UNBOUNDED LADDERS INDUCED BY GORENSTEIN ALGEBRAS

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Abstract. The derived category $D(\mathrm{Mod}A)$ of a Gorenstein triangular matrix algebra A admits an unbounded ladder; and this ladder restricts to $D^-(\mathrm{Mod})$ (resp. $D^b(\mathrm{Mod})$, $D^b(\mathrm{mod})$, $K^b(\mathrm{proj})$). A left recollement of triangulated categories with Serre functors sits in a ladder of period 1; as an application, the singularity category of A admits a ladder of period 1.

Recollements ([BBD]) provide a powerful tool for studying problems in triangulated categories and algebraic geometry. To study mixed categories, ladders have been introduced ([BGS], [AHKLY]). Recollements are ladders of height 1; while ladders of height ≥ 2 give more information ([AHKLY], [HQ]). A fundamental question is when unbounded ladders occur naturally in representation theory. This essentially deals with the existence of infinite adjoint sequences. It is known that if A is an algebra of finite global dimension, then any recollement of derived category $D(\mathrm{Mod}A)$ sits in an unbounded ladder ([AHKLY, 3.7]).

Let $A := \left(\begin{smallmatrix} B & 0 \\ M & C \end{smallmatrix}\right)$ with M a C-B-bimodule. An algebra Λ is of this form if and only if Λ has an idempotent e with $(1-e)\Lambda e = 0$. It is well-known that there is a recollement $(D(\mathrm{Mod}B),\ D(\mathrm{Mod}A),\ D(\mathrm{Mod}C))$ of derived categories ([AHKL], [Han]); and in fact, it sits in a ladder of height 2 ([AHKLY, 3.4]). We further claim that it sits in an unbounded ladder, provided that A, B and C are Gorenstein algebras. This unbounded ladder enjoys pleasant properties in the sense that it restricts to $D^-(\mathrm{Mod})$ (resp. $D^b(\mathrm{Mod})$, $D^b(\mathrm{mod})$, $K^b(\mathrm{proj})$).

For an adjoint pair (F, G) of categories with Serre functors, F (resp. G) always has a left (resp. right) adjoint. So a left recollement of triangulated categories with Serre functors sits in a ladder of period 1. As an application, the singularity category ([B], [O]) of a Gorenstein triangular matrix algebra admits a ladder of period 1 (Thm. 3.2), via the stable category of Gorenstein-projective modules ([EJ], [B], [H2]).

1. **Preliminaries**

1.1. Let $\mathcal{C}'$, $\mathcal{C}$ and $\mathcal{C}''$ be triangulated categories. *A recollement* $(\mathcal{C}', \mathcal{C}, \mathcal{C}'', i^*, i_*, i^!, j_!, j^*, j_*)$ of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$ ([BBD]) is a diagram of triangle functors

$$\mathcal{C}' \begin{array}{c} \xleftarrow{\ i^*\ } \\ \xrightarrow{\ i_*\ } \\ \xleftarrow{\ i^!\ } \end{array} \mathcal{C} \begin{array}{c} \xleftarrow{\ j_!\ } \\ \xrightarrow{\ j^*\ } \\ \xleftarrow{\ j_*\ } \end{array} \mathcal{C}'' \tag{1.1}$$

satisfying the following conditions:

(R1) $(i^*, i_*), (i_*, i^!), (j_!, j^*)$ and (j^*, j_*) are adjoint pairs;

(R2) $i_*, j_!$ and j_* are fully faithful;

(R3) $j^* i_* = 0$ (and thus $i^* j_! = 0 = i^! j_*$);

(R4) for $X \in \mathcal{C}$ there are distinguished triangles $j_! j^* X \xrightarrow{\epsilon_X} X \xrightarrow{\eta_X} i_* i^* X \to (j_! j^* X)[1]$ and $i_* i^! X \xrightarrow{\omega_X} X \xrightarrow{\zeta_X} j_* j^* X \to (i_* i^! X)[1]$, where the marked morphisms are the counits and the units of the adjunctions.

A left (resp. *right*) *recollement* of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$ is the upper (resp. lower) two rows of (1.1) satisfying the same conditions involving only these functors ([P], [Kö]. For other or related names see e.g. [BGS], [M], [BO], [Kr], [IKM]). *An opposed recollement* of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$ is a diagram

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$$\mathcal{C}' \begin{array}{c} \xrightarrow{i_{-1}} \\ \xleftarrow{j_0} \\ \xrightarrow{i_1} \end{array} \mathcal{C} \begin{array}{c} \xrightarrow{j_{-1}} \\ \xleftarrow{i_0} \\ \xrightarrow{j_1} \end{array} \mathcal{C}''$$

such that $(\mathcal{C}'', \mathcal{C}, \mathcal{C}', j_{-1}, i_0, j_1, i_{-1}, j_0, i_1)$ is a recollement of $\mathcal{C}$ relative to $\mathcal{C}''$ and $\mathcal{C}'$.

Lemma 1.1. (1) *Given the upper two rows of* (1.1)*, the following are equivalent*:

(i) *it is a left recollement*;

(ii) (i^*, i_*) *and* $(j_!, j^*)$ *are adjoint pairs,* i_* *and* $j_!$ *are fully faithful, and* $\mathrm{Im} i_* = \mathrm{Ker} j^*$;

(iii) (i^*, i_*) *and* $(j_!, j^*)$ *are adjoint pairs,* i_* *and* $j_!$ *are fully faithful, and* $\mathrm{Im} j_! = \mathrm{Ker} i^*$.

(2) (see e.g. [IKM, 1.7]) *Given diagram* (1.1) *of triangle functors, the following are equivalent*:

(i) *it is a recollement*;

(ii) *it satisfies* (R1) *and* (R2)*, and* $\mathrm{Im} i_* = \mathrm{Ker} j^*$, $\mathrm{Im} j_! = \mathrm{Ker} i^*$ *and* $\mathrm{Im} j_* = \mathrm{Ker} i^!$;

(iii) *it satisfies* (R1) *and* (R2)*, and any one of the equalities in* (2)(ii).

1.2. *A ladder* ([BGS, 1.2], [AHKLY, Sect. 3]) is a finite or an infinite diagram of triangle functors:

$$\mathcal{C}' \begin{array}{c} \vdots \\ \xrightarrow{i_{-2}} \\ \xleftarrow{j_{-1}} \\ \xrightarrow{i_0} \\ \xleftarrow{j_1} \\ \xrightarrow{i_2} \\ \vdots \end{array} \mathcal{C} \begin{array}{c} \vdots \\ \xrightarrow{j_{-2}} \\ \xleftarrow{i_{-1}} \\ \xrightarrow{j_0} \\ \xleftarrow{i_1} \\ \xrightarrow{j_2} \\ \vdots \end{array} \mathcal{C}'' \qquad (1.2)$$

such that any two consecutive rows form a left or right recollement (or equivalently, any three consecutive rows form a recollement or an opposed recollement) of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$. Its *height* is the number of rows minus 2. Ladders of height 0 (resp. 1) are exactly left or right recollements (resp. recollements or opposed recollements). A ladder is *unbounded* if it goes infinitely both upwards and downwards.

A two-sided infinite sequence $(\cdots, F_{-1}, F_0, F_1, \cdots)$ of additive functors is *an infinite adjoint sequence*, if (F_n, F_{n+1}) is an adjoint pair for each $n \in \mathbb{Z}$. In such a sequence if some F_i is a triangle functor then so are all F_n's (see e.g. [Ke1, 6.7]).

Lemma 1.2. *Recollement* (1.1) *sits in an unbounded ladder if and only if there is an infinite adjoint sequence* $(\cdots, F_{-1}, i^*, i_*, i^!, F_1, \cdots)$.

1.3. *An equivalence* of left recollements ([PS, 2.5], [FP]) is a triple (F', F, F'') of triangle-equivalences such that

$$\begin{array}{ccccc} \mathcal{C}' & \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \end{array} & \mathcal{C} & \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \end{array} & \mathcal{C}'' \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F''} \\ \mathcal{D}' & \begin{array}{c} \xleftarrow{i^*_{\mathcal{D}}} \\ \xrightarrow{i^{\mathcal{D}}_*} \end{array} & \mathcal{D} & \begin{array}{c} \xleftarrow{j^{\mathcal{D}}_!} \\ \xrightarrow{j^*_{\mathcal{D}}} \end{array} & \mathcal{D}'' \end{array}$$

commutes. Similarly we have an equivalence of (right, opposed) recollements.

We call $(\mathcal{C}', \mathcal{C}, \mathcal{C}'', j_{2t-1}, i_{2t}, j_{2t+1}, i_{2t-1}, j_{2t}, i_{2t+1})$ in ladder (1.2) *the* t*-th recollement*, $(\mathcal{C}', \mathcal{C}, \mathcal{C}'', i_{2t}, j_{2t+1}, i_{2t+2}, j_{2t}, i_{2t+1}, j_{2t+2})$ *the* t*-th opposed recollement*, and the left (right) recollement sitting in the t-th recollement *the* t*-th left* (*right*) *recollement*. An unbounded ladder (1.2) is *periodic*, if there is an integer $t \geq 1$ such that the t-th left recollement is equivalent to the 0-th one. Such a minimal t is called *the period*. The following describes the period via the associated TTF tuple, and justifies the terminology.

Lemma 1.3. (1) *Given recollements* $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ *and* $(\mathcal{D}', \mathcal{D}, \mathcal{D}'')$*, the following are equivalent*:

(i) *they are equivalent*;

(ii) *there is a triangle-equivalence* $F : \mathcal{C} \to \mathcal{D}$ *such that* $F(\mathrm{Im} j_!) = \mathrm{Im} j^{\mathcal{D}}_!$, $F(\mathrm{Im} i_*) = \mathrm{Im} i^{\mathcal{D}}_*$ *and* $F(\mathrm{Im} j_*) = \mathrm{Im} j^{\mathcal{D}}_*$;

(iii) *there is a triangle-equivalence $F : \mathcal{C} \to \mathcal{D}$ such that one of the equalities in* (ii) *holds.*

(2) *Given a ladder of period t, then the $(qt + l)$-th (left, right, opposed) recollement is equivalent to the l-th (left, right, opposed) recollement for $q \in \mathbb{Z}$ and $l = 0, \cdots, t-1$, under the same equivalence.*

(3) *Given an unbounded ladder* (1.2)*, the following are equivalent*:

(i) *it is of period t;*

(ii) *t is the minimal positive integer such that there is a triangle-equivalence $F : \mathcal{C} \to \mathcal{C}$ satisfying $F(\mathrm{Im} i_{2t+1}) = \mathrm{Im} i_1$, $F(\mathrm{Im} i_{2t}) = \mathrm{Im} i_0$ and $F(\mathrm{Im} i_{2t-1}) = \mathrm{Im} i_{-1}$;*

(iii) *t is the minimal positive integer such that there is a triangle-equivalence $F : \mathcal{C} \to \mathcal{C}$ satisfying one of the equalities in* (ii)*.*

1.4. If no specified, modules are right modules. For algebra A over a field, denote by Mod$A$ (resp. A-Mod) the category of right (resp. left) A-modules. If A is finite-dimensional, then we denote by mod$A$ (resp. A-mod) the category of finitely generated right (resp. left) A-modules, and by $\mathcal{GP}(A)$ the full subcategory of mod$A$ consisting of Gorenstein-projevtive modules ([EJ]). Then $\mathcal{GP}(A)$ is a Frobenius category whose projective-injective objects are exactly projective modules ([Be]), and hence the stable category $\underline{\mathcal{GP}(\mathcal{A})}$ is triangulated ([H1]). A finite-dimensional algebra A is *Gorenstein* if $\mathrm{inj.dim}_A A < \infty$ and $\mathrm{inj.dim} A_A < \infty$.

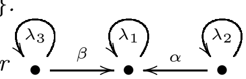
Let $K^b(\mathrm{proj}A)$ (resp. $K^b(\mathrm{inj}A)$) be the homotopy category of bounded complexes of finitely generated projective (resp. injective) right A-modules, $D(\mathrm{Mod}A)$ (resp. $D^-(\mathrm{Mod}A)$, $D^b(\mathrm{Mod}A)$) the unbounded (resp. upper bounded, bounded) derived category of Mod$A$, and $D^b(\mathrm{mod}A)$ the bounded derived category of mod$A$. Note that $D(\mathrm{Mod}A)$ is compactly generated by A_A (see [S]; also [BN]).

For a triangulated category $\mathcal{T}$ with coproducts, denote by $\mathcal{T}^c$ the full subcategory of $\mathcal{T}$ consisting of compact objects. Then $D^c(\mathrm{Mod}A) = K^b(\mathrm{proj}A)$ ([N1]).

2. Main results

Theorem 2.1. *Let B and C be Gorenstein algebras and ${}_CM_B$ a C-B-bimodule, such that $A = \left(\begin{smallmatrix} B & 0 \\ M & C \end{smallmatrix}\right)$ is Gorenstein. Then there is an unbounded ladder $(D(\mathrm{Mod}B),\ D(\mathrm{Mod}A),\ D(\mathrm{Mod}C))$ of derived categories.*

Remark 2.2. *For the Gorensteinness of $A := \left(\begin{smallmatrix} B & 0 \\ M & C \end{smallmatrix}\right)$ we refer to* [C] *and* [Z, Thm. 2.2]*. If B and C are Gorenstein, then A is Gorenstein if and only if* $\mathrm{proj.dim}_C M$ *and* $\mathrm{proj.dim} M_B$ *are finite* ([C, Thm. 3.3])*. Also note that* $\mathrm{gl.dim}A \geq \max\{\mathrm{gl.dim}B,\ \mathrm{gl.dim}C\}$.

For example, let A be the algebra given by quiver  *and relations $\lambda_1^2,\ \lambda_2^2,\ \lambda_3^2,\ \alpha\lambda_2 - \lambda_1\alpha,\ \beta\lambda_3 - \lambda_1\beta$. Then $A = \left(\begin{smallmatrix} B & 0 \\ {}_CM_B & C \end{smallmatrix}\right) = \left(\begin{smallmatrix} C & 0 & 0 \\ C & C & 0 \\ 0 & C & C \end{smallmatrix}\right)$, where $C := k[x]/\langle x^2 \rangle$, $B := T_2(C) := \left(\begin{smallmatrix} C & 0 \\ C & C \end{smallmatrix}\right)$ and ${}_CM_B := {}_C(0, C)_{T_2(C)}$. Since $\mathrm{proj.dim}_C M = 0$ and $\mathrm{proj.dim} M_{T_2(C)} = 1$, A is Gorenstein of $\mathrm{gl.dim}A = \infty$.*

2.1. Let $A = \left(\begin{smallmatrix} B & 0 \\ {}_CM_B & C \end{smallmatrix}\right)$. A right A-module is given by $(X_B, Y_C)_\phi$, where $X_B \in$ mod B, $Y_C \in$ mod C, and $\phi :\ Y \otimes_C M_B \to X_B$ is a right B-map. A right A-map $(X_B, Y_C)_\phi \to (X'_B, Y'_C)_{\phi'}$ is given by (f, g) with $f \in \mathrm{Hom}_B(X_B, X'_B)$ and $g \in \mathrm{Hom}_C(Y_C, Y'_C)$, such that $f\phi = \phi'(g \otimes_B \mathrm{Id}_M)$. A left A-module is given by $\left(\begin{smallmatrix} {}_BX \\ {}_CY \end{smallmatrix}\right)_\phi$, where ${}_BX \in B$-mod, ${}_CY \in C$-mod, and $\phi :\ {}_CM \otimes_B X \to {}_CY$ is a left C-map. A left A-map $\left(\begin{smallmatrix} {}_BX \\ {}_CY \end{smallmatrix}\right)_\phi \to \left(\begin{smallmatrix} {}_BX' \\ {}_CY' \end{smallmatrix}\right)_{\phi'}$ is given by (f, g) with $f \in \mathrm{Hom}_B({}_BX,\ {}_BX')$ and $g \in \mathrm{Hom}_C({}_CY,\ {}_CY')$, such that $g\phi = \phi'(\mathrm{Id}_M \otimes_B f)$. The projective right A-modules are exactly $(P_B, 0)$ and $(Q \otimes_C M, Q_C)_{\mathrm{Id}}$, where $P_B \in \mathrm{proj}B$ and $Q_C \in \mathrm{proj}C$. The projective left A-modules are exactly $\left(\begin{smallmatrix} {}_BP \\ M \otimes_B P \end{smallmatrix}\right)_{\mathrm{Id}}$ and $\left(\begin{smallmatrix} 0 \\ {}_CQ \end{smallmatrix}\right)$, where ${}_BP \in B$-proj and ${}_CQ \in C$-proj. See [ARS, p.73].

2.2. Let A be an algebra over a field with idempotent e. The ideal AeA is *stratifying* ([CPS, 2.1.1]), if the multiplication map $m : Ae \otimes_{eAe} eA \to AeA$ is injective and $\mathrm{Tor}^n_{eAe}(Ae, eA) = 0$ for $n \geq 1$. As pointed out by S. König and H. Nagase [KN, Rem. 3.2], ${}_A(AeA)$ (resp. $(AeA)_A$) is projective if and only if ${}_{eAe}(eA)$ (resp. $(Ae)_{eAe}$) is projective and the map m is injective. Thus, if AeA is projective either as a left or a right A-module, then AeA is a stratifying ideal.

Lemma 2.3. (see e.g. [AHKL, 4.5], [Han, 2.1]) *If AeA is a stratifying ideal, then there is a recollement*

$$D(\mathrm{Mod}A/AeA) \begin{smallmatrix} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{smallmatrix} D(\mathrm{Mod}A) \begin{smallmatrix} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{smallmatrix} D(\mathrm{Mod}\ eAe)$$

where

$$i^* = - \overset{\mathrm{L}}{\otimes}_A A/AeA, \qquad i_* = - \overset{\mathrm{L}}{\otimes}_{A/AeA} A/AeA, \qquad i^! = \mathrm{R}\operatorname{Hom}_A(A/AeA, -),$$

$$j_! = - \overset{\mathrm{L}}{\otimes}_{eAe} eA, \qquad j^* = - \overset{\mathrm{L}}{\otimes}_A Ae, \qquad j_* = \mathrm{R}\operatorname{Hom}_{eAe}(Ae, -).$$

2.3. Let $\mathcal{T}$ be a triangulated category compactly generated by $\mathcal{S}_0$. Denote by $\langle \mathcal{S}_0 \rangle$ the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{S}_0$ and closed under coproducts. Brown representability (A. Neeman [N2, Thm. 3.1]) claims that every cohomological functor $F : \mathcal{T}^{op} \to \mathrm{Ab}$ which sends coproducts to products is representable (i.e., $F \cong \operatorname{Hom}_{\mathcal{T}}(-, X)$ for some $X \in \mathcal{T}$), and that $\mathcal{T} = \langle \mathcal{S}_0 \rangle$. And, Brown representability for the dual (H. Krause [Kr, Thm. A]) claims that $\mathcal{T}$ has products, and that every cohomological functor $F : \mathcal{T} \to \mathrm{Ab}$ which sends products to products is representable (i.e., $F \cong \operatorname{Hom}_{\mathcal{T}}(X, -)$ for some $X \in \mathcal{T}$).

Using Brown representability one has

Lemma 2.4. ([N2, Thm. 4.1 and 5.1]) *Let $F : \mathcal{C} \to \mathcal{D}$ be a triangle functor between compactly generated triangulated categories, with a right adjoint G. Then the following are equivalent*:

(i) *G admits a right adjoint*;
(ii) *F preserves compact objects*;
(iii) *G preserves coproducts.*

Using Brown representability for the dual one has

Lemma 2.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a triangle functor between triangulated categories, where $\mathcal{C}$ is compactly generated. Then F admits a left adjoint if and only if F preserves products* (*we are not assuming that $\mathcal{D}$ has products*).

Proof. The "only if" part is well-known. For the "if" part, applying Brown representability for the dual to functor $\operatorname{Hom}_{\mathcal{D}}(Y, F-) : \mathcal{C} \to \mathrm{Ab}$, for each object $Y \in \mathcal{D}$, we then see that F admits a left adjoint. ■

We need the following result due to P. Balmer, I. Dell'ambrogio and B. Sanders.

Lemma 2.6. ([BDS, Lemma 2.6(b)]) *Let $F : \mathcal{C} \to \mathcal{D}$ be a triangle functor between compactly generated triangulated categories, with a right adjoint G. Assume that F preserves compacts, and the restriction $F|_{\mathcal{C}^c} : \mathcal{C}^c \to \mathcal{D}^c$ admits a left adjoint. Then F preserves products.*

2.4. Let A be a finite-dimensional algebra over field k. Using a hoprojective (resp. hoinjective) resolution of a complex in $D(\mathrm{Mod}A)$ ([S]; [BN]) one has the characterizations:

$$K^b(\mathrm{proj}A) = \{P \in D(\mathrm{Mod}A) \mid \dim_k(\bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{D(\mathrm{Mod}A)}(P, Y[i])) < \infty,\ \forall\ Y \in D^b(\mathrm{mod}A)\},$$

and

$$K^b(\mathrm{inj}A) = \{I \in D(\mathrm{Mod}A) \mid \dim_k(\bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{D(\mathrm{Mod}A)}(Y[i], I)) < \infty,\ \forall\ Y \in D^b(\mathrm{mod}A)\}.$$

One has also the characterization:

$$D^b(\mathrm{mod}A) = \{X \in D(\mathrm{Mod}A) \mid \dim_k(\bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{D(\mathrm{Mod}A)}(P, X[i])) < \infty,\ \forall\ P \in K^b(\mathrm{proj}A)\}.$$

See L. Angeleri Hügel, S. König, Q. H. Liu and D. Yang [AHKLY, Lemma 2.4]. Using these one has

Lemma 2.7. *Let A and B be finite-dimensional algebras, and $F : D(\mathrm{Mod}A) \to D(\mathrm{Mod}B)$ a triangle functor with a right adjoint G. Then*

(i) ([AHKLY, Lemma 2.7]) *F preserves $K^b(\mathrm{proj})$ if and only if G preserves $D^b(\mathrm{mod})$.*
(ii) ([HQ, Lemma 1]) *F preserves $D^b(\mathrm{mod})$ if and only if G preserves $K^b(\mathrm{inj})$.*

2.5. Let $\mathcal{C}$ be a Hom-finite category over field k. A k-linear functor $S : \mathcal{C} \to \mathcal{C}$ is *a right Serre functor*, if for any objects X and Y there is a k-isomorphism $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(Y, SX)^*$ which is natural in X and Y, where $(-)^* = \mathrm{Hom}_k(-, k)$. We say that $\mathcal{C}$ *has a Serre functor* if $\mathcal{C}$ has a right Serre functor which is an equivalence, or equivalently, $\mathcal{C}$ has both a right and left Serre functor ([BK], [RV]). If $\mathcal{C}$ is a Hom-finite Krull-Schmidt triangulated category over an algebraically closed field k, then $\mathcal{C}$ has a Serre functor if and only if $\mathcal{C}$ has Auslander-Reiten triangles (note that the assumption that k is algebraically closed is only used in the "only if" part. See I. Reiten and M. Van den Bergh [RV, Thm. 2.4]).

The following observation will play an important role in this paper.

Lemma 2.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with Serre functors, $F : \mathcal{C} \to \mathcal{D}$ an additive functors with a right adjoint G. Then F admits a left adjoint $S_{\mathcal{C}}^{-1} G S_{\mathcal{D}}$, and G admits a right adjoint $S_{\mathcal{D}} F S_{\mathcal{C}}^{-1}$, where $S_{\mathcal{C}}$ and $S_{\mathcal{D}}$ are the right Serre functors of $\mathcal{C}$ and $\mathcal{D}$, respectively.*

Proof. For $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ we have

$$\mathrm{Hom}_{\mathcal{C}}(S_{\mathcal{C}}^{-1} G S_{\mathcal{D}} Y, X) \cong \mathrm{Hom}_{\mathcal{C}}(X, G S_{\mathcal{D}} Y)^* \cong \mathrm{Hom}_{\mathcal{D}}(FX, S_{\mathcal{D}} Y)^* \cong \mathrm{Hom}_{\mathcal{D}}(Y, FX).$$

Similarly $(G, S_{\mathcal{D}} F S_{\mathcal{C}}^{-1})$ is an adjoint pair. ■

We also need the following result due to D. Happel.

Lemma 2.9. ([Hap2, Lemma 1.5, Thm. 3.4]) *Let A be a finite-dimensional algebra. Then A is Gorenstein if and only if $K^b(\mathrm{proj}A) = K^b(\mathrm{inj}A)$ in $D^b(\mathrm{mod}A)$. In this case $K^b(\mathrm{proj}A)$ has a Serre functor.*

2.6. **Proof of Theorem 2.1.** Put $e := \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in A$. Then $AeA = \left(\begin{smallmatrix} 0 & 0 \\ M & C \end{smallmatrix}\right) \cong (M, C)$ is a projective right A-module, and hence AeA is stratifying. Since $A/AeA \cong B$ and $eAe \cong C$ as algebras, and

$${}_A(A/AeA)_B \cong {}_A\left(\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right)_B, \quad {}_B(A/AeA)_A \cong {}_B(B, 0)_A, \quad {}_C(eA)_A \cong {}_C(M, C)_A, \quad {}_A(Ae)_C \cong {}_A\left(\begin{smallmatrix} 0 \\ C \end{smallmatrix}\right)_C$$

as bimodules, it follows from Lemma 2.3 that there is a recollement

$$D(\mathrm{Mod}B) \underset{\overset{\longleftarrow}{i^!}}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{i_*}}} D(\mathrm{Mod}A) \underset{\overset{\longleftarrow}{j_*}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{j^*}}} D(\mathrm{Mod}C) \qquad (2.1)$$

where $i^* = -\overset{\mathrm{L}}{\otimes}_A \left(\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right)$, $i_* = -\overset{\mathrm{L}}{\otimes}_B (B, 0)$, $i^! = \mathrm{R}\,\mathrm{Hom}_A((B, 0)_A, -)$, $j_! = -\overset{\mathrm{L}}{\otimes}_C (M, C)$, $j^* = -\overset{\mathrm{L}}{\otimes}_A \left(\begin{smallmatrix} 0 \\ C \end{smallmatrix}\right)$, $j_* = \mathrm{R}\,\mathrm{Hom}_C(\left(\begin{smallmatrix} 0 \\ C \end{smallmatrix}\right)_C, -)$.

Claim 1. There is an infinite sequence $(\cdots, F_{-3}, F_{-2}, F_{-1}, i^*)$ such that any two consecutive functors form an adjoint pair.

Since the right adjoint i_* of i^* admits a right adjoint $i^!$, it follows from Lemma 2.4 that i^* preserves compacts (this could be also seen directly: since $\left(\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right)$ is projective as a right B-module, it follows that $i^* = -\overset{\mathrm{L}}{\otimes}_A \left(\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right)$ preserves compacts). Since $(B, 0)$ is projective as a right A-module, it follows that $i_* = -\overset{\mathrm{L}}{\otimes}_B (B, 0)$ preserves compacts. Thus $(i^*|_{K^b(\mathrm{proj}A)}, \ i_*|_{K^b(\mathrm{proj}B)})$ is an adjoint pair. Since A and B are Gorenstein algebras, by Lemma 2.9, $K^b(\mathrm{proj}A)$ and $K^b(\mathrm{proj}B)$ have Serre functors, and hence $i^*|_{K^b(\mathrm{proj}A)}$ has a left adjoint, by Lemma 2.8. Applying Lemma 2.6 to the adjoint pair (i^*, i_*) we know that i^* preserves products, and hence by Lemma 2.5, i^* admits a left adjoint, denoted by F_{-1}.

Repeating the above arguments we get **Claim 1**.

Claim 2. There is an infinite sequence $(i^!, G_1, G_2, G_3, \cdots)$ such that any two consecutive functors form an adjoint pair.

Since i_* preserves compacts, it follows from Lemma 2.4 that $i^!$ admits a right adjoint, denoted by G_1.

Since i^* preserves compacts, i.e., i^* preserves $K^b(\mathrm{proj})$, it follows from Lemma 2.7(i) that i_* preserves $D^b(\mathrm{mod})$, and hence $i^!$ preserves $K^b(\mathrm{inj})$ by Lemma 2.7(ii). Since we are dealing with Gorenstein algebras, by Lemma 2.9 this is exactly to say that $i^!$ preserves $K^b(\mathrm{proj})$, i.e., $i^!$ preserves compacts. It follows from Lemma 2.4 that G_1 admits a right adjoint, denoted by G_2.

By the same argument we know that G_1 preserves compacts, and hence by Lemma 2.4, G_2 admits a right adjoint, denoted by G_3. Also, G_2 and G_3 preserve compacts. Repeating these arguments we get **Claim 2**.

Now Theorem 2.1 follows from Lemma 1.2. ■

Remark 2.10. *The unbounded ladder in Theorem 2.1 restricts to* $D^-(\mathrm{Mod})$*,* $D^b(\mathrm{Mod})$*,* $D^b(\mathrm{mod})$ *and* $K^b(\mathrm{proj})$*. In fact, since A, B and C are Gorenstein, all the functors in recollement* (2.1) *restrict to* $D^-(\mathrm{Mod})$ (*resp.* $D^b(\mathrm{Mod})$, $D^b(\mathrm{mod})$, $K^b(\mathrm{proj})$); *then by Lemmas* 2.4 *and* 2.7 *we see that all the functors in the ladder restrict to* $K^b(\mathrm{proj})$) *and* $D^b(\mathrm{mod})$*, respectively. By* [AHKLY, Prop. 4.11] *and* [AHKLY, Coroll. 4.9]*, we also see that all the functors in the ladder restrict to* $D^-(\mathrm{Mod})$ *and* $D^b(\mathrm{Mod})$*, respectively.*

3. **Ladders of period** 1

3.1. We have

Proposition 3.1. (1) *Let $\mathcal{C}'$, $\mathcal{C}$ and $\mathcal{C}''$ be triangulated categories with Serre functors. Then*

(i) *Any left* (*right*) *recollement* $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ *sits in a ladder of period* 1.

(ii) *Any recollement* $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ *sits in a ladder of period* 1.

(2) *Any recollement of triangulated category $\mathcal{C}$ with Serre functor sits in a ladder of period* 1.

Proof. (1)(i) Let $S_{\mathcal{C}'}$, $S_{\mathcal{C}}$ and $S_{\mathcal{C}''}$ be the right Serre functors of $\mathcal{C}', \mathcal{C}$ and $\mathcal{C}''$, respectively. Let

$$\mathcal{C}' \underset{i_0}{\overset{j_{-1}}{\leftrightarrows}} \mathcal{C} \underset{j_0}{\overset{i_{-1}}{\leftrightarrows}} \mathcal{C}''$$

be a left recollement. Applying Lemma 2.8 to adjoint pair (j_{-1}, i_0) we know that j_{-1} admits a left adjoint $i_{-2} = S_{\mathcal{C}}^{-1} i_0 S_{\mathcal{C}'} : \mathcal{C}' \to \mathcal{C}$, and that i_0 admits a right adjoint $j_1 = S_{\mathcal{C}'} j_{-1} S_{\mathcal{C}}^{-1} : \mathcal{C} \to \mathcal{C}'$. Similarly, i_{-1} admits a left adjoint $j_{-2} = S_{\mathcal{C}''}^{-1} j_0 S_{\mathcal{C}}$, and j_0 admits a right adjoint $i_1 = S_{\mathcal{C}} i_{-1} S_{\mathcal{C}''}^{-1}$. By induction we have

$$i_{2n-1} = S_{\mathcal{C}}^{n} i_{-1} S_{\mathcal{C}''}^{-n} : \mathcal{C}'' \longrightarrow \mathcal{C}, \qquad i_{2n} = S_{\mathcal{C}}^{n} i_0 S_{\mathcal{C}'}^{-n} : \mathcal{C}' \longrightarrow \mathcal{C},$$
$$j_{2n-1} = S_{\mathcal{C}'}^{n} j_{-1} S_{\mathcal{C}}^{-n} : \mathcal{C} \longrightarrow \mathcal{C}', \qquad j_{2n} = S_{\mathcal{C}''}^{n} j_0 S_{\mathcal{C}}^{-n} : \mathcal{C} \longrightarrow \mathcal{C}''.$$

By Lemma 1.1(2) $(\mathcal{C}', \mathcal{C}, \mathcal{C}'', j_{-1}, i_0, i_1, i_{-1}, j_0, j_1)$ is a recollement, and hence by Lemma 1.2 we get the desired unbounded ladder. Since $(S_{\mathcal{C}'}, S_{\mathcal{C}}, S_{\mathcal{C}''})$ is an equivalence from the 1st left recollement to the 0-th left recollement, this ladder is of period 1.

(ii) follows from (i) and the fact that one functor in an adjoint pair uniquely determines another.

(2) Let $(\mathcal{C}', \mathcal{C}, \mathcal{C}'', i^*, i_*, i^!, j_!, j^*, j_*)$ be a recollement, and S a right Serre functor of $\mathcal{C}$. Then $\mathcal{C}'$ has a right Serre functor $S_{\mathcal{C}'} = i^! S i_*$ with $S_{\mathcal{C}'}^{-1} = i^* S^{-1} i_*$; and $\mathcal{C}''$ has a right Serre functor $S_{\mathcal{C}''} = j^* S j_!$ with $S_{\mathcal{C}''}^{-1} = j^* S^{-1} j_*$ (see P. Jørgensen [J]. We stress that this result does not hold for left recollements). Then from (1)(ii) the assertion follows. ■

3.2. If A is Gorenstein, then $\underline{\mathcal{GP}(A)}$ is triangle-equivalent to the singularity category $D^b(\mathrm{mod}A)/K^b(\mathrm{proj}A)$ ([B, 4.4.1]). So the following gives a ladder of singularity categories of period 1.

Theorem 3.2. *Let B and C be Gorenstein algebras and ${}_CM_B$ a C-B-bimodule, such that $A = \left(\begin{smallmatrix} B & 0 \\ M & C\end{smallmatrix}\right)$ is Gorenstein. Then we have a ladder $(\underline{\mathcal{GP}(B)},\ \underline{\mathcal{GP}(A)},\ \underline{\mathcal{GP}(C)})$ of period* 1.

Proof. First, by dévissage each of $\underline{\mathcal{GP}(A)}$, $\underline{\mathcal{GP}(B)}$ and $\underline{\mathcal{GP}(C)}$ has a Serre functor. In fact, since A is Gorenstein, $\mathcal{GP}(A)$ is a resolving contravariantly finite subcategory of A-mod ([EJ, Thm. 11.5.1]; also [AR, Prop. 5.1]), and hence $\mathcal{GP}(A)$ is a resolving functorially finite subcategory of A-mod ([KS, Corol. 0.3]). Then by [AS, Thm. 2.4] $\mathcal{GP}(A)$ has relative Auslander-Reiten sequences. While $\mathcal{GP}(A)$ is a Frobenius category, by a direct argument we see that $\underline{\mathcal{GP}(A)}$ has Auslander-Reiten triangles, and hence by [RV, Thm. I 2.4] $\underline{\mathcal{GP}(A)}$ has a Serre functor.

Second, there is a left recollement

$$\underline{\mathcal{GP}(B)} \underset{i_*}{\overset{i^*}{\leftrightarrows}} \underline{\mathcal{GP}(A)} \underset{j^*}{\overset{j_!}{\leftrightarrows}} \underline{\mathcal{GP}(C)}$$

In fact, ${}_CM_B$ is compatible ([Z, Thm. 2.2(iv)]), and hence by [Z, Thm. 1.4], an A-module $(X_B, Y_C)_\phi$ is in $\mathcal{GP}(A)$ if and only if $\phi : Y \otimes_C M \to X$ is injective, $\mathrm{Coker}\phi \in \mathcal{GP}(B)$, and $Y \in \mathcal{GP}(C)$. So by [Z, Thm. 3.3] we get the left recollement above, where i^* sends $(X,Y)_\phi$ to $\mathrm{Coker}\phi$, i_* sends X to $(X,0)$, $j_!$ sends Y to $(Y \otimes_C M, Y)_{\mathrm{Id}}$, and j^* sends $(X,Y)_\phi$ to Y.

Now the assertion follows from Proposition 3.1(1)(i). ■

3.3. Recollement (1.1) is *splitting*, if $i^! \cong i^*$ and $j_* \cong j_!$. A splitting recollement clearly induces a ladder of period 1. The product $\mathcal{C}' \times \mathcal{C}''$ of triangulated categories $(\mathcal{C}', \mathcal{E}', T')$ and $(\mathcal{C}'', \mathcal{E}'', T'')$ is again triangulated, where the shift $T' \times T''$ is given by $(T' \times T'')(C', C'') := (T'C', T''C'')$, and $\mathcal{E}' \times \mathcal{E}''$ is the collection of triangles of $\mathcal{C}' \times \mathcal{C}''$ of the form $(X', X'') \xrightarrow{(u',u'')} (Y', Y'') \xrightarrow{(v',v'')} (Z', Z'') \xrightarrow{(w',w'')} (T'X', T''X'')$, where $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} T'X'$ belongs to $\mathcal{E}'$, and $X'' \xrightarrow{u''} Y'' \xrightarrow{v''} Z'' \xrightarrow{w''} T''X''$ belongs to $\mathcal{E}''$. Then $(\mathcal{C}', \mathcal{C}' \times \mathcal{C}'', \mathcal{C}'', p_1, \sigma_1, p_1, \sigma_2, p_2, \sigma_2)$ is a splitting recollement, where p_1 and p_2 are the projections, and σ_1 and σ_2 are the embeddings. As we see below, this gives all the splitting recollements, up to equivalences.

Proposition 3.3. *Let $(\mathcal{C}', \mathcal{C}, \mathcal{C}'', i^*, i_*, i^!, j_!, j^*, j_*)$ be a recollement of triangulated categories. Then the following are equivalent*:

(i) *it is splitting*;

(ii) $i^! \cong i^*$;

(iii) $j_* \cong j_!$;

(iv) *There is an equivalence* $(\mathrm{Id}_{\mathcal{C}'},\ F,\ \mathrm{Id}_{\mathcal{C}''}) : (\mathcal{C}', \mathcal{C}, \mathcal{C}'') \longrightarrow (\mathcal{C}', \mathcal{C}' \times \mathcal{C}'', \mathcal{C}'')$ *of recollements.*

A stable t-structure ([M]) on triangulated category $\mathcal{C}$ is a pair $(\mathcal{U}, \mathcal{V})$ of triangulated subcategories such that it is a t-structure ([BBD]), i.e., $\mathrm{Hom}(\mathcal{U}, \mathcal{V}) = 0$, and for $X \in \mathcal{C}$ there is a distinguished triangle $U \to X \to V \to U[1]$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. We call this triangle *the t-decomposition* of X, and U and V *the t-part* and *the t-free part* of X, respectively.

Lemma 3.4. (1) ([M], [IKM]) (i) *Given a diagram of triangle functors* $\mathcal{C}' \underset{i_*}{\overset{i^*}{\leftrightarrows}} \mathcal{C}$ *such that (i^*, i_*) is an adjoint pair and i_* is fully faithful, then* $(\mathrm{Ker}i^*,\ \mathrm{Im}i_*)$ *is a stable t-structure on $\mathcal{C}$, and $Y \to X \xrightarrow{\eta_X} i_*i^*X \to Y[1]$ is the t-decomposition of X, where $\eta : \mathrm{Id}_{\mathcal{C}} \to i_*i^*$ is the unit.*

(ii) *Given a diagram of triangle functors* $\mathcal{C}' \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{C}$ *such that $(i_*, i^!)$ is an adjoint pair and i_* is fully faithful, then* $(\mathrm{Im}i_*,\ \mathrm{Ker}i^!)$ *is a stable t-structure on $\mathcal{C}$, and $i_*i^!X \xrightarrow{\epsilon_X} X \to Z \to (i_*i^!X)[1]$ is the t-decomposition of X, where $\epsilon : i_*i^! \to \mathrm{Id}_{\mathcal{C}}$ is the counit.*

(2) *Let* $(\mathcal{Y}, \mathcal{Z})$ *be a stable* t*-structure on* $\mathcal{C}$ *with* $\operatorname{Hom}(\mathcal{Z}, \mathcal{Y}) = 0$. *Then* $F : \mathcal{C} \to \mathcal{Y} \times \mathcal{Z}$ *given by* $FX = (Y, Z)$ *is a triangle-equivalence, where* $Y \stackrel{u}{\to} X \to Z \to Y[1]$ *is the* t*-decomposition.*

Proof. (2) By assumption $\operatorname{Hom}_{\mathcal{C}}(Z[-1], Y) = 0$. By the exact sequence $\operatorname{Hom}_{\mathcal{C}}(X, Y) \stackrel{\operatorname{Hom}(u,Y)}{\longrightarrow} \operatorname{Hom}_{\mathcal{C}}(Y, Y) \to \operatorname{Hom}_{\mathcal{C}}(Z[-1], Y) = 0$ we see that u is a splitting monomorphism. Thus $X \cong Y \oplus Z$ ([H1, p.7]). It is straightforward that $F : \mathcal{C} \to \mathcal{Y} \times \mathcal{Z}$ given by $FX = (Y, Z)$ is a triangle-equivalence. ■

Proof of Proposition 3.3. (i) $\Longrightarrow$ (ii) and (iv) $\Longrightarrow$ (i) are obvious.

(ii) $\Longrightarrow$ (iii) : Suppose $i^! \cong i^*$. For $X \in \mathcal{C}$ and $Y'' \in \mathcal{C}''$ applying $\operatorname{Hom}_{\mathcal{C}}(-, j_!Y'')$ to the recollement triangle $j_!j^*X \to X \to i_*i^*X \to (j_!j^*X)[1]$ we get the exact sequence

$$\operatorname{Hom}(i_*i^*X, j_!Y'') \longrightarrow \operatorname{Hom}(X, j_!Y'') \longrightarrow \operatorname{Hom}(j_!j^*X, j_!Y'') \longrightarrow \operatorname{Hom}((i_*i^*X)[-1], j_!Y'').$$

By $\operatorname{Hom}(i_*i^*X, j_!Y'') \cong \operatorname{Hom}(i^*X, i^!j_!Y'') \cong \operatorname{Hom}(i^*X, i^*j_!Y'') = 0$ and $\operatorname{Hom}((i_*i^*X)[-1], j_!Y'') = 0$, we have $\operatorname{Hom}_{\mathcal{C}}(X, j_!Y'') \cong \operatorname{Hom}_{\mathcal{C}}(j_!j^*X, j_!Y'') \cong \operatorname{Hom}_{\mathcal{C}''}(j^*X, Y'')$, i.e., $(j^*, j_!)$ is an adjoint pair. While (j^*, j_*) is also an adjoint pair, so $j_* \cong j_!$.

(iii) $\Longrightarrow$ (ii) can be similarly proved.

(i) $\Longrightarrow$ (iv) : Assume that $i^! \cong i^*$ and $j_* \cong j_!$. Since $(i_*, i^!)$ is an adjoint pair, so is (i_*, i^*), and hence by Lemma 3.4(1)(ii) $(\operatorname{Im}i_*, \operatorname{Ker}i^*)$ is a stable t-structure. Since $\operatorname{Hom}(\operatorname{Ker}i^*, \operatorname{Im}i_*) = 0$ and the recollement triangle $i_*i^!X \to X \to j_*j^*X \to (i_*i^!X)[1]$ is the t-decomposition (since $j_*j^*X \in \operatorname{Im}j_* = \operatorname{Ker}i^! = \operatorname{Ker}i^*$ by the assumption), by Lemma 3.4(2) $\widetilde{F} : \mathcal{C} \to \operatorname{Im}i_* \times \operatorname{Ker}i^*$ given by $\widetilde{F}X = (i_*i^!X, j_*j^*X)$ is a triangle-equivalence. Since $\operatorname{Im}i_* \cong \mathcal{C}'$ and $\operatorname{Ker}i^* = \operatorname{Im}j_! \cong \mathcal{C}''$, we get a triangle-equivalence $F : \mathcal{C} \to \mathcal{C}' \times \mathcal{C}''$ with $FX = (i^!X, j^*X)$. Now it is straightforward that $(\operatorname{Id}_{\mathcal{C}'}, F, \operatorname{Id}_{\mathcal{C}''}) : (\mathcal{C}', \mathcal{C}, \mathcal{C}'') \to (\mathcal{C}', \mathcal{C}' \times \mathcal{C}'', \mathcal{C}'')$ is an equivalence of recollements. We omit the details. ■

Remark 3.5. (i) *A Hom-finite* k*-triangulated category* $(\mathcal{C}, [1])$ *is a* d*-Calabi-Yau category* ([Ke2]), *if there is a nonnegative integer* d*, such that the* d*-th shift* $[d]$ *is a right Serre functor of* $\mathcal{C}$*.*

By Lemma 2.8 any left (*right*) *recollement of Calabi-Yau category* $\mathcal{C}$ *sits in a splitting recollement. Thus any recollement of Calabi-Yau category is splitting.*

(ii) *If* $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ *is a recollement with* $\mathcal{C}$ Calabi-Yau, *then obviously so are* $\mathcal{C}'$ *and* $\mathcal{C}''$*. However, the converse is not true*: *otherwise,* $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ *is splitting by* (i); *but there are a lot of examples of non-splitting recollements* $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$*, where* $\mathcal{C}'$ *and* $\mathcal{C}''$ *are* Calabi-Yau. *For example, let* $A = \left(\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right)$ *with* k *a field. Then one has a recollement* $(D^b(k\text{-mod}), D^b(A\text{-mod}), D^b(k\text{-mod}))$ ([PS, Exam. 2.10]). *Note that* $D^b(k\text{-mod})$ *is* 0-Calabi-Yau *and that* $(D^b(k\text{-mod}), D^b(A\text{-mod}), D^b(k\text{-mod}))$ *is not splitting* (*otherwise,* $D^b(A\text{-mod})$ *is the product of two* Calabi-Yau *categories, and hence again* Calabi-Yau; *but* $D^b(A\text{-mod})$ *is not* Calabi-Yau).

Appendix: Proofs of lemmas in Section 1

We include proofs of lemmas in Section 1 only for convenience (although they are well-known, it seems that explicit proofs are not available in the literature).

Proof of Lemma 1.1. Since a right recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$ is a left recollement of $\mathcal{C}$ relative to $\mathcal{C}''$ and $\mathcal{C}'$, it follows that (2) can be deduced from (1). We include a proof of (ii) $\Longrightarrow$ (i) of (1).

Since (i^*, i_*) is an adjoint pair and i_* is fully faithful, by Lemma 3.4(1)(i) $Y \to X \stackrel{\eta_X}{\longrightarrow} i_*i^*X \to Y[1]$ is the t-decomposition of X respect to the t-structure $(\operatorname{Ker}i^*, \operatorname{Im}i_*)$. Similarly, by Lemma 3.4(1)(ii) $j_!j^*X \stackrel{\epsilon_X}{\longrightarrow} X \to Z \to (j_!j^*X)[1]$ is the t-decomposition of X respect to the t-structure $(\operatorname{Im}j_!, \operatorname{Ker}j^*)$. Since both $(\operatorname{Ker}i^*, \operatorname{Im}i_*)$ and $(\operatorname{Im}j_!, \operatorname{Ker}j^*)$ are t-structures and $\operatorname{Im}i_* = \operatorname{Ker}j^*$, it follows that $\operatorname{Ker}i^* = \operatorname{Im}j_!$, and the two t-decompositions above are isomorphic. From this one easily deduces that $j_!j^*X \stackrel{\epsilon_X}{\longrightarrow} X \stackrel{\eta_X}{\longrightarrow} i_*i^*X \to (j_!j^*X)[1]$ is a distinguished triangle. ■

Lemma A.1. (see e.g. [BBD], [M], [N3], [IKM]) *Let* $(\mathcal{U}, \mathcal{V})$ *be a stable* t*-structure on* $\mathcal{C}$*. Then*

(i) *there is a triangle-equivalence* $V_{\mathcal{V}} \circ \sigma_{\mathcal{U}} : \mathcal{U} \to \mathcal{C}/\mathcal{V}$*, where* $\sigma_{\mathcal{U}} : \mathcal{U} \hookrightarrow \mathcal{C}$ *is the embedding, and* $V_{\mathcal{V}} : \mathcal{C} \to \mathcal{C}/\mathcal{V}$ *is the Verdier functor. A quasi-inverse of* $V_{\mathcal{V}} \circ \sigma_{\mathcal{U}}$ *sends object* $X \in \mathcal{C}/\mathcal{V}$ *to its* t*-part.*

(ii) *there is a triangle-equivalence* $V_{\mathcal{U}} \circ \sigma_{\mathcal{V}} : \mathcal{V} \to \mathcal{C}/\mathcal{U}$*, where* $\sigma_{\mathcal{V}} : \mathcal{V} \hookrightarrow \mathcal{C}$ *is the embedding, and* $V_{\mathcal{U}} : \mathcal{C} \to \mathcal{C}/\mathcal{U}$ *is the Verdier functor. A quasi-inverse of* $V_{\mathcal{U}} \circ \sigma_{\mathcal{V}}$ *sends object* $X \in \mathcal{C}/\mathcal{U}$ *to its* t*-free part.*

Lemma A.2. ([AHKLY, Lemma 2.2]) *Let* $\mathcal{C}' \xrightarrow{F} \mathcal{C} \xrightarrow{G} \mathcal{C}''$ *be a sequence of triangle functors, such that* F *is fully faithful,* $\mathrm{Im}F = \mathrm{Ker}G$*, and* G *induces a triangle-equivalence* $\mathcal{C}/\mathrm{Ker}G \cong \mathcal{C}''$*. Then* F *has a right (resp. left) adjoint* F' *if and only if so does* G*.*

In this case, the right (resp. left) adjoint G' *of* G *is also fully faithful,* $\mathrm{Im}G' = \mathrm{Ker}F'$*, and* F' *induces a triangle-equivalence* $\mathcal{C}/\mathrm{Ker}F' \cong \mathcal{C}'$*.*

Proof. Using the opposite category, we only need to prove the right version.

By the universal property, G is the composition of the Verdier functor $\mathcal{C} \longrightarrow \mathcal{C}/\mathrm{Ker}G$ with the equivalence $\mathcal{C}/\mathrm{Ker}G \cong \mathcal{C}''$. Thus, for simplicity, without loss of the generality we may assume that $\mathcal{C}/\mathrm{Ker}G = \mathcal{C}''$ and G is just the Verdier functor $\mathcal{C} \to \mathcal{C}/\mathrm{Ker}G$.

$\Longleftarrow$: Assume that G has a right adjoint pair G', i.e., a Bousefield localization functor exists for the pair $\mathrm{Ker}G \subseteq \mathcal{C}$. Thus for $X \in \mathcal{C}$, by A. Neeman [N3, Prop. 9.1.8] there is a distinguished triangle $Z \to X \xrightarrow{\eta_X} G'GX \to Z[1]$ with $Z \in \mathrm{Ker}G = \mathrm{Im}F$, where $\eta : \mathrm{Id}_{\mathcal{C}} \to G'G$ is the unit. Thus $(\mathrm{Im}F,\ \mathrm{Im}G')$ is a t-structure on $\mathcal{C}$, which induces an adjoint pair $(\sigma, \widetilde{F'})$, where $\sigma : \mathrm{Im}F \to \mathcal{C}$ is the embedding, and $\widetilde{F'} : \mathcal{C} \to \mathrm{Im}F$ sends X to its t-part Z. Since $Z \in \mathrm{Im}F$ and F is fully faithful, there is a unique object (up to isomorphism) $Z' \in \mathcal{C}'$ such that $Z \cong FZ'$. Define $F' :\ \mathcal{C} \to \mathcal{C}'$ to be the functor given by $X \mapsto Z'$. Since $(\sigma, \widetilde{F'})$ is an adjoint pair and F is fully faithful, it is easy to see that (F, F') is an adjoint pair. By construction we have $\mathrm{Im}G' = \mathrm{Ker}F'$. Since $(\mathrm{Im}F,\ \mathrm{Im}G')$ is a t-structure, it follows from Lemma A.1(i) that $X \mapsto Z$ gives an triangle-equivalence $\mathcal{C}/\mathrm{Im}G' \to \mathrm{Im}F$; together with $\mathrm{Im}F \cong \mathcal{C}'$ we see that F' induces a triangle-equivalence $\mathcal{C}/\mathrm{Ker}F' \cong \mathcal{C}'$. Since $G(Z) = 0$, $G(\eta_X)$ is an isomorphism, and hence by $\epsilon_{GX} \circ G(\eta_X) = \mathrm{Id}_{\mathcal{C}''}$ (where $\epsilon : GG' \to \mathrm{Id}_{\mathcal{C}''}$ is the counit) we see that ϵ_{GX} is an isomorphism for each $X \in \mathcal{C}$. Since by assumption G is dense, $\epsilon : GG' \to \mathrm{Id}_{\mathcal{C}''}$ is a natural isomorphism of functors, and thus G' is fully faithful.

$\Longrightarrow$: Assume that F has a right adjoint pair F'. Then by Lemma 3.4(1)(ii) $(\mathrm{Im}F,\ \mathrm{Ker}F')$ is a t-structure on $\mathcal{C}$, with t-decomposition $FF'X \xrightarrow{\omega_X} X \to Y \to (FX')[1]$ of $X \in \mathcal{C}$, where $\omega : FF' \to \mathrm{Id}_{\mathcal{C}}$ is the counit. This t-structure induces an adjoint pair $(\widetilde{G}, \sigma)$, where $\widetilde{G} : \mathcal{C} \to \mathrm{Ker}F'$ sends X to its t-free part Y, and $\sigma : \mathrm{Ker}F' \to \mathcal{C}$ is the embedding. By Lemma A.1(ii) the functor $\widetilde{G'} :\ \mathcal{C}/\mathrm{Im}F \to \mathrm{Ker}F'$, which sends each object X to its t-free part Y, is a triangle-equivalence. Thus $G = \widetilde{G'}^{-1}\widetilde{G}$. Put $G' := \sigma\widetilde{G'} : \mathcal{C}/\mathrm{Im}F \to \mathcal{C}$, i.e., $G' : \mathcal{C}'' \to \mathcal{C}$. By construction G' is fully faithful and $\mathrm{Im}G' = \mathrm{Ker}F'$. By Lemma A.1(i) $\mathcal{C}/\mathrm{Ker}F' \to \mathrm{Im}F$ given by $X \mapsto FF'X$ is an triangle-equivalence; together with $\mathrm{Im}F \cong \mathcal{C}'$ we see that F' induces $\mathcal{C}/\mathrm{Ker}F' \cong \mathcal{C}'$. For $X \in \mathcal{C}$ and $C'' \in \mathcal{C}''$, since $(\widetilde{G}, \sigma)$ is an adjoint pair, we have

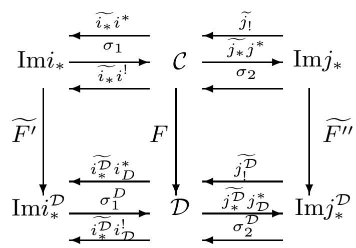
$$\mathrm{Hom}(GX, C'') = \mathrm{Hom}(\widetilde{G'}^{-1}\widetilde{G}X, C'') \cong \mathrm{Hom}_{\mathrm{Ker}F'}(\widetilde{G}X, \widetilde{G'}C'') \cong \mathrm{Hom}_{\mathcal{C}}(X, \sigma\widetilde{G'}C'') = \mathrm{Hom}(X, G'C''),$$

i.e., (G, G') is an adjoint pair. ∎

Proof of Lemma 1.2. It suffices to prove the "if" part. We denote the recollement (1.1) by $(\mathcal{C}', \mathcal{C}, \mathcal{C}'', j_{-1}, i_0, j_1, i_{-1}, j_0, i_1)$ (this labeling coincides with (1.2)), and assume that there is an infinite adjoint sequence $(\cdots, i_{-2},\ j_{-1}, i_0, j_1, i_2, \cdots)$. Since i_1 is fully faithful and j_1 has a right adjoint pair i_2, by applying Lemma A.2 to the sequence $\mathcal{C}'' \xrightarrow{i_1} \mathcal{C} \xrightarrow{j_1} \mathcal{C}'$ we get an adjoint pair (i_1, j_2), such that the right adjoint of j_1 is fully faithful (i.e., i_2 is fully faithful), $\mathrm{Im}i_2 = \mathrm{Ker}j_2$, and that j_2 induces a triangle-equivalence $\mathcal{C}/\mathrm{Ker}j_2 \cong \mathcal{C}''$. Applying Lemma A.2 to the sequence $\mathcal{C}' \xrightarrow{i_2} \mathcal{C} \xrightarrow{j_2} \mathcal{C}''$, and continuing this process we then get a ladder going downwards infinitely, by Lemma 1.1.

Going upwards, and by the same argument we get a ladder going upwards infinitely. Putting together we get an unbounded ladder containing recollement $(\mathcal{C}', \mathcal{C}, \mathcal{C}'', j_{-1}, i_0, j_1, i_{-1}, j_0, i_1)$. ∎

Proof of Lemma 1.3. (1) We only prove (ii) $\Longrightarrow$ (i). Any recollement $(\mathcal{C}', \mathcal{C}, \mathcal{C}'', i^*, i_*, i^!, j_!, j^*, j_*)$ induces an equivalence $(\widetilde{i_*}, \mathrm{Id}_{\mathcal{C}}, \widetilde{j_*}) : (\mathcal{C}', \mathcal{C}, \mathcal{C}'', i^*, i_*, i^!, j_!, j^*, j_*) \to (\mathrm{Im} i_*,\ \mathcal{C},\ \mathrm{Im} j_*,\ \widetilde{i_*} i^*,\ \sigma_1,\ \widetilde{i_*} i^!,\ \widetilde{j_!},\ \widetilde{j_*} j^*,\ \sigma_2)$ of recollements, where $\widetilde{i_*} : \mathcal{C}' \to \mathrm{Im} i_*$ and $\widetilde{j_*} : \mathcal{C}'' \to \mathrm{Im} j_*$ are the equivalences induced by i_* and j_*, respectively, σ_1 and σ_2 are embeddings, and $\widetilde{j_!} : \mathrm{Im} j_* \to \mathcal{C}$ is given by $j_* C'' \mapsto j_! C''$, $\forall\ C'' \in \mathcal{C}''$. By restriction we get $\widetilde{F'} : \mathrm{Im} i_* \xrightarrow{\sim} \mathrm{Im} i_*^{\mathcal{D}}$ and $\widetilde{F''} : \mathrm{Im} j_* \xrightarrow{\sim} \mathrm{Im} j_*^{\mathcal{D}}$. Thus, it suffices to prove that there is an equivalence



i.e., for $C \in \mathcal{C}$ and $j_* C'' \in \mathrm{Im} j_*$ with $C'' \in \mathcal{C}''$, there are natural isomorphisms: $F i_* i^* C \cong i_*^{\mathcal{D}} i_{\mathcal{D}}^* FC$, $F i_* i^! C \cong i_*^{\mathcal{D}} i_{\mathcal{D}}^! FC$, $F j_! C'' \cong \widetilde{j_!^{\mathcal{D}}} F j_* C''$, $F j_* j^* C \cong j_*^{\mathcal{D}} j_{\mathcal{D}}^* FC$. By the recollement triangle $i_* i^! C \to C \to j_* j^* C \to (i_* i^! C)[1]$ we get distinguished triangles

$$F i_* i^! C \to FC \to F j_* j^* C \to (F i_* i^! C)[1], \quad \text{and} \quad i_*^{\mathcal{D}} i_{\mathcal{D}}^! FC \to FC \to j_*^{\mathcal{D}} j_{\mathcal{D}}^* FC \to (i_*^{\mathcal{D}} i_{\mathcal{D}}^! FC)[1].$$

By the assumption, they are both the t-decompositions of FC respect to the t-structure $(\mathrm{Im} i_*^{\mathcal{D}}, \mathrm{Im} j_*^{\mathcal{D}})$, hence $F i_* i^! C \cong i_*^{\mathcal{D}} i_{\mathcal{D}}^! FC$ and $F j_* j^* C \cong j_*^{\mathcal{D}} j_{\mathcal{D}}^* FC$. Similarly, by $j_! j^* C \to C \to i_* i^* C \to (j_! j^* C)[1]$ we get $F j_! j^* C \cong j_!^{\mathcal{D}} j_{\mathcal{D}}^* FC$ and $F i_* i^* C \cong i_*^{\mathcal{D}} i_{\mathcal{D}}^* FC$. It remains to prove $F j_! C'' \cong \widetilde{j_!^{\mathcal{D}}} F j_* C''$. By $C'' \cong j^* j_* C''$ the functor $\widetilde{j_!}$ reads as $\widetilde{j_!} j_* C'' = j_! j^* j_* C''$. Since $F j_* C'' \in \mathrm{Im} j_*^{\mathcal{D}}$, we have $\widetilde{j_!^{\mathcal{D}}} F j_* C'' \cong j_!^{\mathcal{D}} j_{\mathcal{D}}^* F j_* C''$. It follows that $F j_! C'' \cong F j_! j^* j_* C'' \cong j_!^{\mathcal{D}} j_{\mathcal{D}}^* F j_* C'' \cong \widetilde{j_!^{\mathcal{D}}} F j_* C''$.

(2) We claim that the t-th recollement is equivalent to the 0-th one. In fact, by assumption there is equivalence $(F', F, F'') : (\mathcal{C}', \mathcal{C}, \mathcal{C}'', j_{2t-1}, i_{2t}, i_{2t-1}, j_{2t}) \to (\mathcal{C}', \mathcal{C}, \mathcal{C}'', j_{-1}, i_0, i_{-1}, j_0)$ of left recollements. It remains to prove that there are natural isomorphisms $F' j_{2t+1} \cong j_1 F$ and $F i_{2t+1} \cong i_1 F''$. Since $(i_{2t},\ j_{2t+1})$ and $(j_{2t},\ i_{2t+1})$ are adjoint pairs, it suffices to prove $(i_{2t}, F'^{-1} j_1 F)$ and $(j_{2t}, F^{-1} i_1 F'')$ are also adjoint pairs. Indeed, the first adjoint pair can be seen from (and the second one is similarly proved)

$$\mathrm{Hom}(X', F'^{-1} j_1 FY) \cong \mathrm{Hom}(F'X', j_1 FY) \cong \mathrm{Hom}(i_0 F'X', FY) \cong \mathrm{Hom}(F i_{2t} X', FY) \cong \mathrm{Hom}(i_{2t} X', Y).$$

Going downwards (resp. upwards) step by step, by the similar argument we see the assertion.

(3) follows from (1) and (2). ∎

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