海洋科学的数学原理

第八讲 变分原理

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2024 年 8 月 8 日

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任何逻辑上封闭的，其真实性在一定程度上被实验所证实的理论永远不会失去价值.
确定所研究对象的近似程度在理论研究中是极端重要的，最严重的错误是，采用非常精确的理论并详细计算所有的细节修正，同时却忽略了比它们大得多的量.

——朗道，《理论力学》前言（1940）

一. 引言

- 有界闭区间上连续函数取到极值
- $F \in C^1([a,b]), (a,b)$ 内极值点: $f(x) = 0, (f(x) \triangleq F'(x))$.
- 解方程 $\longleftrightarrow$ 求极值 $\longrightarrow$ 梯度下降法
- $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}, \quad$ 极值点 $\nabla F = 0$
- $x \in \Omega$: 状态量, 不是自变量 $\quad F$: 状态空间上函数

二. 有限维系统的 Lagrange 形式

1. 势场中质点运动: 牛顿形式

N 个粒子, $m_1, m_2, \ldots, m_N$, 位置: $q_i = q_i(t) \in \mathbb{R}^3$; 速度: $\dot{q}_i = q_i'(t) \in \mathbb{R}^3$

势场 $v = v(q_1, \ldots, q_N)$

运动方程 $\boxed{m_i \ddot{q}_i = -\dfrac{\partial v}{\partial q_i}}$ m_i : 第 i 个质点的质量

例: N 体问题

$$v(q_1, \ldots, q_N) = \sum_{1 \leqslant i < j \leqslant N} \frac{-m_i m_j}{|q_i - q_j|}$$

能量积分 $\sum_{i=1}^N \dot{q}_i \cdot m_i\ddot{q}_i = -\sum_{i=1}^N \frac{\partial v}{\partial q_i} \cdot \dot{q}_i$ (内积)

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^N \frac{1}{2} m_i |\dot{q}_i|^2\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(v\left(q_1, \ldots, q_N\right)\right) = 0$$

动能 $T = \sum_{i=1}^N \frac{1}{2} m_i \left|\dot{q}_i\right|^2$

势能 $U = v(q_1, \ldots, q_N)$

总能量 $H = T + U, \quad H = H\left(q_1, \ldots, q_N, \dot{q}_1, \cdots \dot{q}_N\right)$

能量守恒 $\frac{\mathrm{d}}{\mathrm{d}t} H\left(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N\right) = 0$

2.Lagrange 形式

Lagrange 密度函数

$$L = L\left(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N, t\right) = \boxed{T - U}$$

Lagrange 作用量泛函

$$J[q] \triangleq \int_{t_1}^{t_2} L\left(q_1, \ldots q_N, \dot{q}_1, \ldots \dot{q}_N\right) \mathrm{d}t$$

Hamilton 原理 对于任何力学系统, 给定初始时刻 $t = t_0$ 的初始状态和终结时刻 $t = t_1$ 的终点状态之后, 其真实的运动区别于其他可能运动之处在于, 真实运动使得泛函 $J = \int_{t_1}^{t_2} L\mathrm{d}t$ 的一阶变分 J' 为 0.

- $J[q]=\int_{t_1}^{t_2} L(q,\dot{q},t)\mathrm{d}t$ 状态空间 $\Lambda=C^1([t_1,t_2]), q\in\Lambda$
- **变分** $\delta q\in\Lambda$, 且 $\delta q\,(t_1)=0=\delta q(t_2)$

  $h(\tau)\triangleq J(q+\tau\delta q)$

  J 在 q 取到极值 $\Rightarrow \forall\delta q,\ h(\tau)$ 在 $\tau=0$ 取极值 $\Rightarrow h'(0)=0$

$$
\begin{aligned}
0=h'(0) &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}h(\tau)\right|_{\tau=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\int_{t_1}^{t_2} L\left(q_i+\tau\delta q_i,\dot{q}_i+\tau\delta\dot{q}_i,t\right)\mathrm{d}t \\
&= \int_{t_1}^{t_2}\left(\sum_{i=1}^{N}\frac{\partial L}{\partial q_i}\delta q_i+\sum_{i=1}^{N}\frac{\partial L}{\partial\dot{q}_i}\delta\dot{q}_i\right)(q_i,\dot{q}_i,t)\,\mathrm{d}t
\end{aligned}
$$

$$
= \int_{t_1}^{t_2} \left(\sum_{i=1}^{N} \frac{\partial L}{\partial q_i} \delta q_i \right) \mathrm{d}t + \sum_{i=1}^{N} \int_{t_1}^{t_2} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) \right] \mathrm{d}t - \int_{t_1}^{t_2} \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} \cdot \delta q_i \right) \mathrm{d}t
$$

$$
= \int_{t_1}^{t_2} \sum_{i=1}^{N} \left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i(t) dt + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Bigg|_{t_1}^{t_2} \qquad \color{blue}{(\delta q_i(t_2) = \delta q_i(t_1) = 0)}
$$

$$
= \int_{t_1}^{t_2} \sum_{i=1}^{N} \left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} \right) (q_i(t), \dot{q}_i(t), t)\, \delta q_i(t) \mathrm{d}t
$$

由 $\delta q_i(t)$ 的任意性 $\Rightarrow$ $\boxed{\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, \ldots, N}$

Euler-Lagrange 方程

$$L=\sum_{i=1}^{N}\frac{1}{2}m_i\left|\dot{q}_i\right|^2-v\left(q_1,\dots q_N\right)$$

$$\frac{\partial L}{\partial \dot{q}_i}=m_i\dot{q}_i,\quad \frac{\partial L}{\partial q_i}=-\frac{\partial v}{\partial \dot{q}_i}\Rightarrow\quad m_i\ddot{q}_i+\frac{\partial v}{\partial q_i}=0,\quad i=1,\dots,N$$

记号

$$\boxed{\delta J\triangleq\int_{t_1}^{t_2}\frac{\delta L}{\delta q_i}\delta q_i\mathrm{d}t=0}$$

泛函导数

$$\boxed{\frac{\delta L}{\delta q_i}\triangleq\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}}$$

Lagrange 形式的优点: 仅知道 T, U 两个标量函数, 不需要分析系统中所有的力, 即可导出运动方程.

带约束系统

约束 $f_\alpha(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N) = 0, \quad \alpha = 1, 2, \ldots, m$

优化问题 $\tilde{L} = L + \sum_{\alpha=1}^{m} \lambda_\alpha(t) f_\alpha$

E-L 方程 $\tilde{J} = \int_{t_1}^{t_2} \tilde{L} \mathrm{d}t$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \quad (\text{广义力})$$

$$
\begin{aligned}
Q_i &= -\sum_{\alpha=1}^{m} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{q}_i}(\lambda_\alpha f_\alpha)\right) + \sum_{\alpha=1}^{m} \lambda_\alpha(t)\frac{\partial f_\alpha}{\partial q_i}.\\
&= \sum_{\alpha=1}^{m}\left\{\lambda_\alpha\left[\frac{\partial f_\alpha}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f_\alpha}{\partial \dot{q}_i}\right)\right] - \frac{\mathrm{d}\lambda_\alpha}{\mathrm{d}t}\frac{\partial f_\alpha}{\partial \dot{q}_i}\right\}
\end{aligned}
$$

其中

$$
\frac{\partial}{\partial \dot{q}_i}(\lambda_\alpha f_\alpha) = \lambda_\alpha \frac{\partial f_\alpha}{\partial \dot{q}_i}
$$

三. 有限维系统的 Hamilton 形式

共轭变量 $\boxed{p_i \triangleq \dfrac{\partial L}{\partial \dot{q}_i}}$ E-L 方程 $\Rightarrow \dot{p}_i = \dfrac{\partial L}{\partial q_i}$

典范变量 (q_i, p_i) $p_i = m_i \dot{q}_i$ (动量)

Hamilton 函数 $H = H(q,p,t) = \sum_{i=1}^{N} (\dot{q}_i p_i) - L(q_i, \dot{q}_i, t)$

注意 由 $p_i = \frac{\partial L}{\partial \dot{q}_i}(q_i, \dot{q}_i, t)$ 反解出 $\dot{q}_i$ 为 p, q 的函数.

$$\mathrm{d}H = \sum_{i=1}^{N} (\dot{q}_i \mathrm{d}p_i + p_i \mathrm{d}\dot{q}_i) - \mathrm{d}L$$

$$\mathrm{d}L = \sum_{i=1}^{N} \left(\frac{\partial L}{\partial q_i} \mathrm{d}q_i + \frac{\partial L}{\partial \dot{q}_i} \mathrm{d}\dot{q}_i \right) + \frac{\partial L}{\partial t} \mathrm{d}t = \sum_{i=1}^{N} (\dot{p}_i \mathrm{d}q_i + p_i \mathrm{d}\dot{q}_i) + \frac{\partial L}{\partial t} \mathrm{d}t$$

$$\Rightarrow \mathrm{d}H = \sum_{i=1}^{N} \left(\dot{q}_i \mathrm{d}p_i - \dot{p}_i \mathrm{d}q_i\right) - \frac{\partial L}{\partial t}\mathrm{d}t$$

又 $\mathrm{d}H = \sum_{i=1}^{N} \left(\frac{\partial H}{\partial p_i}\mathrm{d}p_i + \frac{\partial H}{\partial q_i}\mathrm{d}q_i\right) + \frac{\partial H}{\partial t}\mathrm{d}t$

$$\Rightarrow \boxed{\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial \alpha_i},\quad i = 1, \ldots, N} \qquad \boxed{\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

例 质点运动学系统

$$\begin{cases} \dot{q}_i = \frac{p_i}{m_i} \\ \dot{p}_i = \frac{\partial v}{\partial q_i} \end{cases}$$

$$H = \sum_{i=1}^{N} p_i \dot{q}_i - L = \sum_{i=1}^{N} \frac{|p_i|^2}{m_i} - \frac{1}{2}\sum_{i=1}^{N} m_i \left|\dot{q}_i\right|^2 + v\left(q_1, \ldots, q_N\right)$$

$$= \sum_{i=1}^{N} \frac{|p_i|^2}{2m_i} + v\left(q_1, \ldots, q_N\right)$$

Hamilton 形式对应的变分原理

$$\delta I = \delta \int_{t_1}^{t_2} \left(\sum_{i=1}^{N} p_i \dot{q}_i - H(p, q, t) \right) \mathrm{d}t$$

$$I = I\left[p_i, \dot{p}_i, q_i, \dot{q}_i, t\right] \quad (\text{其中不依赖于 } \dot{p}_i)$$

$\Rightarrow$ E-L 方程为

$$\dot{p}_j + \frac{\partial H}{\partial q_j} = 0, \quad \dot{q}_j - \frac{\partial H}{\partial p_j} = 0$$

例：平面上点涡运动 $L = L\left(z_k, \overline{z_k}, \dot{z}_k, \dot{\bar{z}}_k\right)$

$$I = \int_{t_1}^{t_2} L \mathrm{d}t \qquad \Rightarrow (\mathrm{E-L}) \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}_k}\right) - \frac{\partial L}{\partial z_k} = 0$$

$$L = \frac{1}{\sqrt{-1}} \sum_{k=1}^{n} \Gamma_k \bar{z}_k \dot{z}_k - \frac{1}{2\pi} \sum_{k \neq l} \Gamma_k \Gamma_l \ln\left(z_k - z_l\right) \overline{\left(z_k - z_l\right)}$$

$$= \frac{1}{2\sqrt{-1}} \sum_{k=1}^{N} \Gamma_k \left(\bar{z}_k \dot{z}_k - z_k \dot{\bar{z}}_k\right) - \frac{1}{2\pi} \sum_{k \neq l} \Gamma_k \Gamma_l \ln\left(z_k - z_l\right) \overline{\left(z_k - z_l\right)}$$

$$\boxed{\dot{\bar{z}}_j = \frac{1}{2\pi\sqrt{-1}} \sum_{k \neq l} \frac{\Gamma_j}{z_k - z_l}, \qquad j = 1, \ldots, n}$$

$H = -\frac{1}{2\pi} \sum_{k \neq l} \Gamma_k \Gamma_l \ln((z_k - z_l)\overline{(z_k - z_l)})$

Hamilton 形式

$$q_k = z_k, \quad p_k = \Gamma_k \overline{z_k}$$

$$\mathrm{i}\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \mathrm{i}\dot{p}_k = -\frac{\partial H}{\partial q_k}$$

Posson 括号

$$F = F\left(q_1, \ldots, q_n, p_1, \cdots, p_n, t\right)$$

$$G = G\left(q_1, \ldots, q_n, p_1, \ldots, p_n, t\right)$$

$$\{F, G\} \triangleq \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$$

$$\boxed{\frac{\mathrm{d}F}{\mathrm{d}t}} = \frac{\partial F}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i\right)$$

$$= \frac{\partial F}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right) = \boxed{\frac{\partial F}{\partial t} + \{F, H\}}$$

守恒量 F 与 t 无关且 $\frac{\mathrm{d}F}{\mathrm{d}t} = 0 \Leftrightarrow \{F, H\} = 0.$

Poisson 括号的性质

- $\{f, f\} = 0; \quad \{f, g\} = -\{g, f\}$
- $\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$
- $\{fg, h\} = f\{g, h\} + g\{f, h\}$
- Jacobi 恒等式 $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (非交换 Lie 代数)

Poisson 定理

f,g 为首次积分 $\Rightarrow \{f,g\}$ 也是, 即 $\frac{\mathrm{d}f}{\mathrm{d}t}=0, \frac{\mathrm{d}g}{\mathrm{d}t}=0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\{f,g\}=0.$

证明

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\{f,g\} &= \frac{\partial}{\partial t}\{f,g\} + \{\{f,g\},H\} \\
&= \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\} - \{\{g,H\},f\} - \{\{H,f\},g\} \\
&= \left\{\frac{\partial f}{\partial t} + \{f,H\}, g\right\} + \left\{f, \frac{\partial g}{\partial t} + \{g,H\}\right\} \\
&= \left\{\frac{\mathrm{d}f}{\mathrm{d}t}, g\right\} + \left\{f, \frac{\mathrm{d}g}{\mathrm{d}t}\right\} = 0.
\end{aligned}
$$

$\square$

四. 无限维系统的 Lagrange 形式

- $J: C^1(\bar{\Omega}) \to \mathbb{R}, \quad J[w] = \int_\Omega F(x, w(x), \mathrm{D}w(x))\mathrm{d}x$

  $\Omega \subset \mathbb{R}^n, \quad \partial\Omega \in C^\infty, \quad F \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n)$

- $\arg\min_{w\in\mathcal{A}} J[w]$ 或者 $\arg\max_{w\in\mathcal{A}} J[w]$

  $u \in \mathcal{A}, \quad \forall v \in V, \quad u + tv \in \mathcal{A}, \quad |t| \ll 1$

  $$\delta J = J'(u)v \doteq \lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} = \frac{\mathrm{d}}{\mathrm{d}t} J(u+tv)|_{t=0}$$

  u_*为极值点 $\Rightarrow J'(u_*)\,v = 0, \ \forall v \in V$

- $\mathcal{A}$: 泛函的定义域; V: 测试函数空间

- $F = F(x,z,p) \quad z \in \mathbb{R}, \quad p \in \mathbb{R}^n$
- 散度定理 $\int_\Omega \operatorname{div} V(x)\mathrm{d}x = \int_{\partial\Omega} V(x)\cdot n(x)\mathrm{d}S(x)$

  $\operatorname{div}(gV) = g \operatorname{div} V + V \cdot \mathrm{D}g$
- $\forall u, v \in C^1(\bar{\Omega})$,

$$
\begin{aligned}
J'(u)v &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F(x, u(x) + \mathrm{t}\mathrm{v}(x), \mathrm{D}u(x) + \mathrm{t}\,\mathrm{D}v(x))\mathrm{d}x\Big|_{t=0} \\
&= \int_\Omega \left[F_z\left(x, u(x), \mathrm{D}u(x)\right) v(x) + \langle F_p(x, u(x), \mathrm{D}u(x)), \mathrm{D}v(x)\rangle\right] \mathrm{d}x \\
&= \int_\Omega \left[F_z(x, u(x), \mathrm{D}u(x))v(x) + \operatorname{div}_x\left(v(x)F_p(x, u(x), \mathrm{D}u(x)\right)\right) \\
&\quad - \operatorname{div}_x F_p(x, v(x), \mathrm{D}u(x))v(x)\big]\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\Omega} \left[F_z(x, u(x), \mathrm{D}u(x)) - \mathrm{div}_x F_p(x, u(x), \mathrm{D}u(x))\right] \mathrm{v}(x)\mathrm{d}x \\
&+ \int_{\partial\Omega} F_p(x, u(x), \mathrm{D}u(x)) \cdot n(x) v(x) \mathrm{d}S(x) \\
&= \underbrace{\frac{\delta F}{\delta u}}_{\text{泛函导数}} + \int_{\partial\Omega} F_p(x, u(x), \mathrm{D}u(x)) \cdot n(x) \underbrace{v(x)}_{=0} \mathrm{d}S(x)
\end{aligned}
$$

由 v 的任意性 $\Rightarrow$

$$
\boxed{\mathrm{div}_x F_p(x, u(x), \mathrm{D}u(x)) - F_z(x, u(x), \mathrm{D}u(x)) = 0}
$$

Noether 定理

$I[w] = \int_\Omega L(x, w(x), \mathrm{D}w(x))\mathrm{d}x$

- **记号** $X : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad X(x,\tau) = X, \quad C^\infty$
  $X(x,0) = x, \quad \forall\, x \in \mathbb{R}^n$

  $|\tau| \ll 1$时, $x \mapsto X(x,\tau) \quad C^\infty$ 同胚
- **区域变分** $x \mapsto X(x,t) \quad \Omega_t \triangleq \{X(x,t) \mid x \in \Omega\},\ v(x) \triangleq \frac{\mathrm{d}}{\mathrm{d}\tau}X(x,0)$
- **函数变分** $u : \mathbb{R}^n \to \mathbb{R},\ w : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R},\ w = w(x,\tau),\ w(x,0) = u(x)$
  乘子: $m(x) \triangleq \frac{\mathrm{d}w}{\mathrm{d}\tau}(x,0)$

定义 称泛函 $J[\cdot]$ 在区域变分 X 及函数变分 w 下不变, 若 $\forall \Omega \subset \mathbb{R}^n$, 开集, $\forall |\tau| \ll 1$, 成立

$$\int_\Omega L(x, w(x,\tau), \mathrm{D}w(x,\tau))\mathrm{d}x = \int_{\Omega_\tau} L(x, u(x), \mathrm{D}u(x))\mathrm{d}x.$$

Noether 定理

设 $I[\cdot]$ 在区域变分 X 以及函数变分 w 下不变, 则

(i)

$$\begin{aligned}&\sum_{i=1}^n \left[mL_{p_i}(x, u(x), \mathrm{D}u(x)) - L(x, u(x), \mathrm{D}u(x))v^i\right]_{x_i}\\ =&m\left[\sum_{i=1}^n \left(L_{p_i}(x, u(x), \mathrm{D}u(x))\right)_{x_i} - L_z(x, u(x), \mathrm{D}u(x))\right],\end{aligned}$$

其中 $v = (v^1, \ldots, v^n)$, 由 $v(x) = X_t(x, 0)$ 定义, $m(x) = w_\tau(x, 0)$.

(ii) 若 u 为 $I[\cdot]$ 的临界点, 即成立 (E-L) 方程

$$- \operatorname{div}(\mathrm{D}_p L) + L_z = 0,$$

则

$$\sum_{i=1}^{n} \left(m L_{p_i}(x, u(x), \mathrm{D}u(x)) - L(x, u(x), \mathrm{D}u(x)) v^i \right)_{x_i} = 0.$$

$\longrightarrow$ 散度形式, 守恒律

证明

$$\int_{\Omega} L(x, w(x,\tau), \mathrm{D}w(x,\tau))\mathrm{d}x = \int_{\Omega_\tau} L(x, u(x), \mathrm{D}u(x))\mathrm{d}x$$

两边对 τ 求导 $\Rightarrow$

$$\begin{aligned}\int_{\Omega} (\mathrm{D}_p L \cdot \mathrm{D}m + L_z m)\,\mathrm{d}x &= \int_{\Omega\tau} \mathrm{div}(Lv)\mathrm{d}x \\ = \int_{\partial\Omega\tau} (Lv)\cdot n\mathrm{d}S\bigg|_{\tau=0} &= \int_{\partial\Omega} (Lv)\cdot n\mathrm{d}S\end{aligned}$$

其中 $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} f\mathrm{d}x = \int_{\Omega_t}\left(\frac{\partial f}{\partial t} + \mathrm{div}(f(v))\mathrm{d}x\right. \rightarrow \boxed{\int_{\Omega\tau} \mathrm{div}(Lv)\mathrm{d}x}$

$$
\begin{aligned}
\Rightarrow \int_\Omega \left(-\operatorname{div}\left(\mathrm{D}_p L\right) + L_z\right) m \, \mathrm{d}x &= \int_{\partial\Omega} \left(Lv - m\mathrm{D}_p L\right) \cdot n \, \mathrm{d}S \\
&= \int_\Omega \operatorname{div}\left(-m\mathrm{D}_p L + Lv\right) \mathrm{d}x,
\end{aligned}
$$

由 Ω 的任意性，

$$
\Rightarrow \boxed{\operatorname{div}\left(m\mathrm{D}_p L + Lv\right) = m\left(\operatorname{div}\left(\mathrm{D}_p L\right) - L_z\right)}
$$

$\square$

例: 非线性波动方程 $w = w(x,t),\ (x_{n+1} = t)$

- $I[w] = \int_0^T \int_{\mathbb{R}^n} \left[\frac{1}{2}w_t^2 - \left(\frac{1}{2}|\mathrm{D}w|^2 + F(w)\right)\right] \mathrm{d}x\mathrm{d}t$

  Euler-Lagrange 方程 $L = \frac{1}{2}p_{n+1}^2 - \left(\frac{1}{2}|p|^2 + F(z)\right)$

  $\mathrm{D}_pL = (-p_1, \ldots, -p_n, p_{n+1}) \quad L_z = -F'(z) = -f(z)$

  $\mathrm{div}\,(\mathrm{D}_pL) - L_z = 0 \Rightarrow \partial_t(\partial_t w) - \sum_{i=1}^n \partial_{x_i}(\partial_{x_i} w) + f(w) = 0$

  $\Rightarrow \boxed{\partial_{tt}w - \Delta w + f(w) = 0}$

- **时间平移不变性 ⇒ 能量守恒**

  $X(x,t;\tau) = (x, t+\tau),\ \ w(x,t;\tau) = u(x,t+\tau)$

  $$V = \frac{\mathrm{d}}{\mathrm{d}\tau}X(x,t;\tau)\bigg|_{\tau=0} = (0,1) = e_{n+1}, \quad m = \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} w(x,t;\tau) = u_t(x,t)$$

$$\Rightarrow \sum_{i=1}^{n}(-u_{x_i}u_t)_{x_i} + \underbrace{\left(u_t\cdot u_t - \left(\frac{1}{2}u_t^2 - \frac{1}{2}|\mathrm{D}u|^2 - F(u)\right)\right)_t}_{\frac{1}{2}(u_t^2+|\mathrm{D}u|^2)+F(u)\triangleq e} = 0$$

$$\Rightarrow e_t - \mathrm{div}\,(u_t\mathrm{D}u) = 0$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_{\mathbb{R}^n}\left(\frac{1}{2}\left(u_t^2 + |\mathrm{D}u|^2\right) + F(u)\right)\mathrm{d}x}_{\text{能量}} = 0$$

例: Lagrange 坐标下旋转流体力学方程组 $X = X(a,t), a \in \mathbb{R}^n$,

$I = \int_{t_1}^{t_2}\int_{\mathbb{R}^n}\left[\frac{1}{2}|\dot{X}(a,t)|^2 + \dot{X}(a,t)\cdot(\Omega\times X(a,t)) - e - \phi\right]\mathrm{d}a\mathrm{d}t$, 其中 $\phi = \phi(X,t)$: 势能,

$e = e(\tau,\eta)$: 内能, $\eta = \eta(a)$, $\tau = \det\left(\frac{\partial X}{\partial a}\right) = \frac{\rho a}{P}$.

五. 无限维系统的 Hamilton 形式

- $\mathcal{L} = \mathcal{L}(x^\nu, \eta_\rho, \eta_{\rho,\nu})$, η_ρ : 状态量 (场), $\eta_{\rho,\nu} = \frac{\partial}{\partial x^\nu}\eta_\rho$
- $I[\eta] = \int_\Omega \mathcal{L}(x^\nu, \eta_\rho, \eta_{\rho,\nu})\mathrm{d}x$
- (E-L) $\quad \frac{\mathrm{d}}{\mathrm{d}x^\nu}\left(\frac{\partial \mathcal{L}}{\partial \eta_{\rho,\nu}}\right) - \frac{\partial \mathcal{L}}{\partial \eta_\rho} = 0 \quad$ (重复指标代表求和)
- **记号** $\frac{\delta}{\delta\psi} = \frac{\partial}{\partial\psi} - \frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial}{\partial\psi_{,i}}$, 其中 $\psi_{,i} = \frac{\partial\psi}{\partial x_i}$ $\quad$ (泛函微分)

$$\Rightarrow \quad \boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}_\rho}\right) - \frac{\delta \mathcal{L}}{\delta \eta_\rho} = 0}$$

- **注意** 关于 i 偏导不含对 t 求导. (希腊字母: 含 t; 罗马字母: 不含 t.)

共轭变量 $\pi_\rho = \frac{\partial \mathcal{L}}{\partial \dot{\eta}_\rho}$ $\quad (\Rightarrow \dot{\eta}_\rho$用$\underbrace{\pi_\rho, x, \eta_\rho, \eta_{\rho,i}}_{\text{独立自变量}}$表示$)$

Hamilton 函数 $\mathcal{H} = \sum_\rho \pi_\rho \dot{\eta}_\rho - \mathcal{L}\left(x^\nu, \eta_\rho, \eta_{\rho,\nu}\right)$

$$\boxed{\mathcal{H} = \mathcal{H}\left(x^\nu, \pi_\rho, \eta_\rho, \eta_{\rho,i}\right)} \quad \text{正则变量}(\pi_\rho, \eta_\rho)$$

$$\frac{\partial \mathcal{H}}{\partial \pi_\rho} = \dot{\eta}_\rho + \pi^\lambda \frac{\partial \dot{\eta}_\lambda}{\partial \pi_\rho} - \frac{\partial \mathcal{L}}{\partial \dot{\eta}_\lambda}\frac{\partial \dot{\eta}_\lambda}{\partial \pi_\rho} = \dot{\eta}_\rho$$

$$\frac{\partial \mathcal{H}}{\partial \eta_\rho} = \pi^\lambda \frac{\partial \dot{\eta}_\lambda}{\partial \eta_\rho} - \frac{\partial \mathcal{L}}{\partial \dot{\eta}_\lambda}\frac{\partial \dot{\eta}_\lambda}{\partial \eta_\rho} - \frac{\partial \mathcal{L}}{\partial \eta_\rho} = -\frac{\partial \mathcal{L}}{\partial \eta_\rho}$$

由 E-L 方程 $\Rightarrow$

$$\frac{\partial \mathcal{H}}{\partial \eta_\rho} = -\frac{\partial \mathcal{L}}{\partial \eta_\rho} = -\frac{\partial}{\partial x^u}\left(\frac{\partial \mathcal{L}}{\partial \eta_{\rho,u}}\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\pi_\rho - \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{L}}{\partial \eta_{\rho,i}}\right)$$

又 $$\frac{\partial \mathcal{H}}{\partial \eta_{\rho,i}} = \pi^{\lambda}\frac{\partial \dot{\eta}_{\lambda}}{\partial \mu_{\rho,i}} - \frac{\partial \mathcal{L}}{\partial \dot{\eta}_{\lambda}}\frac{\partial \dot{\eta}_{\lambda}}{\partial \eta_{\rho,i}} - \frac{\partial \mathcal{L}}{\partial \eta_{\rho,i}} = -\frac{\partial \mathcal{L}}{\partial \eta_{\rho,i}}$$

$$\Rightarrow \frac{\partial \mathcal{H}}{\partial \eta_{\rho}} = -\dot{\pi}_{\rho} + \frac{\mathrm{d}}{\mathrm{d}x^{i}}\left(\frac{\partial \mathcal{H}}{\partial \eta_{\rho,i}}\right) \Rightarrow -\dot{\pi}_{\rho} = \frac{\partial \mathcal{H}}{\partial \eta_{\rho}} - \frac{\mathrm{d}}{\mathrm{d}x^{i}}\left(\frac{\partial \mathcal{H}}{\partial \eta_{\rho,i}}\right)$$

$$\Rightarrow \quad \dot{\pi}_{\rho} = -\left(\frac{\partial \mathcal{H}}{\partial \eta_{\rho}} - \frac{\mathrm{d}}{\mathrm{d}x^{i}}\left(\frac{\partial \mathcal{H}}{\partial \eta_{\rho,i}}\right)\right) = -\frac{\delta \mathcal{H}}{\delta \eta_{\rho}}$$

$\Rightarrow$ **Hamilton 方程** $\dot{\eta}_{\rho} = \frac{\partial \mathcal{H}}{\partial \pi_{\rho}} = \frac{\delta \mathcal{H}}{\delta \pi_{\rho}}$ ($\mathcal{H}$ 中不含$\pi_{\rho,i}$)

$$\boxed{\dot{\eta}_{\rho} = \frac{\delta \mathcal{H}}{\delta \pi_{\rho}}, \quad \dot{\pi}_{\rho} = -\frac{\delta \mathcal{H}}{\delta \eta_{\rho}}}$$

例：非线性波动方程 $\eta = \eta(x,t)$

$$I(\eta) = \int_{\mathbb{R}^3} \underbrace{\left[\frac{1}{2}|\eta_\rho|^2 - \left(\frac{1}{2}|\mathrm{D}\eta|^2 + F(\eta)\right)\right]}_{\mathcal{L}(\eta,\eta_t,D\eta)} \mathrm{d}x$$

$$\pi \triangleq \frac{\partial \mathcal{L}}{\partial \eta_t} = \eta_t$$

$$\begin{aligned}\mathcal{H} = \pi\eta_t - \mathcal{L} &= \eta_t^2 - \frac{1}{2}\eta_t^2 + \frac{1}{2}|\mathrm{D}\eta|^2 + F(\eta) = \frac{1}{2}\left((\eta_t)^2 + |\mathrm{D}\eta|^2\right) + F(\eta)\\ &= \frac{1}{2}\pi^2 + \frac{1}{2}|\mathrm{D}\eta|^2 + F(\eta)\end{aligned}$$

Hamilton 方程

$$\dot{\pi} = -\frac{\delta\mathcal{H}}{\delta\eta} = -\left(\frac{\partial\mathcal{H}}{\partial\eta} - \frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial\mathcal{H}}{\partial\eta_i}\right) = -F'(\eta) + \Delta\eta$$

$$\Rightarrow \dot{\pi} = \Delta\eta - f(\eta) \qquad f = F'$$

$$\dot{\eta} = \frac{\delta\mathcal{H}}{\delta\pi} = \frac{\partial\mathcal{H}}{\partial\pi} - \frac{\mathrm{d}}{\mathrm{d}x^j}\cdot\frac{\partial\mathcal{H}}{\partial\pi_j} = \pi$$

$$\Rightarrow \ddot{\eta} = \Delta\eta - f(\eta) \Rightarrow \boxed{\partial_{tt}\eta - \Delta\eta + f(\eta) = 0}$$

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2024 “海洋科学的数学原理” 夏令营

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2024/8/5 — 8/9