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The Hochschild cohomology ring of a Frobenius algebra with semisimple Nakayama automorphism is a Batalin–Vilkovisky algebra

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A B S T R A C T

In analogy with a recent result of N. Kowalzig and U. Krähmer for twisted Calabi–Yau algebras, we show that the Hochschild cohomology ring of a Frobenius algebra with semisimple Nakayama automorphism is a Batalin–Vilkovisky algebra, thus generalizing a result of T. Tradler for finite dimensional symmetric algebras. We give a criterion to determine when a Frobenius algebra given by quiver with relations has semisimple Nakayama automorphism and apply it to some known classes of tame Frobenius algebras. We also provide ample examples including quantum complete intersections, finite dimensional Hopf algebras defined over an algebraically closed field of characteristic zero and the Koszul duals of Koszul Artin–Schelter regular algebras of dimension three.

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Introduction

Let A be an associative algebra over a field k. The Hochschild cohomology $HH^*(A) := \mathrm{Ext}^*_{A^e}(A, A)$ of A has a very rich structure. It is a graded commutative algebra via the cup product or the Yoneda product, and it has a graded Lie bracket of degree -1 so that it becomes a graded Lie algebra; these give $HH^*(A)$ the structure of a Gerstenhaber algebra [16]. Furthermore, Hochschild homology groups are endowed with two actions by the Hochschild cohomology algebra, which give the Hochschild homology groups the structure of graded module and graded Lie module over the Hochschild cohomology algebra. These structures are summarized by the notion of a differential calculus (see [15] and [31]); we explain in detail these structures in the first section.

During several decades, a new structure in Hochschild theory has been extensively studied in topology and mathematical physics, and recently this was introduced into algebra, the so-called Batalin–Vilkovisky structure. Roughly speaking a Batalin–Vilkovisky (aka BV) structure is an operator on Hochschild cohomology which squares to zero and which, together with the cup product, can express the Lie bracket. A BV structure is known to exist over the Hochschild cohomology of certain special classes of algebras only.

T. Tradler first found that the Hochschild cohomology algebra of a finite dimensional symmetric algebra, such as a group algebra of a finite group, is a BV algebra [32]; for later proofs, see e.g. [12,28]. For a Calabi–Yau algebra, V. Ginzburg showed in [17] that the Hochschild cohomology of a Calabi–Yau algebra also has a Batalin–Vilkovisky structure.

Inspired by the result of V. Ginzburg, the first named author introduced in [25] the notion of a differential calculus with duality. This notion explains when BV structure exists and unifies the two known cases of symmetric algebras and Calabi–Yau algebras. Recently as an application of this notion, N. Kowalzig and U. Krähmer [24, Theorem 1.7] proved that the Hochschild cohomology ring of a twisted Calabi–Yau algebra is also a Batalin–Vilkovisky algebra, provided a certain algebra automorphism is semisimple.

The main result of this paper is an analogous statement for Frobenius algebras with semisimple Nakayama automorphism. Our main result reads as follows.

Theorem 0.1. *Let A be a Frobenius algebra with semisimple Nakayama automorphism. Then the Hochschild cohomology ring $HH^*(A)$ of A is a Batalin–Vilkovisky algebra.*

Observe that the semisimplicity is an open condition, and that any finite dimensional self-injective algebra defined over an algebraically closed field is Morita equivalent to its basic algebra which is a Frobenius algebra. Hence our main result shows that the Hochschild cohomology rings of large classes of self-injective algebras are BV algebras.

The paper is organized as follows. In Section 1 we explain the formalism of Tamarkin–Tsygan calculi, calculi with duality and Batalin–Vilkovisky structures. Section 2 develops the Tamarkin–Tsygan structure on the Hochschild cohomology associated with an automorphism of an algebra. We show that when the Nakayama automorphism of a Frobenius algebra is diagonalizable, then there is a differential calculus with duality which is a key ingredient of the proof of our main result. Section 3 then studies the special case of a Frobenius algebra. We provide a proof of the main result in Section 4. Section 5 contains many examples of Frobenius algebras with semisimple Nakayama automorphisms. For a Frobenius algebra given in terms of quiver with relations, we give a very useful combinatorial criterion to guarantee the semisimplicity of the Nakayama automorphism and apply it to some classes of tame Frobenius algebras. We then include other examples such as quantum complete intersections, finite dimensional Hopf algebras and the Koszul duals of Artin–Schelter regular algebras.

Throughout this paper, $\otimes$ is an abbreviation for $\otimes_k$ for k being a chosen base field.

Remark 0.2. After having finished this paper we learned that independently Y. Volkov proved in [34] a similar result with completely different methods. He works directly over Hochschild cohomology by defining an operator analogous to Tradler's operator twisted by the Nakayama automorphism. However, our method uses the concept of Tamarkin–Tsygan calculi.

1. Tamarkin–Tsygan calculus, duality and Batalin–Vilkovisky structure

1.1. Gerstenhaber algebras

First we recall the definition of Gerstenhaber algebras and of differential calculi.

Definition 1.1. A *Gerstenhaber algebra* over a field k is the data $(\mathcal{H}^*, \cup, [\ ,\ ])$, where $\mathcal{H}^* = \oplus_{n\in\mathbb{Z}}\mathcal{H}^n$ is a graded k-vector space equipped with two bilinear maps: a cup product of degree zero

$$\cup\colon \mathcal{H}^n \times \mathcal{H}^m \to \mathcal{H}^{n+m}, \quad (\alpha, \beta) \mapsto \alpha \cup \beta$$

and a Lie bracket of degree -1

$$[\ ,\ ]\colon \mathcal{H}^n \times \mathcal{H}^m \to \mathcal{H}^{n+m-1}, \quad (\alpha, \beta) \mapsto [\alpha, \beta]$$

such that

(i) $(\mathcal{H}^*, \cup)$ is a graded commutative associative algebra with unit $1 \in \mathcal{H}^0$, in particular, $\alpha \cup \beta = (-1)^{|\alpha||\beta|}\beta \cup \alpha$;

(ii) $(\mathcal{H}^*[-1], [\ ,\ ])$ is a graded Lie algebra, that is,

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta, \alpha]$$

and

$$(-1)^{(|\alpha|-1)(|\gamma|-1)}[[\alpha, \beta], \gamma] + (-1)^{(|\beta|-1)(|\alpha|-1)}[[\beta, \gamma], \alpha] + (-1)^{(|\gamma|-1)(|\beta|-1)}[[\gamma, \alpha], \beta] = 0;$$

(iii) for each $\alpha \in \mathcal{H}^*[-1]$ the map $[\alpha, -]$ is a graded derivation of the algebra $(\mathcal{H}^*, \cup)$, or more precisely

$$[\alpha,\, \beta \cup \gamma] = [\alpha,\, \beta] \cup \gamma + (-1)^{(|\alpha|-1)|\beta|} \beta \cup [\alpha,\, \gamma],$$

where α, β, γ are arbitrary homogeneous elements in $\mathcal{H}^*$ and $|\alpha|$ is the degree of the homogeneous element α.

Remark 1.2. Let k' be a field extension of k. Then for a Gerstenhaber algebra $\mathcal{H}^*$ over k, $\mathcal{H}^* \otimes k'$ is a Gerstenhaber algebra over k'. In fact, for homogeneous elements $\alpha, \beta \in \mathcal{H}^*$ and $\lambda, \nu \in k'$, define

$$(\alpha \otimes \lambda) \cup (\beta \otimes \nu) = (\alpha \cup \beta) \otimes (\lambda\nu)$$

and

$$[\alpha \otimes \lambda,\, \beta \otimes \nu] = [\alpha, \beta] \otimes (\lambda\nu).$$

These two operations endow $\mathcal{H}^* \otimes k'$ with a Gerstenhaber algebra structure over k'.

1.2. Tamarkin–Tsygan calculi

Definition 1.3. A *differential calculus* or a *Tamarkin–Tsygan calculus* is the data $(\mathcal{H}^*, \cup, [\ ,\ ], \mathcal{H}_*, \cap, B)$ of $\mathbb{Z}$-graded vector spaces satisfying the following properties:

(i) $(\mathcal{H}^*, \cup, [\ ,\ ])$ is a Gerstenhaber algebra;
(ii) $\mathcal{H}_*$ is a graded module over $(\mathcal{H}^*, \cup)$ via the map $\cap : \mathcal{H}_r \otimes \mathcal{H}^p \to \mathcal{H}_{r-p}$, $z \otimes \alpha \mapsto z \cap \alpha$ for $z \in \mathcal{H}_r$ and $\alpha \in \mathcal{H}^p$. That is, if we denote $\iota_\alpha(z) = (-1)^{rp} z \cap \alpha$, then $\iota_{\alpha \cup \beta} = \iota_\alpha \iota_\beta$;
(iii) There is a map $B : \mathcal{H}_* \to \mathcal{H}_{*+1}$ such that $B^2 = 0$ and we have the Cartan relation

$$[L_\alpha,\ \iota_\beta]_{gr} = (-1)^{|\alpha|-1} \iota_{[\alpha,\beta]}$$

where we denote

$$L_\alpha = [B, \iota_\alpha]_{gr} = B\iota_\alpha - (-1)^{|\alpha|} \iota_\alpha B.$$

One of the first examples of differential calculi is Hochschild theory.

The cohomology theory of associative algebras was introduced by G. Hochschild [20]. Given a k-algebra A, its Hochschild cohomology groups of A with coefficients in a bimodule M are defined as $H^n(A, M) = \mathrm{Ext}^n_{A^e}(A, M)$ for $n \geq 0$, where $A^e = A \otimes A^{\mathrm{op}}$ is the enveloping algebra of A, and the Hochschild homology groups of A with coefficients in M are defined to be $H_n(A, M) = Tor_n^{A^e}(A, M)$ for $n \geq 0$. We shall write $HH_n(A) = H_n(A, A)$ and $HH^n(A) = H^n(A, A)$.

Since A is unitary, denote by 1_A its unity and write $\overline{A} = A/(k \cdot 1_A)$. For $a \in A$, write $\overline{a}$ for its image in $\overline{A}$. There is a projective resolution of A as an A^e-module, the so-called *normalized bar resolution* $\mathrm{Bar}_*(A)$, whose r-th term is given by $\mathrm{Bar}_r(A) = A \otimes \overline{A}^{\otimes r} \otimes A$ for $r \geq 0$ and for which the differential $b'_r : \mathrm{Bar}_r(A) \to \mathrm{Bar}_{r-1}(A)$ sends $a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_r} \otimes a_{r+1}$ to

$$\begin{aligned} &a_0 a_1 \otimes \overline{a_2} \otimes \cdots \otimes \overline{a_r} \otimes a_{r+1} \\ &\quad + \sum_{i=1}^{r-1} (-1)^i a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{a_i a_{i+1}} \otimes \overline{a_{i+2}} \otimes \cdots \otimes \overline{a_r} \otimes a_{r+1} \\ &\quad + (-1)^r a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{r-1}} \otimes a_r a_{r+1} \end{aligned}$$

for all $a_0, \cdots, a_{r+1} \in A$.

The complex which is used to compute the Hochschild cohomology is $C^*(A, M) = \mathrm{Hom}_{A^e}(\mathrm{Bar}_*(A), M)$. Note that for each $r \geq 0$,

$$C^r(A, M) = \mathrm{Hom}_{A^e}(A \otimes \overline{A}^{\otimes r} \otimes A, M) \simeq \mathrm{Hom}_k(\overline{A}^{\otimes r}, M).$$

We identify $C^0(A, M)$ with M. Thus $C^*(A, M)$ has the following form:

$$\begin{aligned} C^*(A, M)\colon M &\xrightarrow{b^0} \mathrm{Hom}_k(\overline{A}, M) \to \cdots \to \mathrm{Hom}_k(\overline{A}^{\otimes r}, M) \\ &\xrightarrow{b^r} \mathrm{Hom}_k(\overline{A}^{\otimes (r+1)}, M) \to \cdots . \end{aligned}$$

Given f in $\mathrm{Hom}_k(\overline{A}^{\otimes r}, M)$, the map $b^r(f)$ is defined by sending $\overline{a_1} \otimes \cdots \otimes \overline{a_{r+1}}$ to

$$\begin{aligned} &(-1)^{r+1} a_1 \cdot f(\overline{a_2} \otimes \cdots \otimes \overline{a_{r+1}}) \\ &\quad + \sum_{i=1}^{r} (-1)^{r+1-i} f(\overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{a_i a_{i+1}} \otimes \overline{a_{i+2}} \otimes \cdots \otimes \overline{a_{r+1}}) \\ &\quad + f(\overline{a_1} \otimes \cdots \otimes \overline{a_r}) \cdot a_{r+1}. \end{aligned}$$

For bimodules M and N, given $\alpha \in C^p(A, M)$ and $\beta \in C^q(A, N)$, the cup product

$$\alpha \cup \beta \in C^{p+q}(A, M \otimes_A N) = \mathrm{Hom}_k(\overline{A}^{\otimes (p+q)}, M \otimes_A N)$$

is given by

$$(\alpha \cup \beta)(\overline{a_1} \otimes \cdots \otimes \overline{a_{p+q}}) := (-1)^{pq} \alpha(\overline{a_1} \otimes \cdots \otimes \overline{a_p}) \otimes_A \beta(\overline{a_{p+1}} \otimes \cdots \otimes \overline{a_{p+q}}).$$

This cup product induces a well-defined product in Hochschild cohomology

$$\cup\colon H^p(A, M) \times H^q(A, N) \longrightarrow H^{p+q}(A, M \otimes_A N)$$

which turns the graded k-vector space $HH^*(A) = \bigoplus_{n\geq 0} HH^n(A)$ into a graded commutative algebra [16, Corollary 1].

The Lie bracket is defined as follows. Let $\alpha \in C^n(A, A)$ and $\beta \in C^m(A, A)$. If $n, m \geq 1$, then for $1 \leq i \leq n$, set $\alpha \,\bar{\circ}_i\, \beta \in C^{n+m-1}(A, A)$ by

$$(\alpha \,\bar{\circ}_i\, \beta)(a_1 \otimes \cdots \otimes a_{n+m-1}) := \alpha(a_1 \otimes \cdots \otimes a_{i-1} \otimes \beta(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1});$$

if $n \geq 1$ and $m = 0$, then $\beta \in A$ and for $1 \leq i \leq n$, set

$$(\alpha \,\bar{\circ}_i\, \beta)(\overline{a_1} \otimes \cdots \otimes \overline{a_{n-1}}) := \alpha(\overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{\beta} \otimes \overline{a_i} \otimes \cdots \otimes \overline{a_{n-1}});$$

for any other case, set $\alpha \,\bar{\circ}_i\, \beta$ to be zero. Now define

$$\alpha \,\bar{\circ}\, \beta := \sum_{i=1}^{n} (-1)^{(m-1)(i-1)} \alpha \,\bar{\circ}_i\, \beta$$

and

$$[\alpha,\, \beta] := \alpha \,\bar{\circ}\, \beta - (-1)^{(n-1)(m-1)} \beta \,\bar{\circ}\, \alpha.$$

Note that $[\alpha,\, \beta] \in C^{n+m-1}(A, A)$. The above $[\ ,\ ]$ induces a well-defined Lie bracket in Hochschild cohomology

$$[\ ,\ ] \colon HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m-1}(A)$$

such that $(HH^*(A), \cup, [\ ,\ ])$ is a Gerstenhaber algebra [16, page 267, Theorem].

The complex used to compute the Hochschild homology $H_*(A, M)$ is $C_*(A, M) = M \otimes_{A^e} \mathrm{Bar}_*(A)$. Notice that for $r \geq 0$, $C_r(A, M) = M \otimes_{A^e} (A \otimes \overline{A}^{\otimes r} \otimes A) \simeq M \otimes \overline{A}^{\otimes r}$ and the differential

$$b_r : C_r(A, M) = M \otimes \overline{A}^{\otimes r} \to C_{r-1}(A, M) = M \otimes \overline{A}^{\otimes (r-1)}$$

sends $x \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_r}$ to

$$xa_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_r} + \textstyle\sum_{i=1}^{r-1} (-1)^i x \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{a_i a_{i+1}} \otimes \overline{a_{i+2}} \otimes \cdots \otimes \overline{a_r} + (-1)^r a_r x \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{r-1}}.$$

There is an A. Connes' B-operator in the Hochschild homology theory which is defined as follows. For $a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_r} \in C_r(A, A)$, let $\mathrm{B}(a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_r}) \in C_{r+1}(A, A)$ be

$$\sum_{i=0}^{r} (-1)^{ir} 1 \otimes \overline{a_i} \otimes \cdots \otimes \overline{a_r} \otimes \overline{a_0} \otimes \cdots \otimes \overline{a_{i-1}}.$$

It is easy to check that B is a chain map satisfying $\mathrm{B} \circ \mathrm{B} = 0$, which induces an operator $\mathrm{B} : HH_r(A) \to HH_{r+1}(A)$.

There is a pairing between the Hochschild cohomology and Hochschild homology, which is called the cap product. For bimodules M and N, there is a bilinear map

$$\cap : C_r(A, N) \otimes C^p(A, M) \to C_{r-p}(A, N \otimes_A M)$$

sending $z \otimes \alpha$ to

$$z \cap \alpha = (-1)^{rp}(x \otimes_A \alpha(\overline{a_1} \otimes \cdots \otimes \overline{a_p})) \otimes \overline{a_{p+1}} \otimes \cdots \otimes \overline{a_r}$$

for $z = x \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_r} \in C_r(A, N)$ and $\alpha \in C^p(A, M)$. One verifies easily that $\cap$ induces a well-defined map on the level of homology, still denoted by $\cap$,

$$\cap : H_r(A, N) \otimes H^p(A, M) \to H_{r-p}(A, N \otimes_A M).$$

I.M. Gelfand, Yu.L. Daletskii and B.L. Tsygan proved the following result; see also [31].

Theorem 1.4. *(See [15].) The data* $(HH^*(A),\ \cup,\ [\ ,\ ],\ HH_*(A),\ \cap,\ \mathrm{B})$ *is a differential calculus.*

1.3. Batalin–Vilkovisky algebras

In the last decade, a new structure in Hochschild theory has been observed, this is the so called Batalin–Vilkovisky structure.

Definition 1.5. A *Batalin–Vilkovisky algebra* (BV algebra for short) is a Gerstenhaber algebra $(\mathcal{H}^*, \cup, [\ ,\ ])$ together with an operator $\Delta \colon \mathcal{H}^* \to \mathcal{H}^{*-1}$ of degree -1 such that $\Delta \circ \Delta = 0$, $\Delta(1) = 0$ and

$$[\alpha, \beta] = (-1)^{|\alpha|+1}(\Delta(\alpha \cup \beta) - \Delta(\alpha) \cup \beta - (-1)^{|\alpha|}\alpha \cup \Delta(\beta)),$$

for homogeneous elements $\alpha, \beta \in \mathcal{H}^*$. The BV-operator $\Delta : \mathcal{H}^* \to \mathcal{H}^{*-1}$ is called a generator of the Gerstenhaber bracket $[\ ,\ ]$.

For a Calabi–Yau algebra, V. Ginzburg showed in [17] that the Hochschild cohomology of a Calabi–Yau algebra has a Batalin–Vilkovisky structure. More precisely, for a Calabi–Yau algebra A of global dimension d, there is a duality $HH^p(A) \simeq HH_{d-p}(A))$ for $p \geq 0$. Via this duality, we obtain an operator $\Delta \colon HH^p(A) \to HH^{p-1}(A)$ which is the dual of Connes' operator. This is just the operator Δ in the Batalin–Vilkovisky structure.

N. Kowalzig and U. Krähmer extended the result of V. Ginzburg to twisted Calabi–Yau algebras under a certain condition. Let A be a twisted Calabi–Yau algebra with semisimple algebra automorphism σ. Then the Hochschild cohomology ring of A is a Batalin–Vilkovisky algebra; see [24, Theorem 1.7].

T. Tradler showed that the Hochschild cohomology algebra of a symmetric algebra is a BV algebra [32]; L. Menichi [28] gave a conceptual proof using the formalism of cyclic operads with multiplications; for another proof, see also [12]. For a symmetric algebra A, he showed that the Δ-operator on the Hochschild cohomology corresponds to the Connes' B-operator on the Hochschild homology via the duality between the Hochschild cohomology and the Hochschild homology.

1.4. Algebras with duality, Tamarkin–Tsygan calculi and BV-structures

Generalizing [25] we define.

Definition 1.6. An algebra with duality is given by $(\mathcal{H}^*, \cup, \mathcal{H}_*, c, \partial)$, where

- $(\mathcal{H}^*, \cup)$ is a graded commutative unitary algebra with unit $1 \in \mathcal{H}^0$,
- $\mathcal{H}_*$ is a graded vector space and c is an element of $\mathcal{H}_d$,
- ∂ is an isomorphism of vector spaces $\partial : \mathcal{H}_* \to \mathcal{H}^{d-*}$ satisfying $\partial(c) = 1$.

Inspired by the result of V. Ginzburg, the first author gave the following result which shows for an algebra with duality there is an equivalence between BV-structure and Tamarkin–Tsygan calculus.

Lemma 1.7. *Let $(\mathcal{H}^*, \cup, \mathcal{H}_*, c, \partial)$ be an algebra with duality.*

(1) *We suppose that*
   (a) *$(\mathcal{H}^*, \cup, [\ ,\ ], \mathcal{H}_*, \cap, B)$ is a Tamarkin–Tsygan calculus,*
   (b) *the duality ∂ is a homomorphism of $\mathcal{H}^*$-right modules, i.e. we have the relation*

$$\partial(z \cap \alpha) = \partial(z) \cup \alpha.$$

Then the Gerstenhaber algebra $(\mathcal{H}^*, \cup, [\ ,\ ])$ is a BV-algebra with generator $\Delta = \partial \circ B \circ \partial^{-1}$.

(2) *We suppose that $(\mathcal{H}^*, \cup, [\ ,\ ], \Delta)$ is a BV-algebra with generator Δ. Then posing $B := \partial^{-1} \circ \Delta \circ \partial$ and $z \cap \alpha := \partial^{-1}(\partial(z) \cup \alpha)$, the data $(\mathcal{H}^*, \cup, [\ ,\ ], \mathcal{H}_*, \cap, B)$ is a Tamarkin–Tsygan calculus.*

Proof. For (1) see [25, Lemme 1.6] and (2) is an easy verification. A similar idea also appeared in [12, Remark 2.3.67]. □

Remark 1.8.

(1) Regardless its simplicity the relation $\partial(z \cap \alpha) = \partial(z) \cup \alpha$, which was first noted by V. Ginzburg in [17, Theorem 34.3], is necessary. For this reason we call it the Ginzburg relation.

(2) This lemma allows to establish the results of V. Ginzburg (for Calabi–Yau algebras), and of Kowalzig and Krähmer (for twisted Calabi–Yau algebras). We shall see that it applies also to the case of Frobenius algebras.

2. Tamarkin–Tsygan calculus associated with an automorphism of an algebra

Let A be an associative, finite dimensional and unitary k-algebra and let $\mathfrak{N} : A \to A$ be an automorphism of this algebra. The aim of this paragraph is to construct a Tamarkin–Tsygan calculus associated to this automorphism $\mathfrak{N}$. Denote by $A_{\mathfrak{N}}$ the A-A-bimodule which is A as a k-vector space, and on which we define the bimodule action as $a \cdot m \cdot b := am\mathfrak{N}(b)$. Kowalzig and Krähmer define in [24, 2.18, 7.2] a morphism of k-vector spaces

$$\beta_{\mathfrak{N}} : C_p(A, A_{\mathfrak{N}}) \to C_{p+1}(A, A_{\mathfrak{N}})$$

by

$$\beta_{\mathfrak{N}}(a_0 \otimes a_1 \otimes \cdots \otimes a_p) = \Sigma_{i=0}^{p}(-1)^{ip}\ 1 \otimes a_i \otimes \cdots \otimes a_p \otimes a_0 \otimes \mathfrak{N}(a_1) \otimes \cdots \otimes \mathfrak{N}(a_{i-1})$$

Let $T : C_p(A, A_{\mathfrak{N}}) \to C_p(A, A_{\mathfrak{N}})$ be the morphism defined by

$$T(a_0 \otimes \cdots \otimes a_p) = \mathfrak{N}(a_0) \otimes \mathfrak{N}(a_1) \otimes \cdots \otimes \mathfrak{N}(a_p).$$

Proposition 2.1. *(See N. Kowalzig and U. Krähmer [24].) On the space $C_p(A, A_{\mathfrak{N}})$, we get the identity*

$$b\beta_{\mathfrak{N}} + \beta_{\mathfrak{N}} b = 1 - T$$

where b is the Hochschild differential.

Proof. See [24, 2.19] in the setup of Hopf algebroids; for a proof in the setup of Hochschild cohomology, see [18, Section 4]. □

2.1. Decomposition of the homology associated with the spectrum of an automorphism

Let Λ be the set of eigenvalues of the automorphism $\mathfrak{N}$. Suppose that $\Lambda \subset k$. Fix an eigenvalue $\mu \in \Lambda$ of $\mathfrak{N}$ and let A_μ be the eigenspace associated with μ. It is trivial to see that for $\mu, \nu \in \Lambda$, we get $A_\mu \cdot A_\nu \subseteq A_{\mu\nu}$. When $\mu\nu \notin \Lambda$, it is understood that $A_{\mu\nu} = 0$. Denote by $\hat{\Lambda} := \langle \Lambda \rangle$ the submonoid of $k^\times$ generated by Λ.

For $\mu \in \Lambda$, write $\overline{A}_\mu = A_\mu$ for $\mu \neq 1$ and $\overline{A}_1 = A_1/(k \cdot 1_A)$, and for each $\mu \in \hat{\Lambda}$ put

$$C_p^\mu(A, A_{\mathfrak{N}}) := \oplus_{\mu_i \in \Lambda, \prod \mu_i = \mu}\ \ A_{\mu_0} \otimes \overline{A}_{\mu_1} \otimes \cdots \otimes \overline{A}_{\mu_p}.$$

The Hochschild differential $b : C_p(A, A_{\mathfrak{N}}) \to C_{p-1}(A, A_{\mathfrak{N}})$ restricts to this subspace and denote its restriction by b^μ, then $(C^\mu_*(A, A_{\mathfrak{N}}), b^\mu)$ is a sub-complex of $(C_*(A, A), b)$. Denote

$$H^\mu_p(A, A_{\mathfrak{N}}) := H_p(C^\mu_*(A, A_{\mathfrak{N}}), b^\mu).$$

We hence obtain a vector space homomorphism $H^\mu_*(A, A_{\mathfrak{N}}) \to H_*(A, A_{\mathfrak{N}})$.

Proposition 2.2.

(1) *For each $\mu \neq 1 \in \hat{\Lambda}$, we get*

$$H^\mu_*(A, A_{\mathfrak{N}}) = 0.$$

(2) *The restriction $\beta^1_{\mathfrak{N}} : C^1_*(A, A_{\mathfrak{N}}) \to C^1_{*+1}(A, A_{\mathfrak{N}})$ of the map $\beta_{\mathfrak{N}}$ to the sub-complex associated to the eigenvalue 1 induces a Connes operator*

$$B_{\mathfrak{N}} : H^1_*(A, A_{\mathfrak{N}}) \to H^1_{*+1}(A, A_{\mathfrak{N}})$$

with coefficients in the twisted bimodule $A_{\mathfrak{N}}$, and this map satisfies $B^2_{\mathfrak{N}} = 0$.

Proof. For each eigenvalue $\mu \in \hat{\Lambda}$ of the automorphism $\mathfrak{N}$ on $C^\mu_p(A, A_{\mathfrak{N}})$, we obtain the identity

$$b^\mu \beta_{\mathfrak{N}} + \beta_{\mathfrak{N}} b^\mu = 1 - T.$$

However, the restriction of T to $C^\mu_*(A, A_{\mathfrak{N}})$ is $\mu \cdot \mathrm{id}$, we get $b^\mu \beta_{\mathfrak{N}} + \beta_{\mathfrak{N}} b^\mu = (1 - \mu) \cdot \mathrm{id}$. Whenever $\mu \neq 1$ the complex $(C^\mu_*(A, A_{\mathfrak{N}}), b^\mu)$ is acyclic with contracting homotopy $\beta_{\mathfrak{N}}$. For $\mu = 1$ we get $b_1 \beta^1_{\mathfrak{N}} + \beta^1_{\mathfrak{N}} b_1 = 0$, which defines $B_{\mathfrak{N}}$. The relation $B^2_{\mathfrak{N}} = 0$ is a consequence of [24, 2.19]. □

An analogous decomposition exists for cohomology. For $\mu \in \hat{\Lambda}$, let $C^p_\mu(A, A)$ be those Hochschild cochains $\varphi \in C^p(A, A)$ such that we have $\varphi(\overline{A}_{\nu_1} \otimes \cdots \otimes \overline{A}_{\nu_p}) \subset A_{\mu\nu_1 \cdots \nu_p}$ for all eigenvalues $\nu_1, \cdots, \nu_p$ of $\mathfrak{N}$. The restriction b_μ of the Hochschild differential $b : C^p(A, A) \to C^{p+1}(A, A)$ to $C^p_\mu(A, A)$ has values in $C^{p+1}_\mu(A)$. Put

$$H^p_\mu(A, A) := H^p(C^*_\mu(A, A), b_\mu).$$

The sub-complex $(C^*_\mu(A, A), b_\mu)$ of $(C^*(A, A), b)$ defines a morphism of graded vector spaces

$$H^*_\mu(A, A) \to HH^*(A).$$

For $\mu, \nu \in \hat{\Lambda}$, we verify that the cup-product $\cup : HH^p(A) \otimes HH^q(A) \to HH^{p+q}(A)$ and the Gerstenhaber bracket $[\ ,\ ] : HH^p(A) \otimes HH^q(A) \to HH^{p+q-1}(A)$ induce restrictions

$$\cup_{\mu,\nu} : H^p_\mu(A, A) \otimes H^q_\nu(A, A) \to H^{p+q}_{\mu\nu}(A, A)$$

and

$$[\ ,\ ]_{\mu,\nu} : H^p_\mu(A, A) \otimes H^q_\nu(A, A) \to H^{p+q-1}_{\mu\nu}(A, A).$$

Analogously, the *cap*-product $\cap : HH_p(A, A_{\mathfrak{N}}) \otimes HH^q(A) \to H_{p-q}(A, A_{\mathfrak{N}})$ induces restrictions

$$\cap_{\mu,\nu} : H^\mu_p(A, A_{\mathfrak{N}}) \otimes H^q_\nu(A, A) \to H^{\mu\nu}_{p-q}(A, A_{\mathfrak{N}}).$$

2.2. The case of eigenvalue 1

Apply the results above to the case $\mu = \nu = 1$. We then get:

Theorem 2.3. *Let $\mathfrak{N}$ be an automorphism of the algebra A. Let Λ be the set of eigenvalues of the automorphism $\mathfrak{N}$. Suppose that $\Lambda \subset k$. Let $\cup_1 := \cup_{1,1}$, $[\ ,\ ]_1 := [\ ,\ ]_{1,1}$ and $\cap_1 := \cap_{1,1}$ be the restrictions of the cup-products, Gerstenhaber bracket and* cap*-product to the homology and cohomology spaces associated with the eigenvalue* 1*. Then Connes' operator $B_{\mathfrak{N}}$ gives*

$$(H^*_1(A, A), \cup_1, [\ ,\ ]_1, H^1_*(A, A_{\mathfrak{N}}), \cap_1, B_{\mathfrak{N}})$$

the structure of a Tamarkin–Tsygan calculus.

Remark 2.4. This Tamarkin–Tsygan calculus applies in diverse types of algebras for which its Hochschild cohomology/homology are naturally equipped with a duality:

- Calabi–Yau algebras for which the dualizing module $\mathcal{D}$ is isomorphic to the module A. In this case the automorphism $\mathfrak{N}$ is the identity and this is the situation studied by V. Ginzburg.
- Twisted Calabi–Yau algebras for which the dualizing module $\mathcal{D}$ is isomorphic to the module $A_{\mathfrak{N}}$. This is the situation studied by N. Kowalzig and U. Krähmer.
- Symmetric algebras for which the Nakayama automorphism is $\mathfrak{N} = \mathrm{id}$. This is the situation studied by T. Tradler.
- Frobenius algebras. This is the situation studied in this paper.

2.3. The diagonalizable case

Proposition 2.5. *If $\mathfrak{N}$ is diagonalizable, then*

$$H^1_*(A, A_{\mathfrak{N}}) \simeq H_*(A, A_{\mathfrak{N}}).$$

Proof. Since $A = \oplus_{\mu\in\Lambda} A_\mu$, we get

$$(C_*(A, A_{\mathfrak{N}}), b) = \oplus_{\mu\in\hat{\Lambda}}(C^\mu_*(A, A_{\mathfrak{N}}), b^\mu)$$

and therefore $H_*(A, A_{\mathfrak{N}}) = \oplus_{\mu\in\hat{\Lambda}} H^\mu_*(A, A_{\mathfrak{N}})$. For $\mu \neq 1$, we get $H^\mu_*(A, A_{\mathfrak{N}}) = 0$. This proves $H_*(A, A_{\mathfrak{N}}) = H^1_*(A, A_{\mathfrak{N}})$. □

3. The Hochschild cohomology ring of a Frobenius algebra

3.1. Algebra with duality associated with a Frobenius algebra

Let k be a field and let A be a finite dimensional k-algebra. Recall (cf. e.g. [37, Section 1.10.1] or [35]) that A is a Frobenius algebra, if there is a non-degenerate associative bilinear form $\langle -, -\rangle : A \times A \to k$. Here the associativity means that $\langle ab, c\rangle = \langle a, bc\rangle$ for all a, b and c in A. Endow $D(A) = \mathrm{Hom}_k(A, k)$, the k-dual of A, with the canonical bimodule structure

$$(afb)(c) = f(bca), \text{ for } f \in D(A), a, b, c \in A.$$

The property of being Frobenius is equivalent to saying that $D(A) = \mathrm{Hom}_k(A, k)$ is isomorphic to A as left or as right modules. It is readily seen that the map $a \mapsto \langle a, -\rangle$ for $a \in A$ gives an isomorphism of right modules between A and $D(A)$, while the map $a \mapsto \langle -, a\rangle$ gives the isomorphism of left modules. For $a \in A$, there exists a unique $\mathfrak{N}(a) \in A$ such that $\langle a, -\rangle = \langle -, \mathfrak{N}(a)\rangle \in D(A)$. It is easy to see that $\mathfrak{N} : A \to A$ is an algebra isomorphism and we call it the Nakayama automorphism of A (associated to the bilinear form $\langle -, -\rangle$). As above we write $A_{\mathfrak{N}}$ for the A-A-bimodule whose underlying space is A and where the left A-module structure is given by left multiplication and the right A-module structure is given by $x \cdot a = x\, \mathfrak{N}(a)$ for $x \in A_{\mathfrak{N}}$ and $a \in A$. Then the map $a \mapsto \langle -, a\rangle$ is an isomorphism of bimodules $A_{\mathfrak{N}} \simeq D(A)$. In fact for $x \in A_{\mathfrak{N}}$ and $a \in A$, via the isomorphism of left modules $A \simeq D(A)$, $x\, \mathfrak{N}(a)$ is sent to

$$\begin{aligned}\langle -,\ x\, \mathfrak{N}(a)\rangle &= \langle \mathfrak{N}^{-1}(x\mathfrak{N}(a)),\ -\rangle \\ &= \langle \mathfrak{N}^{-1}(x)\ a,\ -\rangle = \langle \mathfrak{N}^{-1}(x),\ a\ -\rangle = \langle a\ -,\ x\rangle = \langle -,\ x\rangle a.\end{aligned}$$

Using the isomorphism of bimodules $D(A) \simeq A_{\mathfrak{N}}$, we can establish a well known duality between Hochschild cohomology and Hochschild homology groups. In fact there are isomorphisms of complexes:

$$\begin{aligned} D(C_*(A, A_{\mathfrak{N}})) &= \operatorname{Hom}_k(A_{\mathfrak{N}} \otimes_{A^e} \operatorname{Bar}_*(A), k) \simeq \operatorname{Hom}_{A^e}(\operatorname{Bar}_*(A), D(A_{\mathfrak{N}})) \\ &\simeq \operatorname{Hom}_{A^e}(\operatorname{Bar}_*(A), A) = C^*(A, A), \end{aligned}$$

where the third isomorphism is induced by the isomorphism $A_{\mathfrak{N}} \simeq D(A)$. This induces an isomorphism

$$\partial : D(H_*(A, A_{\mathfrak{N}})) \xrightarrow{\simeq} HH^*(A).$$

This isomorphism comes from the pairing $H_*(A, A_{\mathfrak{N}}) \otimes HH^*(A) \to k$. Explicitly for $a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_p} \in C_p(A, A_{\mathfrak{N}})$ and $\alpha \in C^p(A, A)$, the pairing is given by

$$\langle a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_p}, \alpha \rangle = (-1)^p \langle a_0, \alpha(\overline{a_1} \otimes \cdots \otimes \overline{a_p}) \rangle.$$

Remark 3.1. The isomorphism ∂ is not easy to describe, but its inverse $\partial^{-1} : HH^*(A) \xrightarrow{\simeq} D(H_*(A, A_{\mathfrak{N}}))$ is given by $\partial^{-1}(\alpha) = (-1)^{|\alpha|}\langle -, \alpha \rangle$. In particular for $\alpha = 1_A \in HH^0(A)$ we put $c := \partial^{-1}(1_A) = \langle -, 1_A \rangle$. In other words, the class $c \in D(H_*(A, A_{\mathfrak{N}}))$ is chosen such that $\partial(c) = 1_A$.

Definition 3.2. The element $c \in D(H_*(A, A_{\mathfrak{N}}))$ from Remark 3.1 is called the fundamental class of the Frobenius algebra A.

Proposition 3.3. *Let A be a Frobenius algebra with Nakayama automorphism $\mathfrak{N}$. Put $\mathcal{H}_{-*} := D(H_*(A, A_{\mathfrak{N}}))$, $\mathcal{H}^* = HH^*(A)$ and $c = \langle -, 1_A \rangle \in \mathcal{H}_0$.*

(1) *There is a cap product $\cap : \mathcal{H}_{-p} \otimes \mathcal{H}^q \to \mathcal{H}_{-(p+q)}$ for which the isomorphism $\partial^{-1} : \mathcal{H}^* \to \mathcal{H}_*$ is the cap product by the fundamental class, i.e. for all $\alpha \in \mathcal{H}^*$ it satisfies the equation $\partial^{-1}(\alpha) = c \cap \alpha$.*
(2) *The inverse isomorphism $\partial : \mathcal{H}_{-*} \to \mathcal{H}^*$ is a morphism of $\mathcal{H}^*$-modules i.e. it satisfies the Ginzburg relation $\partial(z \cap \alpha) = \partial(z) \cup \alpha$.*

Proof. (1). For $z \in \mathcal{H}_{-p}$ and $\alpha \in \mathcal{H}^q$, define $z \cap \alpha \in \mathcal{H}_{-(p+q)}$ as follows. For $t \in H_{p+q}(A, A_{\mathfrak{N}})$ we have $t \cap \alpha \in H_p(A, A_{\mathfrak{N}})$. The map $z \cap \alpha : H_{p+q}(A, A_{\mathfrak{N}}) \to k$ is defined by $(z \cap \alpha)(t) := (-1)^{(p+q)q} z(t \cap \alpha)$, that is,

$$z \cap \alpha(-) = (-1)^{(|z|+|\alpha|)\cdot|\alpha|} z(- \cap \alpha).$$

We claim that $\alpha \in \mathcal{H}^p$, the equality $\partial^{-1}(\alpha) = c \cap \alpha$ holds. In fact, we know from the previous remark that $\partial^{-1}(\alpha) = (-1)^{|\alpha|}\langle - , \alpha \rangle$. Suppose that $\alpha = cl(f)$ is the cohomology class of $f : \overline{A}^{\otimes p} \to A$ and $u = cl(a_0 \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_p)$ the homological class of $a_0 \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_p \in A_{\mathfrak{N}} \otimes \overline{A}^{\otimes p}$. Then we get $u \cap \alpha = (-1)^p a_0 f(\overline{a}_1 \otimes \cdots \otimes \overline{a}_p) \in A$ and

$$\begin{aligned} (c \cap \alpha)(u) &= (-1)^{(0+p)p} c(u \cap \alpha) = (-1)^p \langle u \cap \alpha, 1_A \rangle \\ &= (-1)^p \langle (-1)^p a_0 f(\overline{a}_1 \otimes \cdots \otimes \overline{a}_p), 1_A \rangle; \end{aligned}$$

on the other hand,

$$\begin{aligned}\partial^{-1}(\alpha)(u) &= (-1)^p\langle u,\ \alpha\rangle = (-1)^p\langle a_0 \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_p, f\rangle \\ &= (-1)^p(-1)^p\langle a_0, f(\overline{a}_1 \otimes \cdots \otimes \overline{a}_p)\rangle = (c \cap \alpha)(u).\end{aligned}$$

This proves that for all $\alpha \in HH^p(A)$ one has the equality $\partial^{-1}(\alpha) = c \cap \alpha$ in $D(H_p(A, A_{\mathfrak{N}}))$.

(2). For $z \in \mathcal{H}_{-p}$, $\alpha \in \mathcal{H}^r$ and $\beta \in \mathcal{H}^q$, we verify that in $\mathcal{H}_{-p+q+r}$, the equality $z \cap (\alpha \cup \beta) = (z \cap \alpha) \cap \beta$ holds. It follows that ∂^{-1} (and also ∂) is an isomorphism of $\mathcal{H}^*$-modules. □

An alternative and very short proof can be given by *defining* $\cap$ by the Ginzburg relation. Then $\partial(c \cap \alpha) = \partial(c) \cup \alpha = 1 \cup \alpha = \alpha$ and we get 1). The above proof however gives a much more detailed clarification of the structures in the sense that the cap product claimed in the statement of the proposition is indeed the standard cap product of Hochschild cohomology.

As a whole we obtained the following.

Proposition 3.4. *Let A be a Frobenius algebra with Nakayama automorphism $\mathfrak{N}$. Put $\mathcal{H}_{-*} := D(H_*(A, A_{\mathfrak{N}}))$, $\mathcal{H}^* = HH^*(A)$, $\partial : D(H_*(A, A_{\mathfrak{N}})) \xrightarrow{\simeq} HH^*(A)$ and $c = \langle -, 1_A\rangle \in \mathcal{H}_0$. Then $(\mathcal{H}^*, \cup, \mathcal{H}_{-*}, c, \partial)$ is an algebra with duality.*

3.2. The spectrum of the Nakayama automorphism of a Frobenius algebra

Let A be a Frobenius algebra with Nakayama automorphism $\mathfrak{N}$. Let Λ be the set of eigenvalues of $\mathfrak{N}$ (in the algebraic closure $\overline{k}$ of k) considered as a linear transformation of the finite dimensional k-vector space A. Notice that elements of Λ are not necessarily in k.

Since $\mathfrak{N}$ is an automorphism, $0 \notin \Lambda$; since $\mathfrak{N}(1_A) = 1_A$ with 1_A the unit element of A, we have $1 \in \Lambda$; for some eigenvectors $x, y \in A$ with eigenvalues $\mu, \nu \in \Lambda$ respectively, we have $\mathfrak{N}(xy) = \mathfrak{N}(x)\mathfrak{N}(y) = \mu\nu xy$ and therefore, if $xy \neq 0$ then $\mu\nu \in \Lambda$.

Lemma 3.5. *Let A be a Frobenius k-algebra with diagonalizable Nakayama automorphism $\mathfrak{N}$. Let Λ be the set of eigenvalues of $\mathfrak{N}$. For $\mu \in \Lambda$, denote by A_μ the corresponding eigenspace.*

- (i) *For $\mu \in \Lambda$, we have $\mu^{-1} \in \Lambda$.*
- (ii) *The isomorphism of bimodules $D(A) \simeq A_{\mathfrak{N}}$ induces an isomorphism of vector spaces $D(A_\mu) \simeq A_{\mu^{-1}}$, for any $\mu \in \Lambda$.*

Proof. (i). Since for $0 \neq x \in A_\mu$, $\langle x, -\rangle \in D(A)$ is not the zero linear transformation and $A = \oplus_{\nu \in \Lambda} A_\nu$, there exist $\nu \in \Lambda$ and $y \in A_\nu$ such that $\langle x, y\rangle \neq 0$. Now

$$\langle x, y\rangle = \langle y, \mathfrak{N}(x)\rangle = \mu\langle y, x\rangle = \mu\langle x, \mathfrak{N}(y)\rangle = \mu\langle x, \nu y\rangle = \mu\nu\langle x, y\rangle.$$

We see that $\mu\nu = 1$ and $\nu = \mu^{-1}$. This proves (i) in case that $\mathfrak{N}$ is diagonalizable.

(ii). In course of the proof of (i), we showed that for $\mu \in \Lambda$ and $0 \neq x \in A_\mu$, $\langle -, x\rangle$ is zero on A_ν for $\nu \neq \mu^{-1}$. This shows that $D(A_\mu) \subseteq A_{\mu^{-1}}$. By exchanging the role of μ and μ^{-1}, we get that the isomorphism $D(A) \simeq A_{\mathfrak{N}}$ induces an isomorphism $D(A_\mu) \simeq A_{\mu^{-1}}$. □

Remark 3.6. In the spirit of Lemma 3.5, one intends to think that Λ is a group. However, this is not true. A counterexample is given by the algebra

$$A(\mu) = k\langle X, Y\rangle/(X^2, Y^2, XY - \mu YX)$$

with $\mu \in k - \{0\}$. A direct computation shows that $\Lambda = \{1, \mu, \mu^{-1}\}$ which is not a group unless $\mu = 1$, or μ is a square or cubic root of 1, i.e. $(\mu + 1)(\mu^3 - 1) = 0$.

3.3. BV-structure for Frobenius algebras

Let A be a Frobenius algebra with Nakayama automorphism $\mathfrak{N}$. Let Λ be the set of eigenvalues of the automorphism $\mathfrak{N}$ and suppose that $\Lambda \subset k$. In Section 2 we obtained a Tamarkin–Tsygan calculus

$$(H_1^*(A, A), \cup_1, [\ ,\ ]_1, H_*^1(A, A_{\mathfrak{N}}), \cap_1, B_{\mathfrak{N}})$$

associated to the eigenvalue 1 of $\mathfrak{N}$. We have constructed in Section 3.1 the algebra with duality

$$(\mathcal{H}^*, \cup, \mathcal{H}_{-*}, c, \partial).$$

These two structures give an algebra with duality and a Tamarkin–Tsygan calculus satisfying the Ginzburg relation from Lemma 1.7.

Let $\mathcal{H}_1^* := H_1^*(A, A)$ and $\mathcal{H}_{-*}^1 := D(H_*^1(A, A_{\mathfrak{N}}))$. The transpose of the cap product

$$\cap_1 : H_p^1(A, A_{\mathfrak{N}}) \otimes H_1^q(A, A) \to H_{p-q}^1(A, A_{\mathfrak{N}})$$

yields a cap-product, still denoted by $\cap_1$,

$$\cap_1 : \mathcal{H}_{-p}^1 \otimes \mathcal{H}_1^q \to \mathcal{H}_{-(p+q)}^1.$$

We have $c = \langle -, 1_A\rangle \in \mathcal{H}_{-0}^1$ and the restriction $c \cap_1 - : \mathcal{H}_1^p \to \mathcal{H}_{-p}^1$ of $c \cap -$ to $\mathcal{H}_1^p$ is the isomorphism $D(H_p(A, A_{\mathfrak{N}})) \simeq H_1^p(A, A)$. This shows $(\mathcal{H}_1^*,\ \cup_1,\ \mathcal{H}_1^*,\ c,\ \partial_1)$ is an algebra with duality. The transpose of Connes' operator $B_{\mathfrak{N}} : H_p^1(A, A_{\mathfrak{N}}) \to H_{p+1}^1(A, A_{\mathfrak{N}})$ induces a map $B_1 : \mathcal{H}_{-(*+1)}^1 \to \mathcal{H}_{-*}^1$.

Theorem 3.7. *Let A be a Frobenius algebra with Nakayama automorphism $\mathfrak{N}$. Let Λ be the set of eigenvalues of the automorphism $\mathfrak{N}$. Suppose that $\Lambda \subset k$. Let $H_1^*(A, A)$ be the*

Hochschild cohomology space associated to the eigenvalue 1 *of the Nakayama automorphism* $\mathfrak{N}$. *Then the Gerstenhaber algebra* $H_1^*(A,A)$ *is a* BV*-algebra.*

Proof. This is because the algebra with duality $(\mathcal{H}_1^*, \cup_1, \mathcal{H}_*^1, c, \partial_1)$ and the Tamarkin–Tsygan calculus $(\mathcal{H}_1^*, \cup_1, [\ ,\ ]_1, \mathcal{H}_*^1, B_1)$ satisfy the hypotheses of Lemma 1.7. □

Corollary 3.8. *Let* A *be a Frobenius algebra with Nakayama automorphism* $\mathfrak{N}$. *If* $\mathfrak{N}$ *is diagonalizable then the Hochschild cohomology* $HH^*(A)$ *is a* BV *algebra.*

Proof. If $\mathfrak{N}$ is diagonalizable we have seen in Proposition 2.5 that $H_1^*(A,A) = HH^*(A)$. □

4. Proof of the main result

Let us recall the statement of our main result of this paper.

Theorem 4.1. *Let* A *be a Frobenius algebra with semisimple Nakayama automorphism. Then the Hochschild cohomology ring* $HH^*(A)$ *of* A *is a Batalin–Vilkovisky algebra.*

The proof of this theorem occupies the rest of this section.

If the Nakayama automorphism is diagonalizable, this is the statement of Corollary 3.8.

Now suppose that the Nakayama automorphism of a Frobenius algebra is semisimple, that is, it is diagonalizable over the algebraic closure $\overline{k}$ of k.

Let $C = A \otimes \overline{k}$. As is readily verified, C is still a Frobenius algebra with respect to the induced bilinear form

$$\langle a \otimes \mu, b \otimes \nu \rangle = \mu\nu\langle a, b\rangle, \quad a, b \in A, \mu, \nu \in \overline{k}.$$

Therefore, the Nakayama automorphism of C is $\mathfrak{N}_C = \mathfrak{N} \otimes \mathrm{id}_{\overline{k}}$. We shall write $D_{\overline{k}} = \mathrm{Hom}_{\overline{k}}(-, \overline{k})$.

Notice that

$$D_{\overline{k}}(C) = \mathrm{Hom}_{\overline{k}}(A \otimes_k \overline{k}, \overline{k}) \simeq \mathrm{Hom}_k(A, \overline{k}) \simeq \mathrm{Hom}_k(A, k) \otimes \overline{k} = D(A) \otimes \overline{k},$$

where the inverse of the isomorphism $\mathrm{Hom}_k(A, \overline{k}) \simeq \mathrm{Hom}_k(A, k) \otimes \overline{k}$ is given by $f \otimes \mu \mapsto (x \mapsto f(x) \otimes \mu)$ for $f \in \mathrm{Hom}_k(A, k)$ and $\mu \in \overline{k}$. We also have an isomorphism of bimodules $C_{\mathfrak{N}_C} \simeq A_{\mathfrak{N}} \otimes \overline{k}$. For the Frobenius $\overline{k}$-algebra C, the isomorphism of bimodules $D(C) \simeq C_{\mathfrak{N}_C}$ fits into a commutative diagram

$$\begin{array}{ccc} D_{\overline{k}}(C) & \xrightarrow[\simeq]{} & C_{\mathfrak{N}_C} \\ \simeq \big\downarrow & & \big\downarrow \simeq \\ D(A) \otimes \overline{k} & \xrightarrow[\simeq]{} & A_{\mathfrak{N}} \otimes \overline{k} \end{array}$$

where the vertical isomorphisms are explicitly given above.

The diagonalizable case of Theorem 0.1 applies to C and therefore $HH^*_{\overline{k}}(C)$ is a BV algebra, where $HH^*_{\overline{k}}(C)$ is the Hochschild cohomology of C considered as a $\overline{k}$-algebra. Denote by Δ_C the BV-operator over $HH^*_{\overline{k}}(C)$.

Let us explain the idea of the proof. It is true that $HH^*_{\overline{k}}(C) \simeq HH^*(A) \otimes \overline{k}$ as Gerstenhaber algebras; see Proposition 4.2 below. In order to show that $HH^*(A)$ is a BV algebra, we shall prove that the Δ_C-operator sends $HH^*(A) \otimes 1 = HH^*(A)$ into itself (see Lemma 4.3 below), then we have $\Delta_C = \Delta_A \otimes \overline{k}$, where Δ_A denotes the restriction of Δ_C to $HH^*(A)$.

Proposition 4.2. *Let A be an algebra defined over a field k. Denote $C = A \otimes \overline{k}$. Then there is an isomorphism of Gerstenhaber algebras*

$$HH^*_{\overline{k}}(C) \simeq HH^*(A) \otimes \overline{k},$$

where $HH^*_{\overline{k}}(C)$ is the Hochschild cohomology of C considered as a $\overline{k}$-algebra and the Gerstenhaber algebra structure on $HH^*(A) \otimes \overline{k}$ is defined in Remark 1.2.

Proof. In fact for each $p \geq 0$,

$$\begin{aligned} C^p(C,C) = \mathrm{Hom}_{\overline{k}}((C/\overline{k} \cdot 1)^{\otimes_{\overline{k}} p}, C) &\simeq \mathrm{Hom}_{\overline{k}}((A/k \cdot 1)^{\otimes p} \otimes \overline{k}, C) \\ &\simeq \mathrm{Hom}_k((A/k \cdot 1)^{\otimes p}, C) \simeq C^p(A,A) \otimes \overline{k}. \end{aligned}$$

One see easily that this is an isomorphism of complexes. This induces an isomorphism of graded vector spaces $HH^*_{\overline{k}}(C) \simeq HH^*(A) \otimes \overline{k}$.

Moreover, a careful examination on the definition of cup product and Lie bracket shows that this is also an isomorphism of Gerstenhaber algebras. In fact, this can be seen via the isomorphism of the cochain complexes above. We show that the isomorphism preserves cup products. The Lie product is dealt with analogously. For $f \in C^p(A,A)$, $g \in C^q(A,A)$ and $\mu, \nu \in \overline{k}$, $a_i \in A, \gamma_j \in \overline{k}$, $a_{i,j} := a_i \otimes \cdots \otimes a_j$ via the isomorphism, we obtain

$$\begin{aligned} &((f \otimes \mu) \cup (g \otimes \nu))((a_1 \otimes \gamma_1) \otimes \cdots \otimes (a_{p+q} \otimes \gamma_{p+q})) \\ &= (-1)^{pq}(f \otimes \mu)((a_1 \otimes \gamma_1) \otimes \cdots \otimes (a_p \otimes \gamma_p))(g \otimes \nu) \\ &\quad ((a_{p+1} \otimes \gamma_{p+1}) \otimes \cdots \otimes (a_{p+q} \otimes \gamma_{p+q})) \\ &= (-1)^{pq}(f(a_{1,p}) \otimes \mu\gamma_1 \cdots \gamma_p)(g(a_{p+1,p+q}) \otimes \nu\gamma_{p+1} \cdots \gamma_{p+q})) \\ &= (-1)^{pq} f(a_{1,p})g(a_{p+1,p+q}) \otimes \mu\nu\gamma_1 \cdots \gamma_{p+q} \\ &= (f \cup g)(a_{1,p+q}) \otimes \mu\nu\gamma_1 \cdots \gamma_{p+q}. \qquad \square \end{aligned}$$

The proof of the main result then deduces from the following result.

Lemma 4.3.

(i) *There is an isomorphism of complexes* $C_*(C, C_{\mathfrak{N}_C}) \simeq C_*(A, A_{\mathfrak{N}}) \otimes \overline{k}$.
(ii) *There is an isomorphism of complexes* $D_{\overline{k}}(C_*(C, C_{\mathfrak{N}_C})) \simeq D(C_*(A, A_{\mathfrak{N}})) \otimes \overline{k}$.
(iii) *There is a commutative diagram of isomorphisms of complexes*

$$\begin{array}{ccc} D_{\overline{k}}(C_*(C, C_{\mathfrak{N}_C})) & \xrightarrow{\simeq} & D(C_*(A, A_{\mathfrak{N}})) \otimes \overline{k} \\ \simeq \downarrow & & \downarrow \simeq \\ C^p(C, C) & \xrightarrow{\simeq} & C^p(A, A) \otimes \overline{k}, \end{array}$$

where the upper horizontal isomorphism is introduced in (ii), and where the lower horizontal morphism arises from the proof of Proposition 4.2, and the vertical isomorphisms are (induced by) duality isomorphisms. This means that the duality is compatible with extensions of scalars.
(iv) *For each* $p \geq 0$*, there is a commutative diagram involving Connes operators over* C *and* A

$$\begin{array}{ccc} C_p(C, C_{\mathfrak{N}_C})) & \xrightarrow{\mathrm{B}_p} & C_{p+1}(C, C_{\mathfrak{N}_C})) \\ \simeq \downarrow & & \downarrow \simeq \\ C_p(A, A_{\mathfrak{N}}) \otimes \overline{k} & \xrightarrow{\mathrm{B}_p \otimes id_{\overline{k}}} & C_{p+1}(A, A_{\mathfrak{N}}) \otimes \overline{k}, \end{array}$$

where the vertical isomorphisms are introduced in (ii).

Proof. (i). For each $p \geq 0$,

$$C_p(C, C_{\mathfrak{N}_C}) = C_{\mathfrak{N}_C} \otimes_{\overline{k}} (C/\overline{k} \cdot 1)^{\otimes_{\overline{k}} p} \simeq (A_{\mathfrak{N}} \otimes (A/k \cdot 1)^{\otimes p}) \otimes \overline{k} = C_p(A, A_{\mathfrak{N}}) \otimes \overline{k}.$$

One then verifies that these isomorphisms commute with the differential.

(ii). By (i) for each $p \geq 0$,

$$D_{\overline{k}}(C_p(C, C_{\mathfrak{N}_C})) = \mathrm{Hom}_{\overline{k}}(C_p(A, A_{\mathfrak{N}}) \otimes \overline{k}, \overline{k}) \simeq \mathrm{Hom}_k(C_p(A, A_{\mathfrak{N}}), \overline{k}) \simeq DC_p(A, A_{\mathfrak{N}}) \otimes \overline{k},$$

where $D_{\overline{k}}$ denotes the $\overline{k}$-dual $\mathrm{Hom}_{\overline{k}}(-, \overline{k})$.

(iii)–(iv). The proofs can be done by chasing the diagrams. Let us prove (iv) and the proof of (iii) is left to the reader. Let $a_0 \otimes a_{1,p} \in C_p(A, A_{\mathfrak{N}})$ and $\mu \in \overline{k}$. The element $(a_0 \otimes a_{1,p}) \otimes \mu \in C_p(A, A_{\mathfrak{N}}) \otimes \overline{k}$ is sent by the inverse of the left vertical map to $(a_0 \otimes \mu) \otimes_{\overline{k}} (a_1 \otimes 1) \otimes_{\overline{k}} \cdots \otimes_{\overline{k}} (a_p \otimes 1) \in C_p(C, C_{\mathfrak{N}_C})$. Now

$$
\begin{aligned}
&B_p((a_0 \otimes \mu) \otimes_{\overline{k}} (a_1 \otimes 1) \otimes_{\overline{k}} \cdots \otimes_{\overline{k}} (a_p \otimes 1)) \\
&\quad = (1 \otimes 1) \otimes_{\overline{k}} (a_0 \otimes \mu) \otimes_{\overline{k}} (a_1 \otimes 1) \otimes_{\overline{k}} \cdots \otimes_{\overline{k}} (a_p \otimes 1) \\
&\qquad + \sum_{i=1}^{p} (-1)^{ip} (1 \otimes 1) \otimes_{\overline{k}} (a_i \otimes 1) \otimes_{\overline{k}} \cdots \otimes_{\overline{k}} (a_p \otimes 1) \otimes_{\overline{k}} (a_0 \otimes \mu) \otimes_{\overline{k}} (\mathfrak{N}(a_1) \otimes 1) \\
&\qquad \otimes_{\overline{k}} \cdots \otimes_{\overline{k}} (\mathfrak{N}(a_p) \otimes 1) \\
&\quad = \sum_{i=0}^{p} (-1)^{ip} (1 \otimes \mu) \otimes_{\overline{k}} (a_i \otimes 1) \otimes_{\overline{k}} \cdots \otimes_{\overline{k}} (a_p \otimes 1) \otimes (a_0 \otimes 1) \otimes_{\overline{k}} (\mathfrak{N}(a_1) \otimes 1) \otimes_{\overline{k}} \cdots \\
&\qquad \otimes_{\overline{k}} (\mathfrak{N}(a_{i-1}) \otimes 1),
\end{aligned}
$$

which is obviously the image of

$$
(B_p \otimes \mathrm{id}_{\overline{k}})(a_0 \otimes a_{1,p}) = \sum_{i=0}^{p} (-1)^{ip} (1 \otimes a_{i,p} \otimes a_0 \otimes \mathfrak{N}(a_1) \otimes \cdots \otimes \mathfrak{N}(a_{i-1})) \otimes \mu
$$

under the inverse of the right vertical map. □

Now the theorem follows from the diagrams in (iii)–(iv) of the above lemma, since the Δ-operator and Connes operator B are dual to each other.

5. Examples

5.1. *Frobenius algebras given by quiver and relation*

As a first series of examples we shall consider Frobenius algebras given by quiver and relations.

5.1.1. *A criterion of when Frobenius algebras given by quiver with relations have semisimple Nakayama automorphisms*

Let $A = kQ/I$ be a finite dimensional algebra given by quiver with relations. As is well known, we can choose a basis $\mathcal{B}$ of A consisting of paths which also contains a basis for the socle of each indecomposable projective A-module. Suppose now that A is a Frobenius algebra. Then by [22, Proposition 2.8], there is a natural choice of the defining bilinear form $\langle a, b\rangle = tr(ab)$ for $a, b \in A$ induced by the trace map

$$
tr : A \to k, \quad p \in \mathcal{B} \mapsto \begin{cases} 1 \text{ if } p \in Soc(A) \cap \mathcal{B} \\ 0 \text{ otherwise.} \end{cases}
$$

Assume that the basis $\mathcal{B}$ satisfies two further conditions:

(1) for arbitrary two paths $p, q \in \mathcal{B}$, there exist another path $r \in \mathcal{B}$ and a constant $\lambda \in k$ such that $p \cdot q = \lambda r \in A$,

(2) for each path $p \in \mathcal{B}$, there exists a unique element $p^* \in \mathcal{B}$ such that $0 \neq p \cdot p^* \in Soc(A)$.

We can prove the following rather useful result, as we will see in the next subsection.

Criterion 5.1. *Within the above setup, suppose that k is a field of characteristic 0 or of characteristic p with p strictly bigger than the number of arrows of Q. Then the two conditions (1) and (2) imply that the Nakayama automorphism of A is semisimple and the Hochschild cohomology of A is a BV algebra.*

Proof. For $p \in \mathcal{B}$, by (2), let p^* be the unique path in $\mathcal{B}$ such that $p \cdot p^* = \lambda(p) r \in Soc(A)$ with $\lambda(p) \in k \setminus \{0\}$ and $r \in Soc(A) \cap \mathcal{B}$. Then for $p, q \in \mathcal{B}$ we get,

$$\langle p, q\rangle = \begin{cases} \lambda(p) & \text{if } q = p^* \\ 0 & \text{otherwise.} \end{cases}$$

Since $\langle p, q\rangle = \langle q, \mathfrak{N}(p)\rangle$, the Nakayama automorphism sends p to $\frac{\lambda(p)}{\lambda(p^*)} p^{**}$. Since $\mathcal{B}$ is finite, the Nakayama automorphism $\mathfrak{N}$, restricted to $\mathcal{B}$, is a permutation of $\mathcal{B}$, modulo scalars.

We will show that the Nakayama automorphism $\mathfrak{N}$ is diagonalizable if k is an algebraically closed field of characteristic 0 or algebraically closed of characteristic p where $p > \dim(rad(A)/rad^2(A)) =: d$. Recall that the arrows Q_1 of the quiver of A form a k-basis of $rad(A)/rad^2(A)$. Since $\mathfrak{N}$ is an algebra automorphism, and since A satisfies the conditions (1) and (2), for each $p \in Q_1$ we get $p^{**} \in Q_1$. We will show that the action of $\mathfrak{N}$ on the k-vector space $M = rad(A)/rad^2(A)$ generated by Q_1 is diagonalizable. Let G be the infinite cyclic group, generated by c. Then kG acts on $rad(A)/rad^2(A)$ when we define the action of c on M by $\mathfrak{N}$.

Let $\alpha \in Q_1$. Then there is a $t_\alpha \in \mathbb{N}\setminus\{0\}$ such that $c^{t_\alpha} \cdot \alpha = u_\alpha \cdot \alpha$ for some $u_\alpha \in k\setminus\{0\}$. Choose t_α minimal possible. Let $x_i := c^i \cdot \alpha$ for $i \in \{0, 1, \dots, t_\alpha - 1\}$. The k-vector space T_α generated by $x_0, \cdots, x_{t_\alpha-1}$ is then a kG-module and c acts by the matrix C_α, say. Using the basis $\{x_0, \cdots, x_{t_\alpha-1}\}$ of T_α it is easily seen that the characteristic polynomial of C_α is $X^{t_\alpha} \pm u$ and this polynomial has only simple roots in k since the characteristic of k is either 0 or bigger than d and $d \geq t_\alpha$. Now $M = \bigoplus_{\text{some } \alpha} T_\alpha$. Let Q_1' be the basis of M for which the action of $\mathfrak{N}$ is given by a diagonal matrix. This shows that $\mathfrak{N}$ acts diagonally on all paths formed by the elements in Q_1'. We may suppose that A is indecomposable as algebra (i.e. Q is connected) since the Nakayama automorphism acts on each indecomposable factor. Let Q_0 be the set of vertices in the quiver. If A is indecomposable, then $|Q_1| \geq |Q_0| - 1$ and equality holds if and only if Q is a tree. The quiver of a selfinjective algebra is not a tree, and hence $|Q_1| \geq |Q_0|$. Since $\mathfrak{N}$ permutes Q_0, the action of $\mathfrak{N}$ on kQ_0 is diagonalizable, using that the characteristic of the field is 0 or bigger than $|Q_1|$. A basis of A is given by Q_0 and paths of elements of Q_1. Let Q_0' be a basis of kQ_0 and let Q_1' be a basis of M with diagonal action of $\mathfrak{N}$. Then $\mathfrak{N}$ acts diagonally on paths produced by elements of Q'_1 and the set of paths of elements of Q'_1 forms a generating set of $rad(A)$. Eliminating superfluous elements we produce this way a basis $\mathcal{B}_r$ of $rad(A)$ on which $\mathfrak{N}$ acts diagonally. Hence $\mathcal{B}_r \cup Q'_0$ is a basis of A on which $\mathfrak{N}$ acts diagonally. By our main result Theorem 0.1 the Hochschild cohomology of A is a BV algebra. □

These seemingly rather strong conditions (1) and (2) are in fact satisfied by many interesting classes of algebras, as we will see in Section 5.1.2.

5.1.2. Tame Frobenius algebras

In this subsection k denotes an algebraically closed field. We shall apply Criterion 5.1 to deal with tame Frobenius algebras.

Lemma 5.2. *Each self-injective algebra of finite representation type is Morita equivalent to an algebra kQ/I given by a quiver Q modulo admissible relations I verifying the conditions (1) and (2).*

Proof. Each representation-finite algebra has a multiplicative basis (cf. [3]), thus the first condition holds. For the second condition, suppose that for a path $p \in \mathcal{B}$, there exist two paths $q_1, q_2 \in \mathcal{B}$ such that $0 \neq pq_1 = \lambda pq_2 \in Soc(A)$ with $\lambda \in k$. We can assume that p has positive length, otherwise q_1 and q_2 would not be linearly independent in A, using that the socle of each indecomposable projective module is one-dimensional. Now q_1 and q_2 are parallel paths, by reducing suitably their lengths and enlarging p if necessary, one can assume that they have no common arrows. However, this shows that A is of infinite representation type, as there are infinitely many string modules of the form $M((q_1 q_2^{-1})^n)$, $n \in \mathbb{N}$, which is a contradiction.

One can also prove this result using a case-by-case analysis based on the list given in terms of quiver with relations in [2]. □

However, the Nakayama automorphism of a self-injective algebra of finite representation type is not necessarily semisimple.

Example 5.3. Let k be a field of characteristic two. Consider the algebra defined by the quiver with relations

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \qquad\qquad \alpha\beta\alpha\beta = 0 = \beta\alpha\beta\alpha$$

Thus A is a self-injective Nakayama algebra. Then the indecomposable projective A-modules are uniserial and have the following form

$$\begin{array}{c} 1 \\ \downarrow{\scriptstyle \alpha} \\ 2 \\ \downarrow{\scriptstyle \beta} \\ 1 \\ \downarrow{\scriptstyle \alpha} \\ 2 \end{array} \qquad \begin{array}{c} 2 \\ \downarrow{\scriptstyle \beta} \\ 1 \\ \downarrow{\scriptstyle \alpha} \\ 2 \\ \downarrow{\scriptstyle \beta} \\ 1 \end{array}$$

Under the basis $\{e_1, e_2, \alpha, \beta, \alpha\beta, \beta\alpha\}$, the matrix of the Nakayama automorphism is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, the Nakayama automorphism of A is not semisimple. It would be interesting to see whether $HH^*(A)$ is a BV algebra.

For each prime number p, one can construct such a selfinjective Nakayama algebra over a field of characteristic p.

Another class of algebras is the so-called self-injective special biserial algebras. A pair (Q, I) of a quiver Q and admissible relations I is called special biserial, if the following conditions hold:

(a) Each vertex has at most two leaving arrows and at most two entering arrows.
(b) Given an arbitrary arrow α, there exists at most one arrow β such that $t(\alpha) = s(\beta)$ and $\alpha\beta \notin I$ and at most one arrow γ such that $t(\gamma) = s(\alpha)$ and $\gamma\alpha \notin I$.

An algebra is called a special biserial algebra if it is Morita equivalent to kQ/I for a special biserial pair (Q, I).

Lemma 5.4. *For a special biserial pair (Q, I) the algebra kQ/I satisfies the two conditions (1) and (2).*

Proof. It is not difficult to see, and actually well-known (cf. e.g. [9]), that an indecomposable projective module over a self-injective special biserial algebra is either a uniserial module or a module for which the quotient of the radical by its socle is the direct sum of two uniserial modules. The first case is induced by a monomial relation and the second by a commutation relation. For the choice of the basis $\mathcal{B}$, one simply takes representatives of elements in kQ/I given by paths except that for each indecomposable projective non-uniserial module, where we choose one of the two paths from its top to its socle. Now the two conditions hold trivially. $\square$

Now let us look at weakly symmetric algebras of domestic representation type. R. Bocian, T. Holm and A. Skowroński [6,7,21] classified all weakly symmetric algebras of domestic type over k up to derived equivalence and the last two authors of the present paper gave a classification up to stable equivalences [36]. In Bocian–Holm–Skowroński classification, a domestic weakly symmetric standard algebra with singular Cartan matrix is derived equivalent to the trivial extension $T(C)$ of a canonical algebra C of Euclidean type and is thus symmetric; see [6, Theorem 1]. By [6, Theorem 2] a domestic weakly symmetric standard algebra with nonsingular Cartan matrix is derived equivalent to some algebras explicitly given in terms of quiver with relations, denoted by $A(\lambda)$, $A(p,q)$, $\Lambda(n)$ and $\Gamma(n)$. Note that these algebras are symmetric except $A(\lambda)$ with $\lambda \in k \setminus \{0,1\}$. However,

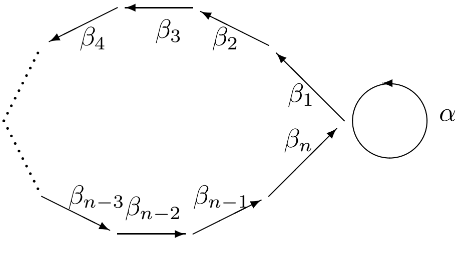
$$A(\lambda) = k\langle X, Y\rangle/(X^2, Y^2, XY - \lambda YX)$$

for $\lambda \notin \{0,1\}$ has a semisimple Nakayama automorphism, given by a diagonal matrix with coefficients $(1, \lambda^{-1}, \lambda, 1)$ with respect to the basis $\{1, X, Y, XY\}$ as is easily verified. One may use the result of the next subsection, as $A(\lambda)$ is a quantum complete intersection.

By [7, Theorem 1] any nonstandard self-injective algebra of domestic type is derived equivalent (and also stably equivalent) to an algebra $\Omega(n)$ with $n \geq 1$. Let us recall the quiver with relations of $\Omega(n)$.

$\Omega(n)$

$n \geq 1$



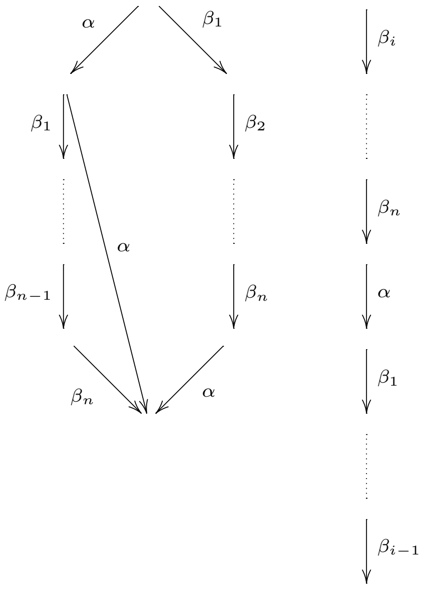
$$\alpha^2 = \alpha\beta_1\beta_2\cdots\beta_n = -\beta_1\beta_2\cdots\beta_n\alpha,$$

$$\beta_n\beta_1 = 0,\ \beta_j\beta_{j+1}\cdots\beta_n\beta_1\cdots\beta_n\alpha\beta_1\cdots\beta_{j-1}\beta_j = 0,\ 2 \leq j \leq n$$

Notice that we cannot use the Criterion 5.1 for the algebra $\Omega(n)$. However, we can still prove the semisimplicity of its Nakayama automorphism.

Lemma 5.5. *The Nakayama automorphism of* $\Omega(n)$ *is diagonalizable.*

Proof. The indecomposable projective modules of $\Omega(n)$ are of the following shape:



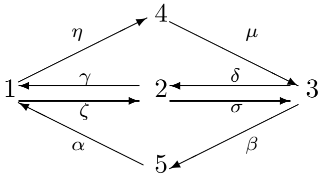
with $2 \le i \le n$. This algebra does not satisfy the two conditions of Criterion 5.1, but we can compute explicitly its Nakayama automorphism. For $\mathcal{B}$, one can take the obvious basis containing $\alpha, \beta_1 \cdots \beta_n, \alpha\beta_1 \cdots \beta_n = \alpha^2 = -\beta_1 \cdots \beta_n \alpha$ etc. However, the dual basis $\mathcal{B}^*$ does not consist of paths. In fact, one obtains $\alpha^* = \beta_1 \cdots \beta_n$ and $(\beta_1 \cdots \beta_n)^* = -\alpha + \beta_1 \cdots \beta_n$ etc. From this, the Nakayama automorphism is given by $\mathfrak{N}(\alpha) = -\alpha + 2\beta_1 \cdots \beta_n$ and for any other path $p \in \mathcal{B}$, we get $\mathfrak{N}(p) = p$. Hence, in characteristic two, the Nakayama automorphism is the identity map (in fact $\Omega(n)$ is symmetric), and in odd characteristic it is diagonalizable. Therefore, the Nakayama automorphism of $\Omega(n)$ is diagonalizable. □

We have shown that each weakly symmetric algebra of domestic type is derived equivalent to a weakly symmetric algebra of domestic type whose Nakayama automorphism is semisimple.

Now we consider self-injective algebras of polynomial growth which are not of domestic type. The derived equivalence classification of the standard (resp. non-standard) non-domestic weakly symmetric (resp. self-injective) algebras of polynomial growth over k is achieved in [4, page 653, Theorem] (resp. [5, Theorem 3.1]).

By [4, page 653, Theorem], an indecomposable standard non-domestic weakly symmetric algebra of polynomial growth is always derived equivalent to a symmetric algebra except that it may be derived equivalent to Λ'_9 in characteristic not two. For the quiver with relations of the algebra Λ'_9, we refer to [4]. From this description, we know that Λ'_9 is the preprojective algebra of type D_4 and that its Nakayama automorphism is diagonalizable (and is of order two) by [10, Section 5.2.1]; see also Example 5.11.

By [5, Theorem 3.1], an indecomposable non-standard non-domestic self-injective algebra of polynomial growth is always derived equivalent to a symmetric algebra except the possibility of Λ_{10} in characteristic two. Let us recall its quiver with relation $\Lambda_{10} = kQ/I$:

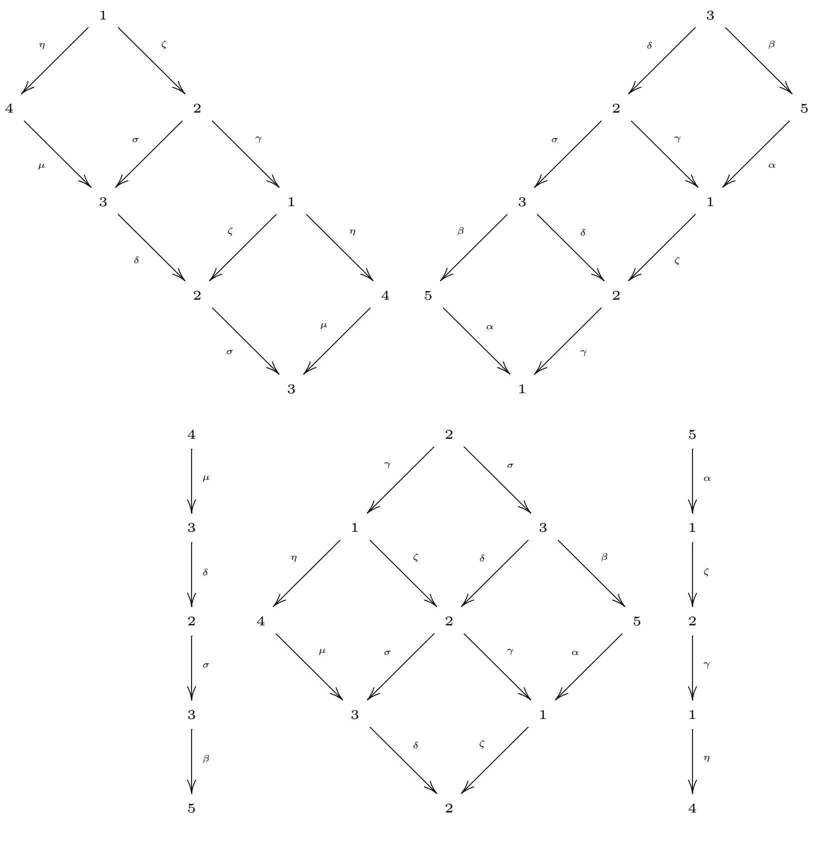


and

$$\Lambda_{10} = KQ/(\beta\alpha - \delta\gamma, \zeta\sigma - \eta\mu, \alpha\eta, \mu\beta, \sigma\delta - \gamma\zeta, \delta\sigma\delta\sigma).$$

Lemma 5.6. *The Nakayama automorphism of Λ_{10} in characteristic two is not semisimple.*

Proof. The indecomposable projective modules of Λ_{10} are of the following shape:



Notice that in the above diagrams, each square is commutative. From this, we observe that $\mathfrak{N}$ is of order 2. However, one sees that the Nakayama automorphism permutes the vertices 1 and 3, hence its matrix under a suitable basis has a block $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and this matrix is not diagonalizable in characteristic two. □

Since derived equivalent algebras have isomorphic Hochschild cohomology rings [30, 23], we have proved in this subsection the following:

Proposition 5.7. *Let A be an algebra falling into one of the following classes of algebras*

- *representation-finite self-injective algebras in characteristic zero;*
- *self-injective special biserial algebras in characteristic zero;*
- *standard weakly symmetric algebras of domestic type which are not representation-finite;*
- *nonstandard self-injective algebras of domestic type which are not representation-finite;*
- *standard non-domestic weakly symmetric algebras of polynomial growth;*
- *nonstandard non-domestic self-injective algebras of polynomial growth over fields of characteristic different from 2, or over fields of characteristic 2 as long as they are not derived equivalent to Λ_{10}.*

Then A is derived equivalent to a Frobenius algebra whose Nakayama automorphism is semisimple. Therefore, the Hochschild cohomology ring of A is a BV algebra.

We actually proved a slightly more precise statement concerning the characteristic of k.

We do not know whether BV structures exist or not on the Hochschild cohomology ring of Example 5.3 or that of Λ_{10} in characteristic 2.

5.2. Quantum complete intersections

In [19], D. Happel asked whether an algebra has finite global dimension whenever its Hochschild cohomology is finite dimensional. Although Happel's conjecture was verified for many classes of algebras, it is wrong in general. A counter-example was exhibited in [8]. This example is in fact our algebra $A(\mu)$ from Remark 3.6.

This example has been generalized the so-called *quantum complete intersections*, which are extensively studied by P.A. Bergh, K. Erdmann, S. Oppermann etc. Let $N \geq 2$ and $\mathbf{a} = (a_1, \cdots, a_N)$ with $a_j \geq 1$. Let $\mathbf{q} = (q_{ij}, 1 \leq i, j \leq N)$ be a family of nonzero constants in k such that $q_{ii} = 1$ and $q_{ij}q_{ji} = 1$. Now define

$$A(\mathbf{q}, \mathbf{a}) = \frac{k\langle X_1, \cdots, X_N\rangle}{(X_i^{a_i+1}, X_iX_j - q_{ij}X_jX_i, 1 \leq i, j \leq N)}.$$

Obviously this algebra is a local weakly symmetric algebra, and is thus a Frobenius algebra. A direct computation shows that for each $1 \le i \le N$, $\mathfrak{N}(X_i) = \big(\prod_{j=1}^{N} q_{ij}^{a_j}\big) X_i$ and so it is diagonalizable.

Corollary 5.8. *The Hochschild cohomology ring of a quantum complete intersection* $A(\mathbf{q}, \mathbf{a})$ *is a BV algebra.*

5.3. Finite dimensional Hopf algebras

Let k be an algebraically closed field of characteristic zero. Let H be a finite dimensional Hopf algebra over k. By [26] we get that H is Frobenius. Indeed, given a right integral $\varphi \in H^*$, a Frobenius bilinear form is given by $\langle a, b\rangle = \varphi(ab)$. Since the antipode S of H has finite order by [29], its Nakayama automorphism also has finite order.

Corollary 5.9. *The Hochschild cohomology ring of a finite dimensional Hopf algebra defined over an algebraically closed field of characteristic zero is a Batalin–Vilkovisky algebra.*

It would be an interesting question to know when the usual cohomology groups of H is a BV subalgebra of $HH^*(H)$; a sufficient condition was provided by L. Menichi in [28, Theorem 50].

5.4. Other examples

There are many other examples of Frobenius algebras related to Calabi–Yau algebras and Artin–Schelter regular algebras.

Example 5.10. In the classical paper [1], M. Artin and W.F. Schelter classified three dimensional Artin–Schelter regular algebras. These algebras are twisted Calabi–Yau algebras, which implies that there is an algebra automorphism σ of A such that $HH^{d-*}(A) \simeq H_*(A, A_\sigma)$. In the classification, they use a generic condition which implies the semisimplicity of the algebra automorphisms σ of these algebras. When these algebras are Koszul, their Koszul duals are Frobenius by [27, Corollary D] and the Nakayama automorphism of $A^!$ and the algebra automorphism σ of A are related by [33, Theorem 9.2]. Therefore, whenever σ is semisimple, the Koszul duals are Frobenius algebras with semisimple Nakayama automorphisms. The Hochschild cohomology ring of the Koszul dual of a three dimensional Artin–Schelter regular algebra is BV algebra. We do not know the explicit BV structure over the Hochschild cohomology rings of these algebras.

Example 5.11. The preprojective algebras of Dynkin quivers ADE are Frobenius algebras whose Nakayama automorphism has finite order; for details see [10]. Except the cases that char $k = 2$, and the type D_n, n odd or E_6, the Nakayama automorphism is diagonalizable. Therefore, except these cases their Hochschild cohomology rings are BV algebras. This is a well known fact (at least over a field of characteristic zero) and our main result gives a structural explanation of the existence of BV structure. This BV structure (over a field of characteristic zero) has been computed by C.-H. Eu in [11].

Example 5.12. Another class of Frobenius algebras, called almost Calabi–Yau algebras, was extensively studied by D.E. Evans and M. Pugh (cf. [13,14]). These algebras are related to $SU(3)$ modular invariants and MacKay correspondence. Their Nakayama automorphisms have also finite order and is thus semisimple over a field of characteristic zero; the authors in fact work over $\mathbb{C}$. Therefore, the Hochschild cohomology ring of an almost Calabi–Yau algebra defined over a field of characteristic zero is a BV algebra. It would be interesting to compute the BV structure over the Hochschild cohomology rings of these algebras.

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