F}$ are nef, it implies that $P\widetilde{F} = N\widetilde{F} = 0$. Hence $\widetilde{F}F' = 0$ for any fiber F' of φ' by Corollary 2.8. So $\tilde{\varphi} = \varphi'$. It implies that $g > 1$ by Corollary 2.8 again. Therefore f is isotrivial from Lemma 2.7 □

Let γ_i be the order of topology monodromy of F_i $(i = 1, \ldots, s)$.

Lemma 4.2. *Take a general fiber F' (resp., F) of $\tilde{\varphi}$ (resp., f). We have*

$$-\frac{2}{d} = 2g(C) - 2 + \sum_{i=1}^{s}\left(1 - \frac{1}{\gamma_i}\right), \tag{4.1}$$

$$\frac{2g-2}{d} = 2g(B) - 2 + \sum_{i=1}^{l}\left(1 - \frac{1}{n_i}\right). \tag{4.2}$$

In particular, the first equality implies that $C \cong \mathbb{P}^1$ and $\sum_{i=1}^{s}\left(1 - \frac{1}{\gamma_i}\right) = 2 - \frac{2}{d} < 2$.

Proof. If $g > 1$, then (4.1) is from Lemma 4.1, Corollary 2.9 and $g(F') = 0$. Now we investigate the case of $g = 1$. Since f is isotrivial, one has $P = 0$ by Lemma 2.7. So $NF' = K_{\mathcal{F}}F' = 0$. Thus one can get (4.1) by a similar proof of Corollary 2.9.

From (2.3), $K_{\widetilde{\mathcal{G}}}F = 2g - 2$. Combining (2.2), Lemma 2.5 and $NF = 0$, one gets

$$K_{\widetilde{\mathcal{G}}}F = \left(2g(B) - 2 + \sum_{i=1}^{l}\left(1 - \frac{1}{n_i}\right)\right)FF'.$$

Thus (4.2) is obtained. □

Proof of Theorem 1.4. Assume that f is a Riccati fibration. By Lemma 4.2, we have $\sum_{i=1}^{s}\left(1 - \frac{1}{\gamma_i}\right) < 2$. It implies that f occurs in one of the cases in Theorem 1.4 by a computation as in Lemma 2.5.

Conversely, for any isotrivial fibration $f : S \to C(\cong \mathbb{P}^1)$ of genus $g > 1$ occurring in one of the cases in Theorem 1.4, Corollary 2.10 implies that it is a Riccati fibration. □

From the proof of Lemma 4.2, we have

Corollary 4.3. *Each principal components Γ_i of F_i $(i = 1, \ldots, s)$ satisfies $\Gamma_i F' = \frac{d}{\gamma_i}$. In particular, for a Riccati fibration of type A_n, both Γ_i's are sections of $\tilde{\varphi}$. Conversely, if a principal component of a Riccati fibration is a section of the corresponding ruling map, then it is of type A_n.*

Similarly, $\Theta_{F_i'}F = \frac{d}{n_i}$ *where* $\Theta_{F_i'}$ *is the* (-1)*-curve as in* (2.1)*. Therefore we have always* $\gamma_i \mid d$ *and* $n_i \mid d$*.*

4.2. **Algebraic Riccati Foliation with Kodaira Dimension Zero.** In this section, we will consider the case of algebraic Riccati foliation with Kodaira Dimension Zero (i.e., $kod(\widetilde{\mathcal{G}}) = 0$). From Lemma 2.7 and Theorem 1.4, $f : X \to C(\cong \mathbb{P}^1)$ is an isotrivial elliptic fibration occurring in one of the cases in Theorem 1.4.

If $s = 0$, f is trivial, i.e., $f : X = E \times \mathbb{P}^1 \to \mathbb{P}^1$. If $s = 2$, then f occurs in one of the cases in Lemma 2.12.

In what follows, we assume that $s = 3$. In this case, the rational fibration $\varphi : Y \to B$ adapted to $\mathcal{G}$ gives s a rational ruled surface (namely, $B \cong \mathbb{P}^1$ and $Y = \mathbb{F}_e$) by Lemma 2.11. Therefore $\mathcal{G}$ occurs in one of the cases in Lemma 2.6 (II):

(1) $l = 3$ and $(n_1, n_2, n_3) = (3, 3, 3)$;
(2) $l = 3$ and $(n_1, n_2, n_3) = (2, 4, 4)$;
(3) $l = 3$ and $(n_1, n_2, n_3) = (2, 3, 6)$;
(4) $l = 4$ and $(n_1, n_2, n_3, n_4) = (2, 2, 2, 2)$.

We exclude the case (2) firstly. Since $n_1 = 4$, the eigenvalues of the singularities on F_1' are $\pm\frac{1}{4}$. So F_1' gives two $\widetilde{\mathcal{G}}$-chains: a (-4)-curve and a Hirzebruch-Jung chain consisting of four (-2)-curves. It implies that f contains two singular fibers of type III and III^* respectively (cf. [BHPV04, Ch. V, Sec. 7]). Thus $\gamma_1 = \gamma_2 = 4$, a contradiction to Lemma 4.2. So the case (2) doesn't occur.

Similarly, one can also exclude the case (3).

Lemma 4.4. *In case (1), up to a proper coordinate,* $\mathcal{G}$ *can be determined uniquely by a differential form*

$$\omega = (3y^2 - 2xy - 1)dx - 6(x^2 - 1)dy \tag{4.3}$$

on $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Let (x, y) be the coordinate of $Y = \mathbb{F}_e$ such that $y = \infty$ is a tautological section Γ_∞ of φ with $\Gamma_\infty^2 = -e$ and $x = \pm 1, \infty$ are all $\mathcal{G}$-invariant fibers. Furthermore, we assume that $(x, y) = (\infty, 0)$ is a singularity with eigenvalue $\frac{1}{3}$.

By our assumption and Corollary 4.3, Γ_∞ is not $\mathcal{G}$-invariant. So $e \le 1$ by Lemma 3.2. We will exclude the case for $e = 1$. Suppose that $e = 1$. By choosing a suitable coordinate, we can assume $(x, y) = (1, 0)$ is another singularity with eigenvalue $\frac{1}{3}$. From Lemma 3.4 and our assumptions, one has

$$\omega = (ay^2 + 4xy + b(x - 1))dx - 6(x^2 - 1)dy$$

for some $a, b \in \mathbb{C}$ $(a \neq 0)$. Since the eigenvalues of both singularities on the fiber $x = -1$ are $\pm\frac{1}{3}$, one gets that $b = 0$. So $y = 0$ defines a $\mathcal{G}$-invariant section, a contradiction. Thus we have $e = 0$.

Without loss of generality, we can choose a suitable coordinate y on a general fibe rof φ such that $(x, y) = (\infty, \infty), (\infty, 0), (1, 1)$ are singularities of $\mathcal{G}$ with eigenvalues $-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ respectively. Thus one gets (4.3) by Lemma 3.4. □

Now we investigate the case (4). In this case, $(\gamma_1, \gamma_2, \gamma_3) = (2, 2, n)$. More precisely, the singular fibers of f are type I_0^*, I_0^* and nI_0 respectively. For the surface $Y = \mathbb{F}_e$, one has $e \le 2$ by Lemma 3.2. By choosing some proper flipping maps, we can always assume that $e = 0$. Furthermore, we can assume that the $\mathcal{G}$-invariant fibers are $x = 0, 1, \lambda, \infty$ $(\lambda \neq 0, 1)$.

We state the following result which will be proved in Sec.5.3.

Lemma 4.5. *In case (4), up to a suitable coordinate and an affine transform* (1.5)*,* $\mathcal{F}$ *can be determined by a differential form*

$$\omega = \frac{\psi'}{\psi(\psi - 1)}(y^2 + n(\psi - 1)y - \psi)dx - 2ndy$$

where $\psi = \frac{xf^2}{(x-1)(x-\lambda)g^2}$ $(f, g \in \mathbb{C}[x]$ *) satisfies*

$$xf^2 - (x-1)(x-\lambda)g^2 = h^n$$

for some $h \in \mathbb{C}[x]$.

Proof of Theorem 1.8. It's from the above discussions, Lemma 4.4 and Lemma 4.5. □

5. Riccati fibrations on a rational surface.

In this section, we investigate a Riccati fibration $f : X \to C$ on a rational ruled surface $\varphi_0 : X \to B(= \mathbb{P}^1)$. We adopt all notations and assumptions in Sec. 4.

Let Γ_∞ be a tautological section of the Hirzebruch surface $\varphi : Y(= \mathbb{F}_e) \to B(= \mathbb{P}^1)$ of degree e with $\Gamma_\infty^2 = -e$ and F' be a general fiber of $\tilde{\varphi}$. Take a general fiber F of f and $d = FF'$.

Let x (resp., y) be the coordinate of $B = \mathbb{P}^1$ (resp., F'). We assume that Γ_∞ is defined by $y = \infty$ and Γ_0 is a section defined by $y = 0$. Each $\mathcal{G}$-invariant fiber F'_i $(i = 1, \ldots, l)$ is of type $\mathrm{I}_{\frac{m_i}{n_i}}$ $(0 < \frac{m_i}{n_i} \leq \frac{1}{2})$ and is defined by $x = a_i$ (for $i < l$) or $x = \infty$ (for $i = l$) respectively.

Let $D_i = \rho(\Gamma_i)$ where Γ_i is the principal component of the singular fiber F_i of f with the order γ_i of periodic topology monodromy $(i = 1, \ldots, s)$. Let $f_i \in \mathbb{C}[x, y]$ be the local equation of $D_i = \mathrm{div}(f_i)$ in Y $(i = 1, \ldots, s)$.

5.1. **Some lemmas.** Note that $\rho(F_i)$ is a sum of D_i and some $\mathcal{G}$-invariant fibers of φ, that is, $\rho_* F_i = \mathrm{div}(u_i f_i^{\gamma_i})$ for some $u_i \in \mathbb{C}[x]$. Since $f : X \to \mathbb{P}^1$ is a pencil of curves, f is determined by the family of the curves on Y.

$$C_t : \ u_1 f_1^{\gamma_i} - t u_2 f_2^{\gamma_2} = 0, \quad \forall t \in \mathbb{P}^1.$$

Without loss of generality, we can assume that $C_1 = \rho_* F_3$ whenever $s = 3$. Thus one gets the relation between f_i's:

$$u_1 f_1^{\gamma_1} - u_2 f_2^{\gamma_2} = u_3 f_3^{\gamma_3}. \tag{5.1}$$

Set $f_i = v_i h_i$ $(i = 1, \ldots, s)$ where $v_i \in \mathbb{C}[x]$ and $h_i \in K[y]$ $(K := \mathbb{C}(x))$ with the leading coefficient 1 as a polynomial of y. Take $\psi := \frac{u_1 v_1^{\gamma_1}}{u_2 v_2^{\gamma_2}}$. Thus the above relation can be rephrase as follows.

$$\psi h_1^{\gamma_1} - h_2^{\gamma_2} = (\psi - 1) h_3^{\gamma_3}, \quad \psi \in K \tag{5.2}$$

and

$$u_1 v_1^{\gamma_1} - u_2 v_2^{\gamma_2} = u_3 v_3^{\gamma_3}. \tag{5.3}$$

Since Γ_i is irreducible, $h_i \in K[y]$ is irreducible. Moreover, by Corollary 4.3, one has $\deg_y h_i = \frac{d}{\gamma_i}$ and $\gcd(h_i, h_j) = 1$ in $K[y]$ $(i \neq j)$.

It's easy to see that the differential form ω of $\mathcal{G}$ is from differential form $d\left(\frac{\psi h_1^{\gamma_1}}{h_2^{\gamma_2}}\right)$ (or $d\left(\frac{(\psi-1)h_3^{\gamma_3}}{h_2^{\gamma_2}}\right)$, etc.).

In what follows, we consider the case for $s = 3$. We assume $2 = \gamma_1 \leq \gamma_2 \leq \gamma_3$.

Lemma 5.1. *There is a* $u \in \mathbb{C}(x)$ *such that*

$$\gamma_1 \psi u h_1^{\gamma_1 - 1} = \frac{h_3}{\gamma_3}\frac{\partial h_2}{\partial y} - \frac{h_2}{\gamma_2}\frac{\partial h_3}{\partial y}, \tag{5.4}$$

$$\gamma_2 u h_2^{\gamma_2 - 1} = \frac{h_3}{\gamma_3}\frac{\partial h_1}{\partial y} - \frac{h_1}{\gamma_1}\frac{\partial h_3}{\partial y}, \tag{5.5}$$

$$\gamma_3 (\psi - 1) u h_3^{\gamma_3 - 1} = \frac{h_1}{\gamma_1}\frac{\partial h_2}{\partial y} - \frac{h_2}{\gamma_2}\frac{\partial h_1}{\partial y}. \tag{5.6}$$

Proof. From (5.2), we have

$$\psi\frac{h_1^{\gamma_1}}{h_3^{\gamma_3}} = \frac{h_2^{\gamma_2}}{h_3^{\gamma_3}} + \psi - 1.$$

Taking $\frac{\partial}{\partial y}$ on both sides of the above equality, we get

$$\psi\frac{h_1^{\gamma_1}}{h_3^{\gamma_3}}\left(\frac{\gamma_1}{h_1}\frac{\partial h_1}{\partial y} - \frac{\gamma_3}{h_3}\frac{\partial h_3}{\partial y}\right) = \frac{h_2^{\gamma_2}}{h_3^{\gamma_3}}\left(\frac{\gamma_2}{h_2}\frac{\partial h_2}{\partial y} - \frac{\gamma_3}{h_3}\frac{\partial h_3}{\partial y}\right),$$

i.e.,

$$\gamma_1\psi h_1^{\gamma_1-1}\left(\frac{h_3}{\gamma_3}\frac{\partial h_1}{\partial y} - \frac{h_1}{\gamma_1}\frac{\partial h_3}{\partial y}\right) = \gamma_2 h_2^{\gamma_2-1}\left(\frac{h_3}{\gamma_3}\frac{\partial h_2}{\partial y} - \frac{h_2}{\gamma_2}\frac{\partial h_3}{\partial y}\right). \tag{5.7}$$

Since $\gcd(h_1, h_2) = 1$, (5.7) implies that

$$h_1^{\gamma_1-1}\Big|\left(\frac{h_3}{\gamma_3}\frac{\partial h_2}{\partial y} - \frac{h_2}{\gamma_2}\frac{\partial h_3}{\partial y}\right), \quad h_2^{\gamma_2-1}\Big|\left(\frac{h_3}{\gamma_3}\frac{\partial h_1}{\partial y} - \frac{h_1}{\gamma_1}\frac{\partial h_3}{\partial y}\right)$$

in $K[y]$.

Note that $\deg_y h_1^{\gamma_1-1} = d - \frac{d}{\gamma_1}$ and

$$\deg_y\left(\frac{h_3}{\gamma_3}\frac{\partial h_2}{\partial y} - \frac{h_2}{\gamma_2}\frac{\partial h_3}{\partial y}\right) \le \frac{d}{\gamma_2} + \frac{d}{\gamma_3} - 2 = d - \frac{d}{\gamma_1}$$

by (4.1). Thus

$$w_1 h_1^{\gamma_1-1} = \left(\frac{h_3}{\gamma_3}\frac{\partial h_2}{\partial y} - \frac{h_2}{\gamma_2}\frac{\partial h_3}{\partial y}\right)$$

for some $w_1 \in \mathbb{C}(x)$. Similarly,

$$w_2 h_2^{\gamma_2-1} = \left(\frac{h_3}{\gamma_3}\frac{\partial h_1}{\partial y} - \frac{h_1}{\gamma_1}\frac{\partial h_3}{\partial y}\right)$$

for some $w_2 \in \mathbb{C}(x)$ satisfying $\gamma_1\psi w_2 = \gamma_2 w_1$ by (5.7). Take $u = \frac{w_1}{\gamma_1\psi} = \frac{w_2}{\gamma_2}$. we get (5.4) and (5.5). The last equality (5.6) can also be obtained similarly. □

Lemma 5.2. *There are $\eta, \xi \in K[y]$, such that*

$$\gamma_3\eta h_2 = \left(\frac{1}{\gamma_3} - \frac{1}{\gamma_2} - \frac{1}{2}\right)\frac{\partial h_2}{\partial y}\frac{\partial h_3}{\partial y} + \left(\frac{h_3}{\gamma_3}\frac{\partial^2 h_2}{\partial y^2} - \frac{h_2}{\gamma_2}\frac{\partial^2 h_3}{\partial y^2}\right), \tag{5.8}$$

$$\gamma_2\xi h_3 = \left(\frac{1}{\gamma_2} - \frac{1}{\gamma_3} - \frac{1}{2}\right)\frac{\partial h_2}{\partial y}\frac{\partial h_3}{\partial y} - \left(\frac{h_3}{\gamma_3}\frac{\partial^2 h_2}{\partial y^2} - \frac{h_2}{\gamma_2}\frac{\partial^2 h_3}{\partial y^2}\right), \tag{5.9}$$

$$\eta h_3 = 2\gamma_2\psi u^2 h_2^{\gamma_2-2} - \frac{1}{2\gamma_2}\left(\frac{\partial h_3}{\partial y}\right)^2, \tag{5.10}$$

$$\xi h_2 = 2\gamma_3\psi(\psi-1)u^2 h_3^{\gamma_3-2} - \frac{1}{2\gamma_3}\left(\frac{\partial h_2}{\partial y}\right)^2. \tag{5.11}$$

Proof. By $\gamma_1 = 2$ and (5.4), we have

$$h_1 = \frac{1}{2\psi u}\cdot\left(\frac{h_3}{\gamma_3}\frac{\partial h_2}{\partial y} - \frac{h_2}{\gamma_2}\frac{\partial h_3}{\partial y}\right).$$

Applying the above equality on (5.5), we obtain

$$h_2\left(2\gamma_2\psi u^2 h_2^{\gamma_2-2} - \frac{1}{2\gamma_2}\left(\frac{\partial h_3}{\partial y}\right)^2\right)$$
$$= \frac{h_3}{\gamma_3}\left(\left(\frac{1}{\gamma_3} - \frac{1}{\gamma_2} - \frac{1}{2}\right)\frac{\partial h_2}{\partial y}\frac{\partial h_3}{\partial y} + \left(\frac{h_3}{\gamma_3}\frac{\partial^2 h_2}{\partial y^2} - \frac{h_2}{\gamma_2}\frac{\partial^2 h_3}{\partial y^2}\right)\right).$$

Since $\gcd(h_2, h_3) = 1$,

$$h_3 \Big| \left(2\gamma_2 \psi u^2 h_2^{\gamma_2 - 2} - \frac{1}{2\gamma_2} \left(\frac{\partial h_3}{\partial y} \right)^2 \right)$$

in $K[x]$. Thus we can find some $\eta \in K[y]$ satisfying (5.10) and get (5.8) by the above equality.

Both (5.9) and (5.11) can be obtained similarly by combining (5.4) and (5.6). □

Lemma 5.3. *We have*

$$\eta = \frac{2}{2\gamma_3 - 2\gamma_2 - \gamma_2\gamma_3} \cdot \frac{\partial^2 h_3}{\partial y^2}, \quad \xi = \frac{2}{2\gamma_2 - 2\gamma_3 - \gamma_2\gamma_3 \cdot} \frac{\partial^2 h_2}{\partial y^2}.$$

Proof. Differentiating both sides of (5.10), one has

$$2\gamma_2(\gamma_2 - 2)\psi u^2 h_2^{\gamma_2 - 3} \frac{\partial h_2}{\partial y} = \eta \frac{\partial h_3}{\partial y} + h_3 \frac{\partial \eta}{\partial y} + \frac{1}{\gamma_2} \left(\frac{\partial h_3}{\partial y} \right) \left(\frac{\partial^2 h_3}{\partial y^2} \right). \tag{5.12}$$

Note that $\gamma_2 = 2, 3$. If $\gamma_2 = 2$, then $d = 2\gamma_3$ by Theorem 1.4 and hence $\deg h_3 = 2$. In this case, (5.10) implies that

$$4\psi u^2 = \eta h_3 + \frac{1}{4} \left(\frac{\partial h_3}{\partial y} \right)^2 \in K.$$

Since $\deg h_3 = 2$ and its leading coefficient is 1, one gets $\eta = -1$.

In what follows, we assume $\gamma_2 = 3$. By (5.10), (5.12) and (5.9), in $K[y]$, we have

$$\begin{cases} h_2 & \equiv \frac{1}{36\psi u^2} \left(\frac{\partial h_3}{\partial y} \right)^2, \\ \frac{\partial h_2}{\partial y} & \equiv \frac{1}{6\psi u^2} \left(\frac{\partial h_3}{\partial y} \right) \left(\eta + \frac{1}{3} \left(\frac{\partial^2 h_3}{\partial y^2} \right) \right), \\ \frac{h_2}{3} \frac{\partial^2 h_3}{\partial y^2} & \equiv \left(\frac{1}{\gamma_3} + \frac{1}{6} \right) \frac{\partial h_2}{\partial y} \frac{\partial h_3}{\partial y}, \end{cases} \pmod{h_3}.$$

respectively. Note that $\gcd(h_3, \frac{\partial h_3}{\partial y}) = 1$. One gets

$$\left(\frac{1}{\gamma_3} + \frac{1}{6} \right) \eta + \frac{1}{3\gamma_3} \frac{\partial^2 h_3}{\partial y^2} \equiv 0 \pmod{h_3} \tag{5.13}$$

from the above equalities.

By (5.10), one can see that

$$\deg \eta \le \max\{\deg h_2 - \deg h_3, \deg h_3 - 2\} = \frac{d}{\gamma_3} - 2 = \deg \frac{\partial^2 h_3}{\partial y^2}.$$

hence (5.13) implies that

$$\left(\frac{1}{\gamma_3} + \frac{1}{6} \right) \eta + \frac{1}{3\gamma_3} \frac{\partial^2 h_3}{\partial y^2} = 0,$$

i.e., $\eta = -\frac{2}{\gamma_3 + 6} \frac{\partial^2 h_3}{\partial y^2}$.

Similarly, we can get the other equality by combining (5.9) and (5.11). □

Lemma 5.4. *If $\gamma_2 = 3$, then we have*

$$h_1 = \frac{1}{216\psi^2 u^3} \left(\frac{18h_3}{\gamma_3 + 6} \cdot \frac{\partial h_3}{\partial y} \cdot \frac{\partial^2 h_3}{\partial y^2} - \frac{36h_3^2}{\gamma_3(\gamma_3 + 6)} \cdot \frac{\partial^3 h_3}{\partial y^3} - \left(\frac{\partial h_3}{\partial y} \right)^3 \right), \tag{5.14}$$

$$h_2 = \frac{1}{36\psi u^2} \left(\left(\frac{\partial h_3}{\partial y} \right)^2 - \frac{12h_3}{\gamma_3 + 6} \cdot \frac{\partial^2 h_3}{\partial y^2} \right), \tag{5.15}$$

$$0 = \frac{3\gamma_3}{2(\gamma_3 + 6)} \left(\frac{\partial^2 h_3}{\partial y^2} \right)^2 - \frac{\partial h_3}{\partial y} \cdot \frac{\partial^3 h_3}{\partial y^3} + \frac{h_3}{\gamma_3 - 2} \cdot \frac{\partial^4 h_3}{\partial y^4}. \tag{5.16}$$

Proof. The equality (5.15) is from (5.10) and Lemma 5.3. Furthermore, it implies that

$$\frac{\partial h_2}{\partial y} = \frac{1}{18\psi u^2(\gamma_3+6)}\left(\gamma_3\frac{\partial h_3}{\partial y}\cdot\frac{\partial^2 h_3}{\partial y^2} - 6h_3\cdot\frac{\partial^3 h_3}{\partial y^3}\right),$$

$$\frac{\partial^2 h_2}{\partial y^2} = \frac{1}{18\psi u^2(\gamma_3+6)}\left(\gamma_3\left(\frac{\partial^2 h_3}{\partial y^2}\right)^2 + (\gamma_3-6)\frac{\partial h_3}{\partial y}\cdot\frac{\partial^3 h_3}{\partial y^3} - 6h_3\cdot\frac{\partial^4 h_3}{\partial y^4}\right).$$

Applying the above equalities on (5.8), one gets (5.16).

(5.14) is from (5.15) and (5.4). □

From Lemma 5.4, it's enough to solve the equation (5.16). Set $m = \frac{d}{r_3}$ and

$$h_3 = \sum_{i=0}^{m}\binom{m}{k}a_k y^{m-k}, \quad a_0 := 1,\ a_k \in K\ (k = 2,\dots,m). \tag{5.17}$$

Since both leading coefficients of h_1, h_2 are 1, (5.14) and (5.15) imply that

$$a_2 = a_1^2 - \psi\left(\frac{6u}{m}\right)^2, \quad a_3 = a_1^3 - 3a_1\psi\left(\frac{6u}{m}\right)^2 + 2\psi^2\left(\frac{6u}{m}\right)^3.$$

Without loss of generality, we can assume $a_1 = 0$ and $u = \frac{m}{6}$ by taking an affine transformation $y = \frac{6u\bar{y}}{m} - a$. Thus $a_2 = \psi$ and $a_3 = -2\psi^2$. By (5.16) and a straightforwards computation, we obtain these undetermined coefficients a_k's. Finally, we have

$$h_3 = \sum_{k=0}^{m}(-1)^{k-1}\binom{m}{k}(k-1)\psi^{[\frac{k+1}{2}]}y^{m-k} - \frac{1}{2}(\gamma_3-3)(\psi-1)(4\psi)^{[\frac{\gamma_3}{2}]+1}\rho_{\gamma_3} \tag{5.18}$$

where $\rho_3 = \rho_4 := 1$ and

$$\rho_5 := \psi^3(1424 - 1600\psi) + 960\psi^3 y - 2079\psi^2 y^2 + 2200\psi^2 y^3 - 990\psi y^4 + 165y^6.$$

Furthermore, one can get h_1, h_2 by (5.14), (5.15) and (5.18).

5.2. **Riccati fibrations of type A_{n-1}.** We assume that f is of type A_{n-1}. The case for A_0 has been discussed in Corollary 3.3. In what follows, we assume $n \geq 2$. In this case, $(\gamma_1, \gamma_2, d) = (n, n, n)$ by Theorem 1.4.

By Corollary 4.3, both Γ_1, Γ_2 are the sections of $\tilde{\varphi}$ and hence both D_1, D_2 are the sections of φ.

From Corollary 2.4, either $e = 0$, or $e > 0$ and $\Gamma_\infty = D_i$ for some i. In the latter case, we can assume that $D_1 = \Gamma_\infty$ and $D_2 = \Gamma_0$ by choosing a suitable coordinate. From Lemma 3.4, the expression (1.2) of $\mathcal{G}$ is as follows:

$$\omega = g_1 y dx - dy, \quad g_1 = \sum_{i=1}^{l-1}\frac{\lambda_i}{x - a_i} \tag{5.19}$$

where $\lambda_i := \pm\frac{m_i}{n_i}$ $(i = 1,\dots,l-1)$.

Note that $n\lambda_i$ is an integer $(i = 1,\dots,l-1)$ by Corollary 4.3. We take

$$\psi = \prod_{i=1}^{l-1}(x - a_i)^{n\lambda_i} \in \mathbb{C}(x). \tag{5.20}$$

Thus $g_1 = \frac{\psi'}{n\psi}$.

Now we consider the case for $e = 0$. If $\Gamma'_\infty = D_1$ or D_2, one can get an expression of ω as above. In what follows, we assume that $\Gamma'_\infty \neq D_1, D_2$. Since D_i's are disjoint sections (Corollary 2.4), one can assume that D_1 (resp., D_2) is defined by $y = -1$ (resp., $y = 0$) by choosing a suitable coordinate. Therefore, we obtain the expression (1.2) of $\mathcal{G}$

$$\omega = \frac{\psi'}{n\psi}(y^2 + y)dx - dy \tag{5.21}$$

where ψ is as in (5.20).

Conversely, (5.19) (resp., (5.21)) gives a pencil defiend by $y^n = t\psi$ (resp., $y^n = t\psi(y+1)^n$) for $t \in \mathbb{C}$. So we get a Riccati fibration of A_{n-1}.

Remark 5.5. By taking $\psi = 1$ in (5.19) or (5.21), one can also get Corollary 3.3.

From the above discussions, we have

Lemma 5.6. *Up to an affine transformation* (1.5)*, an algebraic Riccati foliation of type A_{n-1} has an expression as in* (5.19) *or* (5.21)*. Conversely, a Riccati foliation with such expressions for any non-zero $\psi \in \mathbb{C}(x)$ is of type A_{n-1}.*

5.3. **Riccati foliations of type D_{n+2}.** We consider a Riccati fibration of type D_{n+2} ($n \geq 2$) in this section. In this case, $(\gamma_1, \gamma_2, \gamma_3, d) = (2, 2, n, 2n)$, $\deg h_1 = \deg h_2 = n$ and $\deg h_3 = 2$.

Take $\alpha = \sqrt{\psi}$ and $\overline{K} = K(\alpha)$. In $\overline{K}[y]$, (5.2) implies

$$(\alpha h_1 + h_2)(\alpha h_1 - h_2) = (\alpha^2 - 1)h_3^n. \tag{5.22}$$

Since $\overline{K}[y]$ is a Gaussian integral domain and $\gcd(\alpha h_1 + h_2, \alpha h_1 - h_2) = 1$ in $\overline{K}[y]$,

$$\alpha h_1 + h_2 = (\alpha + 1)\eta_1^n, \quad \alpha h_1 - h_2 = (\alpha - 1)\eta_2^n$$

where both η_1, η_2 are monic polynomials in $K[y]$ satisfying $h_3 = \eta_1\eta_2$. So

$$\eta_1 = y + a + b\alpha, \quad \eta_2 = y + a - b\alpha$$

for some $a, b \in K$. Note that $b \neq 0$ and $\alpha \notin K$ since h_3 is irreducible in $K[y]$.

Therefore we have

$$\begin{cases} h_1 &= \frac{1}{2\alpha}\left((\alpha + 1)(y + a + b\alpha)^n + (\alpha - 1)(y + a - b\alpha)^n\right), \\ h_2 &= \frac{1}{2}\left((\alpha + 1)(y + a + b\alpha)^n - (\alpha - 1)(y + a - b\alpha)^n\right), \\ h_3 &= (y + a)^2 - b^2\psi. \end{cases} \tag{5.23}$$

By the above equalities, we can get the differential expression of the corresponding Riccati foliation as follows:

$$\omega = \left(-\frac{\psi'}{2nb\psi(\psi - 1)}(y + a)^2 + \left(\frac{b'}{b} + \frac{\psi'}{2\psi}\right)(y + a) + \frac{b\psi'}{2n(\psi - 1)} - a'\right)dx - dy. \tag{5.24}$$

Without loss of generality, one can assume that $a = 0$ and $b = 1$ by taking an affine transformation $y = b\bar{y} - a$.

Furthermore, we set $\bar{\psi} = \frac{1}{1-\psi}$. Thus ω has a form as in Theorem 1.9 (2), that is,

$$\omega = \left(\frac{\bar{\psi}'}{2n(\bar{\psi} - 1)}y^2 + \frac{\bar{\psi}'}{2\bar{\psi}(\bar{\psi} - 1)}y - \frac{\bar{\psi}'}{2n\bar{\psi}}\right)dx - dy. \tag{5.25}$$

From the above discussions, we have

Lemma 5.7. *Up to a proper affine transformation* (1.5)*, an algebraic Riccati foliation of type D_{n+2} has an expression as in* (5.25)*. Conversely, a Riccati foliation with such an expression for any non-constant $\bar{\psi} \in \mathbb{C}(x)$ is algebraic.*

Remark 5.8. If $\sqrt{\psi} = \sqrt{1 - 1/\bar{\psi}} \in \mathbb{C}(x)$, then h_3 in (5.22) is reducible and hence (5.25) gives a Riccati foliation of A_{n-1}. The fact can also be found by taking an affine transformation $y = \sqrt{\psi}(2\bar{y} + 1)$ in (5.25). Then one can get an expression (5.21). A similar result can also be got when n is even and one of $\sqrt{\bar{\psi}}$, $\sqrt{\bar{\psi} - 1}$ is in $\mathbb{C}(x)$.

Proof of Lemma 4.5. In this case, by choosing a suitable coordinate x in $B(\cong \mathbb{P}^1)$, we can take $u_1 = x$, $u_2 = (x-1)(x-\lambda)$, $u_3 = 1$ and $\psi = \frac{u_1v_1^2}{u_2v_2^2}$ satisfying (5.3). Set $f = v_1$, $g = v_2$ and $h = v_3$. Thus one has (1.3). □

5.4. **Riccati fibration of E_k.** Combing (5.18) and Lemma 5.4, one can obtain h_1 and h_2. The differential expression ω is from $d(\psi h_1^2/h_2^3)$. By a straightforwards computation, we have

$$\omega = \left(\frac{\psi'}{d\psi(\psi-1)}y^2 + \left(\frac{\psi'}{2\psi} + \frac{\psi'}{6(\psi-1)}\right)y - \left(\frac{1}{6}+\frac{1}{d}\right)\cdot\frac{\psi'}{\psi-1}\right)dx - dy.$$

Furthermore, by taking $y = -\psi\bar{y}$ and $\bar{\psi} = \frac{\psi}{\psi-1}$, ω has an expression as in Theorem 1.9 (2), i.e.,

$$\omega = \left(\frac{\bar{\psi}'}{d(\bar{\psi}-1)}\bar{y}^2 + \left(\frac{\bar{\psi}'}{3(\bar{\psi}-1)} - \frac{\bar{\psi}'}{2\bar{\psi}}\right)\bar{y} - \left(\frac{1}{6}+\frac{1}{d}\right)\cdot\frac{\bar{\psi}'}{\bar{\psi}}\right)dx - d\bar{y}. \tag{5.26}$$

Lemma 5.9. *Up to a proper affine transformation as in* (1.5)*, an algebraic Riccati foliation of type E_k has an expression as in Theorem 1.9 (2), i.e.,* (5.26)*. Conversely, a Riccati foliation with such an expression for any non-constant $\bar{\psi} \in \mathbb{C}(x)$ is algebraic.*

Remark 5.10. If $\gamma_3 = 3$ and $\sqrt[3]{\bar{\psi}-1} \in \mathbb{C}(x)$, then h_1 is reducible and the Riccati foliation gives a fibration of type D_4 or A_1. Similarly, if $\gamma_3 = 4$ and $\sqrt{\bar{\psi}-1} \in \mathbb{C}(x)$, then the Riccati fibration is of type E_6, D_4 or A_1.

5.5. **The proves of main results.** We will prove Theorem 1.9 and Theorem 1.10 firstly.

Proof of Theorem 1.9.

(1) $\Longleftrightarrow$ (2) It's from Lemma 5.6, Lemma 5.7 and Lemma 5.9.

(2) $\Longleftrightarrow$ (3) It's from Lemma 3.5.

(4) $\Longrightarrow$ (1) It's obvious from Miyaoka Theorem [Miy85].

(2) $\Longrightarrow$ (4) Let $\mathcal{F}_0$ be a Riccati foliation w.r.t. $pr_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ defined by

$$\omega_0 = \left(xy^2 + \left(\left(2 - \frac{d}{\gamma_3}\right)x + \left(d - \frac{d}{\gamma_1}\right)\right)y + \left(1 - \frac{d}{\gamma_3}\right)(x-1)\right)dx - d\cdot(x-1)xdy.$$

From Lemma 2.5, $\mathrm{Kod}(\mathcal{F}_0) = -\infty$.

Without loss of generality, we assume the differential form $\mathcal{F}$ is as in (2). It's easy to see that $\mathcal{F}$ is a pulling-back of $\mathcal{F}_0$ by the base change $\psi : \mathbb{P}^1 \to \mathbb{P}^1$.

Up to now, this proof is completed. □

Proof of Theorem 1.10. (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) It's from Corollary 4.3.

(3) $\Longrightarrow$ (4) By choosing a suitable coordinate, we can assume that $y = 0$ is $\mathcal{G}$-invariant section. Thus ω can be written as in (5.19) by Lemma 3.4.

(4) $\Longleftrightarrow$ (5) It's obvious.

(5) $\Longrightarrow$ (1) By Theorem 1.3(2), $\mathcal{F}_0$ is algebraic. So is $\mathcal{F}$. □

Proof of Corollary 1.13.

(1) $\Longrightarrow$ (2) It's from Corollary 2.4 and Corollary 4.3.

(2) $\Longrightarrow$ (3) Let D_1, D_2 be the disjoint $\mathcal{F}$-invariant sections. If $e > 0$, then one of the sections, says D_1, is the tautological section (i.e., $D_1^2 = -e$) defined by $y = \infty$. Hence D_2 is defined by $y = f(x)$ for some $f \in \mathbb{C}[x]$ with $\deg f \le e$. So $g_0 = 0$ and $g_1 f + g_2 = f'$.

If $e = 0$, then we can defined D_i's by $y = a_1$ and $y = a_2$ ($a_1, a_2 \in \mathbb{C} \cup \{\infty\}$, $a_1 \neq a_2$) respectively. If $a_1 = \infty$ (resp., $a_2 = \infty$), then $g_0 = 0$ and $g_2 = -a_2 g_1$ (resp., $g_2 = -a_1 g_1$). If $a_1, a_2 \in \mathbb{C}$, then $g_0 y^2 + g_1 y + g_2 = g_0(y - a_1)(y - a_2)$. Namely, $g_1 = -(a_1 + a_2)g_0$ and $g_2 = a_1 a_2 g_0$. Set $c_1 = -(a_1 + a_2)$ and $c_2 = a_1 a_2$. Since $a_1 \neq a_2$, $c_1^2 - 4c_2 \neq 0$.

(3) $\Longrightarrow$ (1) By Corollary 2.4, we can always assume that both sections are defined by $y = 0$ and $y = \infty$ respectively. Thus $\omega = g_1 y dx - dy$. So it's algebraic from Theorem 1.10. □

Proof of Corollary 1.16.

($\Longrightarrow$) By (5.23), we have a horizontal irreducible $\mathcal{F}$-invariant curve defiend by $h_3 = 0$, i.e.,

$$(y + a)^2 - \mu = 0$$

where $\mu := b^2\psi$ and $b, \psi \in \mathbb{C}(x) \setminus \{0\}$.

By (5.24), $g_0 = -\frac{\psi'}{2nb\psi(\psi-1)} (\neq 0)$. So one has

$$bg_0 = -\frac{1}{2n} \cdot \frac{\psi'}{\psi(\psi-1)} = \frac{1}{2n} \cdot \frac{b(b\mu' - 2b'\mu)}{\mu(b^2-\mu)},$$

i.e.,

$$ng_0 b^2 - \frac{\mu'}{2\mu} \cdot b - ng_0\mu = -b'.$$

Thus $y = -b, -\frac{\mu}{b}$ are the solutions of the differential equation $\tilde{\omega} = 0$ where

$$\tilde{\omega} := \left(ng_0 y^2 + \frac{\mu'}{2\mu} y - ng_0\mu\right) dx - dy.$$

Namely, the Riccati foliation $\widetilde{\mathcal{F}}$ defined by $\tilde{\omega}$ has two $\widetilde{\mathcal{F}}$-invariant sections. By Corollary 1.13 and Corollary 2.4, the standard form of $\widetilde{\mathcal{F}}$ has two disjoint invariant sections and hence $\widetilde{\mathcal{F}}$ is of type A_n.

($\Longleftarrow$) Without loss of generality, we can assume that $a = 0$. Since $y^2 - \mu = 0$ is $\mathcal{F}$-invariant, one has $g_1 = \frac{\mu'}{2\mu}$ and $g_2 = -\mu g_0$. Let $y = y_1(x) \in \mathbb{C}(x)$ be a solution of $\tilde{\omega} = 0$. Take $b = -y_1$ and $\psi = \frac{\mu}{b^2}$. From a straightforwards computation, one gets

$$g_0 = -\frac{\psi'}{2nb\psi(\psi-1)}, \quad g_1 = \frac{b'}{b} + \frac{\psi'}{2\psi}, \quad g_2 = \frac{b\psi'}{2n(\psi-1)}.$$

Namely, ω has an expression as in (5.24). So $\mathcal{F}$ is of type D_{n+2}. □

Proof of Corollary 1.17. We assume that $\mathcal{F}$ is not of type A_{n-1} or D_{n+2}. Suppose that $\mathcal{F}$ be algebraic. From Theorem Theorem 1.9, Theorem 1.10, up to a proper flipping map, $\mathcal{F}$ is from a pulling-back of a Riccati foliation with Kodaira dimension $-\infty$ (more precisely, foliations in Theorem 1.3(4)-(7)). So $\lambda_p \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}\}$, i.e., $n \leq 5$, a contradiction. □

6. Some examples

Example 6.1. Let $\mathcal{F}$ be a Riccati foliations with $\mathrm{Kod}(\mathcal{F}) = -\infty$. Theorem 1.9 and Theorem 1.10 are also valid for $\mathcal{F}$. More precisely, we can find a special Riccati foliation $\mathcal{F}_0$ with $\mathrm{Kod}(\mathcal{F}_0) = -\infty$ such that $\mathcal{F}$ is a pulling-back of $\mathcal{F}_0$ after a base change $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ and a flipping map. Let ω_0 be the differential form of $\mathcal{F}_0$.

(A_{n-1}) $\omega = \lambda y dx - x dy$ ($\lambda = \frac{m}{n} \in \mathbb{Q}^+$), $\omega_0 = y dx - n dy$ and $\psi = x^m$;

(D_{n+2}) $\omega = (xy^2 + y - \lambda^2(x-1))dx - 2x(x-1)dy$ ($\lambda = \frac{m}{n} \in \mathbb{Q}^+$),

$$\omega_0 = (xy^2 + ny - (x-1))dx - 2nx(x-1)dy,$$

$$\psi = (1-x)^{m-2[m/2]} \cdot \left(\sum_{k=0}^{[m/2]} \binom{m}{2k} (x-1)^{[m/2]-k} x^k\right)^2;$$

(E_6) $\omega = \omega_0 = (xy^2 - 2(x-3)y - 3(x-1))dx - 12x(x-1)dy$ and $\psi = x$;

(E_7) $\omega = \omega_0 = (xy^2 - 4(x-3)y - 5(x-1))dx - 24x(x-1)dy$ and $\psi = x$;

(E_8) $\omega = \omega_0 = (xy^2 - 10(x-3)y - 11(x-1))dx - 60x(x-1)dy$ and $\psi = x$;

(E_8') $\omega = (xy^2 - 10(x-3)y - 119(x-1))dx - 60x(x-1)dy$, ω_0 is as in (E_8) and

$$\psi = 1 + \frac{(x-1)(2916x^2 - 3375x - 3125)^3}{(189x - 125)^5}.$$

Example 6.2. Let $\mathcal{F}$ be an algebraic Riccati foliation of type D_{n+2} w.r.t. $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ as in Theorem 1.8 (3).

(D_4) $\lambda = a^2$ and $\psi = -\frac{x(a-1)^2}{(x-1)(x-\lambda)}$;

(D_5) $\lambda = \frac{(a-1)^3(a+1)}{2a-1}$ and

$$\psi = -\frac{x}{(x-1)(x-\lambda)} \cdot \frac{\left(x-1-a^3\right)^2}{(2a-1)(a+1)^2};$$

(D_6) $\lambda = \left(\frac{2a}{a^2-1}\right)^4$ and

$$\psi = -\frac{x}{(x-1)(x-\lambda)} \cdot \left(\frac{\left(x - \frac{4(3a^2+2a+1)}{(a^2-2a+3)(a+1)^4}\right)}{\left(x - \frac{4}{(a+1)^4}\right)} \cdot \frac{(a^2-2a+3)(a^2+2a-1)}{(a^2-1)^2} \right)^2$$

where $a \in \mathbb{C}$ such that $\lambda \neq 0, 1, \infty$.

Example 6.3. Consider the foliation (3.4) in Example 3.6. Assume that $\mathrm{Kod}(\mathcal{F}) \geq 0$. Let $\lambda_i = \frac{m_i}{n_i}$ ($n_i > 1$ and $\gcd(m_i, n_i) = 1$). We claim that $\mathcal{F}$ is not algebraic whenever $n_i \geq 6$ for some i.

By Corollary 1.13, $\mathcal{F}$ is not of type A_{n-1}. We claim that $\mathcal{F}$ is not D_{n+1}. If not, from Corollary 1.16, one can find a horizontal irreducible $\mathcal{F}$-invariant curve Γ defined by $(y + a)^2 - \mu = 0$. Thus $\varphi|_\Gamma : \Gamma \to \mathbb{P}^1$ gives a double cover ramified exactly over two points in $\{0, 1, \infty\}$. Hence there are two $\mathcal{G}$-invariant fibers of type $I_{\frac{1}{2}}$. Thus one gets $g = 0$ from (4.2). Namely, $\mathrm{Kod}(\mathcal{F}) = -\infty$, a contradiction.

Therefore $\mathcal{F}$ is not algebraic from Corollary 1.17.

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