RELATIVE SINGULARITY CATEGORIES AND RELATIVE DEFECT CATEGORIES

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Abstract. We introduced the relative defect category of an abelian category $\mathscr{A}$ with respect to a full additive subcategory $\mathscr{C}$, generalizing Gorenstein defect categories of P. A. Bergh, D. Jorgensen and S. Oppermann ([6]). Under mild conditions, we show that the relative defect category of $\mathscr{A}$ with respect to $\mathscr{C}$ is triangle equivalent to the relative singularity category of $\mathscr{A}$ with respect to the Gorenstein category $\mathscr{G}(\mathscr{C})$. This generalizes a recent result proved by Y.-H. Bao, X.-N. Du and Z.-B. Zhao ([4]) and independently by F. Kong and P. Zhang ([13]).

Introduction

Gorenstein Homological Algebra began with the famous work of M. Auslander and M. Bridger ([2]), where they introduced "modules of G-dimension zero". Later E. E. Enochs and O. M. G. Jenda defined Gorenstein projective/injective modules ([9]). These notions are extensively studied since then. When an algebra is a Gorenstein algebra, that is, a left and right noetherian algebra such that the injective dimension of the left regular module ${}_AA$ and that of the right regular module A_A are both finite, Gorenstein projective modules are well behaved and they are exactly maximal Cohen-Macaulay modules ([8]).

For any left noetherian algebra A, denote by A-mod the category of finitely generated left A-modules and A-Gproj the full subcategory of (finitely generated) Gorenstein projective modules. It is well known that the stable category of (finitely generated) Gorenstein projective modules modulo projective modules, denoted by $A\text{-}\underline{\text{Gproj}}$, is a triangulated category. By R.-O. Buchweitz ([8]) and D. Happel([11]), there exists a fully faithful functor

$$F : A\text{−}\underline{\text{Gproj}} \to D^b_{sg}(A\text{−mod})$$

from the stable category of (finitely generated) Gorestein projective modules to the bounded singularity category of this algebra. A. Beligiannis ([5]), and independently P. A. Bergh, D. Jorgensen and S. Oppermann ([6]), also S.-J. Zhu ([19]), showed that this embedding is an equivalence if and only if A is Gorenstein. Inspired by this result, P. A. Bergh, D. Jorgensen and S. Oppermann ([6]) defined the Gorestein defect category as the Verdier quotient of the bounded singularity category by the essential image of this functor. This category measures the distance for A to be a Gorenstein algebra.

For a left noetherian algebra A, a complex $X^\bullet$ of (finitely generated) left A-modules is Gorenstein-exact, if for any (finitely generated) Gorenstein projective module G, the complex $\mathrm{Hom}_A(G, X^\bullet)$ is exact and the bounded homotopy category of Gorenstein-exact complexes of finitely generated left A-modules is denoted by $K^b_{gp-ac}(A\text{−mod})$. N. Gao and P. Zhang ([10]) introduced the Gorenstein derived category as the Verdier quotient

$$D^b_{gp}(A\text{−mod}) = K^b(A\text{−mod})/K^b_{gp-ac}(A\text{−mod}).$$

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Y.-H. Bao, X.-N. Du and Z.-B. Zhao ([4]) defined the (bounded) Gorenstein singularity category as the Verdier quotient

$$D^b_{gp-sg}(A) = D^b_{gp}(A-\text{mod})/K^b(A-\text{Gproj}).$$

They ([4]) proved, and independently F. Kong and P. Zhang ([13]), the following result.

Theorem 0.1. [13] [4] *Let A be a left noetherian algebra. Suppose that $A-$Gproj is contravariantly finite in $A-$mod. Then there exists a triangle equivalence*

$$D^b_{df}(A-\text{mod}) \simeq D^b_{gp-sg}(A-\text{mod}).$$

Recently, some authors try to generalize the above results to relative homological algebra by replacing projective modules by a full additive subcategory $\mathscr{C}$; for precise definitions, see the main text of this paper.

Given a full additive category $\mathscr{C}$ of an abelian category $\mathscr{A}$, one can define the (bounded) relative derived category $D^b_{\mathscr{C}}(\mathscr{A})$ of $\mathscr{A}$ with respect to $\mathscr{C}$ via the Verdier quotient of the bounded homotopy category $K^b(\mathscr{A})$ by the triangulated subcategory of $\mathscr{C}$-exact complexes. H.-H. Li and Z.-Y. Huang ([14]) defined the relative singularity category $D^b_{\mathscr{C}-sg}(\mathscr{A})$ with respect to $\mathscr{C}$ to be the Verdier quotient of the relative derived category $D^b_{\mathscr{C}}(\mathscr{A})$ by the bounded homotopy category $K^b(\mathscr{C})$ of $\mathscr{C}$.

S. Sather-Wagstaff, T. Sharif and D. White ([16]) introduced $\mathscr{C}$-Gorenstein objects as the zero-th syzygy of a $\mathscr{C}$-biexact complex and denote by $\mathscr{G}(\mathscr{C})$ the category of $\mathscr{C}$-Gorenstein objects. When $\mathscr{C}$ is the full category of projective modules in $A-$mod for a left noetherian algebra, this recovers the notion of Gorenstein projective modules. Similarly, $\mathscr{G}(\mathscr{C})$ is a Frobenius category and there exists a fully faithful functor from the stable category of $\mathscr{G}(\mathscr{C})$ to the bounded relative singularity category $D^b_{\mathscr{C}-sg}(\mathscr{A})$. H.-H. Li and Z.-Y. Huang ([14]) showed that this functor is dense when $\mathscr{A}$ is Gorenstein in the sense that each object of $\mathscr{A}$ has a $\mathscr{C}$-proper $\mathscr{G}(\mathscr{C})$-resolution of finite length. The first main result of this paper says that this is in fact also a necessary condition; see Theorem 2.6 for the precise statement. H.-H. Li informed the authors that he also has a proof of this result.

We define the notion of relative defect categories in this paper and we show an analogous result as Theorem 0.1 holds in much more general setup.

Theorem 0.2. *(see Theorem 3.7) Let $\mathscr{C}$ be an admissible subcategory of an abelian category $\mathscr{A}$. Suppose that $\mathscr{G}(\mathscr{C})$ is contravariantly finite and closed under taking kernels of $\mathscr{C}$-epimorphisms. Then there exists a triangle equivalence*

$$D^b_{\mathscr{C}-df}(\mathscr{A}) \simeq D^b_{\mathscr{G}(\mathscr{C})-sg}(\mathscr{A}),$$

that is, the relative defect category of $\mathscr{A}$ with respect to $\mathscr{C}$ is triangle equivalent to the relative singularity category of $\mathscr{A}$ with respect to $\mathscr{G}(\mathscr{C})$.

This paper is organized as follows. We recall the definition and some basis facts about relative derived categories in the first section. In the second section, we study relative singularity categories and prove the first main result. We introduce relative defect categories in the third section and prove our second main result.

We conclude the introduction by introducing some notations.

For an complex $(X^\bullet, d^\bullet)$ over $\mathscr{A}$, we define four complexes below:

$$\cdots\cdots \xrightarrow{d^{n-3}} X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^n \longrightarrow 0 \longrightarrow 0 \cdots\cdots$$

called the left brutal truncation of $X^\bullet$, denoted as $X^\bullet_{\leq n}$;

$$\cdots\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \xrightarrow{d^{n+2}} \cdots\cdots$$

called the right brutal truncation of $X^\bullet$, denoted as $X^\bullet_{\geq n}$;

$$\cdots\cdots \xrightarrow{d^{n-3}} X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \longrightarrow \mathrm{Ker}(d^n) \longrightarrow 0 \longrightarrow 0 \cdots\cdots$$

called the left good truncation of $X^\bullet$, denoted as $\tau_{\leq n}X^\bullet$

$$\cdots\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{Im}(d^n) \hookrightarrow X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \xrightarrow{d^{n+2}} \cdots\cdots$$

called the right good truncation of $X^\bullet$, denoted as $\tau_{\geq n}X^\bullet$.

In a triangulated category, for a distinguished triangle

$$X \xrightarrow{f} Y \to Z \to X[1],$$

we sometimes write the third term Z as $\mathrm{Cone}(f)$.

1. Relative derived categories

Definition 1.1. *Let $\mathscr{A}$ be an additive category and $\mathscr{C}$ a full additive subcategory of $\mathscr{A}$. A morphism $f : C \to X$ in $\mathscr{A}$ is called a $\mathscr{C}$-epimorphism if the natural induced map*

$$\mathrm{Hom}_{\mathscr{A}}(C', f) : \mathrm{Hom}_{\mathscr{A}}(C', C) \to \mathrm{Hom}_{\mathscr{A}}(C', X)$$

is an epimorphism for all $C' \in \mathscr{C}$; if, moreover, $C \in \mathscr{C}$, then f is called a right $\mathscr{C}$ -approximation of X. We say that $\mathscr{C}$ is contravariantly finite in $\mathscr{A}$ if any object in $\mathscr{A}$ has a right $\mathscr{C}$-approximation; a contravariantly finite subcategory $\mathscr{C}$ is called admissible if any right $\mathscr{C}$-approximation is an epimorphism.

Furthermore, if $\mathscr{A}$ is an abelian category and $\mathscr{C}$ is a full additive subcategory of $\mathscr{A}$, a complex $X^\bullet$ over $\mathscr{A}$ is $\mathscr{C}$-exact if $\mathrm{Hom}_{\mathscr{A}}(C, X^\bullet)$ is exact for any object $C \in \mathscr{C}$; the complex $X^\bullet$ is $\mathscr{C}$-coexact if $\mathrm{Hom}_{\mathscr{A}}(X^\bullet, C)$ is exact for any object $C \in \mathscr{C}$; the complex $X^\bullet$ is $\mathscr{C}$-biexact if it is both $\mathscr{C}$-exact and $\mathscr{C}$-coexact. A chain map $f^\bullet : X^\bullet \to Y^\bullet$ is called a $\mathscr{C}$-quasi-isomorphism if for any object $C \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{A}}(C, f^\bullet)$ is an quasi-isomorphism.

The following result is trivial whose proof is left to the reader.

Lemma 1.2. *For an admissible subcategory $\mathscr{C}$ of an abelian category $\mathscr{A}$ and a complex $X^\bullet$ over $\mathscr{A}$, $X^\bullet$ is exact if it is $\mathscr{C}$-exact.*

From now on, $\mathscr{A}$ denotes an abelian category. For $* \in \{\varnothing, +, -, b\}$, $K^*(\mathscr{A})$ denotes the homotopy category of unbounded (resp. bounded below, bounded above, bounded) complexes. Denote by $K^*_{\mathscr{C}-ac}(\mathscr{A})$ the full subcategory of $K^*(\mathscr{A})$ consisting of $\mathscr{C}$-exact complexes, and it is a thick subcategory of $K^*(\mathscr{A})$. Let $K^{-,\mathscr{C}b}(\mathscr{C})$ denotes the full subcategory of $K^-(\mathscr{C})$ consisting of those complexes which are $\mathscr{C}$-exact except finitely many terms. It's easy to see that $K^{-,\mathscr{C}b}(\mathscr{C})$ is a thick subcategory of $K^*(\mathscr{A})$.

Definition 1.3. *Let $\mathscr{C}$ be a full additive subcategory of an abelian category $\mathscr{A}$. For $* \in \{\varnothing, +, -, b\}$, the Verdier quotient $D^*_{\mathscr{C}}(\mathscr{A}) = K^*(\mathscr{A})/K^*_{\mathscr{C}-ac}(\mathscr{A})$ is called the unbounded (resp. bounded below, bounded above, bounded) relative derived category of $\mathscr{A}$ with respect to $\mathscr{C}$.*

We recall some well known results about relative derived categories which will be used in the following.

Lemma 1.4. [17, Lemma 5.5] *Let $\mathscr{C}$ be a contravariantly finite subcategory of an abelian category $\mathscr{A}$. For arbitrarily object $X^\bullet$ in $K^b(\mathscr{A})$, there exists a $\mathscr{C}$-quasi-isomorphism $C^\bullet_X \longrightarrow X^\bullet$, with $C^\bullet_X \in K^{-,\mathscr{C}b}(\mathscr{C})$.*

Proposition 1.5. [1, Theorem 3.3] *Let $\mathscr{C}$ be a contravariantly finite subcategory of an abelian category $\mathscr{A}$. Then there exists a triangle equivalence*

$$D^b_{\mathscr{C}}(\mathscr{A}) \simeq K^{-,\mathscr{C}b}(\mathscr{C}).$$

2. Relative singularity categories

By Proposition 1.5, we know that $K^b(\mathscr{C})$ can be viewed as a triangulated subcategory of $D^b_{\mathscr{C}}(\mathscr{A})$.

Definition 2.1. *Let $\mathscr{C}$ be a contravariantly finite subcategory of an abelian category $\mathscr{A}$, the Verdier quotient$D^b_{\mathscr{C}-sg}(\mathscr{A}) = D^b_{\mathscr{C}}(\mathscr{A})/K^b(\mathscr{C})$ is called the (bounded) relative singularity category of $\mathscr{A}$ with respect to $\mathscr{C}$.*

Remark 2.2. *When $\mathscr{A} = A$-mod is the category of finitely generated left modules over a finite dimensional algebra, the category $K^b(\mathscr{C})$ is a Krull-Schmidt category (*[14, section 4], [7, propertyA.2]*), thus it is a thick subcategory of $K^{-,b}(\mathscr{C}) \simeq D^b_{\mathscr{C}}(\mathscr{A})$.*

Definition 2.3. *Let M be an object of an abelian category $\mathscr{A}$ and let $\mathscr{C}$ be a full additive subcategory of $\mathscr{A}$. A resolution of M*

$$\cdots \to C^{-2} \xrightarrow{d^{-2}} C^{-1} \xrightarrow{d^{-1}} C^0 \longrightarrow M \longrightarrow 0$$

is a $\mathscr{C}$-resolution of M, if for each $i \leq 0$, $C^i \in \mathscr{C}$;

A coresolution of M of the form $0 \longrightarrow M \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \ldots$ a $\mathscr{C}$-coresolution if for each $i \geq 0$, $C^i \in \mathscr{C}$.

An exact complex $(C^\bullet, d^\bullet)$ is called a complete resolution of M if it is $\mathscr{C}$-biexact and $M \simeq \mathrm{Im}(d^0)$; a complete resolution of M is called a $\mathscr{C}$-complete resolution if for any i, $C^i \in \mathscr{C}$.

Let $\mathscr{D}$ be another subcategory of $\mathscr{A}$ and $M \in \mathscr{A}$. A $\mathscr{C}$-proper $\mathscr{D}$-resolution of M is a $\mathscr{C}$-exact complex

$$\cdots \longrightarrow D^{-n} \longrightarrow D^{-n+1} \longrightarrow \cdots \longrightarrow D^0 \longrightarrow M \longrightarrow 0$$

with $D^n \in \mathscr{D}$ for any $n \leq 0$.The $\mathscr{C}$-proper $\mathscr{D}$-dimension of M, written $\mathscr{C}\mathscr{D}$-$\dim(M)$, is defined as the minimal integer n such that exists $\mathscr{C}$-proper $\mathscr{D}$-resolution

$$0 \longrightarrow D^{-n} \longrightarrow D^{-n+1} \longrightarrow \cdots \longrightarrow D^0 \longrightarrow M \longrightarrow 0.$$

If no such an integer exists, then set $\mathscr{C}\mathscr{D}$-$\dim(M) = \infty$.

An object M in $\mathscr{A}$ is called a $\mathscr{C}$-Gorenstein object if there exists a $\mathscr{C}$-complete resolution of M. The full subcategory of $\mathscr{A}$ consisting of all $\mathscr{C}$-Gorenstein objects called the Gorenstein category of $\mathscr{C}$, denoted by $\mathscr{G}(\mathscr{C})$.

Since for $C \in \mathscr{C}$, the complex $0 \to C \xrightarrow{\mathrm{Id}_C} C \to 0$ is $\mathscr{C}$-biexact, we deduce that $\mathscr{C} \subseteq \mathscr{G}(\mathscr{C})$.

Let $\mathscr{E}$ denote the class of all the $\mathscr{C}$-exact sequence $0 \to L \to M \to N \to 0$ with $L, M, N \in \mathscr{G}(\mathscr{C})$.

Proposition 2.4. [14, Proposition 4.7] *Let $\mathscr{A}$ be an abelian category, $\mathscr{C}$ an admissible subcategory of $\mathscr{A}$, then $(\mathscr{G}(\mathscr{C}), \mathscr{E})$ is Frobenius category, Denote by $\underline{\mathscr{G}(\mathscr{C})}$ denotes the stable category of $(\mathscr{G}(\mathscr{C}), \mathscr{E})$. The functor $F : \underline{\mathscr{G}(\mathscr{C})} \longrightarrow D^b_{\mathscr{C}-sg}(\mathscr{A})$ sending an object to the corresponding stalk complex is fully faithful,*

H.-H. Li and Z.-Y. Huang provides a sufficient condition such that the functor F is an equivalence.

Proposition 2.5. [14, Theorem 4.9] *Let $\mathscr{C}$ be an admissible subcategory of an abelian category $\mathscr{A}$ such that $\mathscr{G}(\mathscr{C})$ is contravariantly finite. Suppose that $\mathscr{C}\mathscr{G}(\mathscr{C})$-$\dim(M) < \infty, \forall M \in \mathscr{A}$. Then the functor $F : \underline{\mathscr{G}(\mathscr{C})} \longrightarrow D^b_{\mathscr{C}-sg}(\mathscr{A})$ is dense.*

We can show that this is in fact an equivalent condition.

Theorem 2.6. *Let $\mathscr{C}$ be an admissible subcategory of an abelian category $\mathscr{A}$. Suppose that $\mathscr{G}(\mathscr{C})$ is contravariantly finite and closed under taking direct summands. Then the functor $F : \underline{\mathscr{G}(\mathscr{C})} \longrightarrow D^b_{\mathscr{C}-sg}(\mathscr{A})$ is dense if and only if $\mathscr{C}\mathscr{G}(\mathscr{C})$-$\dim(M) < \infty, \forall M \in \mathscr{A}$.*

The proof of this result occupies the rest of this section; our method is inspired from the master thesis of S.-J. Zhu ([19]); see also [18]. In the rest of this section, for simplicity, $\mathscr{C}$ denotes an admissible subcategory of an abelian category $\mathscr{A}$.

We can define the relative stable category of an abelian category with respect to a full additive subcategory.

Definition 2.7. *The relative stable category of $\mathscr{A}$ with respect to $\mathscr{C}$, denoted by $\underline{\mathscr{A}}_{\mathscr{C}}$, defined as follows: the objects of $\underline{\mathscr{A}}_{\mathscr{C}}$ are the same objects in $\mathscr{A}$, and the morphism space from X to Y in $\underline{\mathscr{A}}_{\mathscr{C}}$ is the quotient group*

$$\mathrm{Hom}_{\underline{\mathscr{A}}_{\mathscr{C}}}(X,Y) = \mathrm{Hom}_{\mathscr{A}}(X,Y)/\mathscr{C}(X,Y)$$

where $\mathscr{C}(X,Y)$ denotes the subgroup of morphisms in $\mathrm{Hom}_{\mathscr{A}}(X,Y)$ which factor through an object in $\mathscr{C}$.

Clearly, $\underline{\mathscr{A}}_{\mathscr{C}}$ is an additive category, and M is isomorphic to zero in $\underline{\mathscr{A}}_{\mathscr{C}}$ if and only if M is a direct summand of an object in $\mathscr{C}$.

We begins with a well known lemma.

Lemma 2.8. *For objects $X, Y \in \mathscr{A}$, $X \simeq Y$ in $\underline{\mathscr{A}}_{\mathscr{C}}$ if and only if there exists $C \simeq 0 \simeq D$ in $\underline{\mathscr{A}}_{\mathscr{C}}$ such that $X \oplus C \simeq Y \oplus D$ in $\mathscr{A}$.*

Definition 2.9. *Let n_0 be a fixed integer.*

(i) *A complex $(C^\bullet, d^\bullet)$ over $\mathscr{A}$ left splits from n_0 if for each $i \leq n_0+1$, there exists a morphism $s^i : C^i \longrightarrow C^{i-1}$ such that $d^i s^{i+1} d^i = d^i$ for any $i \leq n_0$.*

(ii) *Let $(C^\bullet, d^\bullet)$ and $(D^\bullet, e^\bullet)$ be complexes over $\mathscr{A}$ and let $f : C \longrightarrow D$ be a chain map. We say that f is left null-homotopic from n_0 if there exist morphisms $s^i : C^i \longrightarrow D^{i-1}$ for $i \leq n_0 + 1$ such that $f^i = s^{i+1} d^i + e^{i-1} s^i$ for $i \leq n_0$*

(iii) *A chain map $f : C \longrightarrow D$ is a left homotopy equivalence from n_0 if there exists another chain map $g : D \longrightarrow C$ such that $gf - Id_C$ and $fg - Id_D$ are both left null-homotopic from n_0. In this case, we say that the two complexes C and D are left homotopy equivalent from n_0.*

(iv) *A complex $(C^\bullet, d^\bullet)$ is left null-homotopic from n_0 if the identity map of $(C^\bullet, d^\bullet)$ is left null-homotopic from n_0.*

(v) *We just say that a complex left splits, or a chain map is left null-homotopic or a left homotopy equivalence if it is so from n_0 for a certain natural number n_0.*

We give a list of basic properties of left homotopies whose proofs are similar to the usual ones.

Lemma 2.10. (1) *The class of left splitting complexes and the class of left null-homotopic complexes are closed under taking homotopy equivalences.*

(2) *If two complexes $(C^\bullet, d^\bullet)$ and $(D^\bullet, e^\bullet)$ are left homotopy equivalent from n_0, then the left good truncations $\tau_{\leq n_0} C^\bullet$ and $\tau_{\leq n_0} D^\bullet$ are homotopy equivalent.*

(3) *Given a complex $(C^\bullet, d^\bullet)$ in $K^{-,\mathscr{C}b}(\mathscr{C})$, then $(C^\bullet, d^\bullet)$ left splits if it is homotopy equivalent to a complex from $K^b(\mathscr{C})$. The converse also holds if $\mathscr{C}$ is closed under taking direct summands.*

(4) *Let $P^\bullet$ and $Q^\bullet$ be complexes in $K^{-,\mathscr{C}b}(\mathscr{C})$ and let $f : P \longrightarrow Q$ be a chain map. Then f is a left homotopy equivalence if $\mathrm{Cone}(f) \in K^b(\mathscr{C})$.*

Now we are going to prove Theorem 2.6

Proof. The sufficiency follows from Proposition 2.5. Let us prove the necessity.

Let $M \in \mathscr{A}$. Since the functor F is dense, then there exists $G \in \mathscr{G}(\mathscr{C})$ such that $F(G) \simeq M$ in $D^b_{\mathscr{C}-sg}(\mathscr{A})$. Let $(Q^\bullet, e^\bullet)$ be the left brutal truncation from the zero-th position of a $\mathscr{C}$-complete resolution of G and $(P^\bullet, d^\bullet)$ be a $\mathscr{C}$-resolution of M. Then $P^\bullet \simeq Q^\bullet$ in $D^b_{\mathscr{C}-sg}(\mathscr{A})$, that is, there exists a left roof in $K^{-,\mathscr{C}b}(\mathscr{C})$

$$\begin{array}{ccc} & L^\bullet & \\ {}^{f}\swarrow & & \searrow^{s} \\ P^\bullet & & Q^\bullet \end{array}$$

with $L^\bullet \in K^{-,\mathscr{C}b}(\mathscr{C})$ and $\mathrm{Cone}(f) \in K^b(\mathscr{C})$, $\mathrm{Cone}(s) \in K^b(\mathscr{C})$. By Lemma 2.10 (4), s and f are left homotopy equivalence.

Lemma 2.10 (4) implies that there exists an integer n_0 such that $\tau_{\leq n_0}P^\bullet$ is homotopy equivalent to $\tau_{\leq n_0}Q^\bullet$, thus there exists $f : \mathrm{Ker}(d^{n_0}) \longrightarrow \mathrm{Ker}(e^{n_0})$ and $g : \mathrm{Ker}(e^{n_0}) \longrightarrow \mathrm{Ker}(d^{n_0})$ such that $gf - Id_{\mathrm{Ker}(d^{n_0})}$ factors through P^{n_0-1} and $fg - Id_{\mathrm{Ker}(e^{n_0})}$ factors through Q^{n_0-1}, which means that $\mathrm{Ker}(d^{n_0}) \simeq \mathrm{Ker}(e^{n_0})$ in $\underline{\mathscr{A}}_{\mathscr{C}}$. By Lemma2.8, there exists objects C and D which are direct summands of objects in $\mathscr{C}$, such that $\mathrm{Ker}(d^{n_0}) \oplus C \simeq \mathrm{Ker}(e^{n_0}) \oplus D$. Since $\mathscr{G}(\mathscr{C})$ is closed under taking direct summands, $\mathrm{Ker}(e^{n_0})$, C, $D \in \mathscr{G}(\mathscr{C})$, thus $\mathrm{Ker}(d^{n_0}) \in \mathscr{G}(\mathscr{C})$. This implies that $\tau_{\geq n_0}P^\bullet$ is a $\mathscr{C}$-proper $\mathscr{G}(\mathscr{C})$-resolution of M, thus $\mathscr{C}\mathscr{G}(\mathscr{C})$-dim$(M) < \infty$. □

Remark that it is not true in general that $\mathscr{G}(\mathscr{C})$ is closed under taking direct summands and we conclude this section with the following criteria which assures that $\mathscr{G}(\mathscr{C})$ is closed under taking direct summands.

Definition 2.11. *Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ is a additive subcategory of $\mathscr{A}$. The full subcategory ${}^\perp\mathscr{C} = \{X \in \mathscr{A} | \mathrm{Ext}^{\geq 1}(X, C) = 0, \forall C \in \mathscr{C}\}$ is called the left perpendicular subcategory of $\mathscr{C}$; the full subcategory $\mathscr{C}^\perp = \{X \in \mathscr{A} | \mathrm{Ext}^{\geq 1}(C, X) = 0, \forall C \in \mathscr{C}\}$ is called the right perpendicular subcategory of $\mathscr{C}$; We say that $\mathscr{C}$ is self-perpendicular if $\mathscr{C} \subseteq {}^\perp\mathscr{C}$.*

We say $\mathscr{C}$ is preresolving if it closed under extension and closed under taking kernels of epimorphisms.

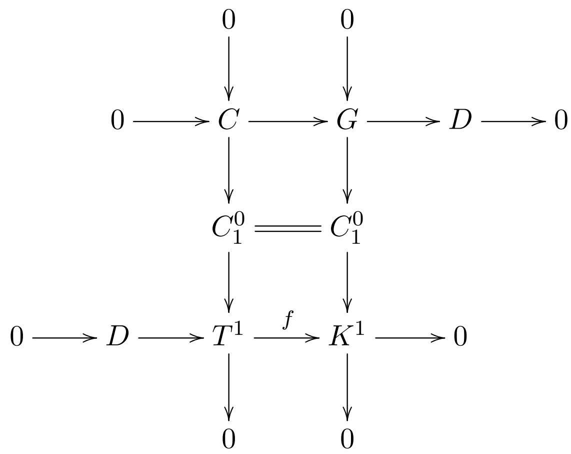
Proposition 2.12. *Let $\mathscr{A}$ be an abelian category with a preresolving and self-perpendicular admissible subcategory $\mathscr{C}$. Then $\mathscr{G}(\mathscr{C})$ is closed under taking direct summands.*

Proof. Let $C \oplus D = G \in \mathscr{G}(\mathscr{C})$. By the proof of Proposition 3.8. $\mathscr{G}(\mathscr{C}) \in^\perp \mathscr{C} \cap \mathscr{C}^\perp$. Since $\mathrm{Ext}^{\geq 1}(G, \mathscr{C}) = 0$, we have $\mathrm{Ext}^{\geq 1}(C, \mathscr{C}) = 0$ and this shows that any $\mathcal{C}$-resolution of C is $\mathscr{C}$-biexact. The same holds for D.

By the relative Horseshoe Lemma ([3, Proposition 3.6]), $\mathscr{G}(\mathscr{C})$ is closed under taking extension.

Now we prove that $\mathscr{G}(\mathscr{C})$ is closed under taking direct summands.

Let $C \oplus D = G \in \mathscr{G}(\mathscr{C})$ and $C_1^\bullet$ is the complete resolution of G with $G = Z^0(C_1^\bullet)$. Consider the commutative diagram with exact rows and columns:



where the first row is the obvious split short exact sequence, the second column is induced by the inclusion $G = Z^0(C_1^\bullet) \to C_1^0$ and the third row exists by the Snake Lemma. Since $C, D \in^\perp \mathscr{C} \cap \mathscr{C}^\perp$ and $K^1 \in \mathscr{G}(\mathscr{C}) \subseteq^\perp \mathscr{C} \cap \mathscr{C}^\perp$, we have $T^1 \in^\perp \mathscr{C} \cap \mathscr{C}^\perp$.

The third row shows that there exists a short exact sequence

$$0 \longrightarrow C \oplus D \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}} C \oplus T^1 \longrightarrow K^1 \longrightarrow 0.$$

Since $\mathscr{G}(\mathscr{C})$ is closed under taking extension, $T^1 \oplus C \in \mathscr{G}(\mathscr{C})$. Choose a complete resolution $C_2^\bullet$ of $T^1 \oplus C$ and again we have the commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & 0 \longrightarrow & T^1 & \longrightarrow & T^1 \oplus C & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & C_2^0 & = & C_2^0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \longrightarrow C \longrightarrow & & T^2 & \longrightarrow & K^2 & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where as above the first row is the obvious split short exact sequence, the second column is induced by the inclusion $T^1 \oplus C = Z^0(C_2^\bullet) \to C_2^0$ and the third row exists by the Snake Lemma. Since $K^2 \in \mathscr{G}(\mathscr{C}) \subseteq {}^\perp\mathscr{C} \cap \mathscr{C}^\perp$, $T^2 \in {}^\perp\mathscr{C} \cap \mathscr{C}^\perp$. Repeating this process gives an exact sequence

$$
\begin{array}{ccccccccccc}
0 \longrightarrow C \longrightarrow & C_1^0 & & \longrightarrow & & C_2^0 & & \longrightarrow & & C_3^0 & \longrightarrow \cdots \\
 & & \searrow & & \nearrow & & \searrow & & \nearrow & & \\
 & & & T^1 & & & & T^2 & & &
\end{array}
$$

with $C, T^i \in {}^\perp\mathscr{C} \cap \mathscr{C}^\perp, \forall i \geq 1$, which means that the sequence is $\mathscr{C}$-biexact.

Combine it with a $\mathscr{C}$-resolution of C and we get a complete resolution of C. □

3. Relative defect categories

Definition 3.1. *Let $\mathscr{C}$ be an admissible subcategory of an abelian category $\mathscr{A}$. The Verdier quotient*

$$D^b_{\mathscr{C}-df}(\mathscr{A}) = D^b_{\mathscr{C}-sg}(\mathscr{A})/\mathrm{Im}(F)$$

is called the (bounded) relative defect category of $\mathscr{A}$ with respect to $\mathscr{C}$, where F is the functor in Proposition 2.4

Lemma 3.2. *Let $\mathscr{A}$ be an abelian category, let $\mathscr{C} \subseteq \mathscr{D}$ be both contravariantly finite subcategories of $\mathscr{A}$. If $\mathscr{C}$ is admissible, so is $\mathscr{D}$, in particular, when $\mathscr{G}(\mathscr{C})$ is contravariantly finite, it is admissible.*

Proof. Given an arbitrary object X of $\mathscr{A}$ and a right $\mathscr{D}$-approximation $D \to X$ of X whose kernel is denoted by K, we have the following exact sequence

$$0 \to K \to D \to X.$$

After applying the functor $\mathrm{Hom}_{\mathscr{A}}(D', -)$ with $D' \in \mathscr{D}$, we get the following exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{A}}(D', K) \to \mathrm{Hom}_{\mathscr{A}}(D', G) \to \mathrm{Hom}_{\mathscr{A}}(D', X) \to 0.$$

Thus for any object C in $\mathscr{C}$,

$$0 \to \mathrm{Hom}_{\mathscr{A}}(C, K) \to \mathrm{Hom}_{\mathscr{A}}(C, G) \to \mathrm{Hom}_{\mathscr{A}}(C, X) \to 0$$

is exact. By Lemma 1.2 this finishes the proof. □

Lemma 3.3. *Let $\mathscr{C}$ be an admissible subcategory of an abelian category $\mathscr{A}$. For any object $C^\bullet$ in $K^{-,\mathscr{C}b}(\mathscr{C})$, there exists a bounded complex $\overline{C}^\bullet$ and a $\mathscr{C}$-quasi-isomorphism*

$$f : C^\bullet \longrightarrow \overline{C}^\bullet.$$

Proof. It suffices to take $\overline{C}^\bullet$ to be the right good truncation

$$\tau_{\geq n}C^\bullet (= \cdots \to 0 \to \mathrm{Im}(d^n) \to C^{n+1} \to C^{n+2} \to \cdots)$$

for $n \ll 0$. □

Lemma 3.4. *Let $\mathscr{C}$ be an admissible subcategory of $\mathscr{A}$. Suppose that $\mathscr{G}(\mathscr{C})$ is contravariantly finite in $\mathscr{A}$. For any object $C^\bullet$ in $K^{-,\mathscr{C}b}(\mathscr{C})$, there exists a $\mathscr{C}$-quasi-isomorphism $f : C^\bullet \longrightarrow G^\bullet$ with $G^\bullet \in K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C}))$.*

Proof. By Lemma 3.2, $\mathscr{G}(\mathscr{C})$ is admissible.

Given a complex $C^\bullet \in K^{-,\mathscr{C}b}(\mathscr{C})$ which is $\mathscr{C}$-exact in $\leq n+2$ position, it can be written in the form:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_C^{n-2}} & C^{n-1} & \xrightarrow{d_C^{n-1}} & C^n & \xrightarrow{\qquad d_C^n \qquad} & C^{n+1} & \xrightarrow{d_C^{n+1}} & \cdots \\ & & & & \searrow & & \nearrow & & \\ & & & & & \mathrm{Ker}(d_C^{n+1}) & & & \end{array}$$

Lemma 1.4 implies that there exists a $\mathscr{G}(\mathscr{C})$-exact complex

$$\cdots \longrightarrow G^{n-1} \longrightarrow G^n \twoheadrightarrow \mathrm{Ker}(d_C^{n+1}) \to 0$$

with $G^i \in \mathscr{G}(\mathscr{C})$ for $i \leq n$. Write $G^\bullet_{\leq n}$ for the truncated complex

$$\cdots \longrightarrow G^{n-1} \longrightarrow G^n \longrightarrow 0.$$

Combining the two complexes $G^\bullet_{\leq n}$ and $C^\bullet_{\geq n+1}$ gives the complex

$$\cdots \longrightarrow G^{n-1} \longrightarrow G^n \longrightarrow C^{n+1} \longrightarrow C^{n+2} \longrightarrow \cdots$$

denoted by$(G^\bullet, d_G^\bullet)$, which obviously belongs to $K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C}))$.

We construct a chain map $f^\bullet : C^\bullet \longrightarrow G^\bullet$ which is a $\mathscr{C}$-quasi-isomorphism..

Let $f^i = Id_{C^i}$ for $i \geq n+1$. Since $G^\bullet_{\leq n} \twoheadrightarrow \mathrm{Ker}(d_C^{n+1})$ is $\mathscr{G}(\mathscr{C})$-exact, after applying functor $\mathrm{Hom}_{\mathscr{A}}(C^n, -)$ we can get a morphism $f^n : C^n \to G^n$ such that $d_C^n = d_G^n \circ f^n$. This induces a morphism $\mathrm{Ker}(d_C^n) \to \mathrm{Ker}(d_G^n)$, denoted by $\tilde{f}^n$. Now the long exact sequence obtained by applying the functor $\mathrm{Hom}_{\mathscr{A}}(C^{n-1}, -)$ gives a morphism $f^{n-1} : C^{n-1} \to G^{n-1}$ such that $\tilde{f}^n \circ d_C^{n-1} = d_G^{n-1} \circ f^{n-1}$ and thus $f^n \circ d_C^{n-1} = d_G^{n-1} \circ f^{n-1}$. Similarly, we can define f^i for each $i \leq n$ by repeating this process. This gives a chain map $f : C^\bullet \longrightarrow G^\bullet$, which is clearly a $\mathscr{C}$-quasi-isomorphism. □

Lemma 3.5. *Let $\mathscr{C}$ be an admissible subcategory of an abelian category $\mathscr{A}$. Suppose that $\mathscr{G}(\mathscr{C})$ is contravariantly finite in $\mathscr{A}$ and is closed under taking kernels of $\mathscr{C}$-epimorphisms. Then*

$$K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C})) \cap K_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C})) = K^b_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C})).$$

Proof. The proof of the inclusion $K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C})) \cap K_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C})) \supseteq K^b_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C}))$ is trivial, so we just need to prove $K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C})) \cap K_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C})) \subseteq K^b_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C}))$.

Given an arbitrary complex $(X^\bullet, d_X^\bullet) \in K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C})) \cap K_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C}))$, let $n \ll 0$ such that $X^\bullet$ is $\mathscr{G}(\mathscr{C})$-exact for $\leq n$ positions. Since $X^\bullet$ is $\mathscr{C}$-exact, and $\mathscr{G}(\mathscr{C})$ is closed under taking kernels of $\mathscr{C}$-epimorphisms, it is easy to see by induction that $\mathrm{Ker}(d_X^i) \in \mathscr{G}(\mathscr{C})$ for each $i \leq n$. Since the natural epimorphism $X^{n-1} \to \mathrm{Ker}(d_X^n)$ is a $\mathscr{G}(\mathscr{C})$-epimorphism, there exists a morphism from $\mathrm{Ker}(d_X^n)$ to X^{n-1} splitting this epimorphism. We have $X^\bullet \simeq \tau_{\leq n-1}X^\bullet \oplus \tau_{\geq n-1}X^\bullet$.

We see that $\tau_{\leq n-1}X^\bullet$ is null-homotopic, which implies the statement. In fact, this follows from the facts that $\tau_{\leq n-1}X^\bullet$ is $\mathscr{G}(\mathscr{C})$-exact and that $\mathrm{Ker}(d_X^i) \in \mathscr{G}(\mathscr{C})$ for each $i \leq n$. □

Proposition 3.6. *Let $\mathscr{C}$ be an admissible subcategory of an abelian category $\mathscr{A}$. Suppose that $\mathscr{G}(\mathscr{C})$ is contravarianty finite and closed under taking kernels of $\mathscr{C}$-epimorphisms. Then there exists a triangle equivalence*

$$K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C}))/K^b_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C})) \simeq D^b_{\mathscr{C}}(\mathscr{A}),$$

which induces a triangle equivalence

$$K^b(\mathscr{G}(\mathscr{C}))/K^b_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C})) \simeq \langle \mathscr{G}(\mathscr{C}) \rangle \subseteq D^b_{\mathscr{C}}(\mathscr{A})$$

where $\langle \mathscr{G}(\mathscr{C}) \rangle$ denotes the smallest triangulated subcategory of $D^b_{\mathscr{C}}(\mathscr{A})$ containing $\mathscr{G}(\mathscr{C})$.

Proof. We consider the functor

$$\eta : K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C})) \hookrightarrow K^-(\mathscr{A}) \longrightarrow D^-_{\mathscr{C}}(\mathscr{A}) = K^-(\mathscr{A})/K_{\mathscr{C}-ac}(\mathscr{A}).$$

Clearly, $\eta(K^b_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C})) = 0$, hence this induces a triangle functor

$$\overline{\eta} : K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C}))/K^b_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C})) \to D^-_{\mathscr{C}}(\mathscr{A}).$$

Notice that the functor acts on the objects by identity.

By Lemma 3.2 , $\mathscr{G}(\mathscr{C})$ is admissible. Apply Lemma 3.3 to $\mathscr{G}(\mathscr{C})$, and we know that every complex in $K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C}))$ is $\mathscr{C}$-quasi-isomorphic to a bounded complex, this shows that the image of $\overline{\eta}$ contained in $D^b_{\mathscr{C}}(\mathscr{A})$. Hence $\overline{\eta}$ induces a functor $K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C}))/K^b_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C})) \to D^b_{\mathscr{C}}(\mathscr{A})$, still denoted by $\overline{\eta}$.

We claim that $\overline{\eta}$ is full and dense, and maps non-zero objects to non-zero objects. By [15, p. 446], we know that $\overline{\eta}$ is a triangle equivalence.

Lemma 3.4 implies that $\overline{\eta}$ is dense.

Let $C^\bullet \in K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C}))$ such that $\eta(C^\bullet) \simeq 0 \in D^b_{\mathscr{C}}(\mathscr{A})$, Then $C^\bullet$ is $\mathscr{C}$-exact. By Lemma 3.5, $C^\bullet \in K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C})) \cap K_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C})) = K^b_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C}))$. we know that $\overline{\eta}$ maps non-zero objects to non-zero objects.

At last we prove $\overline{\eta}$ is full. For two arbitrary objects $G_1^\bullet, G_2^\bullet \in K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C}))$ and $f \in \mathrm{Hom}_{D^-_{\mathscr{C}}(\mathscr{A})}(G_1^\bullet, G_2^\bullet)$, f can be written as a fraction as follows:

$$\begin{array}{ccc} & X^\bullet & \\ {}^{s}\swarrow\!\!\swarrow & & \searrow^{\alpha} \\ G_1^\bullet & & G_2^\bullet \end{array}$$

where $X^\bullet \in K^-(\mathscr{A})$, s is a $\mathscr{C}$-quasi-isomorphism. Since $G_1^\bullet$ is $\mathscr{G}(\mathscr{C})$-exact except finitely many terms, $X^\bullet$ is $\mathscr{C}$-exact except finitely many terms. By Lemma 3.2, there exists a $\mathscr{C}$-quasi-isomorphism $g : C^\bullet \longrightarrow X^\bullet$ with $C^\bullet \in K^{-,\mathscr{C}b}(\mathscr{C})$ and so the following diagram is commutative

$$\begin{array}{ccccc} & & X^\bullet & & \\ & {}^{s}\swarrow\!\!\swarrow & \Uparrow g & \searrow^{\alpha} & \\ G_1^\bullet & \overset{}{\underset{sg}{\Longleftarrow}} & C^\bullet & \xrightarrow{\alpha g} & G_2^\bullet \end{array}$$

Let $n \ll 0$ such that $G_1^\bullet$ is $\mathscr{G}(\mathscr{C})$-exact for $\leq n+1$ positions and $C^\bullet$ is $\mathscr{C}$-exact for $\leq n+2$ positions. By Lemma 3.4, there exists an object $G^\bullet \in K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C}))$ with a $\mathscr{C}$-quasi-isomorphism $t : C^\bullet \longrightarrow G^\bullet$ which is identity for $\geq n+1$ positions . We can construct a chain map h from $G^\bullet$ to $G_1^\bullet$:

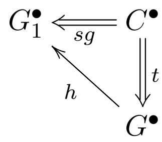
$$\begin{array}{ccccccccccc} \cdots \longrightarrow & G^{n-2} & \longrightarrow & G^{n-1} & \longrightarrow & G^n & \longrightarrow & G^{n+1} = C^{n+1} & \longrightarrow & G^{n+2} = C^{n+2} & \longrightarrow \cdots \\ & \downarrow \exists h^{n-2} & & \downarrow \exists h^{n-1} & & \downarrow \exists h^n & & \downarrow h^{n+1} & & \downarrow h^{n+2} & \\ \cdots \longrightarrow & G_1^{n-2} & \longrightarrow & G_1^{n-1} & \longrightarrow & G_1^n & \longrightarrow & G_1^{n+1} & \longrightarrow & G_1^{n+2} & \longrightarrow \cdots \end{array}$$

where $h^i = s^i \circ g^i$ for $i \geq n+1$. The fact that $G_1^\bullet$ is $\mathscr{G}(\mathscr{C})$-exact in $\leq n+1$ positions guarantees the existence of h^i for $i \leq n$.

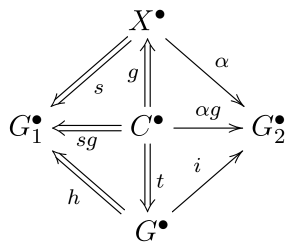
We consider the diagram below, where $\beta^i = h^i \circ t^i - s^i \circ g^i$ for $i \in \mathbb{Z}$:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & C^{n-2} & \xrightarrow{d^{n-2}} & C^{n-1} & \xrightarrow{d^{n-1}} & C^{n} & \xrightarrow{d^{n}} & C^{n+1} & \xrightarrow{d^{n+1}} & C^{n+2} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \beta^{n-2}} & & \downarrow{\scriptstyle \beta^{n-1}} & & \downarrow{\scriptstyle \beta^{n}} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\
\cdots & \longrightarrow & G_1^{n-2} & \xrightarrow{d_1^{n-2}} & G_1^{n-1} & \xrightarrow{d_1^{n-1}} & G_1^{n} & \xrightarrow{d_1^{n+1}} & G_1^{n} & \xrightarrow{d_1^{n+1}} & G_1^{n+2} & \longrightarrow & \cdots
\end{array}$$

Since $G_1^\bullet$ is $\mathscr{G}(\mathscr{C})$ -exact for $\leq n+1$ positions, there exists $r^{n+1} : C^{n+1} \to G_1^n$ satisfies $d_1^{n+1} \circ r^{n+1} = 0$. Replace β^n by $\beta^n - r^{n+1} \circ d^n$ and we get a morphism $r^n : C^n \to G_1^{n-1}$ with $d_1^{n-1} \circ r^n = \beta^n - r^{n+1} \circ d^n$. Hence the following diagram is commutative up to homotopy



It means that h is also $\mathscr{C}$-quasi-isomorphism. Similarly, there exists a morphism $i : P^\bullet \longrightarrow G_2^\bullet$ such that the following diagram is commutative in $K^-(\mathscr{A})$



So Cone(h) is $\mathscr{C}$-exact and

$$\mathrm{Cone}(f) \in K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C})) \bigcap K_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C})).$$

By Lemma 3.5,Cone$(f) \in K^b_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C}))$.

In summary, we have proven that the functor $\overline{\eta}$ is full and dense and maps non-zero objects to non-zero objects, so it is triangle equivalence.

The second statement is immediate from the first one.

□

Theorem 3.7. *Let $\mathscr{C}$ be an admissible subcategory of an abelian category $\mathscr{A}$. Suppose that $\mathscr{G}(\mathscr{C})$ is contravariantly finite and closed under taking kernels of $\mathscr{C}$-epimorphisms. Then we have a triangle equivalence*

$$D^b_{\mathscr{C}-df}(\mathscr{A}) \simeq D^b_{\mathscr{G}(\mathscr{C})-sg}(\mathscr{A}),$$

that is, the relative defect category of $\mathscr{A}$ with respect to $\mathscr{C}$ is triangle equivalent to the relative singularity category of $\mathscr{A}$ with respect to $\mathscr{G}(\mathscr{C})$.

Proof. We have

$$\begin{array}{rcl}
D^b_{\mathscr{C}-df}(\mathscr{A}) & = & D^b_{\mathscr{C}-sg}(\mathscr{A})/\mathrm{Im}(F) \simeq D^b_{\mathscr{C}}(\mathscr{A})/\langle \mathscr{G}(\mathscr{C}) \rangle \\
 & \simeq & (K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C}))/K^b_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C})))/(K^b(\mathscr{G}(\mathscr{C}))/K^b_{\mathscr{C}-ac}(\mathscr{G}(\mathscr{C}))) \\
 & \simeq & K^{-,\mathscr{G}(\mathscr{C})b}(\mathscr{G}(\mathscr{C}))/K^b(\mathscr{G}(\mathscr{C})) \\
 & = & D^b_{\mathscr{G}(\mathscr{C})-sg}(\mathscr{A}).
\end{array}$$

□

Remark that it may not true in general that $\mathscr{G}(\mathscr{C})$ is closed under taking kernel of $\mathscr{C}$-epimorphism. Now we give a sufficient condition of this property.

Proposition 3.8. *Let $\mathscr{A}$ be an abelian category with a preresolving and self-perpendicular admissible subcategory $\mathscr{C}$, and recall $\mathscr{G}(\mathscr{C})$ denote the Gorenstein category of $\mathscr{C}$. Then $\mathscr{G}(\mathscr{C})$ is closed under taking kernels of $\mathscr{C}$-epimorphisms.*

Proof. We form a full subcategory of $\mathscr{A}$:

$$\mathcal{X}_{\mathscr{C}} = \{M | \exists \textit{ a short sequence } 0 \to M \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \textit{ with } \operatorname{Ker}(d_i) \in {}^{\perp}\mathscr{C} \cap \mathscr{C}^{\perp}, i \geq 0\}.$$

Firstly notice a well known fact that $\mathscr{G}(\mathscr{C}) \subseteq \mathcal{X}_{\mathscr{C}}$.

Note that $\mathscr{G}(\mathscr{C}) \supseteq \mathcal{X}_{\mathscr{C}}$ may not true.

Secondly, we can show that $\mathcal{X}_{\mathscr{C}}$ is closed under taking extension using the relative Horseshoe Lemma ([3, Proposition 3.6]).

Now we show that $\mathscr{G}(\mathscr{C})$ is closed under taking kernels of $\mathscr{C}$-epimorphism.

Given a short exact and $\mathscr{C}$-exact sequence $0 \to X \to Y \to Z \to 0$ with $Y, Z \in \mathscr{G}(\mathscr{C}) \subseteq \mathcal{X}_{\mathscr{C}}$, we need to show that $X \in \mathscr{G}(\mathscr{C})$. We have $\operatorname{Ext}^1(\mathscr{C}, X) = 0$. Choose a $\mathscr{C}$-biexact $\mathscr{C}$-resolution of Y

$$0 \to Y \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots,$$

Consider the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C^0 & = & C^0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & K & \longrightarrow & \operatorname{Ker}(d^0) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Note that the third row exists by the Snake Lemma. Since $\operatorname{Ker}(d^0), Z \in \mathcal{X}_{\mathscr{C}}$, we have $K \in \mathcal{X}_{\mathscr{C}}$, so the first column is $\mathscr{C}$-coexact. Moreover, $\operatorname{Ext}^1(\mathscr{C}, X) = 0$ implies the first column is $\mathscr{C}$-exact.

Since $\mathscr{C}$ is preresolving, by [12, theorem1.2] we get the following exact sequence:

$$\cdots \to D^{-2} \xrightarrow{e^{-2}} D^{-1} \xrightarrow{e^{-1}} X \to 0$$

which is $\mathscr{C}$-biexact. Combining it with the sequences

$$0 \to X \longrightarrow C^0 \longrightarrow K \longrightarrow 0$$

and

$$0 \to K \to D^0 \longrightarrow D^1 \longrightarrow \cdots$$

shows $X \in \mathscr{G}(\mathscr{C})$. □

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