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# On the vertices of indecomposable modules over dihedral 2-groups

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ABSTRACT

Let $k$ be an algebraically closed field of characteristic 2. We calculate the vertices of all indecomposable $kD_8$-modules for the dihedral group $D_8$ of order 8. We also give a conjectural formula of the induced module of a string module from $kT_0$ to $kG$ where $G$ is a dihedral group of order $\geqslant 8$ and where $T_0$ is a dihedral subgroup of index 2 of $G$. Some cases where we verified this formula are given.



## 1. Introduction

Let $k$ be an algebraically closed field and let $G$ be a finite group. A subgroup $D \leqslant G$ is called a vertex of an indecomposable $kG$-module $M$ if $M$ is a direct summand of the induced module of a $kD$-module from $D$ to $G$ and if $D$ is minimal for this property. It can be easily seen that two vertices of $M$ are conjugate in $G$. The knowledge of vertices of modules is a central point in modular representation theory of finite groups. In particular, it is important to understand the category of modules over group algebras. It is usually is a hard problem to determine the vertex of an indecomposable module. Much work has been done on this problem and is mainly centered around the vertex of a simple module (see [12,17] for general statements). Only special situations are known. We mention a few of them. According to a theorem of V.M. Bondarenko and Yu.A. Drozd [4], a block $B$ of a group algebra has finitely many isomorphism classes of indecomposable modules (i.e. $B$ has finite representation type) if and only if it has cyclic defect groups. So blocks with cyclic defect groups are the easiest to study, see for example [5,13,19,20,22]. Vertices of simple modules for the case of blocks with cyclic defect groups were calculated in [19] (and also vertices of all indecomposable modules in [20]). The case of tame representation type is a natural continuation to deal with and by the classification of tame

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blocks those of dihedral defect groups are natural candidates. K. Erdmann dealt with some blocks with dihedral defect groups in [6]. In this note, we consider dihedral 2-groups. Some other group algebras of not necessarily tame representation type are also considered in the literature, see [18,21,27], etc.

Let $k$ be an algebraically closed field of characteristic 2. Given $D_8$ the dihedral group of order 8, using purely linear algebra method, we compute the induced modules of all indecomposable modules from each subgroup to $D_8$ and we thus obtain the vertices of *all* indecomposable modules. Roughly speaking, in the Auslander–Reiten quiver of $kD_8$, for a homogeneous tube, all modules have the same vertex, or the module at the bottom has a smaller vertex and all other modules have the same; for a component of type $\mathbb{Z}A_\infty^\infty$, if different vertices appear, there are two $\tau$-orbits which have the same vertex and all the other modules have a larger vertex where $\tau$ is the Auslander–Reiten translation.

Since the pioneering work of P. Webb [26], the relations between inductions from subgroups and the Auslander–Reiten quiver are extensively studied, see [7,14–16]. The distribution of vertices of modules in the Auslander–Reiten quiver becomes an interesting problem. This problem was solved in case of $p$-groups in [8] and in [24,25] in general. In fact, K. Erdmann considered all components except homogeneous tubes in the Auslander–Reiten quiver. Our results thus verify her result in the special case of the dihedral group of order 8 and furthermore complement it by dealing with homogeneous tubes which cause most of the difficulties of calculations.

For dihedral 2-groups of order $\geqslant 16$, we only obtain partial results, but we propose a conjectural formula for the induced module of a string module from a dihedral subgroup of index 2 to the whole dihedral group. This formula should give the vertices of all string modules. More precisely, let

$$G = D_{2^n} = \langle x, y \mid x^{2^{n-1}} = e = y^2,\ \ yxy = x^{-1} \rangle$$

be the dihedral group of order $2^n$ with $n \geqslant 3$ and let $T_0 = \langle x^2, y \rangle$ be a dihedral group of index 2. Let $M(C)$ be a string module over $kT_0$ (for the definition of a string module, see Section 2). Then we construct a new string $\varphi(C)$ over $kG$ (for details see Section 4) and the following formula should hold

$$\mathrm{Ind}_{T_0}^G M(C) := M(C) \otimes_{kT_0} kG \cong M(\varphi(C)).$$

This paper is organized as follows. In Section 2 we present the classification of indecomposable modules over dihedral 2-groups. Vertices of indecomposable modules over the dihedral group of order eight are calculated in the third section, where the main theorems of this paper: Theorem 3.1 and Theorem 3.2 are proved, but we postpone in the final section the proof of Proposition 3.10 which is rather technical. We give the formula for induced modules of string modules in Section 4 and some special cases of this formula are proved.

**Notations and convention.** $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_1 = \{1, 2, 3, \ldots\}$. We always work with right modules.

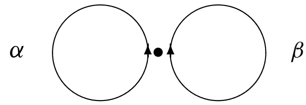
## 2. Classification of indecomposable modules over dihedral 2-groups

Since the pioneering work of P. Gabriel [10], quivers become important in representation theory. A theorem of P. Gabriel says that any finite-dimensional algebra over an algebraically closed field is Morita equivalent to an algebra, its basic algebra, defined by quiver with relations. We will use the presentation by quiver with relations throughout the present paper. For the general theory of quiver with relations, see [3, Chapter 4] or [2, Chapter 3].

Let

$$G = D_{2^n} = \langle x, y \mid x^{2^{n-1}} = e = y^2,\ \ yxy = x^{-1} \rangle$$

be the dihedral group of order $2^n$ with $n \geqslant 2$. The group algebra $kG$ is basic and its quiver with relations can be chosen in the following form [3, Chapter 4, Section 4.11]:



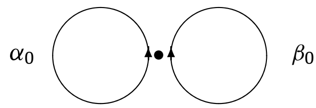

$$\alpha^2 = 0 = \beta^2, \qquad (\alpha\beta)^{2^{n-2}} = (\beta\alpha)^{2^{n-2}}$$

where $\alpha = 1 + y$ and $\beta = 1 + xy$.

For the convenience of later use, we record some subgroups of $G$ and the quivers with relations of the corresponding group algebras. When $n \geqslant 3$, denote

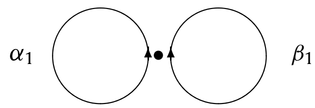
$$H = \langle x \rangle, \qquad T_0 = \langle x^2, y \rangle, \qquad T_1 = \langle x^2, xy \rangle$$

These are all the subgroups of index 2 of $G$. Furthermore, $H \simeq C_{2^{n-1}}$ is the cyclic group of order $2^{n-1}$ and $T_0 \simeq D_{2^{n-1}} \simeq T_1$ are isomorphic to the dihedral group of order $2^{n-1}$. The quivers with relations of $kT_0$, $kT_1$ and $kH$ can be chosen, respectively, as follows:



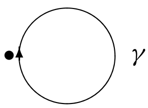

$$\alpha_0^2 = 0 = \beta_0^2, \qquad (\alpha_0\beta_0)^{2^{n-3}} = (\beta_0\alpha_0)^{2^{n-3}}$$

where $\alpha_0 = 1 + y$ and $\beta_0 = 1 + x^2 y$,




$$\alpha_1^2 = 0 = \beta_1^2, \qquad (\alpha_1\beta_1)^{2^{n-3}} = (\beta_1\alpha_1)^{2^{n-3}}$$

where $\alpha_1 = 1 + xy$ and $\beta_1 = 1 + x^3 y$ and




$$\gamma^{2^{n-1}} = 0$$

where $\gamma = 1 + x$.

Inspired by the work [11] of I.M. Gelfand and V.A. Ponomarev on the representation theory of the Lorentz group, C.M. Ringel classified indecomposable modules over $kG$ in [23]. The indecomposable modules excluding the module of the entire group algebra $kG$ can be divided into two families: string modules and band modules. We now recall his classification.

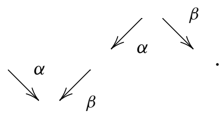
We define two strings $1_\alpha$ and $1_\beta$ of length zero with $1_\alpha^{-1} = 1_\beta$ and $1_\beta^{-1} = 1_\alpha$. Consider now $\alpha, \beta, \alpha^{-1}, \beta^{-1}$ as 'letters' in formal language and define $(\alpha^{-1})^{-1} = \alpha$ and $(\beta^{-1})^{-1} = \beta$. If $l$ is a letter, we write $l^*$ to mean 'either $l$ or $l^{-1}$.' A *string* $C = l_1 l_2 \cdots l_m$ of length $m \geqslant 1$ is given by a sequence $l_1 l_2 \cdots l_m$ of letters subject to

(1) $l_i = \alpha^*$ for $1 \leqslant i \leqslant m-1$ implies $l_{i+1} = \beta^*$ and similarly $l_i = \beta^*$ for $1 \leqslant i \leqslant m-1$ implies $l_{i+1} = \alpha^*$;
(2) neither $l_i \cdots l_j$ nor $l_j^{-1} \cdots l_i^{-1}$ is in the set $\{(\alpha\beta)^{2^{n-2}}, (\beta\alpha)^{2^{n-2}}\}$, for any $1 \leqslant i < j \leqslant m$.

For instance, if $n \geqslant 3$, the word $C = \alpha\beta^{-1}\alpha^{-1}\beta$ is a string of length 4 and if $n = 2$, it is not a string, as

$$\left(\beta^{-1}\alpha^{-1}\right)^{-1} = \alpha\beta \in \{\alpha\beta, \beta\alpha\}.$$

We usually illustrate this string by the following graph:

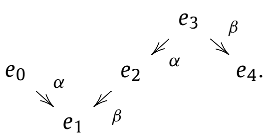



In this graph, we draw an arrow from north-west to south-east for a direct letter, and an arrow from north-east to south-west for an inverse letter. If $C = l_1 \cdots l_m$ is a string, then its inverse is given by $C^{-1} = l_m^{-1} \cdots l_1^{-1}$. Let $\mathcal{S}t$ be the set of all strings. Let $\rho$ be the equivalence relation on $\mathcal{S}t$ which identifies each string to its inverse. If $C = l_1 \cdots l_m$ and $D = f_1 \cdots f_u$ are two strings, their product is given by $CD = l_1 \cdots l_m f_1 \cdots f_u$ provided that this is again a string. Let $\mathcal{B}d$ be the set of strings of even length $\neq 0$ and which are not powers of strings of strictly smaller length. The elements of $\mathcal{B}d$ are called *bands*. If $C = l_1 \cdots l_m$ is a band, then for $1 \leqslant i \leqslant m-1$ denote by $C_{(i)}$ the $i$th cyclic permutation word, thus $C_{(0)} = l_1 \cdots l_m$, $C_{(1)} = l_2 \cdots l_m l_1$, up to $C_{(m-1)} = l_m l_1 \cdots l_{m-1}$. Let $\rho'$ be the equivalence relation which identifies with the band $C$ all its cyclic permutations $C_{(i)}$ and their inverses $C_{(i)}^{-1}$. For each equivalence class under $\rho'$, we fix a representative $C = l_1 \cdots l_m$ whose last letter $l_m$ is inverse.

To every string $C$, we are going to construct an indecomposable module, denoted by $M(C)$ and called a *string module*. Namely, let $C = l_1 \cdots l_m$ be a string of length $m$. Let $M(C)$ be given by a $K$-vector space of dimension $m+1$, say with basis $e_0, e_1, \ldots, e_m$ on which $\alpha$ and $\beta$ operate according to the following schema

$$e_0 \overset{l_1}{\text{———}} e_1 \overset{l_2}{\text{———}} e_2 \overset{l_3}{\text{———}} \cdots\cdots \overset{l_{m-1}}{\text{———}} e_{m-1} \overset{l_m}{\text{———}} e_m .$$

For example, if $C = \alpha\beta^{-1}\alpha^{-1}\beta$, we have the following schema

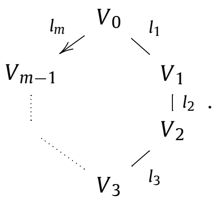



Note that we already use the notation above to adjust the direction of the arrows according to whether the letter $l_i$ is direct or not. This graph indicates how the basis vectors $e_i$ are mapped into each other or into zero, more precisely,

$$e_0\alpha = e_1, \quad e_1\alpha = 0, \quad e_2\alpha = 0, \quad e_3\alpha = e_2, \quad e_4\alpha = 0,$$
$$e_0\beta = 0, \quad e_1\beta = 0, \quad e_2\beta = e_1, \quad e_3\beta = e_4, \quad e_4\beta = 0.$$

It is obvious that $M(C)$ and $M(C^{-1})$ are isomorphic.

Next we construct *band modules*. Let $\lambda \in k^*$ and $m \in \mathbb{N}_1$. Let $C = l_1 \cdots l_m$ be a band such that $l_m$ is inverse. Let $M(C, m, \lambda)$ be given by $M(C, m, \lambda) = \bigoplus_{i=0}^{m-1} V_i$ with $V_i = k^m$ for any $1 \leqslant i \leqslant m$ on which $\alpha$ and $\beta$ operate according to the following schema




This means that the action is given by

(1) $l_s : V_{s-1} \to V_s$ is the identity map, if $l_s$ is direct for $1 \leqslant s \leqslant m-1$;
(2) $l_s^{-1} : V_s \to V_{s-1}$ is the identity map, if $l_s$ is inverse for $1 \leqslant s \leqslant m-1$;
(3) $l_m^{-1} : V_m = V_0 \to V_{m-1}$ is $J_m(\lambda)$ where $J_m(\lambda)$ is the block of Jordan.

**Remark 2.1.** Let $C = l_1 \cdots l_m$ be a band such that $l_m$ is inverse. Take $\lambda \in k^*$ and $m \in \mathbb{N}_1$. Then for $1 \leqslant i \leqslant m$

(1) if $l_i$ is inverse, then $M(C_{(i)}, m, \lambda) \cong M(C, m, \lambda)$;
(2) if $l_i$ is direct, then $M(C^{-1}_{(i-1)}, m, \lambda) \cong M(C, m, \frac{1}{\lambda})$.

**Theorem 2.2.** *(See [23, Section 8].) The strings modules $M(C)$ with $C \in \mathcal{S}t/\rho$ and the band modules $M(C, m, \lambda)$ with $C \in \mathcal{B}d/\rho'$, $m \in \mathbb{N}_1$ and $\lambda \in k^*$, together with $kG$, furnish a complete list of isomorphism classes of indecomposable $kG$-modules for $G$ the dihedral group of order $2^n$ with $n \geqslant 2$ over an algebraically closed field $k$ of characteristic two.*

Now we can describe the Auslander–Reiten quiver of $kG$. For the general theory of Auslander–Reiten quivers, we refer to [3, Chapter 4] and [2]. Let $C$ be a string, we denote by $Q(C)$ the component of the Auslander–Reiten quiver containing $M(C)$. Let $D$ be a band and $\lambda \in k^*$. Then we denote by $Q(D, \lambda)$ the component of the Auslander–Reiten quiver containing $M(D, m, \lambda)$ for $m \in \mathbb{N}_1$. It is well known that $Q(D, \lambda)$ is a homogeneous tube.

**Proposition 2.3.** *(See [3, Chapter 4, Section 4.17].) The Auslander–Reiten quiver of the group algebra $kV_4$ of the Klein-four group $V_4$ is composed by*

- *infinitely many homogeneous tubes $Q(\alpha\beta^{-1}, \lambda)$ with $\lambda \in k^*$ formed by band modules $M(\alpha\beta^{-1}, m, \lambda)$ with $m \in \mathbb{N}_1$;*
- *two homogeneous tubes $Q(\alpha)$ and $Q(\beta)$ consisting of string modules;*
- *one component of type $\mathbb{Z}\tilde{A}_{12}$ consisting of all the syzygies of the trivial module of dimension 1.*

**Proposition 2.4.** *(See [3, Chapter 4, Section 4.17].) Let $G$ be a dihedral group of order $\geqslant 8$. The Auslander–Reiten quiver of $kG$ is composed by*

- *infinitely many homogeneous tubes consisting of band modules;*
- *two homogeneous tubes $Q((\alpha\beta)^{2^{n-2}-1}\alpha)$ and $Q((\beta\alpha)^{2^{n-2}-1}\beta)$ consisting of string modules;*
- *infinitely many components of type $\mathbb{Z}A_\infty^\infty$ consisting of string modules.*

## 3. The dihedral group of order eight

### 3.1. Statement of the main theorem

Now we specialize to the dihedral group of order 8. We fix some notations. Let

$$D_8 = \langle x, y \mid x^4 = e = y^2,\ \ yxy = x^{-1} \rangle$$

be the dihedral group of order 8. The quiver with relations of $kD_8$ is given in Section 2. Note that

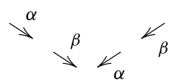
$$H = \langle x \rangle = \{e, x, x^2, x^3\},$$
$$T_0 = \langle x^2, y \rangle = \{e, y, x^2, x^2 y\},$$

and

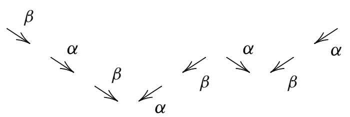
$$T_1 = \langle x^2, xy \rangle = \{e, xy, x^2, x^3 y\}.$$

Recall that $H \simeq C_4$ is the cyclic group of order 4 and $T_0 \simeq V_4 \simeq T_1$ are the Klein-four group. Their quivers with relations are also given in Section 2.

In order to state the main theorems, we introduce some particular bands. For simplicity, we present bands in form of strings and we shall do this from now on. For $n = 1, 2$, we define

$$C_1 = \alpha\beta\alpha^{-1}\beta^{-1} =$$


and

$$C_2 = \beta\alpha\beta\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-1} =$$

.

If $n \geqslant 2$ is even, $C_{n+1} = \alpha\beta\alpha^{-1}C_n\beta^{-1}$; if $n \geqslant 3$ is odd, $C_{n+1} = \beta C_n \alpha\beta^{-1}\alpha^{-1}$. For $n \in \mathbb{N}_1$, we define $D_n$ as the band obtained by exchanging $\alpha$ and $\beta$ in $C_n$. Notice that $C_1 = D_1^{-1}$.

Now we state the main result of this paper which describes the distribution of vertices in the Auslander–Reiten quiver of $kD_8$. The following two theorems specify respectively all indecomposable modules with a cyclic vertex or which have $T_0$ or $T_1$ as a vertex. Obviously the trivial group $\{e\}$ is the vertex of $kD_8$ and except $kD_8$ and the modules appearing in the following theorems any other indecomposable module has $D_8$ as a vertex. In the following, we will denote by $vx(M)$ a vertex of an indecomposable module $M$.

**Theorem 3.1.** *Let $M$ be an indecomposable module over the group algebra $kD_8$ which has a cyclic vertex. Then $M$ is one of the following modules*:

(1) *the module $M(\beta\alpha\beta)$ at the bottom of the homogeneous tube $Q(\beta\alpha\beta)$ with $vx(M(\beta\alpha\beta)) = \langle y \rangle = \{e, y\}$*;

(1′) *the module $M(\alpha\beta\alpha)$ at the bottom of the homogeneous tube $Q(\alpha\beta\alpha)$ with $vx(M(\alpha\beta\alpha)) = \langle yx \rangle = \{e, yx\}$*;

(2) *the module $M(\beta\alpha^{-1}, 1, 1)$ at the bottom of the homogeneous tube $Q(\beta\alpha^{-1}, 1)$ with $vx(M(\beta\alpha^{-1}, 1, 1)) = H$*;

(3) *the module* $M(\beta\alpha\beta^{-1}\alpha^{-1}, 1, 1)$ *with*

$$vx\big(M\big(\beta\alpha\beta^{-1}\alpha^{-1}, 1, 1\big)\big) = \langle x^2\rangle = \{e, x^2\}$$

*which is at the bottom of the homogeneous tube* $Q(\beta\alpha\beta^{-1}\alpha^{-1}, 1)$;

(4) *the module* $M(\beta\alpha\beta^{-1}\alpha^{-1}, 1, 1)$ *with* $vx(M(\beta\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}, 1, 1)) = H$ *which is at the bottom of the homogeneous tube* $Q(\beta\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}, 1)$;

(5) *the module* $M(C_1, 1, 1)$ *at the bottom of the homogeneous tube* $Q(C_1, 1)$ *with*

$$vx\big(M(C_1, 1, 1)\big) = \langle x^2\rangle = \{e, x^2\}.$$

**Theorem 3.2.** *Let* $M$ *be an indecomposable module over the group algebra* $kD_8$ *which has* $T_0$ *or* $T_1$ *as a vertex. Then* $M$ *is one of the following modules*:

(1) *any module* $M \ncong M(\beta\alpha\beta)$ *in the homogeneous tube* $Q(\beta\alpha\beta)$ *which has* $T_0$ *as a vertex*;

(1′) *any module* $M \ncong M(\alpha\beta\alpha)$ *in the homogeneous tube* $Q(\alpha\beta\alpha)$ *which has* $T_1$ *as a vertex*;

(2) *the syzygies* $\Omega^m(M(\beta))$ *with* $m \in \mathbb{Z}$ *which have* $T_0$ *as vertices and which form two* $\tau$*-orbits in the component* $Q(\beta)$ *of type* $\mathbb{Z}A_\infty^\infty$;

(2′) *the syzygies* $\Omega^m(M(\alpha))$ *with* $m \in \mathbb{Z}$ *which have* $T_1$ *as vertices and which form two* $\tau$*-orbits in the component* $Q(\alpha)$ *of type* $\mathbb{Z}A_\infty^\infty$;

(3) *any module in the homogeneous tube* $Q(\beta\alpha\beta\alpha^{-1}, \mu)$ *with* $\mu \in k^*$ *with* $T_0$ *as a vertex*;

(3′) *any module in the homogeneous tube* $Q(\alpha\beta\alpha\beta^{-1}, \mu)$ *with* $\mu \in k^*$ *with* $T_1$ *as a vertex*;

(4) *the module* $M(C_m, 1, 1)$ *at the bottom of the homogeneous tube* $Q(C_m, 1)$ *with* $m \geqslant 2$, *with* $vx(M(C_m, 1, 1)) = T_0$;

(4′) *the module* $M(D_m, 1, 1)$ *at the bottom of the homogeneous tube* $Q(D_m, 1)$ *with* $m \geqslant 2$, *with* $vx(M(D_m, 1, 1)) = T_1$.

We next collect some well-known results about cyclic groups and the Klein-four group which will be needed in the proof of the preceding theorems.

**Lemma 3.3.** *Let* $H = \langle x\rangle$ *be the cyclic subgroup of order* 4 *of* $D_8$.

(1) *Each indecomposable* $kH$*-module is of the form* $M(\gamma^i)$ *for* $0 \leqslant i \leqslant 3$.

(2) *We have*

$$vx\big(M\big(\gamma^0\big)\big) = H = vx\big(M\big(\gamma^2\big)\big),$$
$$vx\big(M\big(\gamma^1\big)\big) = \langle x^2\rangle,$$
$$vx\big(M\big(\gamma^3\big)\big) = vx(kH) = \{e\}.$$

**Proof.** For (1), see [1, Section II.4], we just translate the description there into the context of string modules. For (2), it is sufficient to calculate the induced module of the trivial module from the subgroup $\{e, x^2\}$ to $H$. □

**Lemma 3.4.** *Let* $T_0 = \langle x^2, y\rangle \cong V_4$.

(1) $vx(kT_0) = \{e\}$.

(2) *We have*

$$vx\big(M(\beta_0)\big) = \langle y\rangle = \{e, y\},$$

$$vx\big(M(\alpha_0)\big) = \langle yx^2\rangle = \{e, yx^2\},$$
$$vx\big(M\big(\alpha_0\beta_0^{-1}, 1, 1\big)\big) = \langle x^2\rangle = \{e, x^2\}.$$

(3) *Any other indecomposable module has $T_0$ as a vertex.*

**Proof.** Recall the general method to calculate an induced module. Let $G$ be a finite group and $H$ a subgroup of index $m$. Then we write $G = \coprod_{i=1}^{m} Hg_i$ a coset partition. For a $kH$-module $M$, its induced module is

$$\mathrm{Ind}_H^G M := M \otimes_{kH} kG = \bigoplus_{i=1}^{m} M \otimes g_i$$

and the action is given by $(x \otimes g_i)g = xh \otimes g_j$, for all $g \in G$, $1 \leqslant i \leqslant m$, $x \in M$ and with $h \in H$ such that $g_i g = hg_j$.

Now return to our situation. It is sufficient to compute inductions from its subgroups to $T_0$. Denote by $L$ the subgroup $\{e, x^2\}$. Note that there are only two indecomposable modules over $kL$: the trivial module $k$ and $kL$. If we write $T_0 = L \coprod Ly$, then the induced module of $k$ is

$$\mathrm{Ind}_L^{T_0} k = k \otimes_{kL} kT_0 = k(1 \otimes e) \oplus k(1 \otimes y).$$

We obtain easily

$$(1 \otimes e)\alpha_0 = 1 \otimes e + 1 \otimes y, \qquad (1 \otimes y)\alpha_0 = 1 \otimes y + 1 \otimes e,$$
$$(1 \otimes e)\beta_0 = 1 \otimes e + 1 \otimes y, \qquad (1 \otimes y)\beta_0 = 1 \otimes y + 1 \otimes e.$$

If we write $f_0 = 1 \otimes e$ and $f_1 = 1 \otimes e + 1 \otimes y$, then $f_0\alpha_0 = f_1$, $f_0\beta_0 = f_1$ and $f_1\alpha_0 = 0 = f_1\beta_0$. We obtain the isomorphism

$$\mathrm{Ind}_L^{T_0} k = \big(f_0 \underset{\beta_0}{\overset{\alpha_0}{\rightrightarrows}} f_1\big) \simeq M\big(\alpha_0\beta_0^{-1}, 1, 1\big).$$

The inductions from $\{e, y\}$ to $T_0$ and from $\{e, x^2y\}$ to $T_0$ can be calculated similarly. □

To prove the main theorems, we will calculate the induced module for each indecomposable module over $H$, $T_0$ and $T_1$, respectively.

### 3.2. Induction from $H$ to $D_8$

By Lemma 3.3, all indecomposable $kH$-modules are of the form $M(\gamma^i)$ with $0 \leqslant i \leqslant 3$ and the following lemma computes their induced modules.

**Lemma 3.5.** *We have*

(1) $\mathrm{Ind}_H^{D_8} M(\gamma^0) = \mathrm{Ind}_H^{D_8} k = M(\beta\alpha^{-1}, 1, 1)$;
(2) $\mathrm{Ind}_H^{D_8} M(\gamma^1) = M(\beta\alpha\beta^{-1}\alpha^{-1}, 1, 1)$;
(3) $\mathrm{Ind}_H^{D_8} M(\gamma^2) = M(\beta\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}, 1, 1)$;
(4) $\mathrm{Ind}_H^{D_8} M(\gamma^3) = \mathrm{Ind}_H^{D_8} kH = kD_8$.

The proof uses the same argument as in Lemma 3.4 and is left to the reader.

*3.3. Induction from $T_0$ to $D_8$*

The element $x \in D_8$ acts via conjugation over $T_0$ et therefore induces an automorphism of $kT_0$, say $\sigma$. We have

$$\sigma(\alpha_0) = x\alpha_0 x^{-1} = 1 + xyx^{-1} = 1 + x^2 y = \beta_0$$

and

$$\sigma(\beta_0) = x\beta_0 x^{-1} = 1 + xx^2 yx^{-1} = 1 + y = \alpha_0.$$

So the action of $x$ exchanges $\alpha_0$ and $\beta_0$. Let $M$ be a $kT_0$-module. Denote by $M^\sigma$ the new $kT_0$-module obtained via $\sigma$, that is, for $t \in T_0$ and $x \in M^\sigma$, $t \cdot x = \sigma(t)x$. The following lemma deduces immediately from the argument above et from the constructions of string modules and band modules presented in Section 2.

**Lemma 3.6.** *Let $C$ be a string and denote by $C^\sigma$ the new string by exchanging $\alpha_0$ and $\beta_0$.*

(1) $M(C)^\sigma \cong M(C^\sigma)$.
(2) *If $C$ is a band (such that the last letter is inverse), then for all $m \in \mathbb{N}$ and $\lambda \in k^*$, we have $M(C, m, \lambda)^\sigma \cong M(C^\sigma, m, \lambda)$.*

As a consequence, for the Auslander–Reiten quiver of $kT_0$, we obtain

**Proposition 3.7.** *One has*

(1) *each module $M$ in the component of type $\mathbb{Z}\tilde{A}_{12}$ is stable by $\sigma$ (i.e. $M^\sigma \cong M$);*
(2) *$\sigma$ induces an isomorphism from the homogeneous tube $Q(\alpha_0)$ to the homogeneous tube $Q(\beta_0)$;*
(3) *given $\lambda \in k^*$, the component $Q(\alpha_0\beta_0^{-1}, \lambda)$ is stable by $\sigma$ if and only if $\lambda = 1$.*

**Proof.** (1) The modules in the component of type $\mathbb{Z}\tilde{A}_{12}$ are of the form $\Omega^m(k) = M((\alpha_0\beta_0^{-1})^m)$ or $\Omega^{-m}(k) = M((\alpha_0^{-1}\beta_0)^m)$ for all $m \in \mathbb{N}_0$. A string $C$ in this component verifies $C^\sigma = C^{-1}$ and recall that $M(C^{-1}) \cong M(C)$, Lemma 3.6(1) thus implies the desired result.

(2) As all modules in the component $Q(\alpha_0)$ (resp. $Q(\beta_0)$) are of the form $M(\alpha_0(\beta_0\alpha_0)^m)$ with $m \in \mathbb{N}_0$ (resp. $M(\beta_0(\alpha_0\beta_0)^m)$ with $m \in \mathbb{N}_0$), then by the preceding lemma, $M(\alpha_0(\beta_0\alpha_0)^m)^\sigma \cong M(\beta_0(\alpha_0\beta_0)^m)$. $\sigma$ thus establishes an isomorphism between $Q(\alpha_0)$ and $Q(\beta_0)$.

(3) $M(\alpha_0\beta_0^{-1}, m, \lambda)^\sigma \cong M(\beta_0\alpha_0^{-1}, m, \lambda) \cong M(\alpha_0\beta_0^{-1}, m, 1/\lambda)$ where the last isomorphism follows from Remark 2.1(2). □

**Lemma 3.8.** *We have*

(1) $\mathrm{Ind}_{T_0}^{D_8} \Omega^m(k) \cong \Omega^n M(\beta)$ *for all $m \in \mathbb{Z}$;*
(2)
$$\mathrm{Ind}_{T_0}^{D_8} M(\alpha_0) \cong M(\beta\alpha\beta) \cong \mathrm{Ind}_{T_0}^{D_8} M(\beta_0).$$

*As a consequence, the homogeneous tubes $Q(\alpha_0)$ and $Q(\beta_0)$ are transformed onto the same homogeneous tube $Q(\beta\alpha\beta)$ by the induction from $T_0$ to $D_8$.*

**Proof.** Since the indecomposablity theorem of Green [3, Theorem 3.13.3] implies that for any $m \in \mathbb{Z}$,

$$\Omega^m\big(\mathrm{Ind}_{T_0}^{D_8} k\big) \cong \mathrm{Ind}_{T_0}^{D_8}\big(\Omega^m k\big),$$

it suffices to prove that $\mathrm{Ind}_{T_0}^{D_8} k \cong M(\beta)$ which is an easy calculation. The isomorphism $\mathrm{Ind}_{T_0}^{D_8} M(\alpha_0) \cong M(\beta\alpha\beta)$ can be proved as follows. Suppose that $M(\alpha_0) = (e_0 \xrightarrow{\alpha_0} e_1)$. This means that $e_0\alpha_0 = e_1$, $e_1\alpha_0 = 0$, $e_0\beta_0 = 0$, and $e_1\beta_0 = e_1$. Using the coset partition $D_8 = T_0 \coprod T_0 x$, we compute that

$$(e_0 \otimes e)\alpha = e_1 \otimes e, \qquad (e_0 \otimes x)\alpha = 0, \qquad (e_1 \otimes e)\alpha = 0, \qquad (e_1 \otimes x)\alpha = 0,$$
$$(e_0 \otimes e)\beta = e_0 \otimes e + e_0 \otimes x, \qquad (e_0 \otimes x)\beta = e_0 \otimes x + e_0 \otimes e,$$
$$(e_1 \otimes e)\beta = e_1 \otimes e + e_1 \otimes x, \qquad (e_1 \otimes x)\beta = e_1 \otimes x + e_1 \otimes e.$$

Then it is easy to see that the terms of the sequence

$$\mathcal{S}_1(\alpha_0) := (e_0 \otimes x \xrightarrow{\beta} e_0 \otimes x + e_0 \otimes e \xrightarrow{\alpha} e_1 \otimes e \xrightarrow{\beta} e_1 \otimes e + e_1 \otimes x)$$

form a basis of $\mathrm{Ind}_{T_0}^{D_8} M(\alpha_0)$ and a basis of $M(\beta\alpha\beta)$ as well, thus giving the desired isomorphism, as $(e_0 \otimes x)\alpha = 0$ and $(e_1 \otimes e + e_1 \otimes x)\alpha = 0$.

Recall that for a group $G$ and $H$ a normal subgroup of $G$, the inertia group of a component of the Auslander–Reiten quiver of $kH$ is by definition the set of elements of $G$ whose induced inner automorphisms of $kG$ map this component to itself. As $\sigma$ transforms $Q(\alpha_0)$ into $Q(\beta_0)$, the inertia group of $Q(\alpha_0)$ is $T_0$ and a theorem of S. Kawata [16] implies that induction from $T_0$ to $G$ induces an isomorphism from $Q(\alpha_0)$ (also from $Q(\beta_0)$) to $Q(\beta\alpha\beta)$

**Lemma 3.9.** *If $\lambda \in k - \{0, 1\}$, $\mathrm{Ind}_{T_0}^{D_8} M(\alpha_0\beta_0^{-1}, m, \lambda) \cong M(\beta\alpha\beta\alpha^{-1}, m, \mu)$ with $\mu = \frac{\lambda}{\lambda^2+1}$. Consequently, the component $Q(\alpha_0\beta_0^{-1}, \lambda)$ with $\lambda \in k - \{0, 1\}$ becomes the component $Q(\beta\alpha\beta\alpha^{-1}, \mu)$ after the induction from $T_0$ to $D_8$.*

**Proof.** Write

$$M\big(\alpha_0\beta_0^{-1}, 1, \lambda\big) \cong \big(e_0 \underset{\beta_0=\lambda}{\overset{\alpha_0=1}{\rightrightarrows}} e_1\big).$$

Then

$$e_0\alpha_0 = e_1, \qquad e_1\alpha_0 = 0, \qquad e_0\beta_0 = \lambda e_1, \qquad e_1\beta_0 = 0.$$

Its induced module is

$$\mathrm{Ind}_{T_0}^{D_8} M\big(\alpha_0\beta_0^{-1}, 1, \lambda\big) = k(e_0 \otimes e) \oplus k(e_1 \otimes e) \oplus k(e_0 \otimes x) \oplus k(e_1 \otimes x).$$

Direct calculations yield that

$$(e_0 \otimes e)\alpha = e_1 \otimes e, \qquad (e_1 \otimes e)\alpha = 0, \qquad (e_0 \otimes x)\alpha = \lambda e_1 \otimes x, \qquad (e_1 \otimes x)\alpha = 0,$$
$$(e_0 \otimes e)\beta = e_0 \otimes e + e_0 \otimes x + \lambda e_1 \otimes x, \qquad (e_1 \otimes e)\beta = e_1 \otimes e + e_1 \otimes x,$$
$$(e_0 \otimes x)\beta = e_0 \otimes x + e_0 \otimes e + \lambda e_1 \otimes e, \qquad (e_1 \otimes x)\beta = e_1 \otimes x + e_1 \otimes e.$$

If we impose

$$e'_0 = e_0 \otimes e + \frac{1}{\lambda} e_0 \otimes x, \qquad e'_1 = \frac{\lambda+1}{\lambda}(e_0 \otimes e + e_0 \otimes x) + e_1 \otimes e + \lambda e_1 \otimes x,$$
$$e'_2 = \frac{\lambda+1}{\lambda}(e_1 \otimes e + \lambda e_1 \otimes x), \qquad e'_3 = \frac{\lambda^2+1}{\lambda}(e_1 \otimes e + e_1 \otimes x),$$

then we can verify that

$$e'_0\beta = e'_1, \qquad e'_1\alpha = e'_2, \qquad e'_2\beta = e'_3, \qquad e'_0\alpha = \mu e'_3.$$

The statement now deduces from the following diagram:

$$\begin{array}{ccc} e'_0 & \xrightarrow{\beta=1} & e'_1 \\ {\scriptstyle \alpha=\mu}\downarrow & & \downarrow{\scriptstyle \alpha=1} \\ e'_3 & \xleftarrow{\beta=1} & e'_2 \end{array} .$$

Since $\lambda \neq 1$, $\sigma$ does not stabilize the component $Q(\alpha_0\beta_0^{-1}, \lambda)$ by Proposition 3.7 (3) and the inertia group de $Q(\alpha_0\beta_0^{-1}, \lambda)$ is $T_0$, the theorem of S. Kawata cited above implies that the induction from $T_0$ to $D_8$ induces an isomorphism between $Q(\alpha_0\beta_0^{-1}, \lambda)$ and $Q(\beta\alpha\beta\alpha^{-1}, \mu)$. $\square$

**Proposition 3.10.** *For any $n \in \mathbb{N}_1$, $\mathrm{Ind}_{T_0}^{D_8} M(\alpha_0\beta_0^{-1}, m, 1) = M(C_m, 1, 1)$.*

The proof of this proposition is rather complicated and is postponed to the final section.

### 3.4. Induction from $T_1$ to $D_8$

All the statements in this subsection can be proved using the same method as in the previous subsection, so we omit them.

**Lemma 3.11.** *One has*

(1) $\mathrm{Ind}_{T_1}^{D_8} \Omega^m(k) \cong \Omega^m M(\alpha)$ *for all* $m \in \mathbb{Z}$;
(2) $\mathrm{Ind}_{T_1}^{D_8} M(\alpha_1) \cong M(\alpha\beta\alpha) \cong \mathrm{Ind}_{T_1}^{D_8} M(\beta_1)$.

*As a consequence, the homogeneous tubes $Q(\alpha_1)$ and $Q(\beta_1)$ are transformed onto the same homogeneous tube $Q(\alpha\beta\alpha)$ after the induction from $T_1$ to $D_8$.*

**Lemma 3.12.** *If $\lambda \in k - \{0, 1\}$, $\mathrm{Ind}_{T_1}^{D_8} M(\alpha_1\beta_1^{-1}, m, \lambda) \cong M(\alpha\beta\alpha\beta^{-1}, m, \mu)$ with $\mu = \frac{\lambda}{\lambda^2+1}$. Consequently, the component $Q(\alpha_1\beta_1^{-1}, \lambda)$ with $\lambda \in k - \{0, 1\}$ becomes the component $Q(\alpha\beta\alpha\beta^{-1}, \mu)$ by the induction from $T_1$ to $D_8$.*

**Proposition 3.13.** *For arbitrary $m \in \mathbb{N}_1$,*

$$\mathrm{Ind}_{T_1}^{D_8} M\big(\alpha_1\beta_1^{-1}, m, 1\big) = M(D_m, 1, 1).$$

*3.5. Proof of Theorems 3.1 and 3.2*

Since we have calculated all the induced modules, we can deduce the main theorems from the calculations in Sections 3.2–3.4, taking into account the results recalled at the end of Section 2.

## 4. Induction of string modules

In this section, let $G$ be the dihedral group of order $2^n$ with $n \geqslant 3$. We introduce firstly some notations. A string $D = \alpha_1 \cdots \alpha_s$ of strictly positive length is direct (resp. inverse) if all the $\alpha_i$ are direct arrows (resp. formal inverses). Let $C = C_1 C_2 \cdots C_m$ where the substrings $C_1, \ldots, C_m$ are direct or inverse and such that for each $1 \leqslant i \leqslant m-1$, $C_i$ is direct (resp. inverse) if and only if $C_{i+1}$ is inverse (resp. direct). These substrings $C_i$ are called *segments* of $C$. We denote by $|E|$ the length of a string $E$.

Let $C = C_1 C_2 \cdots C_m$ be a string over $kT_0$ where the substrings $C_i$ are its segments. We use the convention that $|C_j| = -1$ for $j \leqslant 0$ or $j \geqslant m+1$. Now fix $1 \leqslant i \leqslant m$ and we will define a function $\theta : \mathbb{N}_0 \to \{+1, 0, -1\}$ to compare $C_i$ and $C_{i+1}$. Let $s \in \mathbb{N}_0$, then $\theta(s)$ is defined as follows:

(1) (if $|C_{i-s+1}| > |C_{i+s}|$ and $s$ is odd) or (if $|C_{i-s+1}| < |C_{i+s}|$ and $s$ is even), then $\theta(s) = 1$;
(2) if $|C_{i-s+1}| = |C_{i+s}|$, $\theta(s) = 0$;
(3) (if $|C_{i-s+1}| > |C_{i+s}|$ and $s$ is even) or (if $|C_{i-s+1}| < |C_{i+s}|$ and $s$ is odd), then $\theta(s) = -1$.

If for each $s \in \mathbb{N}_0$, $\theta(s) = 0$, this means that $C$ is a symmetric string, that is, $m = 2u$ for $u \in \mathbb{N}_1$, $i = u$ and $|C_j| = |C_{m-j}|$ for each $1 \leqslant j \leqslant u$. In this case, we define $C_i > C_{i+1}$. Otherwise, let $t \in \mathbb{N}_0$ be the first number such that $\theta(t) \neq 0$. If $\theta(t) = 1$, then we define $C_i > C_{i+1}$ and if $\theta(t) = -1$, then $C_i < C_{i+1}$.

With this order at hand, we construct a new string over $kG$, say $\varphi(C) = \tilde{C}_1 \tilde{C}_2 \cdots \tilde{C}_m$ where for all $1 \leqslant i \leqslant m$ the $\tilde{C}_i$ are the segments of $\varphi(C)$ such that

(1) for all $1 \leqslant i \leqslant m$, $\tilde{C}_i$ is direct (resp. inverse) if and only if $C_i$ is direct (resp. inverse);

(2)
$$|\tilde{C}_i| = \begin{cases} 2|C_i| - 1 & \text{if } C_i < C_{i-1}, C_{i+1}, \\ 2|C_i| + 1 & \text{if } C_i > C_{i-1}, C_{i+1}, \\ 2|C_i| & \text{otherwise;} \end{cases}$$

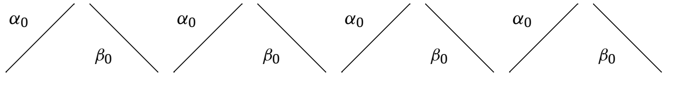
(3) for $i$ such that $1 \leqslant i \leqslant m$ and $C_i > C_{i-1}, C_{i+1}$, we impose that $\tilde{C}_i$ begins with $\beta$ or $\beta^{-1}$ according to whether $\tilde{C}_i$ is direct or not.

We can also construct similarly a new string $\psi(D)$ over $kG$ from a string $D = D_1 \cdots D_m$ over $kT_1$. The difference with the case $kT_0$ is that the last condition becomes

(3′) for $i$ such that $1 \leqslant i \leqslant m$ and $D_i > D_{i-1}, D_{i+1}$, then we impose that $\tilde{D}_i$ begins with $\alpha$ or $\alpha^{-1}$.

**Remark 4.1.** As we expect that $\mathrm{Ind}_{T_0}^G M(C) \cong M(\varphi(C))$, the new string has the 'right' length. In fact, as always $C_0 < C_1$ and $C_m > C_{m+1}$, if there are $t$ segments $C_i$ such that $C_i < C_{i-1}, C_{i+1}$, then there exist $t+1$ segments $C_i$ such that $C_i > C_{i-1}, C_{i+1}$. We thus have $|\varphi(C)| = 2|C| + 1$ which is the 'right' length.

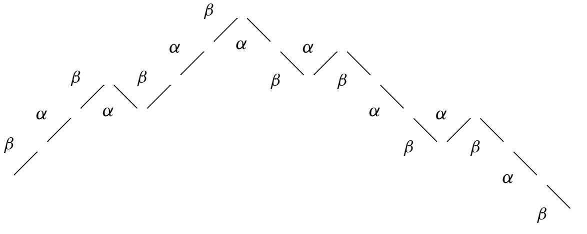
**Example 4.2.** Let $C = C_1 \cdots C_8 = \alpha_0^{-1} \beta_0 \alpha_0^{-1} \beta_0 \alpha_0^{-1} \beta_0 \alpha_0^{-1} \beta_0$.



Then

$$C_1 > C_2 < C_3 > C_4 > C_5 < C_6 > C_7 < C_8$$

and the new string $\varphi(C) = \tilde{C}_1 \cdots \tilde{C}_8$ is of the form:




We give the following small

**Conjecture 4.3.**

$$\mathrm{Ind}_{T_0}^{G} M(C) \cong M\big(\varphi(C)\big)$$

*and*

$$\mathrm{Ind}_{T_1}^{G} M(D) \cong M\big(\psi(D)\big).$$

It is easy to verify this conjecture for the dihedral group of order 8.

**Proposition 4.4.** *The preceding formulae hold when $G = D_8$ is the dihedral group of order* 8.

**Proof.** As we have calculated the induced module from $T_0$ to $D_8$ for each string module over $kT_0$ in Lemma 3.8, we just need to verify that this is just the string module defined above using the order. We only consider $\Omega^m(k)$ with $m \geqslant 1$, the situation being similar for all other string modules.

It is obvious to see that $\Omega^m(k) = M((\alpha_0\beta_0^{-1})^m)$. We write

$$\big(\alpha_0\beta_0^{-1}\big)^m = C_1C_2\cdots C_{2m}$$

with the $C_i$ being its segments. We now compare its segments. The result can be illustrated as follows:

$$C_1 > C_2 < \cdots < C_m > C_{m+1} > \cdots > C_{2m-1} < C_{2m}$$

in which the symbols $>$ and $<$ appear in the alternating way from $C_1$ to $C_m$ with $C_1 > C_2$ and from $C_{2m}$ to $C_{m+1}$ with $C_{2m} > C_{2m-1}$. We thus obtain the following description of $\varphi((\alpha_0\beta_0^{-1})^m)$.

$$\varphi\big(\alpha_0\beta_0^{-1}\big) = \tilde{C}_1\tilde{C}_2 = (\beta\alpha\beta)(\beta\alpha)^{-1}. \tag{1}$$

For $m \geqslant 1$, $\varphi((\alpha_0\beta_0^{-1})^{2m+1})$ is obtained by adding $(\beta\alpha\beta)\alpha^{-1}$ to the left side and the right side of $\varphi((\alpha_0\beta_0^{-1})^{2m-1})$.

$$\varphi\big(\big(\alpha_0\beta_0^{-1}\big)^2\big) = \tilde{C}_1\tilde{C}_2\tilde{C}_3\tilde{C}_4 = (\beta\alpha\beta)(\beta\alpha)^{-1}\alpha(\beta\alpha\beta)^{-1}. \tag{2}$$

For $m \geqslant 2$, $\varphi((\alpha_0\beta_0^{-1})^{2m})$ is obtained by adding $(\beta\alpha\beta)\alpha^{-1}$ to the left side and the right side of $\varphi((\alpha_0\beta_0^{-1})^{2m-2})$.

By the above description and by the construction of Auslander–Reiten translations of string modules given in [9, Chapter 2],

$$M\big(\varphi\big(\big(\alpha_0\beta_0^{-1}\big)^{m+2}\big)\big) \cong \tau M\big(\varphi\big(\big(\alpha_0\beta_0^{-1}\big)^{m}\big)\big).$$

Since $kD_8$ is a symmetric algebra, $\tau = \Omega^2$. We can verify without difficulty that $\Omega^m(M(\beta)) \cong M(\varphi((\alpha_0\beta_0^{-1})^m))$. $\square$

**Remark 4.5.** If in general Conjecture 4.3 is true, iterations of these formulae should give the vertices of all string modules. Furthermore, it would be nice if we could extend this formula to band modules. In [28], a constructive method was developed and we could verify these formulae in the case that there exists at most one segment $C_i$ such that $C_{i-1} > C_i < C_{i+1}$. For the idea of this method, see Example 4.6. However, the general case remains unsolved.

**Example 4.6.** The main idea of the constructive method is the following. Let $C = C_1 \cdots C_m$ be a string and $\varphi(C) = \tilde{C}_1 \cdots \tilde{C}_m$ be the induced string defined above. One has to find a basis of the induced module $\mathrm{Ind}_{T_0}^{G} M(C)$ such that it establishes an isomorphism $\mathrm{Ind}_{T_0}^{G} M(C) \cong M(\varphi(C))$, that is, it is also a basis of $M(\varphi(C))$. For each segment $C_i$, on can find a canonical sequence of linearly independent elements of $\mathrm{Ind}_{T_0}^{G} M(C)$ which should be a basis of the segment $\tilde{C}_i$ in $M(\varphi(C))$ (for the construction of such a sequence, the following examples will give some idea). Next one should combine these sequences to give a basis of $M(\varphi(C))$. However, this generally is not easy. We have to modify eventually some sequences. These necessary modifications make the problem complicated and prevent us to solve the conjecture in general.

(1) Let $C = \alpha_0^{-1}$. Then $\varphi(C) = \beta^{-1}\alpha^{-1}\beta^{-1}$. Suppose $M(\alpha_0^{-1}) = (e_0 \stackrel{\alpha_0}{\leftarrow} e_1)$. Then the sequence

$$\mathcal{S}_1\big(\alpha_0^{-1}\big) := (e_0 \otimes e + e_0 \otimes x \stackrel{\beta}{\leftarrow} e_0 \otimes e \stackrel{\alpha}{\leftarrow} e_1 \otimes x + e_1 \otimes e \stackrel{\beta}{\leftarrow} e_1 \otimes x)$$

gives the isomorphism $\mathrm{Ind}_{T_0}^{D_8} M(\alpha_0^{-1}) \cong M(\beta^{-1}\alpha^{-1}\beta^{-1})$.

(2) Let $C = \beta_0$. Then $\varphi(C) = \beta\alpha\beta$. Let $M(C)$ be given by $M(\beta_0) = (e_1 \stackrel{\beta_0}{\rightarrow} e_2)$ and then

$$\mathcal{S}_1(\beta_0) := (e_1 \otimes e \stackrel{\beta}{\rightarrow} e_1 \otimes e + e_1 \otimes x \stackrel{\alpha}{\rightarrow} e_2 \otimes x \stackrel{\beta}{\rightarrow} e_2 \otimes x + e_2 \otimes e)$$

gives the desired isomorphism $\mathrm{Ind}_{T_0}^{D_8} M(\alpha_0) \cong M(\beta\alpha\beta)$. We denote

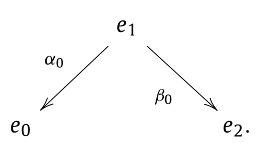
$$\mathcal{S}_2(\beta_0) := (e_1 \otimes x \stackrel{\alpha}{\rightarrow} e_2 \otimes e \stackrel{\beta}{\rightarrow} e_2 \otimes e + e_2 \otimes x)$$

and

$$\mathcal{S}_2'(\beta_0) := (e_1 \otimes x \stackrel{\alpha}{\rightarrow} e_2 \otimes e)$$

which is obtained by deleting the last element $e_2 \otimes x + e_2 \otimes e$ from $\mathcal{S}_2(\beta_0)$.

(3) Let $C = C_1C_2 = \alpha_0^{-1}\beta_0$ with $C_1 = \alpha_0^{-1}$ and $C_2 = \beta_0$. Then $C_1 > C_2$ and $\varphi(C) = \tilde{C}_1\tilde{C}_2 = \beta^{-1}\alpha^{-1}\beta^{-1}\alpha\beta$ with $\tilde{C}_1 = \beta^{-1}\alpha^{-1}\beta^{-1}$ and $\tilde{C}_2 = \alpha\beta$. Suppose that $M(C)$ is given by



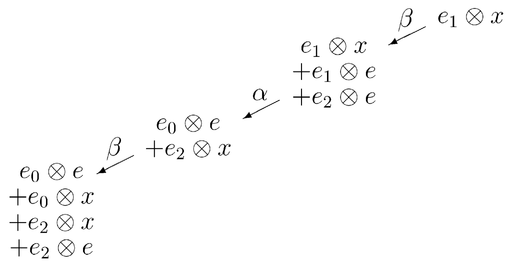



**Fig. 1.** $\mathcal{S}_1(C_1)$.

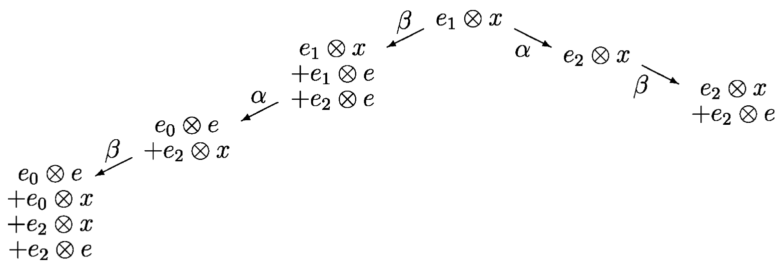



**Fig. 2.** $\alpha_0^{-1}\beta_0$.

For $\tilde{C}_1$, one can take the sequence $\mathcal{S}_1(\alpha_0^{-1})$ introduced in (1), but because of the existence of $C_2$, it is slightly different with that given in (1) although they end with the same term $e_1 \otimes x$. We denote this sequence by $\mathcal{S}_1(C_1)$. (See Fig. 1.)

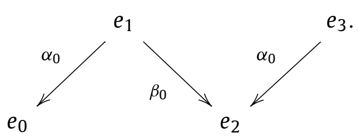
For $\tilde{C}_2$, since it is of length two, we should take $\mathcal{S}_2(C_2)$ which is the same as $\mathcal{S}_2(\beta_0)$ introduced in (2).

$$\mathcal{S}_2(C_2) := (e_1 \otimes x \xrightarrow{\alpha} e_2 \otimes e \xrightarrow{\beta} e_2 \otimes e + e_2 \otimes x).$$

Since the last term of $\mathcal{S}_1(C_1)$ and the first term of $\mathcal{S}_2(C_2)$ are the same, one can combine them to obtain Fig. 2.

One see easily that it gives the desired isomorphism $\mathrm{Ind}_{T_0}^{D_8} M(C) \cong M(\varphi(C))$.

(4) Let $C = C_1C_2C_3 = \alpha_0^{-1}\beta_0\alpha_0^{-1}$ with $C_1 = \alpha_0^{-1}$, $C_2 = \beta_0$ and $C_3 = \alpha_0^{-1}$. Then $C_1 > C_2 < C_3$ and $\varphi(C) = \tilde{C}_1\tilde{C}_2\tilde{C}_3 = \beta^{-1}\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-1}\beta^{-1}$ with $\tilde{C}_1 = \beta^{-1}\alpha^{-1}\beta^{-1}$, $\tilde{C}_2 = \alpha$ and $\tilde{C}_3 = \beta^{-1}\alpha^{-1}\beta^{-1}$. Suppose that $M(C)$ is given by




We write $C^{(1)} = C_1C_2$ and $C^{(2)} = C_3$ and construct respectively some sequences of linearly independent elements corresponding to $\varphi(C^{(1)})$ and $\varphi(C^{(2)})$. For $\varphi(C^{(1)}) = \tilde{C}_1\tilde{C}_2$, one can use the sequence introduced in (3), but since now $\tilde{C}_2$ is of length one, we take $\mathcal{S}'_2(C_2)$ instead of $\mathcal{S}_2(C_2)$. (See Fig. 3.)

For $\tilde{C}_3$, we take $\mathcal{S}_1(C_3)$ which is of the form

$$\mathcal{S}_1(C_3) := (e_2 \otimes e + e_2 \otimes x \xleftarrow{\beta} e_2 \otimes e \xleftarrow{\alpha} e_3 \otimes x + e_3 \otimes e \xleftarrow{\beta} e_3 \otimes x).$$

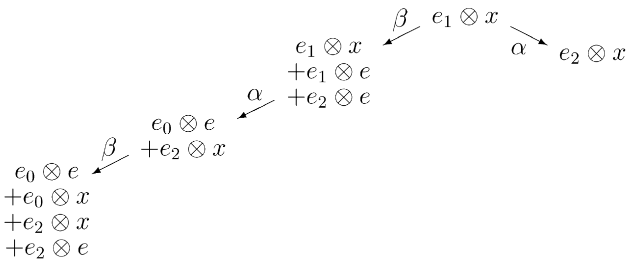



**Fig. 3.** Temporary choice for $\tilde{C}_1\tilde{C}_2$.

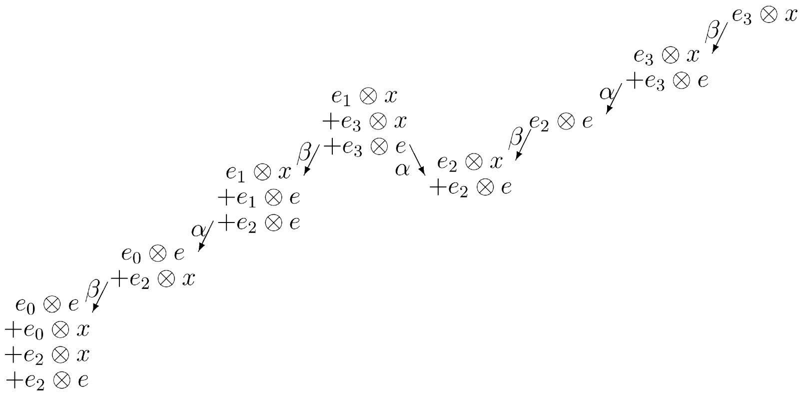



**Fig. 4.** $\alpha_0^{-1}\beta_0\alpha_0^{-1}$.

Notice that the last term of $\mathcal{S}_2'(C_2)$ is different from the first term of $\mathcal{S}_1(C_3)$. Since the difference is just the second term $e_2 \otimes e$ of $\mathcal{S}_1(C_3)$, one needs to add $\mathcal{S}_1'(C_3)^{-1}$ to $\mathcal{S}_2'(C_2)$ where

$$\mathcal{S}_1'(C_3) := (e_2 \otimes e \stackrel{\alpha}{\leftarrow} e_3 \otimes x + e_3 \otimes e \stackrel{\beta}{\leftarrow} e_3 \otimes x)$$

and

$$\mathcal{S}_1'(C_3)^{-1} := (e_3 \otimes x \stackrel{\beta}{\rightarrow} e_3 \otimes x + e_3 \otimes e \stackrel{\alpha}{\rightarrow} e_2 \otimes e).$$

But since $\mathcal{S}_1'(C_3)^{-1}$ is longer than $\mathcal{S}_2'(C_2)$, we just take a part of $\mathcal{S}_1'(C_3)^{-1}$, that is

$$\overline{\mathcal{S}_1'(C_3)}^{-1} := (e_3 \otimes x + e_3 \otimes e \stackrel{\alpha}{\rightarrow} e_2 \otimes e).$$

Finally we obtain Fig. 4.

Notice that $(e_3 \otimes x + e_3 \otimes e)\beta = 0$ and we do not need to change the terms of $\mathcal{S}_1(C_1)$ except the last one. One verifies that it establishes the desired isomorphism $\mathrm{Ind}_{T_0}^{D_8} M(C) \cong M(\varphi(C))$.

## 5. Proof of Proposition 3.10

Before giving the proof of Proposition 3.10, let us consider in detail the structure of the bands $C_m$.

For $m = 2$ (see Fig. 5), denote by $C_2'$ the part with boundary $---$ which is $\alpha^{-1}\beta^{-1}\alpha$, by $C_2^{(1)}$ the part with boundary ⋯⋯⋯ which is $\alpha^{-1}$ and by $C_2^{(2)}$ the part with boundary ——— which is $\alpha$. We see that $(C_2^{(1)})^{-1}$ is equal to $C_2^{(2)}$ as strings (in fact $\alpha$). We then have $C_2 = \beta\alpha\beta C_2^{(1)}\beta^{-1}C_2^{(2)}\beta^{-1}\alpha^{-1}$.

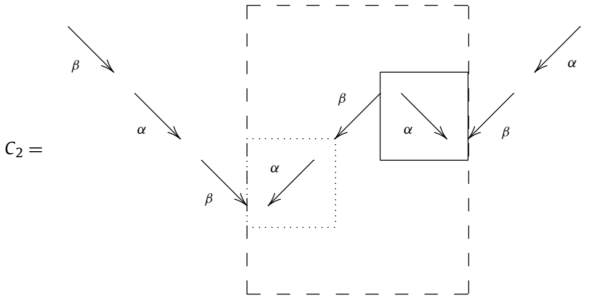


**Fig. 5.** $C_2$.

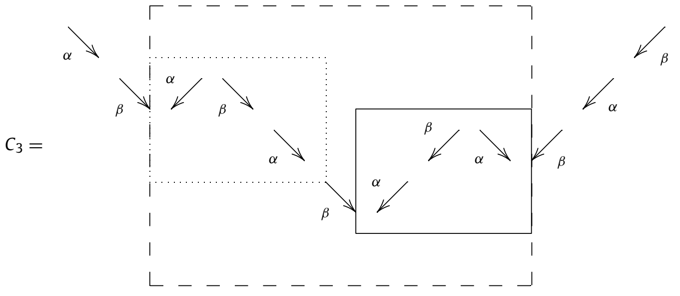


**Fig. 6.** $C_3$.

For $m = 3$, denote by $C_3'$ the part with boundary — — —, by $C_3^{(1)}$ the part with boundary ⋯⋯⋯ and by $C_3^{(2)}$ the part with boundary ———. We see easily (Fig. 6) that $(C_3^{(2)})^{-1}$ is equal to $C_3^{(1)}$ as strings (in fact, $\alpha^{-1}\beta\alpha$). We then have $C_3 = \alpha\beta C_3^{(1)}\beta C_3^{(2)}\beta^{-1}\alpha^{-1}\beta^{-1}$.

If $m$ is even and $m \geqslant 4$, we have $C_m = \beta\alpha\beta C_m^{(1)}\beta^{-1}C_m^{(2)}\beta^{-1}\alpha^{-1}$ with $(C_m^{(1)})^{-1} = C_m^{(2)}$. In fact, by construction of $C_m$, we have

$$\begin{aligned} C_m &= \beta C_{m-1}\alpha\beta^{-1}\alpha^{-1} \\ &= \beta\alpha\beta\alpha^{-1}C_{m-2}\beta^{-1}\alpha\beta^{-1}\alpha^{-1} \\ &= \beta\alpha\beta\alpha^{-1}\beta\alpha\beta C_{m-2}^{(1)}\beta^{-1}C_{m-2}^{(2)}\beta^{-1}\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-1} \end{aligned}$$

where the third equality holds by induction hypothesis. We impose $C_m^{(1)} = \alpha^{-1}\beta\alpha\beta C_{m-2}^{(1)}$ and $C_m^{(2)} = C_{m-2}^{(2)}\beta^{-1}\alpha^{-1}\beta^{-1}\alpha$, then $C_m^{(1)} = (C_m^{(2)})^{-1}$. The situation can be illustrated by Fig. 7. In this diagram, $C_m'$ is the part with boundary — — —, $C_m^{(1)}$ is the part with boundary ⋯⋯⋯ and $C_m^{(2)}$ is the part with boundary ———. Notice that the string $C_2$ appears in the middle of this diagram (and also in the middle of all the diagrams which appear from now on and which contain $C_2$).

If $m$ is odd and $m \geqslant 5$, $C_m = \alpha\beta C_m^{(1)}\beta C_m^{(2)}\beta^{-1}\alpha^{-1}\beta^{-1}$ with $(C_m^{(2)})^{-1} = C_m^{(1)}$. This can be proved by induction similarly as above. The situation can be illustrated by Fig. 8. In this diagram, $C_m'$ is the part with boundary — — —, $C_m^{(1)}$ is the part with boundary ⋯⋯⋯ and $C_m^{(2)}$ is the part with boundary ———.

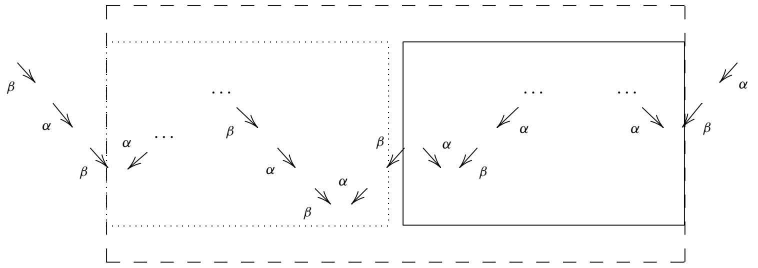


**Fig. 7.** $C_m$: $m$ is even and $m \geqslant 4$.

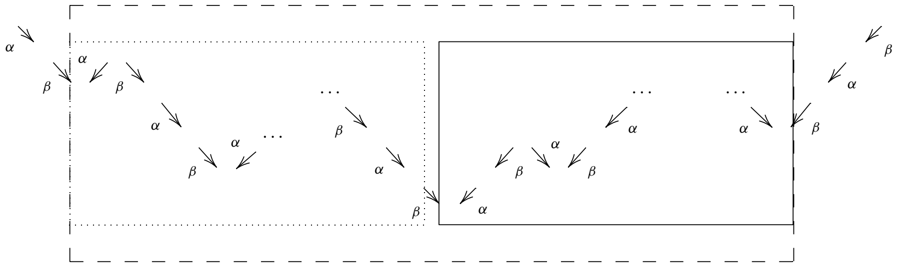


**Fig. 8.** $C_m$: $m$ is odd and $m \geqslant 5$.

We now begin the proof of Proposition 3.10.

Given

$$M\left(\alpha_0\beta_0^{-1}, m, 1\right) = \left(e_1 \underset{\beta_0 = J_m(1)}{\overset{\alpha_0 = \mathrm{Id}}{\rightrightarrows}} e_2\right)$$

where $e_1 = (e_{11}, \ldots, e_{1m})^t$ and $e_2 = (e_{21}, \ldots, e_{2m})^t$ and where Id is the identity matrix of size $m \times m$ and where $J_m(1)$ is the Jordan block. We have for all $1 \leqslant i \leqslant m$, $e_{1i}\alpha_0 = e_{2i}$, $e_{2i}\alpha_0 = 0$, $e_{1i}\beta_0 = e_{2i} + e_{2,i-1}$ and $e_{2i}\beta_0 = 0$ where we use the convention that $e_{2,0} = 0$. The induced module $\mathrm{Ind}_{T_0}^G M(\alpha_0\beta_0^{-1}, m, 1)$ is

$$\left(\bigoplus_{i=1}^m ke_{1i} \otimes e\right) \oplus \left(\bigoplus_{i=1}^m ke_{1i} \otimes x\right) \oplus \left(\bigoplus_{i=1}^m ke_{2i} \otimes e\right) \oplus \left(\bigoplus_{i=1}^m ke_{2i} \otimes x\right).$$

Direct calculations give that for all $1 \leqslant i \leqslant m$, $(e_{1i} \otimes e)\alpha = e_{2i} \otimes e$, $(e_{1i} \otimes x)\alpha = e_{2i} \otimes x + e_{2,i-1} \otimes x$, $(e_{2i} \otimes e)\alpha = 0$, $(e_{2i} \otimes x)\alpha = 0$, $(e_{1i} \otimes e)\beta = e_{1i} \otimes e + e_{1i} \otimes x + e_{2i} \otimes x + e_{2,i-1} \otimes x$, $(e_{1i} \otimes x)\beta = e_{1i} \otimes x + e_{1i} \otimes e + e_{2i} \otimes e + e_{2,i-1} \otimes e$, $(e_{2i} \otimes e)\beta = e_{2i} \otimes e + e_{2i} \otimes x$ and $(e_{2i} \otimes x)\beta = e_{2i} \otimes x + e_{2i} \otimes e$.

To prove Proposition 3.10, we shall construct an explicit basis for the induced module $\mathrm{Ind}_{T_0}^G M(\alpha_0\beta_0^{-1}, m, 1)$ which will establishes the isomorphism

$$\mathrm{Ind}_{T_0}^G M\left(\alpha_0\beta_0^{-1}, m, 1\right) \cong M(C_m, 1, 1)$$

for each $m \in \mathbb{N}_1$. Our method is by induction. The cases $m = 1, 2, 3$ are given below and serve as the base of the induction process.

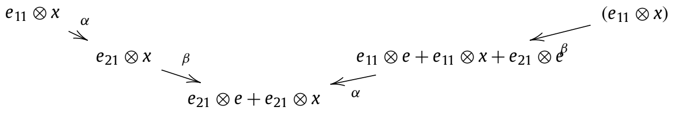



**Fig. 9.** Case $m = 1$.

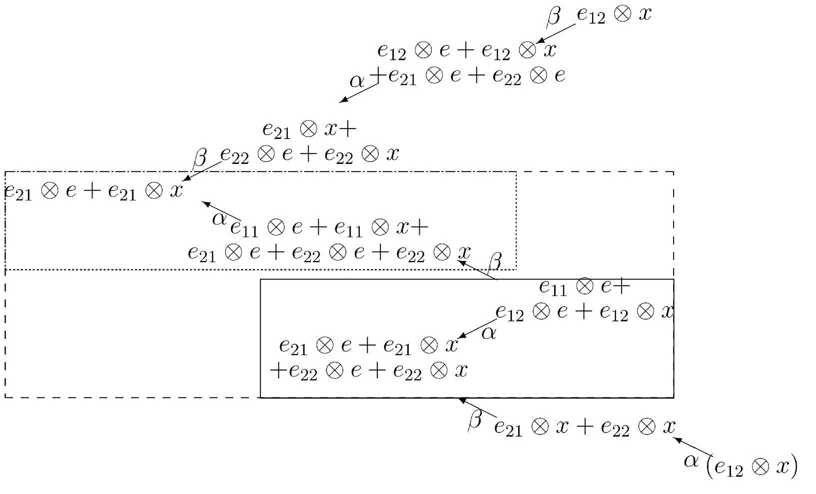



**Fig. 10.** Case $m = 2$.

*Case* $m = 1$. In this diagram (Fig. 9), the element in each position is given and it is easy to see that they form a basis of $\mathrm{Ind}_{T_0}^G M(\alpha_0\beta_0^{-1}, 1, 1)$ (of course, we have to delete one $e_{11} \otimes x$, since we present bands in form of strings. We place the element to delete in parenthesis and we will do this from now on). This diagram gives the desired isomorphism

$$\mathrm{Ind}_{T_0}^G M\big(\alpha_0\beta_0^{-1}, 1, 1\big) \cong M(C_1, 1, 1).$$

*Case* $m = 2$. As in the case $m = 1$, this diagram (Fig. 10, p. 1679) implies the isomorphism

$$\mathrm{Ind}_{T_0}^G M\big(\alpha_0\beta_0^{-1}, 2, 1\big) \cong M(C_2, 1, 1)$$

(notice that we have turned the diagram of 90 degrees in the clockwise direction and we will do this for all diagrams which appear from now on). Remark that the part with boundary $- - -$ is $C_2'$, the part with boundary ············ is $C_2^{(1)}$ and the part with boundary ——— is $C_2^{(2)}$. Since as strings, $(C_2^{(1)})^{-1}$ is equal to $C_2^{(2)}$, if in the diagram $C_2'$, we add to the position of $C_2^{(2)}$ the diagram $(C_2^{(1)})^{-1}$ (with the elements already given in $(C_2^{(1)})^{-1}$), then the diagram $C_2'$ becomes the following diagram (Fig. 11, p. 1680), denoted by $\tilde{C}_2'$,

*Case* $m = 3$. We see easily that the following diagram (Fig. 12, see p. 1680) gives the desired isomorphism. Remark that the part in the box, which is equal to $C_2'$ as strings, is exactly the diagram $\tilde{C}_2'$.

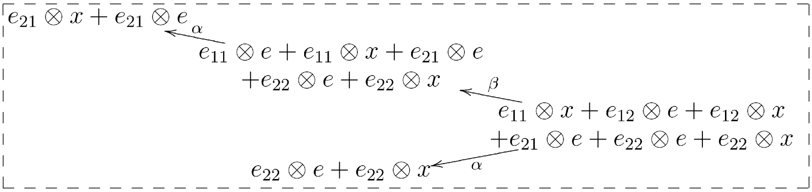



**Fig. 11.** $\tilde{C}_2'$.

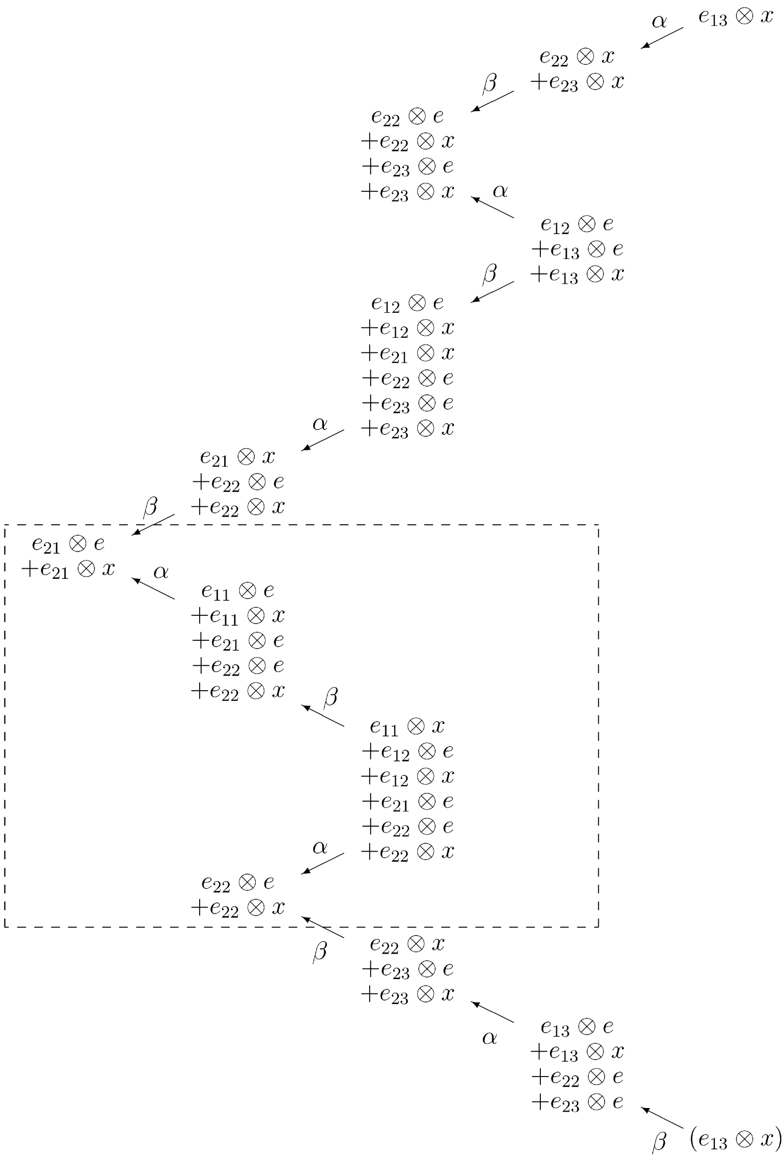



**Fig. 12.** Case $m = 3$.

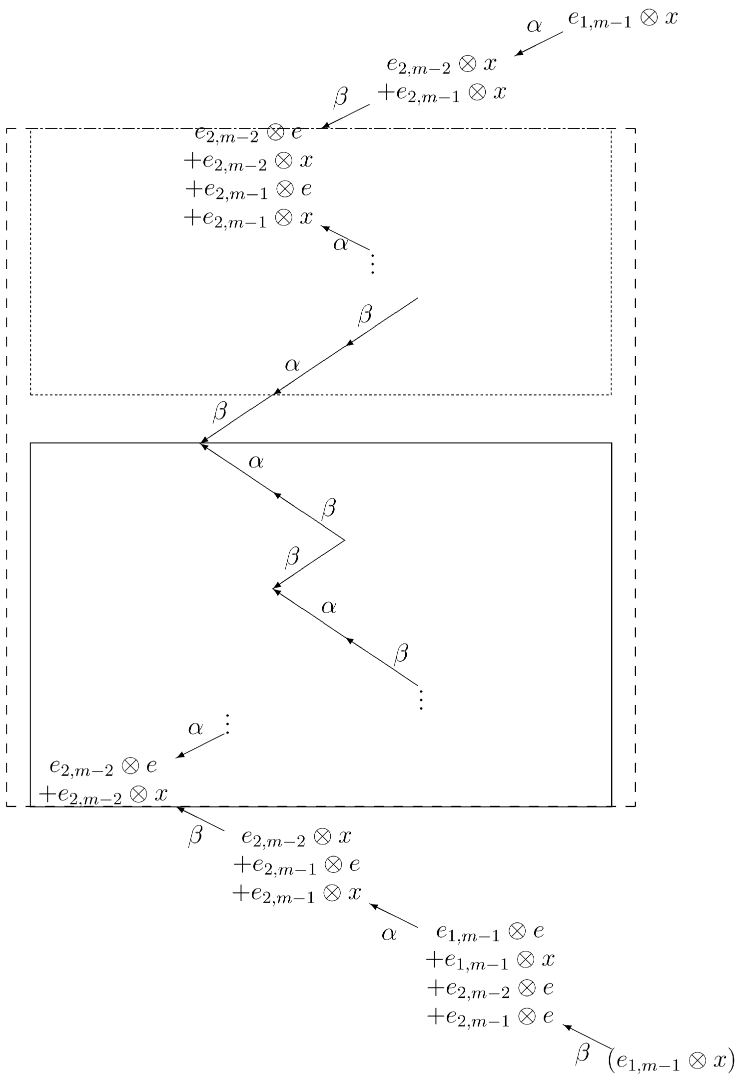



**Fig. 13.** Induction hypothesis: $m-1$ is odd and $m-1 \geqslant 3$.

The induction hypothesis for $m-1 \geqslant 2$ is the following:

(1) $\mathrm{Ind}_{T_0}^{G} M(\alpha_0\beta_0^{-1}, m-1, 1) \cong M(C_{m-1}, 1, 1)$.
(2) There exists a basis of $\mathrm{Ind}_{T_0}^{G} M(\alpha_0\beta_0^{-1}, m-1, 1)$ which gives the isomorphism and which contains the elements already given in the following diagrams:
   (i) If $m-1$ is odd and $m-1 \geqslant 3$, the basis of $\mathrm{Ind}_{T_0}^{G} M(\alpha_0\beta_0^{-1}, m-1, 1)$ is of the form (Fig. 13, p. 1681).

The part with boundary ··········· is $C_{m-1}^{(1)}$, the part with boundary ——— is $C_{m-1}^{(2)}$ and the part with boundary $---$ is $C'_{m-1}$. Since as strings, $(C_{m-1}^{(2)})^{-1}$ is equal to $C_{m-1}^{(1)}$, if we add in $C'_{m-1}$ to the position of $C_{m-1}^{(1)}$ the diagram $(C_{m-1}^{(2)})^{-1}$ (with the given elements), then the diagram $C'_{m-1}$ becomes

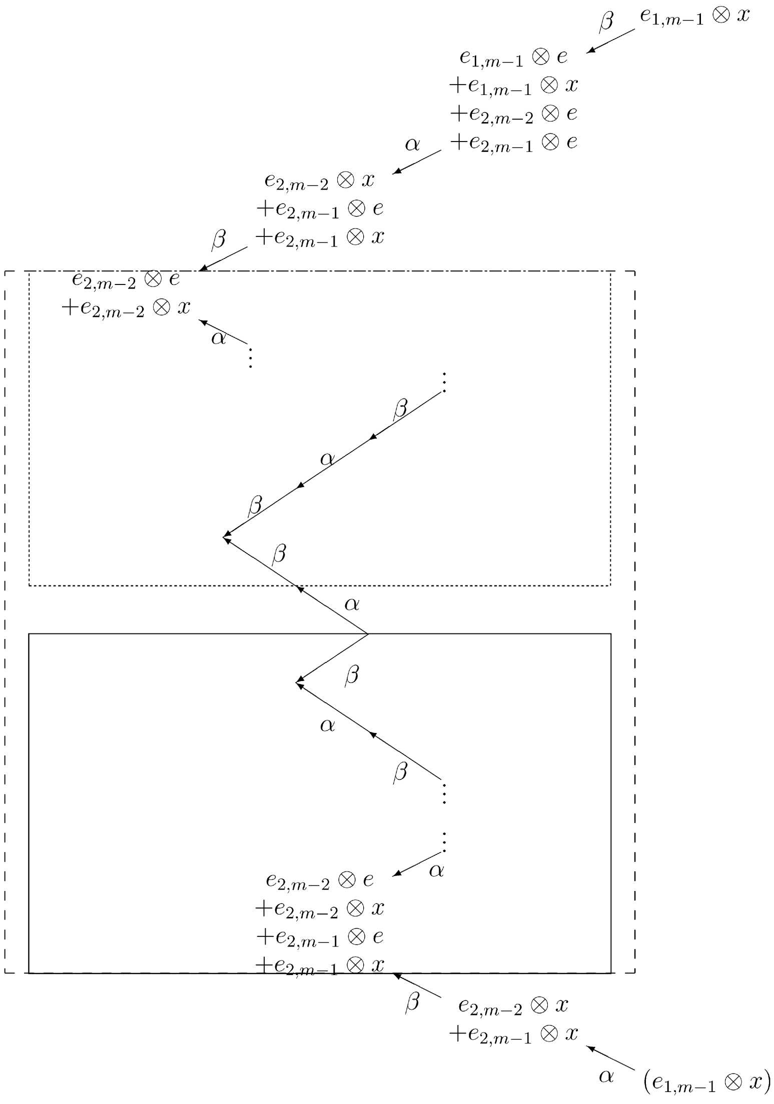



**Fig. 14.** Induction hypothesis: $m-1$ is even and $m-1 \geqslant 4$.

a diagram, denoted by $\tilde{C}'_{m-1}$, whose 'highest' element is $e_{2,m-1} \otimes e + e_{2,m-1} \otimes x$ and whose 'lowest' element is $e_{2,m-2} \otimes e + e_{2,m-2} \otimes x$.

(ii) If $m-1$ is even and $m-1 \geqslant 2$, the basis of $\mathrm{Ind}_{T_0}^{G} M(\alpha_0 \beta_0^{-1}, m-1, 1)$ is of the form (Fig. 14, p. 1682).

The part with boundary ············ is $C^{(1)}_{m-1}$, the part with boundary ——— is $C^{(2)}_{m-1}$ and the part with boundary $---$ is $C'_{m-1}$. Since as strings, $(C^{(1)}_{m-1})^{-1}$ is equal to $C^{(2)}_{m-1}$, if we add in $C'_{m-1}$ to the position of $C^{(2)}_{m-1}$ the diagram $(C^{(1)}_{m-1})^{-1}$ (with the given elements), then the diagram $C'_{m-1}$ becomes a diagram, denoted by $\tilde{C}'_{m-1}$, whose 'highest' element is $e_{2,m-2} \otimes e + e_{2,m-2} \otimes x$ and whose 'lowest' element is $e_{2,m-1} \otimes e + e_{2,m-1} \otimes x$.

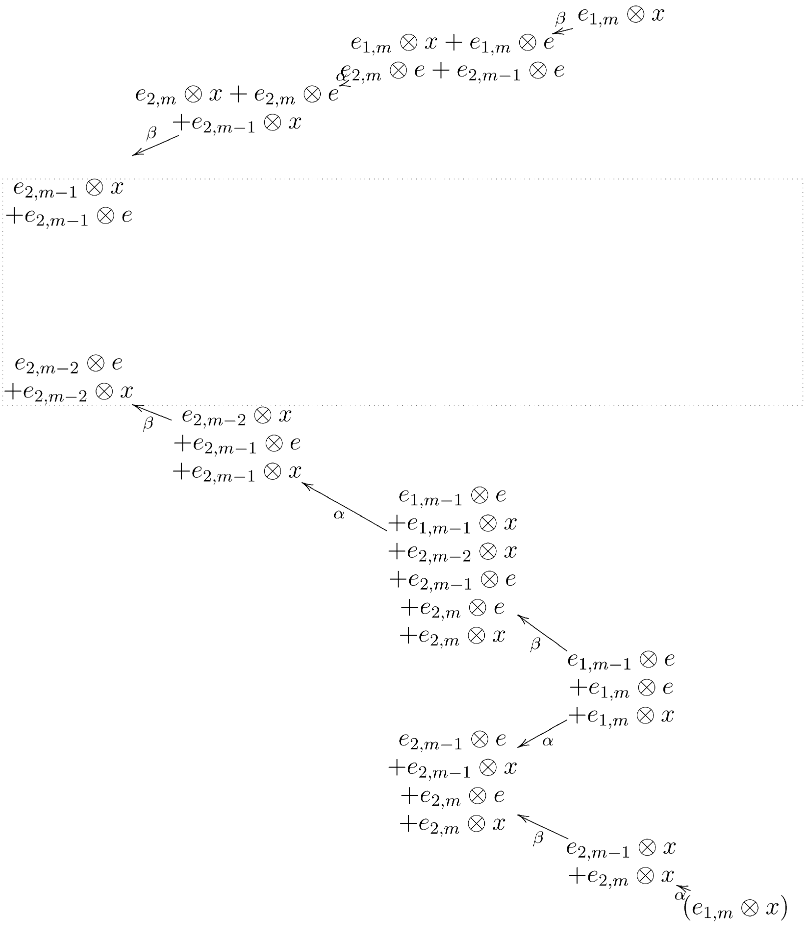


**Fig. 15.** Incomplete diagram: $m$ is even and $m \geqslant 4$.

This finishes the statement of the induction hypothesis. On verifies that the induction hypothesis holds for $m-1=2,3$.

We now construct the diagram $C_m$ which establishes the desired isomorphism.

If $m$ is even and $m \geqslant 4$, at first we construct an incomplete diagram (Fig. 15, p. 1683) which is $C_m$ as a string and which contains some given elements. Since the empty box is equal to $C'_{m-1}$ as a string, we replace the box by the diagram $\tilde{C}'_{m-1}$ constructed in the induction hypothesis (2)(i) and it is easy to see that $\tilde{C}'_{m-1}$ glues with the elements already given. We verify that the complete diagram constructed above gives the desired isomorphism $\mathrm{Ind}_{T_0}^{G} M(\alpha_0\beta_0^{-1}, m, 1) \cong M(C_m, 1, 1)$ and thus satisfies the induction hypothesis for $m$.

If $m$ is odd and $m \geqslant 5$, as above we construct an incomplete diagram (Fig. 16, p. 1684) which is $C_m$ as a string and which contains some given elements. Since the empty box is equal to $C'_{m-1}$ as a string, we replace the box by the diagram $\tilde{C}'_{m-1}$ constructed in the induction hypothesis (2)(ii) and it is easy to see that $\tilde{C}'_{m-1}$ glues with the elements already given. We verify that the complete diagram constructed above gives the desired isomorphism $\mathrm{Ind}_{T_0}^{G} M(\alpha_0\beta_0^{-1}, m, 1) \cong M(C_m, 1, 1)$ and thus satisfies the induction hypothesis for $m$.

This finishes the proof.

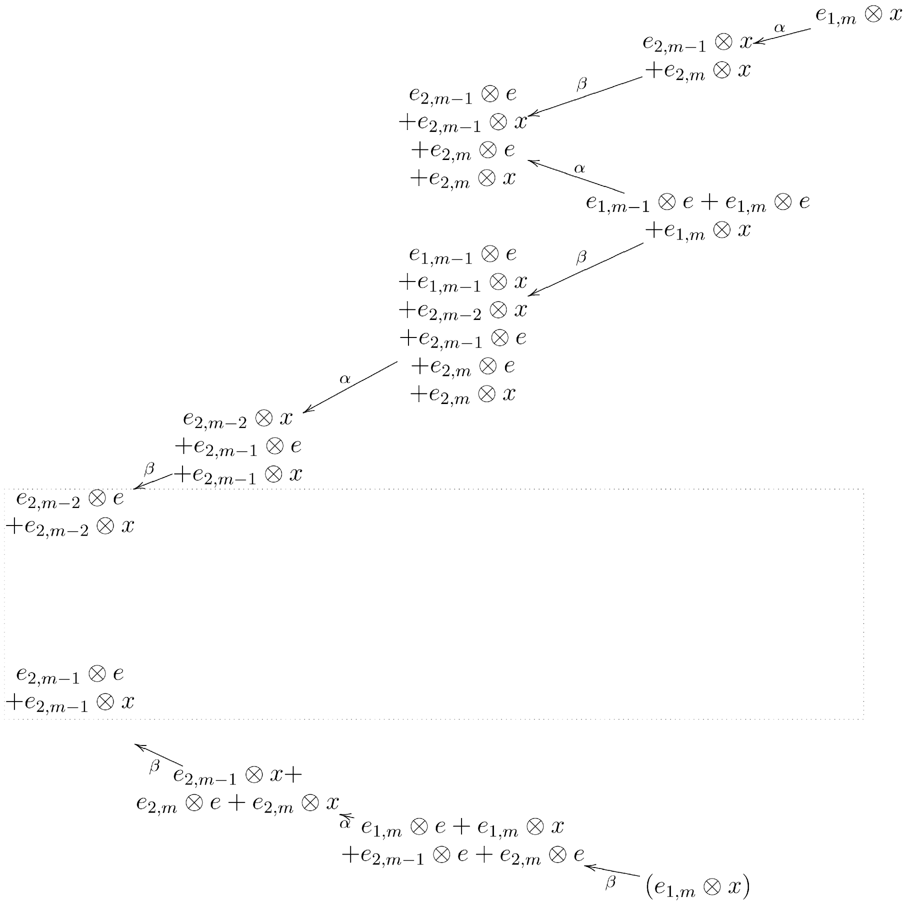



**Fig. 16.** Incomplete diagram: $m$ is odd and $m \geqslant 5$.

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## References

[1] J.L. Alperin, Local Representation Theory, Cambridge Stud. Adv. Math., vol. 11, Cambridge University Press, 1986.
[2] M. Auslander, I. Reiten, S. Smalø, Representation Theory of Artin Algebras, corrected reprint of the 1995 original, Cambridge Stud. Adv. Math., vol. 36, Cambridge University Press, 1997.
[3] D.J. Benson, Representations and Cohomology I: Basic Representation Theory of Finite Groups and Associative Algebras, Cambridge Stud. Adv. Math., vol. 30, Cambridge University Press, 1991.
[4] V.M. Bondarenko, Yu.A. Drozd, The representation type of finite groups. Modules and representations, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 71 (1977) 24–41 (in Russian).
[5] E.C. Dade, Blocks with cyclic defect groups, Ann. of Math. 84 (1966) 20–48.
[6] K. Erdmann, Principal blocks of groups with dihedral Sylow groups, Comm. Algebra 5 (1977) 665–694.
[7] K. Erdmann, On modules with cyclic vertices in the Auslander–Reiten quiver, J. Algebra 104 (1986) 289–300.
[8] K. Erdmann, On the vertices of modules in the Auslander–Reiten quiver of $p$-groups, Math. Z. 203 (1990) 321–334.

[9] K. Erdmann, Blocks of Tame Representations Type and Related Algebras, Lecture Notes in Math., vol. 1428, Springer-Verlag, Berlin, 1990.
[10] P. Gabriel, Unzerlegbare Darstellungen I, Manuscripta Math. 6 (1972) 71–103.
[11] I.M. Gelfand, V.A. Ponomarev, Indecomposable representations of the Lorentz group, Uspekhi Mat. Nauk 23 (1968) 3–60.
[12] W. Hamernik, G.O. Michler, On vertices of simple modules in $p$-solvable groups, Mitt. Math. Sem. Giessen Heft 121 (1976) 147–162.
[13] G. Janusz, Indecomposable modules for finite groups, Ann. of Math. 89 (1969) 209–241.
[14] S. Kawata, The Green correspondence and Auslander–Reiten sequences, J. Algebra 123 (1989) 1–5.
[15] S. Kawata, Module correspondence in Auslander–Reiten quivers for finite groups, Osaka J. Math. 26 (3) (1989) 671–678.
[16] S. Kawata, The modules induced from a normal subgroup and the Auslander–Reiten quiver, Osaka J. Math. 27 (1990) 265–269.
[17] R. Knörr, On the vertices of irreducible modules, Ann. of Math. (2) 110 (1979) 487–499.
[18] P. Landrock, G.O. Michler, Block structure of the smallest Janko group, Math. Ann. 232 (1978) 205–238.
[19] G.O. Michler, Green correspondence between blocks with cyclic defect groups, I, J. Algebra 39 (1976) 26–51.
[20] G.O. Michler, Green correspondence between blocks with cyclic defect groups. II. Representations of algebras, in: Proc. Internat. Conf. Carleton Univ., Ottawa, Ontario, 1974, in: Lecture Notes in Math., vol. 488, Springer, Berlin, 1975, pp. 210–235.
[21] J. Müller, R. Zimmermann, Green vertices and sources of simple modules of the symmetric group labelled by hook partitions, Arch. Math. (Basel) 89 (2007) 97–108.
[22] R.M. Peacock, Blocks with a cyclic defect group, J. Algebra 34 (1975) 232–259.
[23] C.M. Ringel, The indecomposable representations of the dihedral 2-groups, Math. Ann. 21 (1975) 19–34.
[24] T. Okuyama, K. Uno, On the vertices of modules in the Auslander–Reiten quiver II, Math. Z. 217 (1994) 121–141.
[25] K. Uno, On the vertices of modules in the Auslander–Reiten quiver, Math. Z. 208 (1991) 411–436.
[26] P. Webb, The Auslander–Reiten quiver of a finite group, Math. Z. 179 (1982) 97–121.
[27] M. Wildon, Two theorems on the vertices of Specht modules, Arch. Math. (Basel) 81 (2003) 505–511.
[28] G. Zhou, Algèbres courtoises et blocs à défaut diédral, PhD thesis, Université de Picardie Jules Verne, June 2007.