海洋科学的数学原理

第二讲 流体运动控制方程

袁海荣

华东师范大学 数学科学学院

2024 年 8 月 5 日

第二讲 流体运动控制方程

- 预备知识: 微积分、线性代数
- Navier-Stokes (N-S) 方程组
- 地球旋转坐标系下的 N-S 方程组: 坐标变换, 科氏力
- 海水运动的尺度分析
- 雷诺平均的 N-S 方程组

一. 运动的描述: Lagrange 方法与 Euler 方法

参考构型与含参映射

- $\Omega \subset \mathbb{R}^3$: 物理空间, 流体运动占据的区域, 其中的点用 X 表示
- $\varphi_t : \Omega \to \Omega, t \in \mathbb{R}_+ = [0, \infty)$ 描述流体的运动. (数学建模时假设 $X = \varphi_t(x)$ 关于 t, x 可微, 固定 t 时关于 x 可逆, 且逆映射也可微)
- $\Omega_0 \subset \Omega$: 参考构形 (参数区域)
- $\varphi_t(\Omega_0)$: $t = 0$ 时的流体微元 (点) 在 t 时刻所占据的空间区域, $\varphi_t(\Omega_0) \subset \Omega$

  $x \in \Omega_0$: Ω_0 中点 (微元) 用 x 表示 (参数)

  $X = \varphi_t(x) \in \Omega$: $t = 0$ 时在 x 点的流体微元在 t 时刻运动到 $X = \varphi_t(x)$ 点

微分方程: 由局部量确定整体量

- $X = \varphi(t,x) \doteq \varphi_t(x)$ 完全描述了流体运动
- **缺点:** 全局量, 难以实际观测和验证
- **目标:** 引入局部量确定 $X(t,x)$, 进而预测、控制流体运动
- **方法:** 由 $\varphi(t,x)$ 的导出量满足的物理定律, 建立数学模型 (偏微分方程), 求解 $\varphi(t,x)$
- **关键局部量—速度场:** $\boxed{v = v(t,X) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t,x)}$,

  其中 v 是导出的局部量 (可测量). 上式中 $x = \varphi_t^{-1}(X)$.

由速度场确定流体运动

- 假设: $v(t,X)$ 已知且充分光滑 (关于 X 是 Lipschitz 连续的). 由常微分方程组初值问题

$$\begin{cases} \dfrac{\mathrm{d}\varphi_t(x)}{\mathrm{d}t} = v(t,\varphi_t(x)) \\ \varphi_0(x) = x \end{cases} \quad \Rightarrow \quad \exists!\, X = \varphi(t,x) \doteq \varphi_t(x)$$

- 流线: $X = \varphi(\cdot,x)$ (至少局部存在)
- 数学研究: 利用数学工具进一步扩展所得模型适用范围

随体导数 (物质导数)

设 $f(t,X)$ 是 t 时刻位于 $X\in\mathbb{R}^3$ 处流体的某个状态量 (质量密度、温度、盐度等)

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}f(t,\varphi(t,x)) \qquad (X=\varphi(t,x))\\
=&\frac{\partial f}{\partial t}(t,\varphi(t,x))+\sum_{j=1}^{3}\frac{\partial f}{\partial X_j}(t,\varphi(t,x))\frac{\partial X_j}{\partial t}(t,x)\\
=&\left(\frac{\partial f}{\partial t}+v\cdot\nabla f\right)(t,\varphi(t,x))
\end{aligned}
$$

$$
\Rightarrow \boxed{\frac{\mathrm{D}f}{\mathrm{D}t}(t,X)\doteq\frac{\mathrm{d}}{\mathrm{d}t}f(t,\varphi(t,x))=\left(\frac{\partial f}{\partial t}+v\cdot\nabla f\right)(t,X)}
$$

注意 梯度 $\nabla f(t,X)=(\frac{\partial f}{\partial X_1},\frac{\partial f}{\partial X_2},\frac{\partial f}{\partial X_3})(t,X)$; $v\cdot\nabla f$: 向量内积

流体运动的 Lagrange 表述与 Euler 表述

- **Euler 坐标:** 空间坐标, 以 $X \in \mathbb{R}^3$ 为空间自变量表示物理状态量 $f(t, X)$
- **Lagrange 坐标:** 参数坐标, 以 $x \in \Omega_0$ 为空间自变量表示物理状态量 $\tilde{f}(t, x) \doteq f(t, \varphi(t, x))$
- 两种表述的关系: $\boxed{\tilde{f}(t, x) = f(t, \varphi(t, x)), \quad X = \varphi(t, x)}$
- 函数的重要性: 何时何地, 何事
- 测度的重要性: 什么是质量?

二. 含参变量积分求导公式

$$\varphi_t : \Omega_0 \to \Omega_t, \quad x \mapsto \varphi(t,x) = X; \quad v(t,X) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t,x), \quad x = \varphi_t^{-1}(X)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} f(t,X)\,\mathrm{d}X = \int_{\Omega_t}\left(\frac{\partial f}{\partial t} + \mathrm{div}_X(vf)\right)\mathrm{d}X$$

注意 (i) 散度: 对向量场 $v(t,X) = (v_1(t,X), v_2(t,X), v_3(t,X))^\top$, 其散度

$$\mathrm{div}_X v(t,X) \doteq \sum_{j=1}^{3} \frac{\partial v_j(t,X)}{\partial X_j}, \quad X = (X_1, X_2, X_3)^\top$$

(ii) 积分微元: $\mathrm{d}X = \mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3$ (Lebesgue 测度)

证明所用工具

(1) 重积分变量替换公式: $(X = \varphi(t,x))$

$$\boxed{\int_{\Omega_t} f(t,X)\,\mathrm{d}X = \int_{\Omega_0} f(t,\varphi(t,x))|J(t,x)|\,\mathrm{d}x,}$$

$$J(t,x) = \det\left[\left(\frac{\partial X_i}{\partial x_j}\right)_{i,j}\right] \qquad \text{Jacobi 行列式}$$

(2) 行列式求导公式:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}(\det A(t)) = \mathrm{tr}\left(A(t)^{-1}\dot{A}(t)\right)\det A(t)}$$

其中假设 $A(t): t \in \mathbb{R} \to M_{n\times n}(\mathbb{R})$ (n 行 n 列矩阵) 可逆, 而 $\dot{A}(t) = \frac{\mathrm{d}}{\mathrm{d}t}A(t) \doteq (\frac{\mathrm{d}}{\mathrm{d}t}A_{ij}(t))_{i,j}$ (对每个元素求导得到的矩阵); $\mathrm{tr}(A)$: 矩阵 A 的迹

注意 稍后证明上述求导公式

证明 由标准的含参积分求导公式

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} f(t,X)\,\mathrm{d}X \\
=&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0} f(t,\varphi(t,x))|J(t,x)|\,\mathrm{d}x \\
=&\int_{\Omega_0}\left[\frac{\partial f}{\partial t}(t,\varphi(t,x)) + v(t,\varphi(t,x))\cdot\nabla_X f(t,\varphi(t,x))\right]|J(t,x)|\,\mathrm{d}x \\
&\quad + \int_{\Omega_0} f(t,\varphi(t,x))\frac{\partial}{\partial t}|J(t,x)|\,\mathrm{d}x.
\end{aligned}
$$

根据行列式求导公式,

$$\frac{\partial}{\partial t}J(t,x) = \operatorname{tr}\left(\left(\frac{\partial\varphi}{\partial x}(t,x)\right)^{-1}\frac{\partial}{\partial t}\left(\frac{\partial\varphi(t,x)}{\partial x}\right)\right)J(t,x)$$

$$=J(t,x)\operatorname{tr}\left(\left(\frac{\partial\varphi}{\partial x}(t,x)\right)^{-1}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\varphi(t,x)\right)\right) \quad \text{(求导换序)}$$

$$=J(t,x)\operatorname{tr}\left(\left(\frac{\partial\varphi(t,x)}{\partial x}\right)^{-1}\frac{\partial}{\partial x}v(t,X(t,x))\right) \quad (v\ \text{的定义})$$

$$=J(t,x)\operatorname{tr}\left(\left(\frac{\partial\varphi(t,x)}{\partial x}\right)^{-1}\frac{\partial v(t,X)}{\partial X}\frac{\partial\varphi(t,x)}{\partial x}\right) \quad \text{(链式法则)}$$

$$=J(t,x)\operatorname{tr}\left(\frac{\partial v(t,X)}{\partial X}\right) \quad \text{(相似矩阵有相同的迹)}$$

$$=J(t,x)\operatorname{div}_X v(t,X) \quad \text{(定义)}$$

设 φ_t 为 C^1 双射, 可设 $J(t,x)$ 不变号 $\Rightarrow$

$$\boxed{\frac{\partial}{\partial t}|J(t,x)| = |J(t,x)| \operatorname{div}_X v(t,\varphi(t,x)).}$$

意义 散度为 0 的速度场 $v(t,X)$ $(\operatorname{div}_X v(t,X) \equiv 0)$ 对应的流体运动保持体积不变 (不可压缩性):

$$\text{此时} \quad |J(t,x)| = |J(0,x)| = 1 \quad \Rightarrow$$

$$\int_{\Omega_t} \mathrm{d}X = \int_{\Omega_0} |J| \,\mathrm{d}x = \int_{\Omega_0} \mathrm{d}x \Rightarrow |\Omega_t| = |\Omega_0|\,.$$

由上述 Jacobi 行列式求导公式,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} f(t,X)\,\mathrm{d}X = \int_{\Omega_0}\left(\frac{\partial f}{\partial t} + \underbrace{v\cdot\nabla_X f + f\,\mathrm{div}_X v}_{\mathrm{div}_X(fv)}\right)(t,\varphi(t,x))|J(t,x)|\,\mathrm{d}x$$

$$= \int_{\Omega_0}\left(\frac{\partial f}{\partial t} + \mathrm{div}_X(fv)\right)(t,\varphi(t,x))\,\underbrace{|J(t,x)|\,\mathrm{d}x}_{\mathrm{d}X}$$

$$= \int_{\Omega_t}\left(\frac{\partial f}{\partial t} + \mathrm{div}(fv)\right)(t,X)\,\mathrm{d}X.$$

□

推论 当 $F(t,X)$ 是向量时,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} F(t,X)\,\mathrm{d}X = \int_{\Omega_t}\left(\frac{\partial F}{\partial t} + \operatorname{div}(F\otimes v)\right)(t,X)\,\mathrm{d}X,}$$

其中 $F\otimes v = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} F_1v_1 & F_1v_2 & F_1v_3 \\ F_2v_1 & F_2v_2 & F_2v_3 \\ F_3v_1 & F_3v_2 & F_3v_3 \end{pmatrix}$, 对矩阵

$B(x) = \begin{pmatrix} B_1(x) \\ B_2(x) \\ B_3(x) \end{pmatrix}$, $B_i(x)$ 为行向量, 定义其散度 $\operatorname{div} B(x) \doteq \begin{pmatrix} \operatorname{div} B_1(x) \\ \operatorname{div} B_2(x) \\ \operatorname{div} B_3(x) \end{pmatrix}$.

行列式求导公式

设 $A(t) \in C^1$ 且非奇异, 则

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det A(t)) = \operatorname{tr}\left(A(t)^{-1}\dot{A}(t)\right)\det A(t)$$

证明 (1) $A(0) = I_n$ (n 阶单位阵) 情形:

$A(t) = (A_1(t), \cdots, A_n(t))$, $A(0) = (e_1, \cdots, e_n)$.

由行列式定义 (按列展开)

$$\det A(t) = \sum_{(i_1 \cdots i_n)} (-1)^{\tau(i_1 \cdots i_n)} a_{i_1,1}(t) a_{i_2,2}(t) \cdots a_{i_n,n}(t)$$

$$
\begin{aligned}
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\det A(t)\Big|_{t=0} &= \det\left(\dot{A}_1(t), A_2(t), \cdots, \mathrm{A_n}(t)\right)\big|_{t=0} \\
&\quad + \det\left(A_1(t), \dot{A}_2(t), \cdots, A_n(t)\right)\big|_{t=0} \\
&\quad + \cdots + \det\left(A_1(t), A_2(t), \cdots, \dot{A}_n(t)\right)\big|_{t=0} \\
&= \det\left(\dot{A}_1(0), e_2, \cdots, e_n\right) + \det\left(e_1, \dot{A}_2(0), \cdots, e_n\right) \\
&\quad + \cdots + \det\left(e_1, e_2, \cdots, \dot{A}_n(0)\right) \\
&= \dot{a}_{11}(0) + \dot{a}_{22}(0) + \cdots + \dot{a}_{nn}(0) = \operatorname{tr}\dot{A}(0).
\end{aligned}
$$

(2) 若 $A(0) \neq I_n$, 令 $B(t) \doteq A(0)^{-1}A(t)$, 则 $B(0) = I_n$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \det B(t)\bigg|_{t=0} = \operatorname{tr} \dot{B}(0) = \operatorname{tr}\left(A(0)^{-1}\dot{A}(0)\right)$$

又
$$\frac{\mathrm{d}}{\mathrm{d}t} \det B(t)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\det(A(0)^{-1}) \det A(t)\right)\bigg|_{t=0} = \left(\det A(0)\right)^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \det A(t)\bigg|_{t=0}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} \det A(t)\bigg|_{t=0} = \operatorname{tr}\left(A(0)^{-1}\dot{A}(0)\right) \det A(0).$$

(3) 令 $C(h)=A(t+h)$, 则 $\frac{\mathrm{d}}{\mathrm{d}h}C(h)\big|_{h=0}=\dot{A}(t)$,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}h}\det C(h)|_{h=0}=\operatorname{tr}\left(C(0)^{-1}\dot{C}(0)\right)\det C(0)\\
\Rightarrow&\frac{\mathrm{d}}{\mathrm{d}h}\det A(t+h)|_{h=0}=\operatorname{tr}\left(A(t)^{-1}\dot{A}(t)\right)\det A(t)\\
\Rightarrow&\frac{\mathrm{d}}{\mathrm{d}t}\det A(t)=\operatorname{tr}(A(t)^{-1}\dot{A}(t))\det A(t).
\end{aligned}
$$

□

推论 设 $\dot{B}(t) = CB(t)$, C 为常数矩阵, 则

$$(\det B(t))' = (\operatorname{tr} C) \det B(t).$$

证明

$$\begin{aligned}(\det B(t))' &= \operatorname{tr}\left(B(t)^{-1}\dot{B}(t)\right) \det B(t) \\ &= \operatorname{tr}(B(t)^{-1}CB(t)) \det B(t) = (\operatorname{tr}C) \det B(t).\end{aligned}$$

$$\boxed{B(t) = \mathrm{e}^{Ct}B_0 \Rightarrow \det B(t) = \mathrm{e}^{(\operatorname{tr}C)t} \det B_0.}$$

□

注意 矩阵指数函数 $\mathrm{e}^C = \sum_{k=0}^{\infty} \frac{1}{k!} C^k$.

三. 速度场的散度与旋度

速度场的分解

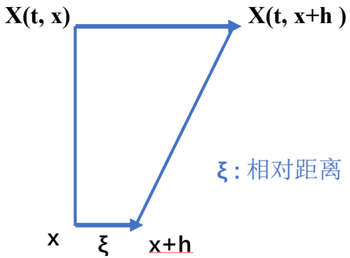
以下记 $X = X(t,x) \doteq \varphi(t,x)$.

$$
\begin{aligned}
\frac{\mathrm{d}X(t,x)}{\mathrm{d}t} &= v(t, X(t,x)),\\
\frac{\mathrm{d}X(t,x+h)}{\mathrm{d}t} &= v(t, X(t,x+h)) = v(t, X(t,x))\\
&\quad + \frac{\partial v}{\partial X}(t, X(t,x))(X(t,x+h) - X(t,x))\\
&\quad + O\left(|X(t,x+h) - X(t,x)|^2\right)
\end{aligned}
$$

$$
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(X(t,x+h) - X(t,x)) = \frac{\partial v}{\partial X}(t, X(t,x))(X(t,x+h) - X(t,x)) + O(|h|^2).
$$

令 $\xi = \xi(t,x;h) = X(t,x+h) - X(t,x)$, 线性化方程:

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(t,x;h) = \frac{\partial v}{\partial X}(t,X)\xi(t,x;h) = \mathrm{D}\,v(t,X)\xi + \mathrm{S}\,v(t,X)\xi,$$



其中

$$\mathrm{D}\, v(t,X) = \frac{1}{2}\left(\frac{\partial v}{\partial X}(t,X) + \left(\frac{\partial v}{\partial X}(t,X)\right)^{\top}\right) \qquad \text{(实对称阵)}$$

$$\operatorname{tr} \mathrm{D} v = \operatorname{div}_X v \qquad \text{(散度)}$$

$$\mathrm{S}\, v(t,X) = \frac{1}{2}\left(\frac{\partial v}{\partial X}(t,X) - \left(\frac{\partial v}{\partial X}(t,X)\right)^{\top}\right) \qquad \text{(反对称阵)}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

旋度

$$\omega = \nabla \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_1 & \partial_2 & \partial_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{pmatrix} \frac{\partial v_3}{\partial X_2} - \frac{\partial v_2}{\partial X_3} \\ \frac{\partial v_1}{\partial X_3} - \frac{\partial v_3}{\partial X_1} \\ \frac{\partial v_2}{\partial X_1} - \frac{\partial v_1}{\partial X_2} \end{pmatrix}$$

$$\boxed{\underbrace{Sv(t,X)\xi}_{\text{矩阵乘法}} = \underbrace{\frac{1}{2}\omega \times \xi}_{\text{外积}}}$$

$$\Rightarrow \frac{\mathrm{d}\xi}{\mathrm{d}t} = (\mathrm{D}v)\xi + (\mathrm{S}v)\xi.$$

算子分裂 $\mathrm{e}^{t(A+B)} - \mathrm{e}^{tA}\mathrm{e}^{tB} = -\frac{1}{2}[A,B]t^2 + O(t^3), \quad [A,B] \doteq AB - BA$

散度的意义

散度部分诱导的运动 $\boxed{\frac{\mathrm{d}\xi}{\mathrm{d}t} = (\mathrm{D}v)\xi}$

作为线性近似, Dv 的自变量固定为 (t_0, X_0). 将 $\mathrm{D}v(t_0, X_0)$ 对角化: $\exists\, \mathbb{R}^3$ 中标准正交基 $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$, 使得

$$\mathrm{D}v = \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix},$$

在新坐标系下

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = (\mathrm{D}v)\xi \;\Rightarrow\; \frac{\mathrm{d}\xi_1}{\mathrm{d}t} = d_1\xi_1, \quad \frac{\mathrm{d}\xi_1}{\mathrm{d}t} = d_2\xi_2, \quad \frac{\mathrm{d}\xi_3}{\mathrm{d}t} = d_3\xi_3.$$

伸缩变形: 微元体积相对变化率

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\xi_1\xi_2\xi_3) &= d_1\xi_1\xi_2\xi_3 + \xi_1 d_2\xi_2\xi_3 + \xi_1\xi_2 d_3\xi_3 \\
&= (d_1 + d_2 + d_3)\,\xi_1\xi_2\xi_3 \\
&= (\mathrm{tr}\,\mathrm{D}v)\xi_1\xi_2\xi_3 = (\mathrm{div}_X\, v)\xi_1\xi_2\xi_3.
\end{aligned}
$$

$$\Rightarrow \frac{\frac{\mathrm{d}}{\mathrm{d}t}|\Omega_t|}{|\Omega_t|} \sim \mathrm{div}_X v \quad \text{即: 散度为微元流动时体积的相对变化率,}$$

与

$$\frac{\partial}{\partial t}|J(t,x)| = \mathrm{div}_X v(t, X(t,x))|J(t,x)|$$

相符.

旋度的意义

旋度部分诱导的运动 $\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\xi = (\mathrm{S}\,v)\xi = \frac{1}{2}\,\omega \times \xi \;\Rightarrow\; \xi(t) = \mathrm{e}^{(\mathrm{S}v)t}\xi_0.}$

利用 $\mathrm{S}v = \mathrm{S}v(t_0, X_0)$ 的反对称性以及 $\mathrm{S}v$ 与 $\mathrm{S}v^\top$ 可交换,

$$\left(\mathrm{e}^{(\mathrm{S}v)t}\right)\left(\mathrm{e}^{(\mathrm{S}v)t}\right)^\top = \mathrm{e}^{\mathrm{S}vt}\cdot \mathrm{e}^{(\mathrm{S}v)^\top t}$$

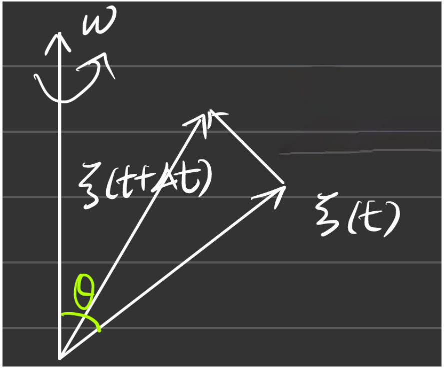
$$= \mathrm{e}^{\left(\mathrm{S}v+\mathrm{S}v^\top\right)t}$$

$$= \mathrm{e}^{0t} = I_n.$$

$\Rightarrow \quad \mathrm{e}^{(\mathrm{S}v)t}$ 为正交阵 $\Rightarrow \xi(t)$ 是 ξ_0 的旋转.

旋转轴

$$\frac{\mathrm{d}\xi}{\mathrm{d}t}\Delta t = \frac{1}{2}\omega \times \xi(t)\Delta t \Rightarrow \begin{cases} |\xi(t+\Delta t)| = |\xi(t)| \\ \frac{\mathrm{d}\xi}{\mathrm{d}t}\Delta t \perp \omega, \xi \end{cases} \Rightarrow \omega \text{ 为旋转轴;}$$



角速度

单位时间转过的角度 (弧度 = 弧长/半径):

$$\frac{\left|\frac{d\xi}{dt}\Delta t\right|}{|\xi(t)|\sin\theta}\frac{1}{\Delta t} = \frac{\frac{1}{2}|\omega\times\xi(t)|}{|\xi(t)|\sin\theta} = \frac{1}{2}\frac{|\omega||\xi(t)|\sin\theta}{|\xi(t)|\sin\theta} = \frac{1}{2}|\omega|,$$

即旋度诱导的微元运动是以向量 ω 为旋转轴, 角速度为 $\frac{1}{2}|\omega|$ 的旋转.

四. Navier-Stokes 方程组

质量守恒定律

$\rho(t,X)$: 质量密度函数. 质量守恒定律 $\int_{\Omega_t}\rho(t,X)\mathrm{d}X = \Omega_0$ 中流体质量 $\Leftrightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\rho(t,X)\,\mathrm{d}X = 0 \Leftrightarrow \int_{\Omega_t}\left(\frac{\partial\rho}{\partial t} + \mathrm{div}_X(\rho v)\right)(t,X)\,\mathrm{d}X = 0.$$

Lagrange 表述

由 Ω_t 任意性, 以及 $\int_{\Omega_t}\rho(t,X)\,\mathrm{d}X = \int_{\Omega_0}\rho_0(x)\,\mathrm{d}x$,

左端 $= \int_{\Omega_0}\rho(t,X)|J(t,x)|\,\mathrm{d}x \Rightarrow \rho(t,X(t,x))|J(t,x)| = \rho_0(x)$.

Euler 表述

$$\boxed{\frac{\partial\rho}{\partial t} + \mathrm{div}_X(\rho v) = 0} \quad \text{(连续性方程)}$$

动量守恒定律 (牛顿第二定律)

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \rho v(t,X)\,\mathrm{d}X}_{\text{动量变化率}} = \underbrace{\int_{\Omega_t} \rho b(t,X)\,\mathrm{d}X}_{\text{体积力}} + \underbrace{\int_{\partial\Omega_t} \tau(n)(t,Y)\,\mathrm{d}S_Y}_{\text{表面力}}$$

$b(t,X)$: 单位质量流体受到的体积力 (如重力、电场力)

n: $\partial\Omega_t$ 的外单位法向量

$\mathrm{d}S_Y$: Y 点处曲面 $\partial\Omega_t$ 的面积元 (曲面上面积测度)

$\tau(n)(t,Y)$: 法向为 n 的 $\partial\Omega_t$ 上 Y 点处单位面积微元所受外力 (例: 气体压强)

注意: 此力一般来讲不与 n 平行

Cauchy 应力定理

$\tau(n)(t,Y)=\sigma(t,Y)n$, 其中 $\sigma(t,Y)$ 为 3×3 阶矩阵 (应力张量).

注意 σ 的三列分别对应在 e_1, e_2, e_3 三个方向的表面力.

散度定理

$$\int_\Omega \mathrm{div}_X\, v(X)\,\mathrm{d}X=\int_{\partial\Omega} v(Y)\cdot n(Y)\,\mathrm{d}S_Y$$

动量守恒 $\Rightarrow$

$$\int_{\Omega_t}\left(\frac{\partial}{\partial t}(\rho v)+\mathrm{div}_X(\rho v\otimes v)\right)\mathrm{d}X=\int_{\Omega_t}\rho b\,\mathrm{d}X+\int_{\Omega_t}\mathrm{div}_X(\sigma(t,X))\,\mathrm{d}X$$

$$\Rightarrow\quad \boxed{\partial_t(\rho v)+\mathrm{div}_X(\rho v\otimes v)=\rho b+\mathrm{div}_X\sigma}$$

应力张量的确定

- 角动量守恒 $\Rightarrow \sigma$ 为对称阵 (推导略) $\sigma = F(\mathrm{D}v) - PI_3$

  $P(t,X)$: (标量函数) 流体静止时压强 $F(\mathrm{D}v)$: (实对称阵) 流体有相对运动时粘性力 $(\mathrm{div}(PI_3) = \nabla_X P.)$

- Newton 流体: 此处假设了 F 只依赖于 $\mathrm{D}v$,

$$\mathrm{D}v = \frac{1}{2}\left(\nabla v + (\nabla v)^\top\right) \text{ 或 } (\mathrm{D}v)_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right), \quad 1 \leqslant i,j \leqslant 3.$$

- 各向同性、客观性(与旋转、平移、Galileo 变换无关)$\Rightarrow F = 2\mu\mathrm{D}v + \lambda(\mathrm{div}v)I_3$

  μ : shear viscosity; λ : bulk viscosity

- 热力学第二定律: $\mu \geqslant 0, \quad 2\mu + 3\lambda \geqslant 0$

动量守恒方程

$$\boxed{\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla P - \mu \Delta v - (\lambda + \mu)\nabla \operatorname{div} v = \rho b}$$

可压缩等熵 Navier-Stokes 方程组

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \quad P = P(\rho) \quad \text{状态方程}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) - \mu \Delta v + \nabla (P - (\lambda + \mu) \operatorname{div} \mathrm{v}) = \rho b,$$

($\lambda = \mu = 0$ 时: 等熵可压缩欧拉方程组. $P = \text{const}.\rho^2$ 时: 浅水波方程,此时 $\Omega \subset \mathbb{R}^2$, ρ 代表水高.)

向量场公式 $\Delta v = \nabla \operatorname{div} v - \nabla \times (\nabla \times v)$, Laplace 算子: $\Delta v(X) = \operatorname{div}\nabla v = \sum_{i=1}^{3} \frac{\partial^2 v}{\partial X_i^2}$

不可压 Navier-Stokes 方程组

不可压: $\operatorname{div} v = 0$

质量守恒: $\partial_t \rho + v \cdot \nabla \rho = 0$

动量守恒: $\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) - \mu \Delta v + \nabla P = \rho b$

$$\Rightarrow \partial_t v + v \cdot \nabla v - \frac{\mu}{\rho} \Delta v + \frac{\nabla P}{\rho} = b$$

注意 (i) $v \cdot \nabla v = \sum_{j=1}^{3} v_j \partial_{X_j} v$. 此处函数 v, ρ, P 的自变量均为 (t, X).

(ii) 在质量守恒成立时, 密度 $\rho =$ 常数 $\Rightarrow$ 不可压, 但反之不一定.

标准不可压 Navier-Stokes 方程组

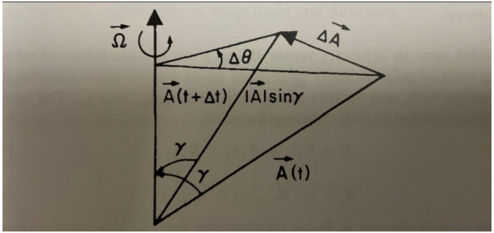
假设 $\rho \equiv$ 常数,

$$\left.\begin{aligned} \operatorname{div} v = 0 \\ v_t + v \cdot \nabla v - \varepsilon \Delta v + \nabla P = b \end{aligned}\right\} \text{(N-S)}$$

第二个方程:

$$\underbrace{\frac{\mathrm{D}v}{\mathrm{D}t}}_{\text{物质导数}} - \varepsilon \Delta v + \nabla P = b$$

五. 地转坐标下的 N-S 方程组



标架旋转方程

- Ω 为旋转轴, $|\Omega|$ 为角速度.　(不要和区域 Ω 混淆!)
- 向量 A 绕 Ω 旋转. 在旋转系中看 A 是常向量. 设 γ 为 A 与 Ω 之夹角, 保持不变. 相对于惯性系, A 依赖于时间 t, 但长度不变, 记为 $A(t)$.

Δt 时间: $\quad \Delta A = A(t+\Delta t) - A(t)$

旋转角度: $\quad \Delta\theta = |\Omega|\Delta t$

$\triangle A$ 的方向: $\quad \vec{n} = \frac{\Omega\times A(t)}{|\Omega\times A(t)|}$

$\triangle A$ 的大小: $\quad |\Delta A| = \Delta\theta \cdot |A| \sin\gamma$

$$\Rightarrow \quad \lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\theta|A|\sin\gamma}{\Delta t}\cdot\frac{\Omega\times A(t)}{|\Omega\times A(t)|}$$

$$= |\Omega||A|\sin\gamma\frac{\Omega\times A}{|\Omega\times A|} = \Omega\times A$$

旋转方程

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} = \Omega\times A}$$

旋转方程 $\frac{\mathrm{d}A}{\mathrm{d}t} = \Omega \times A.$

记 $\Omega = (\omega_1, \omega_2, \omega_3)$,

$$S \doteq \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\frac{\mathrm{d}A}{\mathrm{d}t} = SA}$$

$$\Rightarrow \ A(t) = \mathrm{e}^{St} A(0)$$

变长度向量的旋转方程

设 $B = B_1\vec{e}_1 + B_2\vec{e}_2 + B_3\vec{e}_3$, $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ 为旋转标架 (单位长度, 正交且随 Ω 旋转), 则相对于惯性系:

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \underbrace{\frac{\mathrm{d}B_1}{\mathrm{d}t}\vec{e}_1 + \frac{\mathrm{d}B_2}{\mathrm{d}t}\vec{e}_2 + \frac{\mathrm{d}B_3}{\mathrm{d}t}\vec{e}_3}_{\text{在旋转系中观察看到的现象, 记为}\left(\frac{\mathrm{d}B}{\mathrm{d}t}\right)_R} + B_1\frac{\mathrm{d}\vec{e}_1}{\mathrm{d}t} + B_2\frac{\mathrm{d}\vec{e}_2}{\mathrm{d}t} + B_3\frac{\mathrm{d}\vec{e}_3}{\mathrm{d}t}$$

$$= \left(\frac{\mathrm{d}B}{\mathrm{d}t}\right)_R + B_1(\Omega \times \vec{e}_1) + B_2(\Omega \times \vec{e}_2) + B_3(\Omega \times \vec{e}_3)$$

$$= \left(\frac{\mathrm{d}B}{\mathrm{d}t}\right)_R + \Omega \times B$$

即 $\boxed{\frac{\mathrm{d}B}{\mathrm{d}t} = \left(\frac{\mathrm{d}B}{\mathrm{d}t}\right)_R + \Omega \times B.}$

速度与加速度转换关系

速度: $\frac{\mathrm{d}X}{\mathrm{d}t} = \left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)_R + \Omega \times X$, 其中 $X = \varphi(t, x)$, 注意 $\Omega \times X$ 与坐标系无关.

于是 $\boxed{v = v_R + \Omega \times X,}$ (注意: 以下式子中的重复指标从 1 到 3 自动求和)

$$v = \underbrace{v_i \vec{e}_i}_{=v_R} + X_i\, \Omega \times \vec{e_i} \quad \Rightarrow$$

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{\mathrm{d}v_i}{\mathrm{d}t}\vec{e}_i + v_i\frac{\mathrm{d}\vec{e_i}}{\mathrm{d}t} + v_i\Omega \times \vec{e_i} + v_i\frac{\mathrm{d}\Omega}{\mathrm{d}t} \times \vec{e}_i + X_i\Omega \times \frac{\mathrm{d}e_i}{\mathrm{d}t} \\
&= \frac{\mathrm{d}v_i}{\mathrm{d}t}\vec{e}_i + v_i\left(\Omega \times \vec{e}_i\right) + v_i\Omega \times \vec{e}_i + \frac{\mathrm{d}\Omega}{\mathrm{d}t} \times v_R + X_i\Omega \times \left(\Omega \times \vec{e}_i\right) \\
&= \frac{\mathrm{d}v_i}{\mathrm{d}t}\vec{e_i} + 2\Omega \times v_R + \frac{\mathrm{d}\Omega}{\mathrm{d}t} \times v_R + \Omega \times (\Omega \times X)
\end{aligned}$$

- 即 (对质点 (微元) 而言)

$$\underbrace{\frac{\mathrm{d}v}{\mathrm{d}t}}_{\text{绝对加速度}} = \underbrace{\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_R}_{\text{相对加速度 (旋转系中观察看)}} + \underbrace{2\Omega \times v_R}_{\text{Coriolis 加速度}} + \underbrace{\frac{\mathrm{d}\Omega}{\mathrm{d}t} \times v_R}_{\text{旋转系加速度}} + \underbrace{\Omega \times (\Omega \times X)}_{\text{向心加速度}}$$

其中 $\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_R \doteq \sum_{i=1}^{3} \frac{\mathrm{d}v_i}{\mathrm{d}t}\vec{e}_i$. 以下设 $\frac{\mathrm{d}\Omega}{\mathrm{d}t} = 0$.

- 由于变换

$$y = \mathrm{e}^{St}x : \ x \mapsto y, \quad (S \text{ 反对称}, \mathrm{e}^{St} \text{ 正交})$$

保距离 (度量), 关于空间变量的散度、梯度、Laplace 算子形式不变.

例: $\mathrm{div}_y\, v(\mathrm{e}^{-St}y) = \mathrm{div}_x\, v(x)$.

旋转系下 N-S 方程 (以下将 v_R 写为 v)

$$\operatorname{div} v = 0$$

$$\underbrace{\frac{\mathrm{d}v}{\mathrm{d}t}}_{\text{相对加速度}} + 2\Omega \times v + \Omega \times (\Omega \times X) + \nabla P - \varepsilon \Delta V = b$$

因为 $\Omega \times (\Omega \times X)$ 与流动状态无关, 可以作为体积力吸收到 b 中, 从而得到

$$\begin{cases} \operatorname{div} v = 0 \\ \frac{\mathrm{D}v}{\mathrm{D}t} - \varepsilon \Delta v + \nabla P + 2\Omega \times v = b \end{cases} \qquad \varepsilon: \text{分子粘性}$$

其中 $\frac{\mathrm{D}v}{\mathrm{D}t} = \frac{\partial v}{\partial t} + v \cdot \nabla v.$

旋转系下标量的随体导数 (用于旋转系下盐度等物质输运平流方程的推导)

设 x 为空间固定一点, 其在旋转坐标系中表示为 $x = \sum_{i=1}^{3} \vec{e}_i(t)y_i(t)$,

$$\underbrace{f(t,x)}_{\text{惯性系}} = f(t, \vec{e}_i(t)y_i(t)) = \underbrace{\tilde{f}(t, y_1(t), y_2(t), y_3(t))}_{\text{旋转系}}$$

因为 $x = \vec{e}_j y_j \Rightarrow 0 = \frac{\mathrm{d}\vec{e}_j}{\mathrm{d}t} y_j + \vec{e}_j \frac{\mathrm{d}y_j}{\mathrm{d}t}$, 并且 $\frac{\mathrm{d}\vec{e}_j}{\mathrm{d}t} = \Omega \times \vec{e}_j$

$$\Rightarrow \vec{e}_j \frac{\mathrm{d}y_j}{\mathrm{d}t} = -\Omega \times x \Rightarrow \boxed{\frac{\mathrm{d}y_j}{\mathrm{d}t} = -(\Omega \times x) \cdot \vec{e}_j}$$

于是

$$\begin{aligned}\frac{\partial f}{\partial t}(t,x) &= \frac{\partial \tilde{f}}{\partial t} + \frac{\partial \tilde{f}}{\partial y_j}\frac{\mathrm{d}y_j}{\mathrm{d}t} \\ &= \frac{\partial \tilde{f}}{\partial t} - (\Omega \times x)\cdot\left(\frac{\partial \tilde{f}}{\partial y_j}\vec{e}_j\right) \\ &= \frac{\partial \tilde{f}}{\partial t} - (\Omega \times x)\cdot(\nabla_y \tilde{f}).\end{aligned}$$

由

$$x = \vec{e}_k y_k \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = Q\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

其中 $Q = Q(t)$ 为正交阵, 则

$$\frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial y_j} = \frac{\partial \tilde{f}}{\partial y_j}$$

$$\frac{\partial x_i}{\partial y_j} = Q_{ij} \Rightarrow Q_{ij}\frac{\partial f}{\partial x_i} = \frac{\partial \tilde{f}}{\partial y_j} \Rightarrow \left(Q^\top \frac{\partial f}{\partial x}\right)_j = \left(\frac{\partial \tilde{f}}{\partial y}\right)_j,$$

即

$$Q^\top \nabla_x f = \nabla_y \tilde{f} \ \Rightarrow \ \nabla_x f = Q\nabla_y \tilde{f} = \vec{e}_i \partial_{y_i} \tilde{f},$$

于是在惯性系下,

$$v \cdot \nabla_x f = (v_R + \Omega \times x) \cdot \left(\vec{e}_i \partial_{y_i} \tilde{f}\right)$$

- $v_R = v_{R_i}\vec{e}_i \Rightarrow v_R \cdot (\vec{e}_i \partial_{y_i} f) = v_{R_j}\partial_{y_j}\tilde{f} = v_R \cdot \nabla_y \tilde{f}.$
- $(\Omega \times x) \cdot \left(\vec{e}_i \partial_{y_i} \tilde{f}\right) = (\Omega \times x) \cdot \nabla_y \tilde{f}$

$$\begin{aligned}\Rightarrow \frac{\partial f}{\partial t} + v \cdot \nabla_x f &= \frac{\partial \tilde{f}}{\partial t} - (\Omega \times x) \cdot \nabla_y \tilde{f} + v_R \cdot \nabla_y \tilde{f} + (\Omega \times x) \cdot \nabla_y \tilde{f} \\ &= \frac{\partial \tilde{f}}{\partial t} + v_R \cdot \nabla_y \tilde{f}.\end{aligned}$$

从而, 在惯性系与旋转系下, 随体导数不变:

$$\boxed{\left(\frac{\mathrm{D}f}{\mathrm{D}t}\right)_I = \left(\frac{\mathrm{D}f}{\mathrm{D}t}\right)_R}$$

六、尺度分析(无量纲化)

- 特征长度 L，特征时间 T，特征速度 L/T
- 无量纲量 $\bar{x}=\frac{x}{L},\ \bar{t}=\frac{t}{T},\ \bar{v}=\frac{v}{L/T}=\frac{vT}{L} \quad \Rightarrow \quad x=L\bar{x},\ t=T\bar{t},\ v=L\bar{v}/T$
- 方程各项的尺度变换

$$\frac{\partial v}{\partial t}=\frac{L}{T^2}\frac{\partial \bar{v}}{\partial \bar{t}}, \quad v\cdot\nabla_x v=\left(\frac{L}{T}\right)^2\bar{v}\cdot\nabla_{\bar{x}}\bar{v}\frac{1}{L}=\frac{L}{T^2}\bar{v}\cdot\nabla_{\bar{x}}\bar{v},$$

$$-\varepsilon\Delta_x v=-\varepsilon\Delta_{\bar{x}}\bar{v}\left(\frac{L}{T}\right)\frac{1}{L^2}=-\varepsilon\frac{1}{LT}\Delta_{\bar{x}}\bar{v},$$

$$\nabla P=\frac{1}{L}\nabla_{\bar{x}}P, \quad \Omega\times v=\Omega\times\bar{v}\frac{L}{T}$$

- 无量纲量的动量方程 $\Rightarrow \frac{L}{T^2}\frac{\mathrm{D}\bar{v}}{\mathrm{D}\bar{t}}-\varepsilon\frac{1}{LT}\Delta_{\bar{x}}\bar{v}+\frac{1}{L}\nabla_{\bar{x}}P+2\frac{L}{T}\Omega\times\bar{v}=b$

Reynolds 数

- 无量纲量的动量守恒方程

$$\Rightarrow \frac{\mathrm{D}\bar{v}}{\mathrm{D}\bar{t}} - \varepsilon\frac{T}{L^2}\Delta_{\bar{x}}\bar{v} + \frac{T^2}{L^2}\nabla_{\bar{x}}P + 2T\Omega \times \bar{v} = \frac{T^2}{L}b$$

- 特征速度: $u = L/T$
- Reynolds 数: $\boxed{\mathrm{Re} \doteq \frac{1}{\varepsilon}\frac{L^2}{T} = \frac{L^2}{\varepsilon T} = \frac{Lu}{\varepsilon}}$
- 无量纲压强和无量纲加速度

$$\bar{P} = PT^2/L^2 = P/u^2, \quad \bar{b} = \frac{T^2}{L}b = \frac{(L/u)^2}{L}b = \frac{L}{u^2}b$$

Rossby 数

- $\bar{\Omega} = \frac{\Omega}{|\Omega|}$, $2T\Omega \times \bar{v} = 2\frac{L}{u}|\Omega|\frac{\Omega}{|\Omega|} \times \bar{v} = \frac{1}{\varepsilon}\bar{\Omega} \times \bar{v} \Rightarrow \varepsilon \doteq \frac{1}{\frac{2L|\Omega|}{u}} = \frac{u}{2|\Omega|L}$
- 无量纲量 N-S 方程组

$$\Rightarrow \begin{cases} \operatorname{div} \bar{v} = 0 \\ \frac{\mathrm{D}\bar{v}}{\mathrm{D}t} - \frac{1}{\mathrm{Re}}\Delta\bar{v} + \nabla\bar{P} + \frac{1}{\varepsilon}\bar{\Omega} \times \bar{v} = \bar{b} \end{cases}$$

注意: $\bar{\Omega}$ 是单位常值向量

- ε 充分小 $\Rightarrow$ 旋转起主导作用, 地转 $\sim$ 惯性
- Re 充分小 $\Rightarrow$ 粘性主导，粘性 $\sim$ 惯性

七、旋度方程

$\omega \doteq \nabla \times v$, $\operatorname{div} v = 0$. 计算 $\nabla \times$ $\boxed{\frac{\partial}{\partial t} v + v \cdot \nabla v - \frac{1}{\mathrm{Re}} \Delta v + \nabla P + \frac{1}{\varepsilon} \bar{\Omega} \times v = b}$.

- 公式: $\boxed{\nabla \times (A \times B) = A \nabla \cdot B + (B \cdot \nabla) A - B \nabla \cdot A - (A \cdot \nabla) B}$

$$\Rightarrow \nabla \times (\bar{\Omega} \times v) = \bar{\Omega} \underbrace{\nabla \cdot v}_{=0} + v \cdot \nabla \bar{\Omega} - v \underbrace{\nabla \cdot \bar{\Omega}}_{=0} - \bar{\Omega} \cdot \nabla v = -\bar{\Omega} \cdot \nabla v, \ (\bar{\Omega} \text{ 常值})$$

- 公式: $\boxed{v \cdot \nabla v = \omega \times v + \nabla \left(\frac{|v|^2}{2} \right); \nabla \times (\nabla P) = 0; \nabla \cdot \omega = 0}$

$$\Rightarrow \nabla \times (v \cdot \nabla v) = \nabla \times (\omega \times v) = \omega \nabla \cdot v + v \cdot \nabla \omega - v \nabla \cdot \omega - \omega \cdot \nabla v$$

$$= v \cdot \nabla \omega - \omega \cdot \nabla v$$

旋度方程

$$\Rightarrow \frac{\partial}{\partial t}\omega + v \cdot \nabla \omega - \omega \cdot \nabla v - \frac{1}{\mathrm{Re}}\Delta \omega + \frac{1}{\varepsilon}(-\bar{\Omega} \cdot \nabla v) = \nabla \times b$$

$$\Rightarrow \boxed{\frac{\mathrm{D}}{\mathrm{D}t}\omega - \frac{1}{\mathrm{Re}}\Delta \omega - \omega \cdot \nabla v - \frac{1}{\varepsilon}\bar{\Omega} \cdot \nabla v = \nabla \times b}$$

八. Reynolds 平均 N-S 方程

湍流

- Re $\gg$ 1 时: “流动具有随机性”
- **N-S 方程**

$$\operatorname{div} v = 0$$

$$\frac{\partial}{\partial t}v + \underbrace{v\cdot\nabla v}_{\text{唯一的非线性项: 对流项}} \underbrace{-\frac{1}{\mathrm{Re}}\Delta v}_{\text{扩散项}} + \underbrace{\nabla P}_{\text{压差}} + \underbrace{\frac{1}{\varepsilon}\bar{\Omega}\times V}_{\text{地转}} = \underbrace{b}_{\text{外力}}$$

- 数学描述: 设解是随机的, $\omega \in \Omega$, 概率空间 $(\Omega, \mathcal{F}, \mathbb{P})$,

  $v = v(t, x; \omega), P = P(t, x; \omega)$, 可观测的物理量是随机函数 (v, P) 的统计量

平均场 + 脉动场分解

- 设 $v = U(x,t) + u(x,t;\omega)$, 其中平均场

$$U(x,t) = \underbrace{\mathbb{E}[v(x,t;\omega)]}_{\text{均值 (期望)}} = \int_\Omega v(x,t;\omega)\,\mathrm{d}\mathbb{P}(\omega) \doteq \langle v\rangle = \bar{v}$$

- 脉动场期望为零: $\langle u\rangle = \bar{u} = 0$
- 质量守恒方程: $0 = \operatorname{div} v = \operatorname{div}(U+u) = \operatorname{div} U + \operatorname{div} u$

$$\text{(取期望)} \;\Rightarrow\quad 0 = \langle \operatorname{div} U\rangle + \langle \operatorname{div} u\rangle = \operatorname{div} U + \underbrace{\operatorname{div}\langle u\rangle}_{=0}$$

$$\Rightarrow\quad \operatorname{div} U = 0 \quad \text{(平均速度场不可压)}$$

动量守恒方程 (*)

$$\partial_t (U_i + u_i) + \sum_{j=1}^{3} (U_j + u_j)\, \partial_j (U_i + u_i) = -\partial_i (P + p) + \frac{1}{\mathrm{Re}} \Delta (U_i + u_i) + b_i,\ i = 1, 2, 3.$$

取期望 $\Rightarrow$ $\boxed{\partial_t U_i + U_j \partial_j U_i + \partial_i P = \frac{1}{\mathrm{Re}} \Delta U_i + \bar{b}_i - \langle u_j \partial_j u_i \rangle}$

注意 (重复指标从 1 到 3 求和)

- $\langle \partial_i p \rangle = \partial_i \langle p \rangle = 0$
- $\langle u_j \partial_j u_i \rangle = \langle \partial_j (u_i u_j) \rangle = \partial_j \langle u_i u_j \rangle$ (利用 $\langle \partial_i u_i \rangle = 0$)
- $\langle (U_j + u_j)\, \partial_j (U_i + u_i) \rangle = \langle U_j \partial_j U_i + U_j \partial_j u_i + u_j \partial_j U_i + u_j \partial_j u_i \rangle =$

  $U_j \partial_j U_i + \langle u_j \partial_j u_i \rangle + \underbrace{U_j \partial_j \langle u_i \rangle}_{=0} + \underbrace{\langle u_j \rangle \partial_j U_i}_{=0}$

Reynolds 平均 N-S 方程 (RANS)

$$\operatorname{div} U = 0$$

$$\partial_t U + U \cdot \nabla U + \nabla P = \frac{1}{\mathrm{Re}} \Delta U - \underbrace{\operatorname{div}(\overline{u \otimes u})}_{\partial_j(\overline{u_i u_j})} + \bar{b} \tag{1}$$

- 四个方程, $(P, U, \underbrace{\overline{u \otimes u}}_{\text{实}3\times3\text{对称阵}})$ 共 10 个未知量.
- Reynolds 张量: $\overline{u \otimes u}$

RANS 方程组的闭合问题

速度脉动 u_i 满足的方程: (*) - (1) $\Rightarrow$ (取 $b=0$), 对 $i=1,2,3$,

$$\partial_t u_i + U_k \partial_k u_i + u_k \partial_k U_i + \partial_k \left(u_k u_i - \overline{u_k u_i} \right) = -\partial_i p + \frac{1}{\mathrm{Re}} \Delta u_i \tag{2}$$

Reynolds 张量满足的方程: $u_j \times (2)$, 加上 i,j 互换后得到的方程, 取期望 $\Rightarrow$

$$\partial_t \left(\overline{u_i u_j} \right) + U_k \partial_k \left(\overline{u_i u_j} \right) = -\underbrace{\left(\overline{u_j \partial_i p} + \overline{u_i \partial_j p} \right)}_{\text{redistribution}}$$

$$-\underbrace{2\frac{1}{\mathrm{Re}} \overline{\partial_k u_i \partial_k u_j}}_{\text{dissipation}} - \underbrace{\partial_k \left(\overline{u_k u_i u_j} \right)}_{\text{turbulent transport}} - \underbrace{\overline{u_j u_k} \partial_k U_i - \overline{u_i u_k} \partial_k}_{\text{production}} U_j + \underbrace{\frac{1}{\mathrm{Re}} \Delta \left(\overline{u_i u_j} \right)}_{\text{diffusion}} \tag{3}$$

湍流动能 $k \doteq \frac{1}{2}\overline{u_i u_i} = \frac{1}{2}\overline{|u^2|}$

(3) 式中取 $i=j$ 且求和 $\Rightarrow$

$$\partial_t k + U_m \partial_m k = \underbrace{-\partial_i \overline{(u_i p)}}_{\text{pressure diffusion}} - \underbrace{\frac{1}{\mathrm{Re}}\overline{\partial_k u_i \partial_k u_i}}_{\text{dissipation}\doteq\varepsilon} \underbrace{-\frac{1}{2}\partial_k \overline{u_k u_i u_i}}_{\text{turbulent transport}} \underbrace{-\overline{u_i u_m}\partial_m U_i}_{\text{production}\doteq\wp} + \frac{1}{\mathrm{Re}}\Delta k \tag{4}$$

湍动能的耗散率 $\varepsilon \doteq \frac{1}{\mathrm{Re}}\overline{\partial_k u_i \partial_k u_i} = \frac{1}{\mathrm{Re}}\overline{|\nabla u|^2}$

RANS 的 k-ε 闭合模型

假设 1: $-\overline{u_i u_j} = 2\nu_T s_{ij} - \frac{2}{3}k\delta_{ij}$, ν_T : 湍流粘性系数, k : 湍动能

$s_{ij} \doteq \frac{1}{2}\left(\partial_i U_j + \partial_j U_i\right)$ 注意 $\operatorname{div} U = 0 \Rightarrow \operatorname{tr} s = 0$.

假设 2: $\nu_T = C_\mu k^2/\varepsilon$, ε 为湍动能耗散率 , C_μ = 常数 (经验值 10.09)

假设 3: $-\partial_j\left(\frac{1}{2}\overline{u_j u_i u_i} + \overline{u_j p}\right) = \partial_j\left(\nu_T \partial_j k\right)$

假设 4: $\partial_t \varepsilon + u_j \partial_j \varepsilon = \frac{C_{\varepsilon_1} p - C_{\varepsilon_2}\varepsilon}{k/\varepsilon} + \partial_j\left(\left(\frac{1}{\mathrm{Re}} + \frac{\nu_T}{\sigma_\varepsilon}\right)\partial_j \varepsilon\right)$

$C_{\varepsilon_1} = 1.44, \quad C_{\varepsilon_2} = 1.92, \quad \sigma_\varepsilon = 1.3$ (经验常数)

- 假设 1 $\Rightarrow$

$$\wp = -\overline{u_i u_j}\partial_j U_i = \left(2\nu_T s_{ij} - \tfrac{2}{3}k\delta_{ij}\right)\partial_j U_i = 2\nu_T s_{ij}\partial_j U_i - \underbrace{\frac{2}{3}k\partial_i U_i}_{=0} = 2\nu_T s_{ij}\partial_j U_i.$$

互换 $i, j \Rightarrow \wp = 2\nu_T s_{ji}\partial_i U_j = 2\nu_T s_{ij}\partial_i U_j \Rightarrow \wp = \frac{1}{2}\cdot 2\nu_T s_{ij}\left(\partial_i U_j + \partial_j U_i\right) =$
$2\nu_T s_{ij}s_{ij} \Rightarrow \boxed{\wp = 2\nu_T|s|^2}$

- (4) 结合假设 3 $\Rightarrow \partial_t k + U_j\partial_j k = 2\nu_T|s|^2 - \varepsilon + \partial_j\left(\left(\frac{1}{\mathrm{Re}} + \nu_T\right)\partial_j k\right)$
- 注意: 假设 4 来源于类比上式
- 将假设代入 RANS $\Rightarrow$

$$\partial_t U_i + U_j\partial_j U_i = -\partial_i\left(\wp + \frac{2}{3}k\right) + \partial_j\left(\left(\frac{1}{\mathrm{Re}} + \nu_T\right)\left(\partial_j U_i + \partial_i U_j\right)\right)$$

- 注意: 引入湍流粘性 ν_T, 增加粘性, 方程组为抛物型, 性质更好!

不可压湍流的 k-ε 模型

$$\nu_T = C_\mu k^2/\varepsilon, \quad \wp = 2\nu_T|s|^2, \quad s = (s_{ij}) = (\frac{1}{2}(\partial_i U_j + \partial_j U_i))$$

$$\partial_j U_j = 0,$$

$$\partial_t U_i + U_j \partial_j U_i = -\partial_i(\wp + \frac{2}{3}k) + \partial_j[(\frac{1}{\mathrm{Re}} + \nu_T)(\partial_j U_i + \partial_i U_j)]$$

$$\partial_t k + U_j \partial_j k = \wp - \varepsilon + \partial_j((\frac{1}{\mathrm{Re}} + \nu_T)\partial_j k)$$

$$\partial_t \varepsilon + U_j \partial_j \varepsilon = \frac{c_{\varepsilon_1}\wp - c_{\varepsilon_2}\varepsilon}{k/\varepsilon} + \partial_j((\frac{1}{\mathrm{Re}} + \frac{\nu_T}{\sigma_\varepsilon})\partial_j \varepsilon)$$

其中 $C_\mu, C_{\varepsilon_1}, C_{\varepsilon_2}, \sigma_\varepsilon$ 均为常数, Re 为 Reynolds 数.

注意 未知函数是 $U, \wp, k, \varepsilon$ (六个), 六个方程

hryuan@math.ecnu.edu.cn

2024 “海洋科学的数学原理” 夏令营

华东师范大学数学科学学院 / “数学与工程应用” 教育部重点实验室

2024/8/5 — 8/9