Beitr Algebra Geom (2012) 53:349–364
DOI 10.1007/s13366-011-0036-8

Auslander–Reiten conjecture for symmetric algebras of polynomial growth

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Received: 28 September 2010 / Accepted: 22 December 2010 / Published online: 6 May 2011
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Abstract This paper studies self-injective algebras of polynomial growth. We prove that two indecomposable weakly symmetric algebras of domestic type are derived equivalent if and only if they are stably equivalent. Furthermore we prove that for indecomposable weakly symmetric non-domestic algebras of polynomial growth, up to some scalar problems, the derived equivalence classification coincides with the classification up to stable equivalences of Morita type. As a consequence, we get the validity of the Auslander–Reiten conjecture for stable equivalences between weakly symmetric algebras of domestic type and for stable equivalences of Morita type between weakly symmetric algebras of polynomial growth.

Keywords Auslander–Reiten conjecture · Derived equivalence · Reynolds ideal · Self-injective algebra of polynomial growth · Stable equivalence · Stable equivalence of Morita type

Mathematics Subject Classification (2000) 16G10 · 16G60 · 16E40

G. Zhou was supported by DFG SPP 1388. Part of this work was done during the visit of G. Zhou to the Univerisité de Picardie Jules Verne in March 2010. G. Zhou would like to thank A. Zimmermann for his kind hospitality.

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0 Introduction

Drozd (1980) showed that finite dimensional algebras A over an algebraically closed field K are either of finite representation type, or of tame representation type or of wild representation type. Here, an algebra is of finite representation type if there are only a finite number of isomorphism classes of indecomposable modules. An algebra A is of wild representation type if for every algebra B there is an exact functor $B-mod \longrightarrow A-mod$ that respects isomorphism classes and carries indecomposable modules to indecomposable modules. An algebra is of tame representation type if for every positive integer d there are $A - K[X]$-bimodules $M_i(d)$ for $i \in \{1, \ldots, n_d\}$ so that $M_i(d)$ is finitely generated free as $K[X]$-module and so that all but a finite number of isomorphism classes of indecomposable d-dimensional A-modules are isomorphic to $M_i(d) \otimes_{K[X]} K[X]/(X - \lambda)$ for $\lambda \in K$ and $i \in \{1, \ldots, n_d\}$. If n_d is minimal with respect to d satisfying this property, A is of polynomial growth if there is an integer m so that $\lim_{d\to\infty} \frac{n_d}{d^m} = 0$, and A is domestic if there is an integer m so that $n_d \leq m$ for all d.

In the last three decades, Skowroński and his co-authors made much progress concerning the classification of tame self-injective algebras (see the surveys Skowroński 2006; Erdmann and Skowroński 2008). In this article we are interested in the classification of symmetric algebras of polynomial growth. Bocian et al. (2004, 2007) and Holm and Skowroński (2006) classified all weakly symmetric algebras of domestic type up to derived equivalence and Białkowski et al. (2003a,b) and Holm and Skowroński (2011) did the same thing for all weakly symmetric non-domestic algebras of polynomial growth. The result is a finite number of families of algebras given by quivers and relations, depending on certain parameters. In the case of domestic algebras the classification is complete, whereas in case of weakly symmetric algebras of polynomial growth it is not known if certain choices of parameters may lead to the same derived equivalence class. The question of whether certain parameters lead to derived equivalent algebras will be called the scalar problem in the sequel.

The main result of this paper is that the derived equivalence classification of weakly symmetric algebras of polynomial growth coincides (up to some scalar problems) with the classification up to stable equivalences of Morita type as given in Bocian et al. (2004, 2007), Holm and Skowroński (2006), Białkowski et al. (2003a,b) and Holm and Skowroński (2011). As a consequence, the Auslander–Reiten conjecture holds for a stable equivalence of Morita type between two weakly symmetric algebras of polynomial growth. We should mention that we build largely on the derived equivalence classification, which is actually the most important step. Nevertheless, the stable equivalence classification was still open.

For weakly symmetric algebras of domestic type we can show more. We prove that the derived equivalence classification coincides with the classification up to stable equivalences (no scalar problem occurs in this case) and that the Auslander–Reiten conjecture holds for stable equivalences between weakly symmetric algebras of domestic types.

As mentioned above the classification of symmetric algebras of polynomial growth up to derived equivalence leaves open if for certain choices of parameters the algebras are derived equivalent or not. The same questions concerning the same choices of parameters remain open also for the classification up to stable equivalences of Morita type.

Moreover, using our result (Zhou and Zimmermann 2010), we show that for tame symmetric algebras with periodic modules, the derived equivalence classification coincides with the classification up to stable equivalence of Morita type, up to some scalar problem, and that the Auslander–Reiten conjecture holds for stable equivalences of Morita type between tame symmetric algebras with periodic modules (that is, algebras so that all modules without projective direct summands are periodic with respect to the action of the syzygy operator).

The paper is organised as follows. In Sect. 1 we recall basic definitions and recall the stable invariants we use in the sequel. In Sect. 2 we prove the main result Theorem 2.5 for weakly symmetric algebras of domestic representation type and in Sect. 3 we show the main result Theorem 3.5 for stable equivalences of Morita type for weakly symmetric algebras of polynomial growth. For the convenience of the reader we present the algebras occurring in the classification in the Appendix (Sect. 5).

As soon as we use statements on representation type we shall always assume that K is an algebraically closed field. An algebra is always assumed to be indecomposable.

1 Stable categories; definitions, notations and invariants

1.1 The definitions

Let K be a field and let A be a finite dimensional K-algebra. The stable category $A-\underline{mod}$ has as objects all finite dimensional A-modules and morphisms from an A-module X to another A-module Y are the equivalence classes of A-linear homomorphisms $Hom_A(X,Y)$, where two morphisms of A-modules are declared to be equivalent if their difference factors through a projective A-module.

Two algebras are called *stably equivalent* if $A-\underline{mod} \simeq B-\underline{mod}$ as additive categories. Let M be an A-B-bimodule and let N be a B-A-bimodule. Following Broué the couple (M,N) defines a *stable equivalence of Morita type* if

- M is projective as A-module and is projective as B-module,
- N is projective as B-module and is projective as A-module,
- $M \otimes_B N \simeq A \oplus P$ as A-A-bimodules, where P is a projective A-A-bimodule,
- $N \otimes_A M \simeq B \oplus Q$ as B-B-bimodules, where Q is a projective B-B-bimodule.

Of course, if (M,N) defines a stable equivalence of Morita type, then $M \otimes_B - : B-\underline{mod} \simeq A-\underline{mod}$ is an equivalence. However, it is known that there are algebras which are stably equivalent but for which there is no stable equivalence of Morita type.

1.2 Invariants under stable equivalences

Auslander and Reiten conjectured that if A and B are stably equivalent, then the number of isomorphism classes of non-projective simple A-modules is equal to the number of isomorphism classes of non-projective simple B-modules. The conjecture is open in general, but we shall use in this article the following positive result (Pogorzały 1994) of Pogorzały: if A is stably equivalent to B and if A is self-injective special biserial that is not Nakayama, then B is self-injective special biserial as well and the Auslander–Reiten conjecture holds in this case.

Krause and Zwara (2000) showed that stable equivalences preserve the representation type. Suppose A and B are two finite dimensional stably equivalent K-algebras. If A is of domestic representation type (respectively of polynomial growth, respectively of tame representation type), then so is B.

Reiten (1976) shows that being self-injective is almost invariant under stable equivalences. More precisely, let A be a self-injective finite dimensional K-algebra of Loewy length at least three which is stably equivalent to a finite dimensional indecomposable K-algebra B. Then B is self-injective as well.

Finally, if A and B are stably equivalent finite dimensional indecomposable algebras with Loewy length at least three, then the stable Auslander–Reiten quivers are isomorphic as translation quivers.

1.3 Invariants under stable equivalence of Morita type

Much more is known for stable equivalences of Morita type. We shall cite only those properties used in the sequel.

Xi (2008) has shown that if two finite dimensional K-algebras A and B are stably equivalent of Morita type, then the absolute values of elementary divisors, including their multiplies, of the Cartan matrices different from ± 1 coincide for A and for B, and therefore also the absolute value of the Cartan determinants are equal.

Liu (2008) showed that if A is an indecomposable finite dimensional K-algebra, and B is a finite dimensional K-algebra stably equivalent of Morita type to A, then B is indecomposable as well.

If K is an algebraically closed field of characteristic $p > 0$ and A a finite dimensional K-algebra, define $[A, A]$ the K-space generated by all elements $ab - ba \in A$ for all $a, b \in A$. Then,

$$\{x \in A \mid x^{p^n} \in [A, A]\} =: T_n(A)$$

is a K-subspace of A containing $[A, A]$. Liu et al. (2011) have shown that whenever A and B are stably equivalent of Morita type, then

$$dim_K(T_n(A)/[A, A]) = dim_K(T_n(B)/[B, B])$$

for all $n \in \mathbb{N}$. If A is symmetric, since $[A, A]^\perp = Z(A)$ which is the centre of A, taking the orthogonal space with respect to the symmetrising form gives an decreasing sequence of ideals $T_n(A)^\perp$ of $Z(A)$

$$Z(A) = T_0(A)^\perp \supseteq T_1(A)^\perp \supseteq T_2(A)^\perp \supseteq \cdots \supseteq R(A)$$

whose intersection is the Reynolds ideal $R(A) = \mathrm{Soc}(A) \cap Z(A)$, where $\mathrm{Soc}(A)$ is the socle of A. These ideals are called Külshammer ideals. If A and B are symmetric and stably equivalent of Morita type, then König et al. (2009) have shown that $Z(A)/T_n(A)^{\perp} \simeq Z(B)/T_n(B)^{\perp}$ as rings, in particular, $Z(A)/R(A) \simeq Z(B)/R(B)$.

2 Weakly symmetric algebras of domestic representation type

Self-injective algebras of domestic type split into two subclasses: standard algebras and non-standard ones.

A self-injective algebra of tame representation type is called *standard* if its basic algebra admit simply connected Galois coverings. A self-injective algebra of tame representation type is called *non-standard* if it is not standard.

2.1 The Bocian–Holm–Skowroński derived equivalence classification of domestic weakly symmetric algebras and immediate consequences

The derived equivalence classification of Bocian–Holm–Skowroński is given in the following.

Theorem 2.1 • (Bocian et al. 2004, Theorem 1) *For an algebra A the following statements are equivalent*:

(1) *A is representation-infinite domestic weakly symmetric standard algebra and the Cartan matrix C_A is singular.*
(2) *A is derived equivalent to the trivial extension $T(C)$ of a canonical algebra C of Euclidean type.*
(3) *A is stably equivalent to the trivial extension $T(C)$ of a canonical algebra C of Euclidean type.*

Moreover, the trivial extensions $T(C)$ and $T(C')$ of two canonical algebras C and C' of Euclidean type are derived equivalent (*respectively, stably equivalent*) *if and only if the algebras C and C' are isomorphic.*

- (Bocian et al. 2004, Theorem 2) *For a domestic standard self-injective algebra A, the following statements are equivalent:*
  (1) *A is weakly symmetric and the Cartan matrix C_A is non-singular.*
  (2) *A is derived equivalent to an algebra of the form $A(\lambda)$, $A(p,q)$, $\Lambda(n)$, $\Gamma(n)$.*
  (3) *A is stably equivalent to an algebra of the form $A(\lambda)$, $A(p,q)$, $\Lambda(n)$, $\Gamma(n)$.*
  Moreover, two algebras of the forms $A(\lambda)$, $A(p,q)$, $\Lambda(n)$ or $\Gamma(n)$ are derived equivalent (*respectively, stably equivalent*) *if and only if they are isomorphic.*
- (Bocian et al. 2007, Theorem 1) *Any non-standard representation-infinite self-injective algebra of domestic type is derived equivalent* (*resp. stably equivalent*) *to an algebra $\Omega(n)$ with $n \geq 1$. Moreover, two algebras $\Omega(n)$ and $\Omega(m)$ are derived equivalent* (*respectively, stably equivalent*) *if and only if $m = n$.*
- (Holm and Skowroński 2006, Theorem 1.1) *A standard symmetric algebra of domestic representation type cannot be derived equivalent to a non-standard symmetric one.*

For the precise quiver with relations of the algebras appeared in the above theorem, we refer the reader to the Appendix.

From the first part of this theorem (Bocian et al. 2004, Theorem 1), we know that a standard weakly symmetric algebra of domestic representation type with *singular Cartan matrix* is symmetric. Note that a finite dimensional algebra which is derived equivalent to a symmetric algebra is itself symmetric Rickard (1991, Corollary 5.3).

The algebras $A(p,q)$, $\Lambda(n)$, $\Gamma(n)$ and $A(1)$ are symmetric, whereas

$$A(\lambda) = k\langle X, Y\rangle/(X^2, Y^2, XY - \lambda YX), \lambda \notin \{0, 1\}$$

is not symmetric. Remark that except $\Gamma(n)$, the algebras $A(\lambda)$, $A(p,q)$, $\Lambda(n)$ are all special biserial algebras.

We remark that the algebra $\Omega(n)$ is always weakly symmetric, but it is symmetric only when the characteristic of the base field is 2. Note that $\Omega(n)$ is not special biserial.

2.2 The stable equivalence classification of weakly symmetric domestic algebras

For the rest of the section we shall give a generalisation of Theorem 2.1 to stable equivalences.

Corollary 2.2 (1) *The class of indecomposable standard weakly symmetric algebras of domestic representation type with singular Cartan matrices is closed under stable equivalences.*

(2) *Two standard weakly symmetric algebras of domestic representation type are stably equivalent if and only if they are derived equivalent.*

Proof For the first statement, let A be an indecomposable algebra stably equivalent to an indecomposable standard weakly symmetric algebras of domestic representation type. Then by Theorem 2.1, A is stably equivalent to the trivial extension $T(C)$ of a canonical algebra C of Euclidean type. Again by Theorem 2.1, A is standard weakly symmetric algebras of domestic representation type.

The second statement follows from Theorem 2.1 and the first statement. □

Now we compare weakly symmetric standard domestic algebras with non-standard ones. In course of the proof of Holm and Skowroński (2006, Theorem 1.1) one needs to compare the algebras $\Omega(n)$ with $A(1,n)$ in case the characteristic of K is 2. It is proved in Holm and Skowroński (2006) that the dimensions of the centre modulo the first Külshammer ideal, for $\Omega(n)$ and $A(1,n)$, are different, but this dimension is in fact invariant under stable equivalences of Morita type (Liu et al. 2011). So a standard symmetric algebra of domestic representation type cannot be stably equivalent of Morita type to a non-standard symmetric one. In fact, we can even prove more.

Lemma 2.3 *A standard weakly symmetric algebra of domestic type cannot be stably equivalent to a non-standard one.*

Proof Stably equivalent algebras of Loewy length at least 3 have isomorphic stable Auslander–Reiten quivers (Auslander et al. 1995, Corollary X. 1.9). Notice that the stable Auslander–Reiten quiver of $A(\lambda)$ for $\lambda \neq 0$ consists of an Euclidean component of type $\mathbb{Z}\tilde{A}_1$ and a $\mathbb{P}_1(K)$-family of homogenous tubes. Since all algebras in the above list of Theorem 2.1 are of Loewy length at least 3, by comparing the shapes of the stable Auslander–Reiten quivers as in Holm and Skowroński (2006, Section 4), we know that a standard weakly symmetric algebra of domestic type A in the list of Theorem 2.1 is stably equivalent to a non-standard one $B = \Omega(n)$ only when $A = A(1, n)$ and $B = \Omega(n)$ for $n \geq 1$. However, notice that $A(1, n)$ is special biserial and $\Omega(n)$ is not. By the result of Pogorzały (1994, Theorem 7.3), they can never be stably equivalent.

Proposition 2.4 (1) *The following statements are equivalent for two symmetric algebras A and B of domestic representation type.*
- *A and B are derived equivalent.*
- *A and B are stably equivalent of Morita type.*
- *A and B are stably equivalent.*

(2) *The class of symmetric algebras of domestic representation type is closed under stable equivalences, hence is closed under stable equivalences of Morita type and derived equivalences.*

Proof Corollary 2.2 shows that for standard weakly symmetric algebras the notions of derived equivalence and of stable equivalence are the same. Lemma 2.3 shows that standard weakly symmetric algebras cannot be stably equivalent to non-standard weakly symmetric algebras. The third part of Theorem 2.1 shows that for non-standard domestic weakly symmetric algebras derived and stable equivalence are the same notion. This proves (1).

Now let A be an indecomposable algebra stably equivalent to an indecomposable symmetric algebra B of domestic representation type in the list of algebras in Theorem 2.1. Then by a result of Reiten (1976, Theorem 2.6) A itself is also self-injective or A is isomorphic to a radical square zero algebra whose quiver is of type A with linear orientation. But the latter algebra has finite representation type, and thus cannot be stably equivalent to a representation-infinite algebra. We infer that A is self-injective and now by Bocian et al. (2004, Proposition 5.2), A is weakly symmetric. By Krause and Zwara (2000, Corollary 2 and the discussions afterwards), A is of domestic type. We thus proved that A is weakly symmetric of domestic type.

To finish the proof of the second assertion, one needs to show that A is symmetric and thus to exclude the cases where A is stably equivalent to $A(\lambda)$ with $\lambda \in K\backslash\{0, 1\}$ or to $\Omega(n)$ with $n \geq 1$ in case of $char(K) \neq 2$, since these are the only cases where non-symmetric algebras may occur.

If A is stably equivalent to $A(\lambda)$ with $\lambda \in K\backslash\{0, 1\}$, then by Pogorzały (1994, Theorem 7.3), B is special biserial. Since the Auslander–Reiten conjecture is proved for self-injective special biserial algebras (Pogorzały 1994, Theorem 0.1), B is also local and is thus necessarily isomorphic to $A(1)$. However, $A(1)$ is never stably equivalent to $A(\lambda)$ with $\lambda \neq 1$ by an unpublished but well-known result of Rickard (see for instance, Auslander et al. 1995, Chapter X Exercise 3).

If $char(K) \neq 2$ and A is stably equivalent to $\Omega(n)$ with $n \geq 1$, then by Lemma 2.3, B is necessarily non-standard and $B = \Omega(n)$ for some n. But the fact that B is symmetric implies that $char(K) = 2$ which is a contradiction. □

In fact we can extend (at least partially) the above result to all weakly symmetric algebras of domestic representation type.

Theorem 2.5 (1) *The following statements are equivalent for two weakly symmetric algebras A and B of domestic representation type.*
- *A and B are derived equivalent.*
- *A and B are stably equivalent of Morita type.*
- *A and B are stably equivalent.*

(2) *The class of weakly symmetric algebras of standard domestic representation type is closed under stable equivalences, hence under stable equivalences of Morita type, and derived equivalences.*

Proof Since by Lemma 2.3, a standard weakly symmetric algebra of domestic type cannot be stably equivalent to a non-standard one, Theorem 2.1 implies the first assertion.

For the second assertion, let A be an indecomposable algebra stably equivalent to a weakly symmetric standard algebra B of domestic representation type in the list of algebras in Theorem 2.1. By Krause and Zwara (2000, Corollary 2 and the discussions afterwards), A is of domestic type. As in the proof of Proposition 2.4, A is still self-injective. Since by Proposition 2.4(2), the class of symmetric algebras of domestic type is closed under stable equivalences, one can assume that B is not symmetric. Now $B = A(\lambda)$ with $\lambda \notin \{0, 1\}$. Since $B = A(\lambda)$ is a local special biserial algebra, so is A by Pogorzały (1994, Theorems 0.1 and 7.3). The algebra A is thus weakly symmetric. □

Remark 2.6 (1) If the class of self-injective non-standard algebras of domestic type is closed under stable equivalences (hence under stably equivalences of Morita type and derived equivalences), so is the class of weakly symmetric algebras of domestic representation type.

(2) The second assertion of the above theorem answers a question raised in Bocian et al. (2004, Page 47).

As a consequence, the Auslander–Reiten conjecture holds for this class of algebras.

Corollary 2.7 (1) *Let A be an indecomposable algebra stably equivalent to an indecomposable symmetric algebra B of domestic type. Then A has the same number of simple modules as B.*

(2) *Let A be an indecomposable algebra stably equivalent to an indecomposable weakly symmetric standard algebra B of domestic type. Then A has the same number of simple modules as B.*

Remark 2.8 If the class of self-injective non-standard algebras of domestic type is closed under stable equivalences, then the Auslander–Reiten conjecture holds for a stable equivalence between two weakly symmetric algebras of domestic type.

3 Non-domestic self-injective algebras of polynomial growth

The derived equivalence classification of the standard (resp. non-standard) non-domestic symmetric algebras of polynomial growth is achieved in Białkowski et al. (2003a, Page 653 Theorem) (resp. Białkowski et al. 2003b, Theorem 3.1). There a complete list of representatives of derived equivalence classes in terms of quiver with relations is given. We reproduce some of them in the Appendix.

3.1 The Białkowski–Holm–Skowroński derived equivalence classification of weakly symmetric non-domestic polynomial growth algebras

We recall the results of Białkowski, Holm and Skowroński on the derived equivalence classification of weakly symmetric polynomial growth algebras.

Theorem 3.1 • (Białkowski et al. 2003a, Page 653 Theorem) *Let A be an indecomposable standard non-domestic weakly symmetric algebra of polynomial growth. Then A is derived equivalent to one of the following algebras:*
- *two simple modules:* Λ_2' *and* $\Lambda_3'(\lambda)$, $\lambda \in K\backslash\{0, 1\}$;
- *three simple modules:* Λ_5' *and* $A_1(\lambda)$, $\lambda \in K\backslash\{0, 1\}$;
- *four simple modules:* Λ_9' *and* A_4;
- *six simple modules:* $\Lambda(2, 2, 2, 2, \lambda)$, $\lambda \in K\backslash\{0, 1\}$;
- *eight simple modules:* $\Lambda(3, 3, 3)$;
- *nine simple modules:* $\Lambda(2, 4, 4)$;
- *ten simple modules:* $\Lambda(2, 3, 6)$.

• (Białkowski et al. 2003b, Theorem 3.1) *Let A be an indecomposable non-standard non-domestic self-injective algebra of polynomial growth. Then we get the following.*
- *If the base field is of characteristic* 3, *then A is derived equivalent to* Λ_2.
- *else the base field is of characteristic* 2 *and then*
  - *if A has two simple modules, A is derived equivalent to* $\Lambda_3(\lambda)$, $\lambda \in K\backslash\{0, 1\}$
  - *if A has three simple modules, A is derived equivalent to* Λ_5
  - *if A has four simple modules, A is derived equivalent to* Λ_9
  - *else A has five simple modules and then A is derived equivalent to* Λ_{10}

• (Holm and Skowroński 2011, Main Theorem) *Let A be a standard self-injective algebra of polynomial growth and let Λ be an indecomposable, non-standard, non-domestic, symmetric algebra of polynomial growth. Then A and Λ are not derived equivalent.*

If the base field is of characteristic 2 and $\lambda \neq \lambda'$, we do not know if $\Lambda_3(\lambda)$ is derived equivalent to $\Lambda_3(\lambda')$ or not. We call this question the scalar problem.

The above classification in the first part of the theorem is complete up to the scalar problems in $\Lambda_3'(\lambda)$, in $\Lambda_3(\lambda)$ and in $A_1(\lambda)$. Remark that except Λ_9' in case the characteristic of the base field K is different from 2, all algebras in the standard case are symmetric.

The above classification in the second part of the theorem is complete up to the scalar problem in case the base field is of characteristic 2 for the algebra $\Lambda_3(\lambda)$, $\lambda \in K\backslash\{0, 1\}$.

Remark that the algebra Λ_{10} is not symmetric (even not weakly symmetric), so it is not stably equivalent of Morita type to any other algebra in the above list.

3.2 The stable equivalence classification of weakly symmetric non-domestic polynomial growth algebras

In the sequel we shall give a generalisation of Theorem 3.1 to a classification up to stable equivalences of Morita type.

We shall prove first

Proposition 3.2 *The classification of indecomposable standard non-domestic weakly symmetric algebras of polynomial growth up to stable equivalences of Morita type coincide with the derived equivalence classification, modulo the scalar problems in* $\Lambda_3'(\lambda), \lambda \in K\backslash\{0,1\}$ *and in* $A_1(\lambda), \lambda \in K\backslash\{0,1\}$.

Proof Since a derived equivalence between self-injective algebras induces a stable equivalence of Morita type (Keller and Vossieck 1987; Rickard 1989), we only need to show that two algebras from the above list which are not derived equivalent are not stably equivalent of Morita type, either.

Since the property of being symmetric is invariant under a stable equivalence of Morita type (Liu 2008, Corollary 2.4), in case the characteristic of the base field is different from 2, Λ_9' cannot be stably equivalent of Morita type to any of the other algebras. We can concentrate on the remaining symmetric algebras in the list.

Now the algebras of at least six simple modules are trivial extensions of canonical algebras of tubular type $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, $(2,3,6)$, respectively. They have singular Cartan matrices, while all other algebras have non-singular Cartan matrix. Since, by a result of Xi (2008, Proposition 5.1) the absolute value of the Cartan determinant is preserved by a stable equivalence of Morita type, we can consider separately those algebras of at most four simple modules and those of at least six simple modules.

For trivial extension cases, by Białkowski et al. (2003a, Propositions 5.1 and 5.2), two algebras of this type are stably equivalent if and only if they are derived equivalent. So the number of simple modules, which is an invariant under derived equivalence, distinguishes them.

Now suppose two indecomposable standard non-domestic weakly symmetric algebras of polynomial growth are stably equivalent of Morita type and have non-singular Cartan matrices. Then the algebras are algebras in the above list. The following table gives some stable invariants of Morita type which distinguish two algebras in the above list.

Algebra A	Λ_2'	$\Lambda_3'(\lambda)$	Λ_5'	$A_1(\lambda)$	Λ_9'	A_4
$det\ C_A$	6	12	6	16	4	12
$dim(Z(A)/R(A))$	3	4	2	2	1	2

□

We can now prove

Proposition 3.3 *The classification of indecomposable non-standard non-domestic self-injective algebras of polynomial growth up to stable equivalences of Morita type coincides with the derived equivalence classification, modulo the scalar problem in* $\Lambda_3(\lambda)$.

Proof We only need to consider the case of characteristic two, since if the characteristic of K is 3, there is only one algebra in the list. But notice $det\ C_{\Lambda_3(\lambda)} = 12$, $det\ C_{\Lambda_5} = 6$ and $det\ C_{\Lambda_9} = 4$. □

Proposition 3.4 *Let A be an indecomposable standard weakly symmetric algebra and Λ be an indecomposable non-standard non-domestic self-injective algebra of polynomial growth. Then A and Λ are not stably equivalent of Morita type.*

Proof Suppose that A and Λ are stably equivalent of Morita type. By Krause and Zwara (2000, Corollary 2), A is also non-domestic of polynomial growth.

We only need to consider symmetric algebras. In fact, the only non-symmetric non-standard, non-domestic self-injective algebra of polynomial growth Λ_{10} is only defined when the characteristic of the base field is 2. In case of characteristic 2, all standard non-domestic weakly symmetric algebras of polynomial growth are symmetric, as Λ_9' is non-symmetric if and only if characteristic of the base field is different from 2.

Now we consider symmetric algebras. In fact, the method of the proof of Holm and Skowroński (2011, Main Theorem) works. Since two algebras which are stably equivalent of Morita type have isomorphic stable Auslander–Reiten quivers, by the shape of the stable Auslander–Reiten quivers [see the second paragraph of Section 4 of Holm and Skowroński (2011)], we proceed by remarking the following facts.

(1) If K is of characteristic 3, the algebras Λ_2 and Λ_2' are not stably equivalent of Morita type. Indeed Holm and Skowroński (2011) have shown that

$$dim\ Z(\Lambda_2')/T_1(\Lambda_2')^{\perp} = 3 \neq 2 = dim\ Z(\Lambda_2)/T_1(\Lambda_2)^{\perp}.$$

(2) If K is of characteristic 2 and $\lambda, \mu \in K\setminus\{0, 1\}$, the algebras $\Lambda_3(\lambda)$ and $\Lambda_3'(\mu)$ are not stably equivalent of Morita type. Indeed Holm and Skowroński (2011) have shown that

$$dim\ Z(\Lambda_3(\lambda))/T_1(\Lambda_3(\lambda))^{\perp} = 3 \neq 2 = dim\ Z(\Lambda_3'(\mu))/T_1(\Lambda_3'(\mu))^{\perp}.$$

(3) If K is of characteristic 2, the algebras Λ_5 and Λ_5' are not stably equivalent of Morita type. Indeed Holm and Skowroński (2011) have shown that

$$dim Z(\Lambda_5')/T_1(\Lambda_5')^{\perp} = 0 \neq dim Z(\Lambda_5)/T_1(\Lambda_5)^{\perp}.$$

(4) If K is of characteristic 2, the algebras Λ_9 and Λ_9' are not stably equivalent of Morita type. Indeed Al-Kadi (2010) (cf. also Holm and Skowroński 2011) has shown that

$$dim HH^2(\Lambda_9) = 4 \neq 3 = dim HH^2(\Lambda_9').$$

Since $Z(A)/T_1(A)^{\perp}$ and $HH^2(A)$ are invariants under stable equivalences of Morita type, this completes the proof. □

Combining Propositions 3.2–3.4, we have

Theorem 3.5 *The classification of indecomposable non-domestic weakly symmetric algebras of polynomial growth up to stable equivalences of Morita type coincides with the derived equivalence classification, up to the above mentioned scalar problems.*

As a consequence, we can prove a special case of Auslander–Reiten conjecture.

Theorem 3.6 (1) *Let A be an indecomposable algebra which is stably equivalent of Morita type to an indecomposable non-domestic symmetric algebra Λ of polynomial growth. Then A and Λ have the same number of simple modules.*

(2) *Let A and Λ be two indecomposable algebras which are both non-domestic weakly symmetric algebra of polynomial growth or which are both non-standard self-injective algebras of polynomial growth. If they are stably equivalent of Morita type, then A and Λ have the same number of simple modules.*

Proof (1) By Krause and Zwara (2000, Corollary 2) and Liu (2008, Corollary 2.4), A is also symmetric, non-domestic, and of polynomial growth. Then one can apply Theorem 3.5, by noticing that the existing scalar problems all occur in families of algebras for which different scalars yield algebras having the same number of simple modules.

(2) is a direct consequence of Theorem 3.5 and Proposition 3.3. □

3.3 Periodic algebras

Recently, Erdmann and Skowroński (2011) have shown that a non-simple indecomposable symmetric algebra A is tame with periodic modules if and only if A belongs to one of the following classes of algebras: a representation-finite symmetric algebra, a non-domestic symmetric algebra of polynomial growth, or an algebra of quaternion type (in the sense of Erdmann 1980). Note that these three classes of algebras are closed under stable equivalences of Morita type, by Krause and Zwara (2000, Corollary 2) and Liu (2008, Corollary 2.4).

In Asashiba (1999) Asashiba, in combination with Holm and Skowroński (2006), the statement displayed in Erdmann and Skowroński (2008, Theorems 3.4–3.6) proved that two representation-finite self-injective indecomposable algebras are derived equivalent if and only if they are stably equivalent. In our recent paper (Zhou and Zimmermann 2010, Theorem 7.1), we proved that for the class of algebras of quaternion type, derived equivalence classification coincide with the classification up to stable equivalences of Morita type (up to some scalar problems). Therefore, combining the above Theorem 3.6, we showed the following

Theorem 3.7 (1) *The class of tame symmetric algebras with periodic modules is closed under stable equivalences of Morita type, in particular under derived equivalences*;

(2) *For tame symmetric algebras with periodic modules derived equivalence classification coincide with the classification up to stable equivalences of Morita type* (*up to some scalar problems*);

(3) *The Auslander–Reiten conjecture holds for a stable equivalence of Morita type between two tame symmetric algebras with periodic modules.*

4 Concluding remarks

In Sect. 3 we classified symmetric algebras of polynomial growth only up to stable equivalences of Morita type, whereas the results in Sect. 2 concern stable equivalences in general. It would be most interesting to get a stable equivalence classification for general self-injective algebras of polynomial growth, in particular the validity of the Auslander–Reiten conjecture in general.

The results of this paper present a second occurrence of a phenomenon of the same kind: in our previous paper (Zhou and Zimmermann 2010) we also obtained that the classification of a class of tame symmetric algebras up to stable equivalence coincides with the derived equivalence classification. One might ask if two indecomposable tame symmetric algebras are stably equivalent of Morita type if and only if they are derived equivalent.

5 Appendix

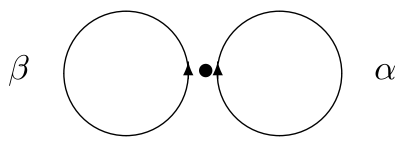
5.1 Self-injective algebras of domestic representation type

5.1.1 Algebras of standard type

For standard weakly symmetric algebras of domestic representation type with *non-singular Cartan matrices*, we have the following derived norm forms.

$A(\lambda)$

$\lambda \in K\backslash\{0\}$

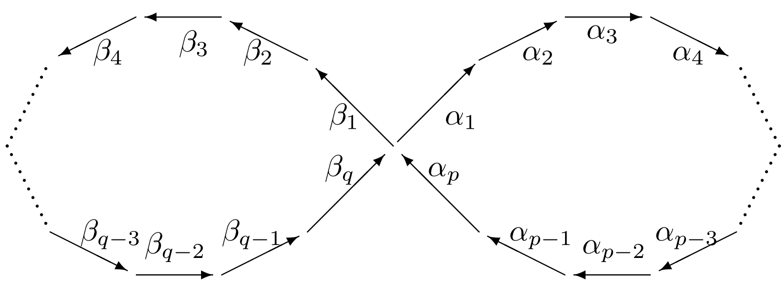


$\alpha^2 = 0, \beta^2 = 0, \alpha\beta = \lambda\beta\alpha$

$A(p, q)$

$1 \leq p \leq q$

$p + q \geq 3$



$\alpha_1\alpha_2 \cdots \alpha_p\beta_1\beta_2 \cdots \beta_q = \beta_1\beta_2 \cdots \beta_q\alpha_1\alpha_2 \cdots \alpha_p$

$\alpha_p\alpha_1 = 0, \beta_q\beta_1 = 0$

$\alpha_i\alpha_{i+1} \cdots \alpha_p\beta_1 \cdots \beta_q\alpha_1 \cdots \alpha_{i-1}\alpha_i = 0, 2 \leq i \leq p$

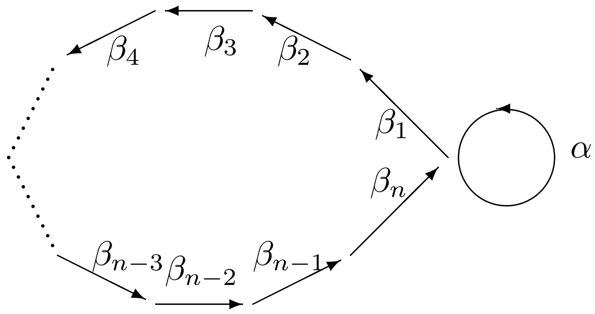
$\beta_j\beta_{j+1} \cdots \beta_q\alpha_1 \cdots \alpha_p\beta_1 \cdots \beta_{j-1}\beta_j = 0, 2 \leq j \leq q$

5.1.2 Algebras of non-standard type

The quiver with relations of $\Omega(n)$ is as follows.

$\Lambda(n)$

$n \geq 2$

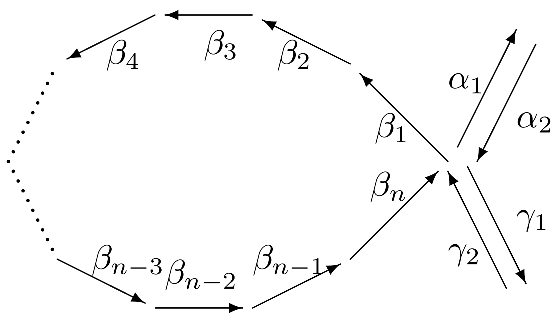


$$\alpha^2 = (\beta_1\beta_2\cdots\beta_n)^2, \alpha\beta_1 = 0, \beta_n\alpha = 0$$

$$\beta_j\beta_{j+1}\cdots\beta_n\beta_1\cdots\beta_n\beta_1\cdots\beta_{j-1}\beta_j = 0, 2 \leq j \leq n$$

$\Gamma(n)$

$n \geq 1$



$$\alpha_1\alpha_2 = (\beta_1\beta_2\cdots\beta_n)^2 = \gamma_1\gamma_2,$$

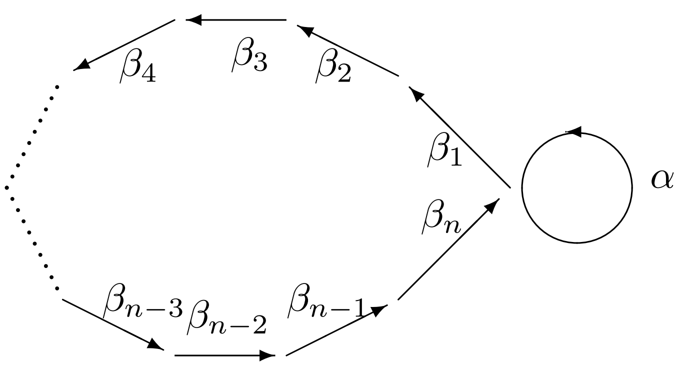
$$\alpha_2\beta_1 = 0, \gamma_2\beta_1 = 0, \beta_n\alpha_1 = 0$$

$$\beta_n\gamma_1 = 0, \alpha_2\gamma_1 = 0, \gamma_2\alpha_1 = 0$$

$$\beta_j\beta_{j+1}\cdots\beta_n\beta_1\cdots\beta_n\beta_1\cdots\beta_{j-1}\beta_j = 0, 2 \leq j \leq n$$

$\Omega(n)$

$n \geq 1$



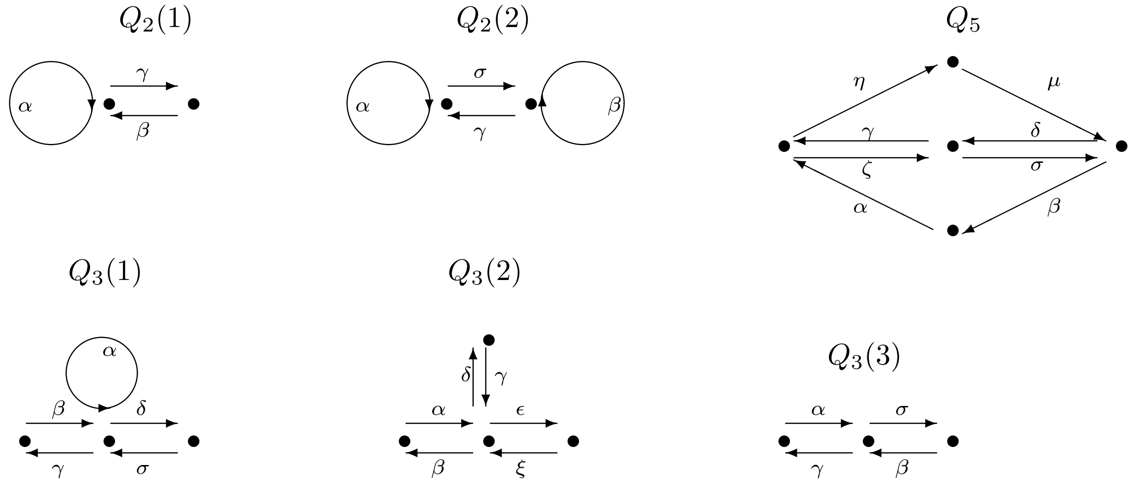
$$\alpha^2 = \alpha\beta_1\beta_2\cdots\beta_n, \alpha\beta_1\beta_2\cdots\beta_n + \beta_1\beta_2\cdots\beta_n\alpha = 0,$$

$$\beta_n\beta_1 = 0, \beta_j\beta_{j+1}\cdots\beta_n\beta_1\cdots\beta_n\beta_1\cdots\beta_{j-1}\beta_j = 0, 2 \leq j \leq n$$

5.2 Self-injective algebras of polynomial growth

We recall the derived norm forms following Holm and Skowroński (2011).

We have five quivers and various relations.



Define for any $\lambda \in K\setminus\{0,1\}$ the algebras

$$\begin{aligned}
\Lambda_2 &:= KQ_2(1)/(\alpha^2\gamma, \beta\alpha^2, \gamma\beta\gamma, \beta\gamma\beta, \beta\gamma-\beta\alpha\gamma, \alpha^3-\gamma\beta)\\
\Lambda_2' &:= KQ_2(1)/(\alpha^2\gamma, \beta\alpha^2, \beta\gamma, \alpha^3-\gamma\beta)\\
\Lambda_3(\lambda) &:= KQ_2(2)/(\alpha^4, \gamma\alpha^2, \alpha^2\sigma, \alpha^2-\sigma\gamma-\alpha^3, \lambda\beta^2-\gamma\sigma, \gamma\alpha-\beta\gamma, \sigma\beta-\alpha\sigma)\\
\Lambda_3'(\lambda) &:= KQ_2(2)/(\alpha^2-\sigma\gamma, \lambda\beta^2-\gamma\sigma, \gamma\alpha-\beta\gamma, \sigma\beta-\alpha\sigma)\\
\Lambda_5 &:= KQ_3(1)/(\alpha^2-\gamma\beta, \alpha^3-\delta\sigma, \beta\delta, \sigma\gamma, \alpha\delta, \sigma\alpha, \gamma\beta\gamma, \beta\gamma\beta, \beta\gamma-\beta\alpha\gamma)\\
\Lambda_5' &:= KQ_3(1)/(\alpha^2-\gamma\beta, \alpha^3-\delta\sigma, \beta\delta, \sigma\gamma, \alpha\delta, \sigma\alpha, \beta\gamma)\\
\Lambda_9 &:= KQ_3(2)/(\beta\alpha+\delta\gamma+\epsilon\xi, \gamma\delta, \xi\epsilon, \alpha\beta\alpha, \beta\alpha\beta, \alpha\beta-\alpha\delta\gamma\beta)\\
\Lambda_9' &:= KQ_3(2)/(\beta\alpha+\delta\gamma+\epsilon\xi, \gamma\delta, \xi\epsilon, \alpha\beta)\\
A_1(\lambda) &:= KQ_3(3)/(\alpha\gamma\alpha-\alpha\sigma\beta, \beta\gamma\alpha-\lambda\cdot\beta\sigma\beta, \gamma\alpha\gamma-\sigma\beta\gamma, \gamma\alpha\sigma-\lambda\sigma\beta\sigma)\\
A_4 &:= KQ_3(2)/(\beta\alpha+\delta\gamma+\epsilon\xi, \alpha\beta, \gamma\epsilon, \xi\delta)\\
\Lambda_{10} &= KQ_5/(\beta\alpha-\delta\gamma, \zeta\sigma-\eta\mu, \alpha\eta, \mu\beta, \sigma\delta-\gamma\zeta, \delta\sigma\delta\sigma)
\end{aligned}$$

Further, denote the trivial extensions of tubular canonical algebras as usual, in particular

$$\begin{aligned}
\Lambda(2,2,2,2,\lambda) &= T(C(2,2,2,\lambda)) \text{ for } \lambda \in K\setminus\{0,1\}\\
\Lambda(3,3,3) &= T(C(3,3,3))\\
\Lambda(2,4,4) &= T(C(2,4,4))\\
\Lambda(2,3,6) &= T(C(2,3,6)).
\end{aligned}$$

The precise quivers with relations of the trivial extensions of tubular canonical algebras in question are displayed in Holm and Skowroński (2011). We will refrain from presenting them here since we do not really need this information in such details.

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