THE GORENSTEIN-PROJECTIVE MODULES OVER A MONOMIAL ALGEBRA

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Abstract. We introduce the notion of a perfect path for a monomial algebra. We classify indecomposable non-projective Gorenstein-projective modules over the given monomial algebra via perfect paths. We apply the classification to a quadratic monomial algebra and describe explicitly the stable category of its Gorenstein-projective modules.

1. Introduction

Let A be a finite-dimensional algebra over a field. We consider the category of finite-dimensional left A-modules. The study of Gorenstein-projective modules goes back to [1] under the name "modules of G-dimension zero". The current terminology is taken from [10]. Due to the fundamental work [5], the stable category of Gorenstein-projective A-modules is closely related to the singularity category of A. Indeed, for a Gorenstein algebra, these two categories are triangle equivalent; also see [12].

We recall that projective modules are Gorenstein-projective. For a selfinjective algebra, all modules are Gorenstein-projective. Hence for the study of Gorenstein-projective modules, we often consider non-selfinjective algebras. However, there are algebras which admit no nontrivial Gorenstein-projective modules, that is, any Gorenstein-projective module is actually projective; see [8].

There are very few classes of non-selfinjective algebras, for which an explicit classification of indecomposable Gorenstein-projective modules is known. In [17], such a classification is obtained for Nakayama algebras; compare [9]. Using the representation theory of string algebras, there is also such a classification for gentle algebras in [15]; compare [6].

We are interested in the Gorenstein-projective modules over a monomial algebra A. It turns out that there is an explicit classification of indecomposable Gorenstein-projective A-modules, so that we unify the results in [17] and [15] to some extent. We mention that we rely on a fundamental result in [18], which implies in particular that an indecomposable Gorenstein-projective A-module is isomorphic to a cyclic module generated by paths. Then the classification pins down to the following question: for which path, the corresponding cyclic module is Gorenstein-projective. The main goal of this work is to answer this question.

Let us describe the content of this paper. In Section 2, we recall basic facts on Gorenstein-projective modules. In Section 3, we introduce the notion of a perfect pair of paths for a monomial algebra A. The basic properties of a perfect pair are studied. We introduce the notion of a perfect path in A; see Definition 3.7. We prove

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the main classification result in Section 4, which claims that there is a bijection between the set of all perfect paths and the set of isoclasses of indecomposable non-projective Gorenstein-projective A-modules; see Theorem 4.1. As an application, we show that for a connected truncated algebra A, either A is seflinjective, or any Gorenstein-projective A-module is projective; see Example 4.4.

We specialize Theorem 4.1 to a quadratic monomial algebra A in Section 5, in which case any perfect path is an arrow. We introduce the notion of a relation quiver for A, whose vertices are given by arrows in A and whose arrows are given by relations in A; see Definition 5.2. We prove that an arrow in A is perfect if and only if the corresponding vertex in the relation quiver belongs to a connected component which is a basic cycle; see Lemma 5.3. Using the relation quiver, we obtain a characterization result on when a quadratic monomial algebra is Gorenstein, which includes the well-known result in [13] that a gentle algebra is Gorenstein; see Proposition 5.5. We describe explicitly the stable category of Gorenstein-projective A-modules, which is proved to be a semisimple abelian category; see Theorem 5.7. This theorem generalizes the main result in [15].

In Section 6, we study a Nakayama monomial algebra A, where the quiver of A is a basic cycle. Following the idea in [17], we describe explicitly in Proposition 6.2 all the perfect paths for A. As a consequence, we recover a key characterization result for indecomposable Gorenstein-projective modules over a Nakayama algebra in [17].

The standard reference on the representation theory of finite-dimensional algebras is [3].

2. The Gorenstein-projective modules

In this section, we recall some basic facts on Gorenstein-projective modules over a finite-dimensional algebra.

Let A be a finite-dimensional algebra over a field k. We consider the category A-mod of finitely generated left A-modules, and denote by A-proj the full subcategory consisting of projective A-modules. We will identify right A-modules as left $A^{\rm op}$-modules, where $A^{\rm op}$ is the opposite algebra of A.

For two left A-modules X and Y, we denote by $\mathrm{Hom}_A(X,Y)$ the space consisting of module homomorphisms from X to Y, and by $P(X,Y)$ the subspace formed by those homomorphisms factoring through a projective module. Write $\underline{\mathrm{Hom}}_A(X,Y) = \mathrm{Hom}_A(X,Y)/P(X,Y)$ to be the quotient space, which is the Hom-space in the stable category A-$\underline{\mathrm{mod}}$. Indeed, the stable category A-$\underline{\mathrm{mod}}$ is defined as follows: the objects are left A-modules, and the Hom-space for two objects X and Y are defined to be $\underline{\mathrm{Hom}}_A(X,Y)$, where the composition of morphisms is induced by the composition of module homomorphisms.

Let M be a left A-module. Then $M^* = \mathrm{Hom}_A(M,A)$ is a right A-module. Recall that an A-module M is *Gorenstein-projective* provided that there is an acyclic complex $P^\bullet$ of projective A-modules such that the Hom-complex $(P^\bullet)^* = \mathrm{Hom}_A(P^\bullet, A)$ is still acyclic and that M is isomorphic to a certain cocycle $Z^i(P^\bullet)$ of $P^\bullet$. We denote by A-Gproj the full subcategory of A-mod formed by Gorenstein-projective A-modules. We observe that A-proj $\subseteq$ A-Gproj. We recall that the full subcategory A-Gproj $\subseteq$ A-mod is closed under direct summands, kernels of epimorphisms and extensions; compare [1, (3.11)] and [2, Lemma 2.3].

The Gorenstein-projective modules are also called Cohen-Macaulay modules in the literature. Following [4], an algebra A is *CM-finite* provided that up to isomorphism there are only finitely many indecomposable Gorenstein-projective A-modules. As an extreme case, we say that the algebra A is *CM-free* [8] provided that A-proj $=$ A-Gproj.

Let M be a left A-module. Recall that its *syzygy* $\Omega(M) = \Omega^1(M)$ is defined to be the kernel of its projective cover $P \to M$. Then we have the d-th syzygy $\Omega^d(M)$ of M defined inductively by $\Omega^d(M) = \Omega(\Omega^{d-1}M)$ for $d \geq 2$. Set $\Omega^0(M) = M$. We observe that for a Gorenstein-projective module M, all its syzygies $\Omega^d(M)$ are Gorenstein-projective.

Since A-Gproj $\subseteq A$-mod is closed under extensions, it becomes naturally an exact category in the sense of Quillen [16]. Moreover, it is a *Frobenius category*, that is, it has enough (relatively) projective and enough (relatively) injective objects, and the class of projective objects coincides with the class of injective objects. In fact, the class of the projective-injective objects in A-Gproj equals A-proj. In particular, we have $\mathrm{Ext}^i_A(M, A) = 0$ for any Gorenstein-projective A-module M and each $i \geq 1$. For details, we refer to [4, Proposition 3.8(i)].

We denote by A-$\underline{\mathrm{Gproj}}$ the full subcategory of A-$\underline{\mathrm{mod}}$ consisting of Gorenstein-projective A-modules. Then the assignment $M \mapsto \Omega(M)$ induces an auto-equivalence $\Omega\colon A\text{-}\underline{\mathrm{Gproj}} \to A\text{-}\underline{\mathrm{Gproj}}$. Moreover, the stable category A-$\underline{\mathrm{Gproj}}$ becomes a triangulated category such that the translation functor is given by a quasi-inverse of Ω, and that the triangles are induced by short exact sequences in A-Gproj. These are consequences of a general result in [11, Chapter I.2].

We observe that the stable category A-$\underline{\mathrm{Gproj}}$ is Krull-Schimidt. We denote by ind A-$\underline{\mathrm{Gproj}}$ the set of isoclasses of indecomposable objects inside. There is a natural identification between ind A-$\underline{\mathrm{Gproj}}$ and the set of isoclasses of indecomposable non-projective Gorenstein-projective A-modules.

The following facts are well known.

Lemma 2.1. *Let M be a Gorenstein-projective A-module which is indecomposable and non-projective. Then the following three statements hold.*

1. *The syzygy $\Omega(M)$ is also an indecomposable non-projective Gorenstein-projective A-module.*
2. *There exists an indecomposable A-module N which is non-projective and Gorenstein-projective such that $M \simeq \Omega(N)$.*
3. *If the algebra A is CM-finite with precisely d indecomposable non-projective Gorenstein-projective modules up to isomorphism, then we have an isomorphism $M \simeq \Omega^{d!}(M)$.* □

Proof. We observe that the auto-equivalence $\Omega\colon A\text{-}\underline{\mathrm{Gproj}} \to A\text{-}\underline{\mathrm{Gproj}}$ induces a permutation on the set of isoclasses of indecomposable non-projective Gorenstein-projective A-modules. Then all the statements follow immediately. For a detailed proof of (1), we refer to [8, Lemma 2.2]. □

Let $d \geq 0$. We recall from [5, 12] that an algebra A is *d-Gorenstein* provided that the injective dimension of the regular module A on both sides is at most d. By a *Gorenstein algebra* we mean a d-Gorenstein algebra for some $d \geq 0$. We observe that 0-Gorenstein algebras coincide with selfinjective algebras.

The following result is also well known; compare [4, Proposition 3.10] and [2, Theorem 3.2].

Lemma 2.2. *Let A be a finite-dimensional algebra and $d \geq 0$. Then the algebra A is d-Gorenstein if and only if for each A-module M, the module $\Omega^d(M)$ is Gorenstein-projective.* □

For an element a in A, we consider the left ideal Aa and the right ideal aA generated by a. We have the following well-defined monomorphism of right A-modules

$$\theta_a\colon aA \longrightarrow (Aa)^* = \mathrm{Hom}_A(Aa, A), \tag{2.1}$$

which is defined by $\theta_a(ax)(y) = yx$ for $ax \in aA$ and $y \in Aa$. Dually, we have the following monomorphism of left A-modules

$$\theta'_a\colon Aa \longrightarrow (aA)^* = \mathrm{Hom}_{A^{\mathrm{op}}}(aA, A), \tag{2.2}$$

which is defined by $\theta'_a(xa)(y) = xy$ for $xa \in Aa$ and $y \in aA$. If e is an idempotent in A, both θ_e and θ'_e are isomorphisms; see [3, Proposition I.4.9].

The following fact will be used later.

Lemma 2.3. *Let a be an element in A satisfying that θ_a is an isomorphism, and let $b \in A$. Then the isomorphism θ_a induces a k-linear isomorphism*

$$\frac{aA \cap Ab}{aAb} \xrightarrow{\sim} \underline{\mathrm{Hom}}_A(Aa, Ab). \tag{2.3}$$

Proof. For a left ideal $K \subseteq A$, we identify $\mathrm{Hom}_A(Aa, K)$ with the subspace of $\mathrm{Hom}_A(Aa, A)$, that consists of homomorphisms with images in K. Therefore, the isomorphism θ_a induces an isomorphism $aA \cap K \xrightarrow{\sim} \mathrm{Hom}_A(Aa, K)$. In particular, we have an isomorphism $aA \cap Ab \simeq \mathrm{Hom}_A(Aa, Ab)$.

Consider the surjective homomorphism $\pi\colon A \to Ab$ satisfying $\pi(1) = b$. Recall that $P(Aa, Ab)$ denotes the subspace of $\mathrm{Hom}_A(Aa, Ab)$ consisting of homomorphisms factoring through a projective module. Then $P(Aa, Ab)$ is identified with the image of $\mathrm{Hom}_A(Aa, \pi)$. Identifying $\mathrm{Hom}_A(Aa, A)$ with aA, we compute that the image of $\mathrm{Hom}_A(Aa, \pi)$ equals aAb. Then the required isomorphism follows immediately. □

3. Monomial algebras and perfect pairs

In this section, we recall some basic notions and results on a monomial algebra. We introduce the notions of a perfect pair and of a perfect path. Some basic properties of a perfect pair are studied.

Let Q be a finite quiver. We recall that a finite quiver $Q = (Q_0, Q_1; s, t)$ consists of a finite set Q_0 of vertices, a finite set Q_1 of arrows and two maps $s, t\colon Q_1 \to Q_0$ which assign to each arrow α its starting vertex $s(\alpha)$ and its terminating vertex $t(\alpha)$.

A path p of length n in Q is a sequence $p = \alpha_n \cdots \alpha_2\alpha_1$ of arrows such that $s(\alpha_i) = t(\alpha_{i-1})$ for $2 \leq i \leq n$; moreover, we define its starting vertex $s(p) = s(\alpha_1)$ and its terminating vertex $t(p) = t(\alpha_l)$. We observe that a path of length one is just an arrow. For each vertex i, we associate a trivial path e_i of length zero, and set $s(e_i) = i = t(e_i)$. A path of length at least one is said to be *nontrivial*. A nontrivial path is called an *oriented cycle* if its starting vertex equals its terminating vertex.

For two paths p and q with $s(p) = t(q)$, we write pq their concatenation. As convention, we have $p = pe_{s(p)} = e_{t(p)}p$. For two paths p and q in Q, we say that q is a *sub-path* of p provided that $p = p''qp'$ for some paths p'' and p'. The sub-path q is *proper* if further $p \neq q$.

Let S be a set of paths in Q. A path p in S is *left-minimal in S* provided that there is no path q such that $q \in S$ and $p = qp'$ for some path p'. Dually, one defines a *right-minimal path in S*. A path p in S is *minimal in S* provided that there is no proper sub-path q of p inside S.

Let k be a field. The path algebra kQ of a finite quiver Q is defined as follows. As a k-vector space, it has a basis given by all the paths in Q. For two paths p and q, their multiplication is given by the concatenation pq if $s(p) = t(q)$, and it is zero, otherwise. The unit of kQ equals $\sum_{i \in Q_0} e_i$. Denote by J the two-sided ideal of kQ generated by arrows. Then J^d is spanned by all the paths of length at least d for each $d \geq 2$. A two-sided ideal I of kQ is *admissible* if $J^d \subseteq I \subseteq J^2$ for some $d \geq 2$. In this case, the quotient algebra $A = kQ/I$ is finite-dimensional.

We recall that an admissible ideal I of kQ is *monomial* provided that it is generated by some paths of length at least two. In this case, the quotient algebra $A = kQ/I$ is called a *monomial algebra*.

Let $A = kQ/I$ be a monomial algebra as above. We denote by $\mathbf{F}$ the set formed by all the minimal paths among the paths in I; it is a finite set. Indeed, the set $\mathbf{F}$ generates I as a two-sided ideal. Moreover, any set consisting of paths that generates I necessarily contains $\mathbf{F}$.

We make the following convention as in [18]. A path p is said to be a *nonzero path in A* provided that p does not belong to I, or equivalently, p does not contain a sub-path in $\mathbf{F}$. For a nonzero path p, we abuse p with its canonical image $p + I$ in A. On the other hand, for a path p in I, we write $p = 0$ in A. We observe that the set of nonzero paths forms a k-basis of A.

For a nonzero path p, we consider the left ideal Ap and the right ideal pA. We observe that Ap has a basis given by all nonzero paths q such that $q = q'p$ for some path q'. Similarly, pA has a basis given by all nonzero paths γ such that $\gamma = p\gamma'$ for some path γ'. If $p = e_i$ is trivial, then Ae_i and e_iA are indecomposable projective left and right A-modules, respectively.

For a nonzero nontrivial path p, we define $L(p)$ to be the set of right-minimal paths in the set $\{\text{nonzero paths } q \mid s(q) = t(p) \text{ and } qp = 0\}$ and $R(p)$ the set of left-minimal paths in the set $\{\text{nonzero paths } q \mid t(q) = s(p) \text{ and } pq = 0\}$.

The following well-known fact is straightforward; compare the first paragraph in [18, p.162].

Lemma 3.1. *Let p be a nonzero nontrivial path in A. Then we have the following exact sequence of left A-modules*

$$0 \longrightarrow \bigoplus_{q\in L(p)} Aq \xrightarrow{\text{inc}} Ae_{t(p)} \xrightarrow{\pi_p} Ap \longrightarrow 0, \tag{3.1}$$

where "inc" is the inclusion map and π_p is the projective cover of Ap with $\pi_p(e_{t(p)}) = p$. Similarly, we have the following exact sequence of right A-modules

$$0 \longrightarrow \bigoplus_{q\in R(p)} qA \xrightarrow{\text{inc}} e_{s(p)}A \xrightarrow{\pi'_p} pA \longrightarrow 0, \tag{3.2}$$

where π'_p is the projective cover of pA with $\pi'_p(e_{s(p)}) = p$. □

We will rely on the following fundamental result contained in [18, Theorem I].

Lemma 3.2. *Let M be a left A-module which fits into an exact sequence $0 \to M \to P \to Q$ of A-modules with P, Q projective. Then M is isomorphic to a direct sum $\bigoplus Ap^{(\Lambda(p))}$, where p runs over all the nonzero paths in A and each $\Lambda(p)$ is some index set.* □

The main notion we need is as follows.

Definition 3.3. Let $A = kQ/I$ be a monomial algebra as above. We call a pair (p, q) of nonzero paths in A *perfect* provided that the following conditions are satisfied:

(P1) both of the nonzero paths p, q are nontrivial satisfying $s(p) = t(q)$ and $pq = 0$ in A;
(P2) if $pq' = 0$ for a nonzero path q' with $t(q') = s(p)$, then $q' = qq''$ for some path q''; in other words, $R(p) = \{q\}$;
(P3) if $p'q = 0$ for a nonzero path p' with $s(p') = t(q)$, then $p' = p''p$ form some path p''; in other words, $L(q) = \{p\}$. □

Let (p, q) be a perfect pair. Applying (P3) to (3.1), we have the following exact sequences of left A-modules

$$0 \longrightarrow Ap \xrightarrow{\text{inc}} Ae_{t(q)} \xrightarrow{\pi_q} Aq \longrightarrow 0. \tag{3.3}$$

In particular, we have that $\Omega(Aq) \simeq Ap$.

The following result seems to be convenient for computing perfect pairs.

Lemma 3.4. *Recall that* $\mathbf{F}$ *denotes the finite set of minimal paths contained in* I*. Let* p *and* q *be nonzero nontrivial paths in* A *satisfying* $s(p) = t(q)$*. Then the pair* (p, q) *is perfect if and only if the following three conditions are satisfied:*

- (P′1) *the concatenation* pq *lies in* $\mathbf{F}$*;*
- (P′2) *if* q' *is a nonzero path in* A *satisfying* $t(q') = s(p)$ *and* $pq' = \gamma\delta$ *for a path* γ *and some path* $\delta \in \mathbf{F}$*, then* $q' = qx$ *for some path* x*;*
- (P′3) *if* p' *is a nonzero path satisfying* $s(p') = t(q)$ *and* $p'q = \delta\gamma$ *for a path* γ *and some path* $\delta \in \mathbf{F}$*, then* $p' = yp$ *for some path* y*.*

Proof. For the "only if" part, we assume that (p, q) is a perfect pair. By (P1) we have $pq = \gamma_2\delta\gamma_1$ with $\delta \in \mathbf{F}$ and some paths γ_1 and γ_2. We claim that $\gamma_1 = e_{s(q)}$. Otherwise, $q = q'\gamma_1$ for a proper sub-path q', and thus $pq' = \gamma_2\delta$ which equals zero in A. This contradicts (P2). Similarly, we have $\gamma_2 = e_{t(p)}$. Then we infer (P′1). The conditions (P′2) and (P′3) follow from (P2) and (P3) immediately.

For "if" part, we observe that the condition (P1) is immediate. For (P2), assume that $pq' = 0$ in A, that is, $pq' = \gamma\delta\gamma_1$ with $\delta \in \mathbf{F}$ and some paths γ and γ_1. We assume that $q' = q''\gamma_1$. Then we have $pq'' = \gamma\delta$. By (P′2) we infer that $q'' = qx$ and thus $q' = q(x\gamma_1)$. This proves (P2). Similarly, we have (P3). □

We study the basic properties of a perfect pair in the following lemmas.

Lemma 3.5. *Let* (p, q) *and* (p', q') *be two perfect pairs. Then the following statements are equivalent:*

1. $(p, q) = (p', q')$*;*
2. *there is an isomorphism* $Aq \simeq Aq'$ *of left* A*-modules;*
3. *there is an isomorphism* $pA \simeq p'A$ *of right* A*-modules.*

Proof. We only prove "(2)⇒(1)". Assume that $\phi\colon Aq \to Aq'$ is an isomorphism. Consider the projective covers $\pi_q\colon Ae_{t(q)} \to Aq$ and $\pi_{q'}\colon Ae_{t(q')} \to Aq'$. Then there is an isomorphism $\psi\colon Ae_{t(q)} \to Ae_{t(q')}$ such that $\pi_{q'} \circ \psi = \phi \circ \pi_q$. In particular, we have $t(q) = t(q')$.

Assume that $\psi(e_{t(q)}) = \lambda e_{t(q)} + \sum \lambda(\gamma)\gamma$, where λ and $\lambda(\gamma)$ are in k and γ runs over all the nonzero nontrivial paths that start at $t(q)$. Since ψ is an isomorphism, we infer that $\lambda \neq 0$. We observe that $\psi(p) = \lambda p + \sum \lambda(\gamma)p\gamma$. Recall that $\pi_q(p) = pq = 0$. By $\pi_{q'} \circ \psi = \phi \circ \pi_q$, we have $\psi(p)q' = 0$. We then infer that $pq' = 0$ and thus $p = \delta p'$ for some path δ. Similarly, we have $p' = \delta' p$. We conclude that $p = p'$. Since $R(p) = \{q\}$ and $R(p') = \{q'\}$, we infer that $q = q'$. Then we are done. □

We recall the homomorphisms in (2.1) and (2.2).

Lemma 3.6. *Let* (p, q) *be a perfect pair. Then both* θ_q *and* θ'_p *are isomorphisms.*

Proof. We have mentioned that the map θ_q is a monomorphism.

To show that it is epic, take a homomorphism $f\colon Aq \to A$ of left A-modules. Since $q = e_{t(q)}q$, we infer that $f(q)$ belongs to $e_{t(q)}A$. We assume that $f(q) = \sum \lambda(\gamma)\gamma$, where each $\lambda(\gamma)$ is in k and γ runs over all nonzero paths terminating at $t(q)$. By $pq = 0$, we deduce that $p\gamma = 0$ for those γ with $\lambda(\gamma) \neq 0$. By (P2) each γ lies in qA. Therefore, we infer that $f(q)$ lies in qA, and thus $\theta_q(f(q)) = f$. Then we infer that θ_q is an isomorphism. Dually, one proves that θ'_p is also an isomorphism. □

The following notion plays a central role in this paper.

Definition 3.7. Let $A = kQ/I$ be a monomial algebra. We call a nonzero path p in A a *perfect path*, provided that there exists a sequence

$$p = p_1, p_2, \cdots, p_n, p_{n+1} = p$$

of nonzero paths such that (p_i, p_{i+1}) are perfect pairs for all $1 \leq i \leq n$. If the given nonzero paths p_i are pairwise distinct, we refer to the sequence $p = p_1, p_2, \cdots, p_n, p_{n+1} = p$ as a *relation-cycle* for p. □

We mention that by Definition 3.3(P2) a perfect path has a unique relation-cycle.

Let $n \geq 1$. By a *basic (n-)cycle*, we mean a quiver consisting of n vertices and n arrows which form an oriented cycle.

Example 3.8. Let Q be a connected quiver and let $d \geq 2$. Recall that J denotes the two-sided ideal of kQ generated by arrows. The monomial algebra $A = kQ/J^d$ is called a *truncated algebra*. We claim that A has a perfect path if and only if the quiver Q is a basic cycle.

Indeed, let (p, q) be a perfect pair. Assume that $p = \alpha_n \cdots \alpha_2\alpha_1$ and $q = \beta_m \cdots \beta_2\beta_1$. We infer that each α_i is the unique arrow starting at $s(\alpha_i)$, otherwise $L(q)$ has at least two elements, contradicting to (P3). Similarly, we have that each β_j is the unique arrow terminating at $t(\beta_j)$. Let p be a perfect path with a relation-cycle $p = p_1, p_2, \cdots, p_n, p_{n+1} = p$. We apply the above observations to the perfect pairs (p_i, p_{i+1}). Then we obtain that the quiver Q is a basic cycle.

On the other hand, if Q is a basic cycle, then any nonzero nontrivial path p of length strictly less than $d - 1$ is perfect.

4. The Gorenstein-projective modules over a monomial algebra

In this section, we parameterize the isoclasses of indecomposable non-projective Gorenstein-projective modules over a monomial algebra by perfect paths.

Recall that for a finite-dimensional algebra A, the set ind $A\text{-}\underline{\mathrm{Gproj}}$ of isoclasses of indecomposable objects in the stable category $A\text{-}\underline{\mathrm{Gproj}}$ is identified with the set of isoclasses of indecomposable non-projective Gorenstein-projective A-modules.

The main result of this paper is as follows.

Theorem 4.1. *Let A be a monomial algebra. Then there is a bijection*

$$\{\textit{perfect paths in } A\} \stackrel{1:1}{\longleftrightarrow} \mathrm{ind}\ A\text{-}\underline{\mathrm{Gproj}}$$

sending a perfect path p to the A-module Ap.

Proof. The well-definedness of the map is due to Proposition 4.3(4). The surjectivity follows from Proposition 4.3(3). The injectivity follows from Lemma 3.5. □

Remark 4.2. We mention that by Theorem 4.1 considerable information on the stable category $A\text{-}\underline{\mathrm{Gproj}}$ is already obtained. For example, the syzygy functor Ω on indecomposable objects is computed by (3.3), and the Hom-spaces between indecomposable objects are computed by Lemma 2.3.

The map θ_p in the following proposition is introduced in (2.1).

Proposition 4.3. *Let $A = kQ/I$ be a monomial algebra, and let M be an indecomposable non-projective Gorenstein-projective A-module. Then the following statements hold.*

1. *For a nonzero nontrivial path p satisfying that the morphism θ_p is an isomorphism, if the A-module Ap is non-projective Gorenstein-projective, then the path p is perfect.*

(2) *For a nonzero nontrivial path p, if the A-module Ap is non-projective Gorenstein-projective, then there is a unique perfect path q such that $L(p) = \{q\}$.*
(3) *There exists a perfect path q satisfying that $M \simeq Aq$.*
(4) *For a perfect path p, the A-module Ap is non-projective Gorenstein-projective.*

Proof. We observe that each indecomposable non-projective Gorenstein-projective A-module X is of the form $A\gamma$ for some nonzero nontrivial path γ. Indeed, there exists an exact sequence $0 \to X \to P \to Q$ of A-modules with P, Q projective. Then we are done by Lemma 3.2. In particular, this observation implies that a monomial algebra A is CM-finite.

(1) By (3.1) we have $\Omega(Ap) = \bigoplus_{q\in L(p)} Aq$, which is indecomposable and non-projective by Lemma 2.1(1). We infer that $L(p) = \{q\}$ for some nonzero nontrivial path q. Consider the exact sequence (3.1) for p

$$\eta\colon 0 \longrightarrow Aq \xrightarrow{\text{inc}} Ae_{t(p)} \xrightarrow{\pi_p} Ap \longrightarrow 0.$$

Recall that the A-module Ap is Gorenstein-projective, in particular, $\mathrm{Ext}^1_A(Ap, A) = 0$. Therefore, the lower row of the following commutative diagram is exact.

$$\begin{array}{lccccccccc}
\epsilon\colon & 0 \longrightarrow & pA & \xrightarrow{\text{inc}} & e_{s(q)}A & \xrightarrow{\pi'_q} & qA & \longrightarrow 0 \\
 & & \downarrow{\theta_p} & & \downarrow{\theta_{e_{s(q)}}} & & \downarrow{\theta_q} & \\
\eta^*\colon & 0 \longrightarrow & (Ap)^* & \xrightarrow{(\pi_p)^*} & (Ae_{t(p)})^* & \xrightarrow{\text{inc}^*} & (Aq)^* & \longrightarrow 0
\end{array}$$

Recall that $\theta_{e_{s(q)}}$ is an isomorphism and θ_q is a monomorphism. However, since inc^* is epic, we infer that θ_q is also epic and thus an isomorphism. We note that the A-module Aq is also non-projective Gorenstein-projective; see Lemma 2.1(1).

We claim that (q, p) is a perfect pair. Indeed, we already have (P1) and (P2). It suffices to show that $R(q) = \{p\}$. By assumption, the map θ_p is an isomorphism. Then the upper sequence ϵ in the above diagram is also exact. Comparing ϵ with (3.2), we obtain that $R(q) = \{p\}$.

We have obtained a perfect pair (q, p) and also proved that θ_q is an isomorphism. We mention that $\Omega(Ap) \simeq Aq$. Set $q_0 = p$ and $q_1 = q$. We now replace p by q and continue the above argument. Thus we obtain perfect pairs (q_{m+1}, q_m) for all $m \geq 0$ satisfying $\Omega(Aq_m) \simeq Aq_{m+1}$. By Lemma 2.1(3), we have for a sufficiently large m, an isomorphism $Aq_m \simeq Aq_0 = Ap$. By Lemma 3.5 we have $q_m = p$. Thus we have a required sequence $p = q_m, q_{m-1}, \cdots, q_1, q_0 = p$, proving that p is perfect.

(2) By the first paragraph in the proof of (1), we obtain that $L(p) = \{q\}$ and that θ_q is an isomorphism. Since $Aq \simeq \Omega(Ap)$, we infer that Aq is non-projective Gorenstein-projective. Then the path q is perfect by (1), proving (2).

(3) By Lemma 2.1(2), there is an indecomposable non-projective Gorenstein-projective A-module N such that $M \simeq \Omega(N)$. By the observation above, we may assume that $N = Ap$ for a nonzero nontrivial path p. Recall from (3.1) that $\Omega(N) \simeq \bigoplus_{p'\in L(p)} Ap'$. Then we have $L(p) = \{q\}$ for some nonzero path q and an isomorphism $M \simeq Aq$. The path q is necessarily perfect by (2).

(4) We take a relation-cycle $p = p_1, p_2, \cdots, p_n, p_{n+1} = p$ for the perfect path p. We define $p_m = p_j$ if $m = an + j$ for some integer a and $1 \leq j \leq n$. Then each pair (p_m, p_{m+1}) is perfect. By (3.3) we have an exact sequence of left A-modules

$$\eta_m\colon 0 \longrightarrow Ap_m \xrightarrow{\text{inc}} Ae_{t(p_{m+1})} \xrightarrow{\pi_{p_{m+1}}} Ap_{m+1} \longrightarrow 0.$$

Gluing all these η_m's together, we obtain an acyclic complex $P^\bullet = \cdots \to Ae_{t(p_m)} \to Ae_{t(p_{m+1})} \to \cdots$ such that Ap is isomorphic to one of the cocycles. We observe that $Ap = Ap_1$ is non-projective, since η_1 does not split.

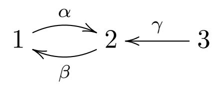
It remains to prove that the Hom-complex $(P^\bullet)^* = \mathrm{Hom}_A(P^\bullet, A)$ is also acyclic. For this, it suffices to show that for each m, the sequence $\mathrm{Hom}_A(\eta_m, A)$ is exact, or equivalently, the morphism $\mathrm{inc}^* = \mathrm{Hom}_A(\mathrm{inc}, A)$ is epic. We observe the following commutative diagram

$$\begin{array}{ccc} e_{s(p_m)}A & \xrightarrow{\pi'_{p_m}} & p_mA \\ \Big\downarrow{\scriptstyle \theta_{e_{s(p_m)}}} & & \Big\downarrow{\scriptstyle \theta_{p_m}} \\ (Ae_{t(p_{m+1})})^* & \xrightarrow{\mathrm{inc}^*} & (Ap_m)^*, \end{array}$$

where we use the notation in (3.2). Recall that $\theta_{e_{s(p_m)}}$ is an isomorphism. By Lemma 3.6 the morphism θ_{p_m} is an isomorphism. Since π'_{p_m} is a projective cover, we infer that the morphism inc^* is epic. We are done with the whole proof. □

The following example shows that the condition that θ_p is an isomorphism is necessary in the proof of Proposition 4.3(1).

Example 4.4. Let Q be the following quiver.



Let I be the ideal generated by $\beta\alpha$ and $\alpha\beta$, and let $A = kQ/I$. We denote by S_i the simple A-module corresponding to the vertex i for $1 \leq i \leq 3$.

The corresponding set $\mathbf{F}$ of minimal paths contained in I equals $\{\beta\alpha, \alpha\beta\}$. Then we observe that there exist precisely two perfect pairs, which are (β, α) and (α, β). Hence, the set of perfect paths equals $\{\alpha, \beta\}$. Then by Theorem 4.1, up to isomorphism, all the indecomposable non-projective Gorenstein-projective A-modules are given by $A\alpha$ and $A\beta$. We observe two isomorphisms $A\alpha \simeq S_2$ and $A\beta \simeq S_1$.

Consider the nonzero path $p = \beta\gamma$. Then there is an isomorphism $Ap \simeq A\beta$ of left A-modules. In particular, the A-module Ap is non-projective Gorenstein-projective. However, the path p is not perfect.

The following example shows that a connected truncated algebra is either selfinjective or CM-free; compare [8, Theorem 1.1].

Example 4.5. Let $A = kQ/J^d$ be the truncated algebra in Example 3.8. If Q is not a basic cycle, then there is no perfect path. Then by Theorem 4.1, the algebra A is CM-free. On the other hand, if Q is a basic cycle, then the algebra A is well known to be selfinjective.

5. The Gorenstein-projective modules over a quadratical monomial algebra

In this section, we specialize Theorem 4.1 to a quadratic monomial algebra. We describe explicitly the stable category of Gorenstein-projective modules over such an algebra.

Let $A = kQ/I$ be a monomial algebra. We say that the algebra A is *quadratic monomial* provided that the ideal I is generated by paths of length two, or equivalently, the corresponding set $\mathbf{F}$ consists of certain paths of length two. By Lemma 3.4(P′1), for a perfect pair (p, q) in A, both p and q are necessarily arrows. In particular, a perfect path is an arrow and its relation-cycle consists entirely of arrows.

Hence, we have the following immediate consequence of Theorem 4.1.

Proposition 5.1. *Let A be a quadratic monomial algebra. Then there is a bijection*

$$\{\text{perfect arrows in } A\} \stackrel{1:1}{\longleftrightarrow} \text{ind } A\text{-}\underline{\text{Gproj}}$$

sending a perfect arrow α to the A-module $A\alpha$. □

We will give a more convenient characterization of a perfect arrow. For this end, we introduce the following notion.

Definition 5.2. Let $A = kQ/I$ be a quadratic monomial algebra. We define its *relation quiver* $\mathcal{R}_A$ as follows: the vertices are given by the arrows in Q, and there is an arrow $[\alpha\beta]\colon \alpha \to \beta$ if $s(\alpha) = t(\beta)$ and $\alpha\beta$ lies in I, or equivalently in $\mathbf{F}$.

Let $\mathcal{C}$ be a connected component of $\mathcal{R}_A$. We call $\mathcal{C}$ a *perfect component* (*resp.* an *acyclic component*) if $\mathcal{C}$ is a basic cycle (*resp.* contains no oriented cycles). □

It turns out that an arrow is perfect if and only if the corresponding vertex belongs to a perfect component of the relation quiver of A.

Lemma 5.3. *Let $A = kQ/I$ be a quadratic monomial algebra, and let α be an arrow. Then the following statements hold.*

(1) *We have $L(\alpha) = \{\beta \in Q_1 \mid s(\beta) = t(\alpha) \text{ and } \beta\alpha \in \mathbf{F}\}$, and $R(\alpha) = \{\beta \in Q_1 \mid t(\beta) = s(\alpha) \text{ and } \alpha\beta \in \mathbf{F}\}$.*
(2) *Assume that β is an arrow with $t(\beta) = s(\alpha)$. Then the pair (α, β) is perfect if and only if there is an arrow $[\alpha\beta]$ from α to β in $\mathcal{R}_A$, which is the unique arrow starting at α and also the unique arrow terminating at β.*
(3) *The arrow α is perfect if and only if the corresponding vertex belongs a perfect component of $\mathcal{R}_A$.*

Proof. For (1), we observe the following fact: for a nonzero path p with $s(p) = t(\alpha)$, then $p\alpha = 0$ if and only if $p = p'\beta$ with $\beta\alpha \in \mathbf{F}$. This fact implies the equation on $L(\alpha)$. Similarly, we have the equation on $R(\alpha)$.

We mention that by (1), the set $L(\alpha)$ consists of all immediate predecessors of α in $\mathcal{R}_A$, and $R(\alpha)$ consists of all immediate successors of α. Then (2) follows immediately from the definition of a perfect pair. The statement (3) is an immediate consequence of (2). □

The following result concerns the homological property of the module $A\alpha$. We call a vertex j in a quiver *bounded*, if the lengths of all the paths starting at j are uniformly bounded. In this case, the length is strictly less than the number of vertices in the quiver.

Lemma 5.4. *Let $A = kQ/I$ be a quadratic monomial algebra, and let α be an arrow. Then the following statements hold.*

(1) *The A-module $A\alpha$ is non-projective Gorenstein-projective if and only if the corresponding vertex of α belongs a perfect component of $\mathcal{R}_A$.*
(2) *The A-module $A\alpha$ has finite projective dimension if and only if α is a bounded vertex in $\mathcal{R}_A$. In this case, the projective dimension of $A\alpha$ is strictly less than the number of arrows in Q.*
(3) *If the corresponding vertex of α in $\mathcal{R}_A$ is not bounded and does not belong to a perfect component, then each syzygy module $\Omega^d(A\alpha)$ is not Gorenstein-projective.*

Proof. The "if" part of (1) follows from Lemma 5.3(3) and Proposition 4.3(4). For the "only if" part, assume that the A-module $A\alpha$ is non-projective Gorenstein-projective. By Proposition 4.3(2) there is a perfect arrow β such that $L(\alpha) = \{\beta\}$. In particular, there is an arrow from β to α in $\mathcal{R}_A$. By Lemma 5.3(3) β belongs to a perfect component $\mathcal{C}$ of $\mathcal{R}_A$. It follows that α also belongs to $\mathcal{C}$.

For (2), we observe that by Lemma 5.3(1) and (3.1) there is an isomorphism

$$\Omega(A\alpha) \stackrel{\sim}{\longrightarrow} \bigoplus A\beta, \tag{5.1}$$

where β runs over all the immediate predecessors of α in $\mathcal{R}_A$. Then (2) follows immediately.

For (3), we assume on the contrary that $\Omega^d(A\alpha)$ is Gorenstein-projective for some $d \geq 1$. We know already by (2) that $\Omega^d(A\alpha)$ is not projective. By iterating the formula (5.1), we obtain an arrow β such that $A\beta$ is non-projective Gorenstein-projective and that there is a path from β to α of length d in $\mathcal{R}_A$. However, by (1) β belongs to a perfect component $\mathcal{C}$. It follows that α also belongs to $\mathcal{C}$, which is a desired contradiction. □

The following result studies the Gorenstein homological properties of a quadratic monomial algebra. In particular, we obtain a characterization of a quadratic monomial algebra being Gorenstein, which contains the well-known result that a gentle algebra is Gorenstein; see [13, Theorem 3.4] and compare Example 5.6.

Proposition 5.5. *Let $A = kQ/I$ be a quadratic monomial algebra with d the number of arrows. Then the following statements hold.*

1. *The algebra A is Gorenstein if and only if any connected component of its relation quiver $\mathcal{R}_A$ is either perfect or acyclic. In this case, the algebra A is $(d+1)$-Gorenstein.*
2. *The algebra A is CM-free if and only if the relation quiver $\mathcal{R}_A$ contains no perfect component.*
3. *The algebra A has finite global dimension if and only if any component of the relation quiver $\mathcal{R}_A$ is acyclic.*

Proof. The statement (2) is an immediate consequence of Proposition 5.1 and Lemma 5.3(3), while the final statement is an immediate consequence of (1) and (2). Here, we recall a well-known consequence of Lemma 2.2: an algebra A has finite global dimension if and only if it is Gorenstein and CM-free.

We now prove (1). Recall from Lemma 2.2 that the algebra A is Gorenstein if and only if there exists a natural number n such that $\Omega^n(M)$ is Gorenstein-projective for any A-module M.

For "only if" part, we assume the contrary. Then there is an arrow α, whose corresponding vertex in $\mathcal{R}_A$ is not bounded and does not belong to a perfect component. By Lemma 5.4 (3), we infer that A is not Gorenstein. A contradiction!

For the "if" part, let α be an arrow. Then the A-module $A\alpha$ is either Gorensein-projective or has finite projective dimension at most $d-1$; see Lemma 5.4(1) and (2). We infer that the A-module $\Omega^{d-1}(A\alpha)$ is Gorenstein-projective. We observe that for a nonzero path $p = \alpha p'$ of length at least two, we have an isomorphism $A\alpha \simeq Ap$, sending x to xp'. So we conclude that for any nonzero path p, the A-module $\Omega^{d-1}(Ap)$ is Gorenstein-projective.

Let M be any A-module. Then by Lemma 3.2 we have an isomorphism between $\Omega^2(M)$ and a direct sum of the modules Ap for some nonzero paths p. It follows by above that the syzygy module $\Omega^{d+1}(M)$ is Gorenstein-projective. This proves that the algebra A is $(d+1)$-Gorenstein.

□

Example 5.6. Let A be a quadratic monomial algebra. We assume that for each arrow α, there exists at most one arrow β with $\alpha\beta \in \mathbf{F}$ and at most one arrow γ with $\gamma\alpha \in \mathbf{F}$. Then the algebra A is Gorenstein. In particular, a gentle algebra satisfies these conditions. As a consequence, we recover the main part of [13, Theorem 3.4].

Indeed, the assumption implies that at each vertex in $\mathcal{R}_A$, there is at most one arrow starting and at most one arrow terminating. It forces that each connected component is either perfect or acyclic.

We recall from [7, Lemma 3.4] that for a semisimple abelian category $\mathcal{A}$ and an auto-equivalence Σ on $\mathcal{A}$, there is a unique triangulated structure on $\mathcal{A}$ with Σ the translation functor. Indeed, all the triangles are split. We denote the resulting triangulated category by $(\mathcal{A}, \Sigma)$.

Let $n \geq 1$. Consider the algebra automorphism $\sigma\colon k^n \to k^n$ defined by

$$\sigma(\lambda_1, \lambda_2, \cdots, \lambda_n) = (\lambda_2, \cdots, \lambda_n, \lambda_1). \tag{5.2}$$

Then σ induces an automorphism σ^* on the category k^n-mod by twisting the module actions. We denote by $\mathcal{T}_n = (k^n\text{-mod}, \sigma^*)$ the resulting triangulated category.

The main result in this section is as follows. It is inspired by the work [15], and extends [15, Theorem 2.5(b)].

Theorem 5.7. *Let $A = kQ/I$ be a quadratic monomial algebra. Assume that $\mathcal{C}_1$, $\mathcal{C}_2, \cdots, \mathcal{C}_m$ are all the perfect components of $\mathcal{R}_A$, and that each d_i denotes the number of vertices in $\mathcal{C}_i$. Then there is a triangle equivalence*

$$A\text{-}\underline{\mathrm{Gproj}} \xrightarrow{\sim} \mathcal{T}_{d_1} \times \mathcal{T}_{d_2} \times \cdots \times \mathcal{T}_{d_m}.$$

Proof. Let α be a perfect arrow. In particular, the morphism θ_α in (2.1) is an isomorphism; see Lemma 3.6. Let β be a different arrow. Then we have $A\beta \cap \alpha A = \alpha A \beta$. We apply Lemma 2.3 to infer that $\underline{\mathrm{Hom}}_A(A\alpha, A\beta) = 0$. We observe that $A\alpha \cap \alpha A = k\alpha \oplus \alpha A \alpha$. Applying Lemma 2.3 again, we infer that $\underline{\mathrm{Hom}}_A(A\alpha, A\alpha) = k\mathrm{Id}_{A\alpha}$.

We recall from Proposition 5.1 that up to isomorphism, all the indecomposable objects in A-$\underline{\mathrm{Gproj}}$ are of the form $A\alpha$, where α is a perfect arrow. Recall from Lemma 5.3(3) that an arrow α is perfect if and only if the corresponding vertex in $\mathcal{R}_A$ belongs to a perfect component. From the above calculation on Hom-spaces, we deduce that the categories A-$\underline{\mathrm{Gproj}}$ and $\mathcal{T}_{d_1} \times \mathcal{T}_{d_2} \times \cdots \times \mathcal{T}_{d_m}$ are equivalent. In particular, both categories are semisimple abelian. To complete the proof, it suffices to verify that such an equivalence respects the translation functors.

Recall that the translation functor Σ on A-$\underline{\mathrm{Gproj}}$ is a quasi-inverse of the syzygy functor Ω. For a perfect arrow α lying in the perfect component $\mathcal{C}_i$, its relation-cycle is of the form $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_{d_i}, \alpha_{d_i+1} = \alpha$. Recall from (5.1) that $\Omega(A\alpha_i) \simeq A\alpha_{i-1}$, and thus $\Sigma(A\alpha_i) = A\alpha_{i+1}$. On the other hand, the translation functor on $\mathcal{T}_{d_i}$ is induced by the algebra automorphism σ in (5.2). By comparing these two translation functors, we infer that they are respected by the equivalence. $\square$

Let us illustrate the results by an example.

Example 5.8. Let Q be the following quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\delta}{\overset{\gamma}{\rightleftarrows}} 3.$$

Let I be the two-sided ideal of kQ generated by $\{\beta\alpha, \alpha\beta, \delta\gamma\}$, and let $A = kQ/I$. Then the relation quiver $\mathcal{R}_A$ is as follows.

$$\alpha \underset{[\beta\alpha]}{\overset{[\alpha\beta]}{\rightleftarrows}} \beta \qquad \delta \xrightarrow{[\delta\gamma]} \gamma$$

By Proposition 5.5(1) the algebra A is Gorenstein. We mention that A is not a gentle algebra.

By Lemma 5.3(3), all the perfect paths in A are $\{\alpha, \beta\}$. Hence, there are only two indecomposable non-projective Gorenstein-projective A-modules $A\alpha$ and $A\beta$. By Theorem 5.7 we have a triangle equivalence $A\text{-}\underline{\mathrm{Gproj}} \xrightarrow{\sim} \mathcal{T}_2$.

6. An Example: the Nakayama monomial case

In this section, we describe another example for Theorem 4.1, where the quiver is a basic cycle. In this case, the monomial algebra A is Nakayama. We recover a key characterization result of Gorenstein-projective A-modules in [17].

Let $n \geq 1$. Let $\mathbb{Z}_n$ be a basic n-cycle with the vertex set $\{1, 2, \cdots, n\}$ and the arrow set $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, where $s(\alpha_i) = i$ and $t(\alpha_i) = i + 1$. Here, we identify $n + 1$ with 1. Indeed, the vertices are indexed by the cyclic group $\mathbb{Z}/n\mathbb{Z}$. For each integer m, we denote by $[m]$ the unique integer satisfying $1 \leq [m] \leq n$ and $m \equiv [m]$ modulo n. Hence, for each vertex i, $t(\alpha_i) = [i + 1]$. We denote by p_i^l the unique path in $\mathbb{Z}_n$ starting at i of length l. We observe that $t(p_i^l) = [i + l]$.

Let I be a monomial ideal of $k\mathbb{Z}_n$, and let $A = k\mathbb{Z}_n/I$ be the corresponding monomial algebra. Then A is a connected Nakayama algebra which is elementary and has no simple projective modules. Indeed, any connected Nakayama algebra which is elementary and has no simple projective modules is of this form.

For each $1 \leq i \leq n$, we denote by $P_i = Ae_i$ the indecomposable projective A-module corresponding to i. Set $c_i = \dim_k P_i$. Following [14], we define a map $\gamma \colon \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ such that $\gamma(i) = [i + c_i]$. An element in $\cap_{d\geq 0}\mathrm{Im}\, \gamma^d$ is called *γ-cyclic*. We observe that γ restricts to a permutation on the set of γ-cyclic elements.

Following [17] we call the projective A-module P_i *minimal* if its radical rad P_i is non-projective, or equivalently, each nonzero proper submodule of P_i is non-projective. Recall that the projective cover of rad P_i is $P_{[i+1]}$. Hence, the projective A-module P_i is minimal if and only if $c_i \leq c_{[i+1]}$. We observe that if P_i is non-minimal, we have $c_i = c_{[i+1]} + 1$.

We denote by S_i the simple A-module corresponding to i. The following terminology is taken from [17]. If P_i is minimal, we will say that the vertex i, or the corresponding simple module S_i, is *black*. The vertex i, or S_i, is *γ-cyclically black* if i is γ-cyclic and $\gamma^d(i)$ is black for each $d \geq 0$.

We recall that $\mathbf{F}$ denotes the set of minimal paths contained in I.

Lemma 6.1. *Let $1 \leq i \leq n$ and $l \geq 0$. Let p, q be two nonzero paths in A such that $s(p) = t(q)$. Then we have the following statements.*

1. *The path p_i^l belongs to I if and only if $l \geq c_i$.*
2. *The path p_i^l belongs to $\mathbf{F}$ if and only if the A-module P_i is minimal and $l = c_i$.*
3. *The pair (p, q) is perfect if and only if the concatenation pq lies in $\mathbf{F}$. In this case, the vertex $s(q)$ is black.*
4. *If (p, q) is a perfect pair, then $t(p) = \gamma(s(q))$.*

Proof. Recall that $P_i = Ae_i$ has a basis given by $\{p_i^j \mid 0 \leq j < c_i\}$. Then (1) follows trivially.

For the "only if" part of (2), we assume that p_i^l belongs to $\mathbf{F}$. By the minimality of p_i^l, we have $l = c_i$. Moreover, if P_i is not minimal, we have $c_{[i+1]} = c_i - 1$ and thus $p_{[i+1]}^{l-1}$ belongs to I. This contradicts to the minimality of p_i^l. By reversing the argument, we have the "if" part.

The "only if" part of (3) follows from Lemma 3.4(P$'$1). Then $pq = p_i^l$ for $i = s(q)$ belongs to $\mathbf{F}$. By (2), the vertex i is black. For the "if" part, we apply Lemma 3.4. We only verify (P$'$2). Assume that $pq' = \gamma\delta$ with $\delta \in \mathbf{F}$. In particular, the path pq' lies in I which shares the same terminating vertex with pq. By the minimality of pq, we infer that q' is longer than q. By $t(q') = t(q)$, we infer that $q' = qx$ for some nonzero path x.

We observe by (3) and (2) that the length of pq equals c_i. Hence, we have $t(p) = [s(q) + c_i] = \gamma(s(q))$, which proves (4). □

The following result describes explicitly all the perfect paths in $A = k\mathbb{Z}_n/I$. It is in spirit close to [17, Lemma 5].

Proposition 6.2. *Let $A = k\mathbb{Z}_n/I$ be as above, and p be a nonzero nontrivial path in A. Then the path p is perfect if and only if both vertices $s(p)$ and $t(p)$ are γ-cyclically black.*

Proof. For "only if" part, we assume that the path p is perfect. We take a relation-cycle $p = p_1, p_2, \cdots, p_m, p_{m+1} = p$. We apply Lemma 6.1(3) and (4) to each perfect pair (p_i, p_{i+1}), and deduce that $s(p_{i+1})$ is black and $t(p_i) = \gamma(s(p_{i+1}))$ is also black because of $t(p_i) = s(p_{i-1})$. Moreover, we have $\gamma(s(p_{i+1})) = s(p_{i-1})$, where the subindex is taken modulo m. Then each $s(p_i)$ is γ-cyclic and so is $t(p_i)$. Indeed, they are all γ-cyclically black.

For the "if" part, we assume that both vertices $s(p)$ and $t(p)$ are γ-cyclically black. We claim that there exists a perfect pair (q, p) with both $s(q)$ and $t(q)$ γ-cyclically black.

Since the vertex $i = s(p)$ is black, by Lemma 6.1(2) the path $p_i^{c_i}$ belongs to $\mathbf{F}$. We observe that $p_i^{c_i} = qp$ for a unique nonzero path q. Then (q, p) is a perfect pair by Lemma 6.1(3). By Lemma 6.1(4), we have $t(q) = \gamma(s(p))$. Hence, both vertices $s(q) = t(p)$ and $t(q)$ are γ-cyclically black, proving the claim.

Set $q_0 = p$ and $q_1 = q$. We apply the claim repeatedly and obtain perfect pairs (q_{i+1}, q_i) for each $i \geq 0$. We assume that $q_l = q_{m+l}$ for some $l \geq 0$ and $m > 0$. Then applying Lemma 3.5 repeatedly, we infer that $q_0 = q_m$. Then we have the desired relation-cycle $p = q_m, q_{m-1}, \cdots, q_1, q_0 = p$ for the given path p. □

As a consequence, we recover a key characterization result of Gorenstein-projective A-modules in [17, Lemma 5]. We denote by top X the top of an A-module X.

Corollary 6.3. *Let $A = k\mathbb{Z}_n/I$ be as above, and M be an indecomposable non-projective A-module. Then the module M is Gorenstein-projective if and only if both* top M *and* top $\Omega(M)$ *are γ-cyclically black simple modules.*

Proof. For the "only if" part, we assume by Theorem 4.1 that $M = Ap$ for a perfect path p. We take a perfect pair (q, p) with q a perfect path. Then by (3.3) we have $\Omega(M) \simeq Aq$. We infer that top $M \simeq S_{t(p)}$ and top $\Omega(M) \simeq S_{t(q)}$. By Proposition 6.2, both simple modules are γ-cyclically black.

For the "if" part, we assume that top $M \simeq S_i$. Take a projective cover $\pi \colon P_i \to M$. Recall that each nonzero proper submodule of P_i is of the form Ap for a nonzero nontrivial path p with $s(p) = i$. Take such a path p with $Ap = \mathrm{Ker}\ \pi$, which is isomorphic to $\Omega(M)$. Therefore, by the assumption both $S_i = S_{s(p)}$ and $S_{t(p)} \simeq$ top $\Omega(M)$ are γ-cyclically black. Then by Proposition 6.2, the path p is perfect. Take a perfect pair (p, q) with q a perfect path. In particular, by (3.3) Aq is isomorphic to P_i/Ap, which is further isomorphic to M. Then we are done, since by Proposition 4.3(4) Aq is a Gorenstein-projective module. □

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