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TWO FINITE-ELEMENT SCHEMES FOR STEADY CONVECTIVE HEAT TRANSFER WITH SYSTEM ROTATION AND VARIABLE THERMAL PROPERTIES

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We develop standard and mixed finite-element schemes for steady convective heat transfer with system rotation, temperature-dependent density and viscosity. Existence and convergence are proven for the two schemes. They are shown to be globally convergent and capable of capturing both nonsingular and singular solutions. The convergence rate is found to be optimal in H^1-norm and L^2-norm for nonsingular solutions.

1. INTRODUCTION

Convective heat transfer in rotating systems is encountered in many branches of engineering and science. Examples are those in compressor impellers, centrifuges, rotating heat exchangers, cooling channels of rotating machinery, particle separation devices, and atmosphere/ocean circulation [1–4]. The heating or cooling of fluids yields a nonisothermal temperature field, which in turn generates variations of fluid density and viscosity. The system rotation, in conjunction with heating or cooling, introduces centrifugal force, Coriolis force, and buoyancy force (due to the density variation in gravitational and centrifugal fields) into the momentum equations, which describe the relative motion of fluids with respect to an inertial frame of reference. Such body forces may induce a secondary flow in a plane perpendicular to the streamwise main flow. This could significantly affect the resistance to the fluid flow and convective heat transfer. As well, these forces may either enhance or impede each other in a nonlinear manner, depending on the directions of rotation and heat flux. This could result in a complicated bifurcation structure of the solution and the

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flow [5–10]. Therefore, the development of effective numerical schemes plays a critical role in capturing such complicated structures, including both nonsingular and singular solutions. In the present work, we develop two finite-element schemes and perform an analysis of existence and convergence for them.

Consider a noninertial frame of reference with an angular rotating velocity ω. Let Ω be a bounded domain in R^d, $d = 2$ or 3, with the smooth boundary Γ or a bounded convex domain with piecewise smooth boundary Γ. By using the usual Boussinesq approximation to deal with the density variation [1], the Navier-Stokes, continuity, and energy equations governing the steady flow and heat transfer are given as follows [1–3, 11–13]:

$$\begin{aligned} -2\nabla \cdot [\mu(T)\mathcal{D}(\mathbf{u})] + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\omega \times \mathbf{u} + \nabla p \\ = \beta(T - T_0)\nabla\phi \qquad \text{in } \Omega \end{aligned} \tag{1.1a}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega \tag{1.1b}$$

$$-\triangle T + \lambda \mathbf{u} \cdot \nabla T = g \qquad \text{in } \Omega \tag{1.1c}$$

with boundary conditions

$$\mathbf{u} = 0, \quad T = 0 \qquad \text{on } \Gamma \tag{1.2}$$

where

$$\nabla\phi = \omega \times (\omega \times \mathbf{r}) + \mathbf{f}_0 \tag{1.3}$$

$\mathbf{u} = (u_1, \ldots, u_d)^T$ and T are the velocity and the temperature, respectively; μ is the kinematic viscosity of the fluid (depending on the temperature); λ is the rotational Groshoff number; β is the coefficient of thermal expansion, and $\mathbf{r}$ is the position vector. T_0 is the reference temperature, $\mathbf{f}_0$ is the transitional acceleration of the origin of the noninertial frame, and p is a pseudo-pressure which absorbs any force residual implied by using the Boussinesq approximation. ∇ and $\nabla\cdot$ are the standard gradient and divergence operators. $\mathcal{D}(\mathbf{u})$ is a tensor defined by

$$\mathcal{D}(\mathbf{u}) = \left[\mathcal{D}_{ij}(\mathbf{u})\right]_{d\times d} \qquad \mathcal{D}_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad 1 \le i,\ j \le d$$

and

$$(\mathbf{u} \cdot \nabla)\mathbf{v} = \left(\sum_{j=1}^{d} u_j \frac{\partial v_1}{\partial x_j}, \ldots, \sum_{j=1}^{d} u_j \frac{\partial v_d}{\partial x_j}\right)^T$$

for any d-dimensional vector-valued functions $\mathbf{u}$ and $\mathbf{v}$.

There are extensive studies on the mathematical theory and finite-element schemes of classical Stokes equations and Navier-Stokes equations for isothermal flows of incompressible fluids. The reader is referred to [13–15] for mathematical

theory, to [16–19] for standard conforming and nonconforming finite-element schemes, to [20–25] for mixed finite-element schemes, to [26] for the penalty approximation, and to [13, 14, 27] for detailed discussion of finite-element methods in general. Most analyses of convergence were made for the Stokes equations. Breezi, Rappaz, and Raviart [28] developed a general theory for the finite-dimensional approximation of nonsingular solution of nonlinear problems. Their theory is mainly for those nonlinear problems with the nonlinearity not appearing in the terms of highest-order derivatives, and concludes that a majority of effective numerical schemes for solving linear Stokes equations can be applied to obtain nonsingular solution branches of nonlinear Navier-Stokes problems with an optimal convergence rate.

While some mathematical theory and analysis for convective heat transfer are available in the literature [11–13], the development and analysis of finite-element schemes are very limited [29]. In particular, the Breezi-Rappaz-Raviart theory fails to work for the general system (1.1) with temperature-dependent viscosity. There also appears no convergence analysis of numerical schemes for (1.1). This, with wide applications of rotating convective heat transfer, motivates the present work to develop two finite-element schemes for (1.1) and to analyze the existence and convergence for them. In particular, we address questions such as whether finite-element schemes are capable of capturing all solutions of (1.1) and under what conditions the approximate solutions possess an optimal order of accuracy.

In Section 2, we first develop the standard and mixed finite-element schemes for solving (1.1), and then establish the existence of solutions of nonlinear discrete finite-element systems for the two schemes. In Sections 3 and 4, we analyze the convergence of two schemes. It is shown that the solutions of nonlinear discrete finite-element systems from both schemes converge to different solutions of the system (1.1) as the mesh size of the finite-element triangulation tends to zero. This enables us to capture all solutions of (1.1) by finding all solutions of the nonlinear discrete finite-element systems. Furthermore, the approximate solution possesses an optimal accuracy of approximate for nonsingular solutions. We conclude the article by some concluding remarks in Section 5.

As in [30], $W^{k,q}(G)$ [for $k \geq 0$ and $1 \leq q \leq \infty$] represents the Sobolev spaces defined on G with the norm $\|\cdot\|_{W^{k,q}(G)}$. $W^{k,2}(\Omega) = H^k(\Omega)$ and $W^{0,q}(\Omega) = L^q(\Omega)$. $H_0^1(\Omega) = \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma\}$. $(\cdot,\cdot)$ denotes the standard inner products in $L^2(\Omega)$ for scale functions, in $[L^2(\Omega)]^d$ for d-dimensional vector-valued functions, or in $[L^2(\Omega)]^{d\times d}$ for $d \times d$-order matrix functions. For the sake of convenience, we omit G if $G = \Omega$ without causing confusion.

2. FINITE-ELEMENT SCHEMES

In this section, we first develop standard and mixed finite-element schemes for (1.1), and then establish the existence for them. Define bilinear forms

$$a(z;\mathbf{w},\mathbf{v}) = 2\int_\Omega \mu(z) \sum_{i,j=1}^{d} \mathcal{D}_{ij}(\mathbf{w})\mathcal{D}_{ij}(\mathbf{v})\, dx$$

$$b(\mathbf{u};\mathbf{w},\mathbf{v}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}\,dx = \sum_{i,j=1}^{d}\int_\Omega u_j\frac{\partial w_i}{\partial x_j}v_i\,dx$$

$$c(\mathbf{u};T,z) = \int_\Omega \mathbf{u}\cdot\nabla T z\,dx = \sum_{i=1}^{d}\int_\Omega u_i\frac{\partial T}{\partial x_i}z\,dx$$

$$d(T,z) = (\nabla T,\nabla z) = \sum_{i=1}^{d}\int_\Omega \frac{\partial T}{\partial x_i}\frac{\partial z}{\partial x_i}\,dx$$

and function spaces

$$\mathcal{X} = [H_0^1]^d \qquad \mathcal{X}_0 = \{\mathbf{v}\in\mathcal{X};\ \nabla\cdot\mathbf{v} = 0 \text{ in } \Omega\}$$

$$\mathcal{Y} = \{q\in L^2;\ (q,1) = 0\} \qquad \mathcal{Z} = H_0^1$$

The system (1.1) may thus be rewritten as a weakly variational form (mixed variational formulation):

Seek $(\mathbf{u},p,T)\in\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}$ such that

$$\begin{aligned} &a(T;\mathbf{u},\mathbf{v}) + b(\mathbf{u};\mathbf{u},\mathbf{v}) + 2(\omega\times\mathbf{u},\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) \\ &\quad = (\mathbf{f}+\beta T\nabla\phi,\mathbf{v}) \qquad \forall\mathbf{v}\in\mathcal{X} \end{aligned} \tag{2.1a}$$

$$(\nabla\cdot\mathbf{u},q) = 0 \qquad \forall q\in\mathcal{Y} \tag{2.1b}$$

$$d(T,z) + \lambda c(\mathbf{u};T,z) = (g,z) \qquad \forall z\in\mathcal{Z} \tag{2.1c}$$

where $\mathbf{f} = -\beta T_0\nabla\phi$.

Because $(\nabla p,\mathbf{v}) = 0$ for all $\mathbf{v}\in\mathcal{X}_0$, the term $(\nabla p,\mathbf{v})$ can be eliminated from (2.1) if we limit $\mathbf{v}\in\mathcal{X}_0$. Therefore, we have the standard variational formulation:

Seek $(\mathbf{u},T)\in\mathcal{X}_0\times\mathcal{Z}$ such that

$$a(T;\mathbf{u},\mathbf{v}) + b(\mathbf{u};\mathbf{u},\mathbf{v}) + 2(\omega\times\mathbf{u},\mathbf{v}) = (\mathbf{f}+\beta T\nabla\phi,\mathbf{v}) \quad \forall\mathbf{v}\in\mathcal{X}_0 \tag{2.2a}$$

$$d(T,z) + \lambda c(\mathbf{u};T,z) = (g,z) \quad \forall z\in\mathcal{Z} \tag{2.2b}$$

Let $\mathcal{K}^h$ be a family of finite-element triangulations of the domain Ω, with h denoting the largest mesh size of elements in $\mathcal{K}^h$. Introduce finite-element spaces $\mathcal{X}_0^h\times\mathcal{Z}^h\subset\mathcal{X}_0\times\mathcal{Z}$. Assume that the finite-element space $\mathcal{X}_0^h\times\mathcal{Z}^h$ possesses properties of approximation so that there exist constants r (≥ 1) and C such that, for $0\leq s\leq r$,

$$\inf_{\mathbf{v}^h\in\mathcal{X}_0^h}\left[\|\mathbf{v}-\mathbf{v}^h\|_{L^2} + h\|\mathbf{v}-\mathbf{v}^h\|_{H^1}\right] \leq Ch^{s+1}\|\mathbf{v}\|_{H^{s+1}} \qquad \forall\mathbf{v}\in\mathcal{X}_0\bigcap[H^{s+1}]^d \tag{2.3a}$$

$$\inf_{z^h \in \mathcal{Z}^h} [\|z - z^h\|_{L^2} + h\|z - z^h\|_{H^1}] \leq Ch^{s+1}\|z\|_{H^{s+1}} \qquad \forall z \in \mathcal{Z} \bigcap H^{s+1} \tag{2.3b}$$

A standard finite-element scheme is thus defined by:

Seek $(\mathbf{u}^h, T^h) \in \mathcal{X}_0^h \times \mathcal{Y}^h$ such that

$$a(T^h; \mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{u}^h; \mathbf{u}^h, \mathbf{v}^h) + 2(\omega \times \mathbf{u}^h, \mathbf{v}^h) = (\mathbf{f} + \beta T^h \nabla \phi, \mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathcal{X}_0^h \tag{2.4a}$$

$$d(T^h, z^h) + \lambda c(\mathbf{u}^h; T^h, z^h) = (g, z^h) \qquad \forall z^h \in \mathcal{Y}^h \tag{2.4b}$$

As (2.4) involves no pressure terms explicitly, it can be easily solved. However, it is often difficult to construct divergence-free finite-element spaces. The reader is referred to [16–19] for detailed discussion of divergence-free finite-element spaces.

Introduce finite-element spaces $\mathcal{X}^h \times \mathcal{Y}^h \times \mathcal{Z}^h \subset \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ defined on the triangulation $\mathcal{K}^h$ such that for a positive constant β_0, the inf-sup condition

$$\inf_{q^h \in \mathcal{Y}^h} \sup_{\mathbf{v}^h \in \mathcal{X}^h} \frac{(q_h, \nabla \cdot \mathbf{v}^h)}{\|q_h\|_{L^2} \|\mathbf{v}^h\|_{H^1}} \geq \beta_0 \tag{2.5}$$

holds. Assume that the finite-element space $\mathcal{X}^h \times \mathcal{Y}^h \times \mathcal{Z}^h$ possesses properties of approximation so that there exist constants r (≥ 1) and C such that for $0 \leq s \leq r$,

$$\inf_{\mathbf{v}^h \in \mathcal{X}^h} \left[\|\mathbf{v} - \mathbf{v}^h\|_{L^2} + h\|\mathbf{v} - \mathbf{v}^h\|_{H^1}\right] \leq Ch^{s+1}\|\mathbf{v}\|_{H^{s+1}} \qquad \forall \mathbf{v} \in \mathcal{X} \bigcap [H^{s+1}]^d \tag{2.6a}$$

$$\inf_{q^h \in \mathcal{Y}^h} \|q - q^h\|_{L^2} \leq Ch^s\|q\|_{H^s} \qquad \forall q \in \mathcal{Y} \bigcap H^s \tag{2.6b}$$

$$\inf_{z^h \in \mathcal{Z}^h} \left[\|z - z^h\|_{L^2} + h\|z - z^h\|_{H^1}\right] \leq Ch^{s+1}\|z\|_{H^{s+1}} \qquad \forall z \in \mathcal{Z} \bigcap H^{s+1} \tag{2.6c}$$

A mixed finite-element scheme can therefore be defined by:

Seek $(\mathbf{u}^h, p^h, T^h) \in \mathcal{X}^h \times \mathcal{Y}^h \times \mathcal{Z}^h$ such that

$$\begin{aligned} &a\left(T^h; \mathbf{u}^h, \mathbf{v}^h\right) + b\left(\mathbf{u}^h; \mathbf{u}^h, \mathbf{v}^h\right) + 2(\omega \times \mathbf{u}^h, \mathbf{v}^h) - \left(p^h, \nabla \cdot \mathbf{v}^h\right) \\ &\quad = \left(\mathbf{f} + \beta T^h \nabla \phi, \mathbf{v}^h\right) \qquad \forall \mathbf{v}^h \in \mathcal{X}^h \end{aligned} \tag{2.7a}$$

$$(\nabla \cdot \mathbf{u}^h, q^h) = 0 \qquad \forall\, q^h \in \mathcal{Y}^h \tag{2.7b}$$

$$d(T^h, z^h) + \lambda c(\mathbf{u}^h; T^h, z^h) = (g, z^h) \qquad \forall z^h \in \mathcal{Z}^h \tag{2.7c}$$

Examples of mixed finite-element spaces satisfying the inf-sup condition (2.5) are Bernardi-Raugel elements in [21], Hood-Taylor elements in [31], and Arnold-Breezi's "mini" elements in [32]. The reader is referred to [13, 14, 27] for discussion

of mixed finite-element schemes for solving Stokes equations and Navier-Stokes equations.

The existence of solutions for the standard finite-element scheme (2.4) can be established by Theorem 1.

Theorem 1. Assume that $0 < \mu_0 \leq \mu(z) \leq \mu^*$ for all $z \in R$ and $\mu(z)$ is uniformly Lipschitz continuous with respect to z. The finite-element scheme (2.4) has at least one solution.

Proof. By Korn's and Poincare's inequalities, there exist constants a_0 and C_0 such that

$$a_0\mu_0\|\nabla \mathbf{v}\|_{L^2}^2 \leq a(\mathbf{w};\mathbf{v},\mathbf{v}) \qquad \forall \mathbf{v} \in \mathcal{X} \tag{2.8}$$

and

$$\|\mathbf{v}\|_{H^1} \leq C_0\|\nabla \mathbf{v}\|_{L^2} \qquad \|z\|_{H^1} \leq C_0\|\nabla z\|_{L^2} \qquad \forall \mathbf{v} \in \mathcal{X},\ z \in \mathcal{Z} \tag{2.9}$$

Note that

$$b(\mathbf{w};\mathbf{v},\mathbf{v}) = 0 \qquad c(\mathbf{w};z,z) = 0 \qquad (\omega \times \mathbf{v},\mathbf{v}) = 0 \qquad \forall \mathbf{w} \in \mathcal{X}_0,\ \mathbf{v} \in \mathcal{X},\ z \in \mathcal{Z} \tag{2.10}$$

Therefore, if $(\mathbf{u}, T)$ is a solution of (2.2), then

$$a_0\mu_0\|\nabla \mathbf{u}\|_{L^2} \leq C_0\|\mathbf{f}\|_{H^{-1}} + \beta\|\nabla\phi\|_{L^\infty}\|T\|_{L^2} \qquad \|\nabla T\|_{L^2} \leq C_0\|g\|_{H^{-1}} \tag{2.11}$$

which implies that all solutions of (2.2) are in the closed set $\mathcal{M}(K_1, K_2) = \{(\mathbf{v}, z) \in \mathcal{X}_0 \times \mathcal{Z};\ \|\nabla \mathbf{v}\|_{L^2} \leq K_1,\ \|\nabla z\|_{L^2} \leq K_2\}$ for

$$K_2 \geq C_0\|g\|_{H^{-1}} \qquad K_1 \geq a_0^{-1}\mu_0^{-1}C_0[\|\mathbf{f}\|_{H^{-1}} + \beta K_2\|\nabla\phi\|_{L^\infty}] \tag{2.12}$$

Similarly, all solutions of (2.4) are also in the closed set $\mathcal{M}^h(K_1, K_2) = \mathcal{M}(K_1, K_2) \bigcap [\mathcal{X}_0^h \times \mathcal{Z}^h]$.

Define the nonlinear operator $\mathcal{L}(\mathbf{w}^h, z^h) = (\hat{\mathbf{w}}^h, \hat{z}^h)$ from $\mathcal{X}_0^h \times \mathcal{Z}^h$ into $\mathcal{X}_0^h \times \mathcal{Z}^h$ such that

$$\begin{aligned} &a(z^h;\hat{\mathbf{w}}^h,\mathbf{v}^h) + b(\mathbf{w}^h;\hat{\mathbf{w}}^h,\mathbf{v}^h) + 2(\omega \times \hat{\mathbf{w}}^h,\mathbf{v}^h) \\ &\quad = (\mathbf{f} + \beta z^h\nabla\phi,\mathbf{v}^h) \qquad \forall \mathbf{v}^h \in \mathcal{X}_0^h \end{aligned} \tag{2.13a}$$

$$d(\hat{z}^h,\psi^h) + \lambda c(\mathbf{w}^h;\hat{z}^h,\psi^h) = (g,\psi^h) \qquad \forall \psi^h \in \mathcal{Z}^h \tag{2.13b}$$

The operator $\mathcal{L}$ is uniquely defined and maps $\mathcal{M}^h(K_1, K_2)$ into itself if K_1 and K_2 satisfy (2.12). It is also continuous in $\mathcal{M}^h(K_1, K_2)$. By Bronwer's theorem of fixed points, the operator $\mathcal{L}$ has at least one fixed point. Therefore, (2.4) has at least one solution.

For the mixed finite-element scheme (2.7), we have following existence theorem.

Theorem 2. Assume that $0 < \mu_0 \le \mu(z) \le \mu^*$ for all $z \in R$ and $\mu(z)$ is uniformly Lipschitz continuous with respect to z. The mixed finite-element scheme (2.7) has at least one solution.

Proof. Let $\mathcal{X}_0^h = \{\mathbf{v}^h \in \mathcal{X}_0^h;\ \nabla \cdot \mathbf{v}^h = 0 \text{ in } \Omega\}$. Define the nonlinear operator $\mathcal{L}(\mathbf{w}^h, q^h, z^h) = (\hat{\mathbf{w}}^h, \hat{q}^h, \hat{z}^h)$ from $\mathcal{X}_0^h \times \mathcal{Y}^h \times \mathcal{Z}^h$ into $\mathcal{X}^h \times \mathcal{Y}^h \times \mathcal{Z}^h$ such that

$$\begin{aligned} &a(z^h; \hat{\mathbf{w}}^h, \mathbf{v}^h) + b(\mathbf{w}^h; \hat{\mathbf{w}}^h, \mathbf{v}^h) + 2(\omega \times \hat{\mathbf{w}}^h, \mathbf{v}^h) - (\hat{q}^h, \nabla \cdot \mathbf{v}^h) \\ &\quad = (\mathbf{f} + \beta z^h \nabla \phi, \mathbf{v}^h) \qquad \forall \mathbf{v}^h \in \mathcal{X}^h \end{aligned} \tag{2.14a}$$

$$(\nabla \cdot \hat{\mathbf{w}}^h, \varphi^h) = 0 \qquad \forall \varphi^h \in \mathcal{Y}^h \tag{2.14b}$$

$$d(\hat{z}^h, \psi^h) + \lambda c(\mathbf{w}^h; \hat{z}^h, \psi^h) = (g, \psi^h) \qquad \forall \psi^h \in \mathcal{Z}^h \tag{2.14c}$$

It is clear that the operator $\mathcal{L}$ is uniquely defined. Note that the inf-sup condition (2.5) and $(\nabla \cdot \hat{\mathbf{w}}^h, \varphi^h) = 0$ for any $\varphi^h \in \mathcal{Y}^h$ imply that $\nabla \cdot \hat{\mathbf{w}}^h \equiv 0$. Therefore, we have, from (2.7), the estimate

$$a_0 \mu_0 \|\nabla \hat{\mathbf{w}}^h\|_{L^2}^2 \le a(T^h; \hat{\mathbf{w}}^h, \hat{\mathbf{w}}^h) \le [\|\mathbf{f}\|_{H^{-1}} + \beta \|\nabla \phi\|_{L^\infty} \|\hat{z}^h\|_{L^2}] \|\hat{\mathbf{w}}^h\|_{H^1} \tag{2.15}$$

and

$$\|\nabla \hat{z}^h\|_{L^2} = d(\hat{z}^h, \hat{z}^h) \le \|g\|_{H^{-1}} \|\hat{z}^h\|_{H^1} \tag{2.16}$$

These lead to

$$a_0 \mu_0 \|\nabla \hat{\mathbf{w}}^h\|_{L^2} \le C_0 [\|\mathbf{f}\|_{H^{-1}} + \beta C_0 \|\nabla \phi\|_{L^\infty} \|g\|_{H^{-1}}] \qquad \|\nabla \hat{z}^h\|_{L^2} \le C_0 \|g\|_{H^{-1}} \tag{2.17}$$

On the other hand, the inf-sup condition yields another estimate:

$$\|\hat{q}^h\|_{L^2} \le \frac{1}{\beta_0} \sup_{\mathbf{v}^h \in \mathcal{X}^h} \frac{(\hat{q}^h, \nabla \cdot \mathbf{v}^h)}{\|\mathbf{v}^h\|_{H^1}} \tag{2.18}$$

For each $\mathbf{v}^h \in \mathcal{X}^h$,

$$\begin{aligned} (\hat{q}^h, \nabla \cdot \mathbf{v}^h) &= a(z^h; \hat{\mathbf{w}}^h, \mathbf{v}^h) + b(\mathbf{w}^h; \hat{\mathbf{w}}^h, \mathbf{v}^h) + 2(\omega \times \hat{\mathbf{w}}^h, \mathbf{v}^h) - (\mathbf{f}, \mathbf{v}^h) - (\beta \hat{z}^h \nabla \phi, \mathbf{v}^h) \\ &\le C_1 [\|\hat{\mathbf{w}}^h\|_{H^1} + \|\mathbf{w}^h\|_{H^1} \|\hat{\mathbf{w}}^h\|_{H^1} + \|\mathbf{f}\|_{H^{-1}} + \|\hat{z}^h\|_{L^2}] \|\mathbf{v}^h\|_{H^1} \end{aligned} \tag{2.19}$$

Therefore,

$$\|\hat{q}^h\|_{L^2} \le C_2 (1 + \|\mathbf{w}^h\|_{H^1})(\|\mathbf{f}\|_{H^{-1}} + \|g\|_{H^{-1}}) \tag{2.20}$$

Let $\mathcal{N}^h(K_1, K_2, K_3) = \{(\mathbf{v}^h, q^h, z^h) \in \mathcal{X}_0^h \times \mathcal{Y}^h \times \mathcal{Z}^h; \|\nabla \mathbf{v}\|_{L^2} \le K_1,\ \|q^h\|_{L^2} \le K_2, \|\nabla z^h\|_{L^2} \le K_3\}$. Therefore, the nonlinear operator $\mathcal{L}$ maps $\mathcal{N}^h(K_1, K_2)$ into itself if $K_3 \ge C_0\|g\|_{H^{-1}}, K_1 \ge a_0^{-1}\mu_0^{-1}C_0^2[\|\mathbf{f}\|_{H^{-1}} + \beta\|\nabla\phi\|_{L^\infty}K_3]$, and $K_2 \ge C_1(1+K_1)[\|\mathbf{f}\|_{H^{-1}} + \|g\|_{H^{-1}}]$. It is ready to show that the operator $\mathcal{L}$ is continuous in $\mathcal{N}^h(K_1, K_2, K_3)$. By Bronwer's theorem of fixed points, the operator $\mathcal{L}$ has at least one fixed point. The system (2.4) has thus at least one solution.

Remark. The nonlinear system (1.1) may have multiple solutions. Convergence analyses in the next two sections show that the solution of (2.4) or (2.7) converges to different solutions of the system (1.1) as $h \to 0$.

3. CONVERGENCE ANALYSIS FOR STANDARD FINITE-ELEMENT SCHEME

In this section, we analyze the convergence of the standard finite-element scheme (2.4). In particular, we prove that the solution sequence of (2.4) can be divided into several subsequences which strongly converge to different solutions of the system (1.1) in H^1-norm. For a nonsingular solution, the solution sequence strongly converges with an optimal accuracy in H^1-norm and L^2-norm.

Theorem 3. If the system (1.1) has multiple solutions, solution sequence $(\mathbf{u}^h, T^h)$ of the standard finite-element procedure (2.4) can be divided into several subsequences which strongly converge to different solutions $(\mathbf{u}, T)$ of the system (1.1) in $[H^s]^d \times H^1$ for any s in $0 \le s < 1$. If $\mathbf{u} \in [W^{1,q}]^d$ for some $q > d$, $\mathbf{u}^h$ is strongly convergent in $[H^1]^d$.

Proof. By (2.11), the sequence $(\mathbf{u}^h, T^h)$ is bounded in $[H^1]^d \times H^1$. It follows from the compact imbedding theory that the sequence $(\mathbf{u}^h, T^h)$ may be divided into several subsequences which are strongly convergent in $[H^s]^d \times H^s$ for any s in $0 \le s < 1$ and weakly convergent in $[H^1]^d \times H^1$. Let $\lim_{h\to 0}(\mathbf{u}^h, T^h) = (\bar{\mathbf{u}}, \bar{T})$. For each $\mathbf{v} \in \mathcal{X}_0$, we have

$$\begin{aligned}
&a(\bar{T}; \bar{\mathbf{u}}, \mathbf{v}) + b(\bar{\mathbf{u}}; \bar{\mathbf{u}}, \mathbf{v}) + 2(\omega \times \bar{\mathbf{u}}, \mathbf{v}) - (\mathbf{f} + \beta\bar{T}\nabla\phi, \mathbf{v}) \\
&\quad = [a(\bar{T}; \bar{\mathbf{u}}, \mathbf{v}) - a(\bar{T}; \mathbf{u}^h, \mathbf{v})] + [b(\bar{\mathbf{u}}; \bar{\mathbf{u}}, \mathbf{v}) - b(\bar{\mathbf{u}}; \mathbf{u}^h, \mathbf{v})] \\
&\qquad + [a(\bar{T}; \mathbf{u}^h, \mathbf{v}) - a(T^h; \mathbf{u}^h, \mathbf{v})] + [b(\bar{\mathbf{u}}; \mathbf{u}^h, \mathbf{v}) - b(\mathbf{u}^h; \mathbf{u}^h, \mathbf{v})] \\
&\qquad + 2(\omega \times (\bar{\mathbf{u}} - \mathbf{u}^h), \mathbf{v}) - (\beta(\bar{T} - T^h)\nabla\phi, \mathbf{v}) \\
&\qquad + \inf_{\mathbf{v}_h \in \mathcal{X}_0^h} [a(\bar{T}^h; \mathbf{u}^h, \mathbf{v} - \mathbf{v}^h) + b(\mathbf{u}^h; \mathbf{u}^h, \mathbf{v} - \mathbf{v}^h) \\
&\qquad + 2(\omega \times \mathbf{u}^h, \mathbf{v} - \mathbf{v}^h) + (\mathbf{f} + \beta T^h\nabla\phi, \mathbf{v}^h - \mathbf{v})]
\end{aligned} \tag{3.1}$$

By the strong convergence in H^s for any s in $0 \le s < 1$ and weak convergence in H^1, all terms in the brackets on the right-hand sides of (3.1) tend to zero as $h \to 0$. This means that

$$a(\bar{T}; \bar{\mathbf{u}}, \mathbf{v}) + b(\bar{\mathbf{u}}; \bar{\mathbf{u}}, \mathbf{v}) + 2(\omega \times \bar{\mathbf{u}}, \mathbf{v}) = (\mathbf{f} + \beta\bar{T}\nabla\phi, \mathbf{v}) \qquad \forall\ \mathbf{v} \in \mathcal{X}_0 \tag{3.2a}$$

$$\nabla \cdot \bar{\mathbf{u}} = 0 \qquad \text{in } \Omega \tag{3.2b}$$

Similarly,

$$d(\bar{T}, z) + \lambda c(\bar{\mathbf{u}}; \bar{T}, z) = (g, z) \qquad \forall z \in \mathcal{Z} \tag{3.3}$$

Define $\bar{p}$ such that

$$(\bar{p}, \nabla \cdot \mathbf{v}) = (\mathbf{f} + \beta \bar{T} \nabla \phi, \mathbf{v}) - a(\bar{T}; \bar{\mathbf{u}}, \mathbf{v}) - b(\bar{\mathbf{u}}; \bar{\mathbf{u}}, \mathbf{v}) - 2(\omega \times \bar{\mathbf{u}}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{X} \tag{3.4}$$

$(\bar{\mathbf{u}}, \bar{p}, \bar{T})$ is thus one solution of the system (1.1).

On the other hand, corresponding to the solution $(\bar{\mathbf{u}}, \bar{p}, \bar{T})$, we have, by (2.4) and (1.1),

$$\begin{aligned} &a(T^h; \mathbf{u}^h - \bar{\mathbf{u}}, \mathbf{v}^h) + b(\mathbf{u}^h; \mathbf{u}^h - \bar{\mathbf{u}}, \mathbf{v}^h) + 2(\omega \times (\mathbf{u}^h - \bar{\mathbf{u}}), \mathbf{v}^h) \\ &\quad = a(\bar{T}; \bar{\mathbf{u}}, \mathbf{v}^{\mathbf{h}}) - a(T^h; \bar{\mathbf{u}}, \mathbf{v}^{\mathbf{h}}) + b(\bar{\mathbf{u}} - \mathbf{u}^h; \bar{\mathbf{u}}, \mathbf{v}^h) \\ &\qquad + (\beta(T^h - \bar{T}) \nabla \phi, \mathbf{v}^h) \qquad \forall \mathbf{v}^h \in \mathcal{X}_0^h \end{aligned} \tag{3.5a}$$

$$d(T^h - \bar{T}, z^h) + \lambda c(\mathbf{u}^h; T^h - \bar{T}, z^h) = \lambda c(\bar{\mathbf{u}} - \mathbf{u}^h; \bar{T}, z^h) \qquad \forall z^h \in \mathcal{Y}^h \tag{3.5b}$$

which lead to

$$\begin{aligned} a_0 \mu_0 \|\nabla(\mathbf{u}^h - \bar{\mathbf{u}})\|_{L^2}^2 \le C \Bigg\{ &\sum_{i,j=1}^{d} \|(\mu(\bar{T}) - \mu(T^h)) \mathcal{D}_{ij}(\bar{\mathbf{u}})\|_{L^2}^2 + \sum_{i,j=1}^{d} \|\bar{u}_j - u_j^h\|_{L^4}^2 \left\| \frac{\partial \bar{u}_i}{\partial x_j} \right\|_{L^2}^2 \\ &+ \|T^h - \bar{T}\|_{L^2}^2 + \inf_{\mathbf{v}^h \in \mathcal{X}_0^h} \|\bar{\mathbf{u}} - \mathbf{v}^h\|_{H^1}^2 \Bigg\} \end{aligned} \tag{3.6a}$$

$$\|\nabla(T^h - \bar{T})\|_{L^2}^2 \le C \left\{ \sum_{i=1}^{d} \|\bar{u}_i - u_i^h\|_{L^4}^2 \left\| \frac{\partial \bar{T}}{\partial x_i} \right\|_{L^2}^2 + \inf_{z^h \in \mathcal{Z}^h} \|\bar{T} - z^h\|_{H^1}^2 \right\} \tag{3.6b}$$

Note that $\|\bar{\mathbf{u}} - \mathbf{u}^h\|_{L^4} \to 0$ and $\inf_{z^h \in \mathcal{Z}^h} \|\bar{T} - z^h\|_{H^1} \to 0$ as $h \to 0$. Therefore, $T^h \to \bar{T}$ strongly in H^1 by (3.6*b*). Furthermore, if $\bar{\mathbf{u}} \in W^{1,p}$ with $p > d$, the embedding theory concludes that

$$\sum_{i,j=1}^{d} \|(\mu(\bar{T}) - \mu(T^h)) \mathcal{D}_{ij}(\bar{\mathbf{u}})\|_{L^2}^2 \le C \|\bar{T} - T^h\|_{H^1}^2 \|\bar{\mathbf{u}}\|_{W^{1,p}}^2 \tag{3.7}$$

This, with (3.6*b*), implies that $\mathbf{u}^h \to \bar{\mathbf{u}}$ strongly in $[H^1]^d$, since $\|\bar{\mathbf{u}} - \mathbf{u}^h\|_{L^4} \to 0$, $\|\bar{T} - T^h\|_{H^1} \to 0$ and $\inf_{\mathbf{v}^h \in \mathcal{X}_0^h} \|\bar{\mathbf{u}} - \mathbf{v}^h\|_{H^1} \to 0$ as $h \to 0$.

For the analysis of convergence rate, name the solution $(\bar{\mathbf{u}}, \bar{p}, \bar{T})$ of the system (1.1) as a nonsingular solution if it is an isolated solution for $(\mathbf{f}, g)$ and the first-order differential approximation of the system (1.1) at the solution $(\bar{\mathbf{u}}, \bar{p}, \bar{T})$ is nonsingular.

This means that the linear system

$$
\begin{aligned}
&-2\sum_{j=1}^{d}\frac{\partial}{\partial x_j}[\mu(\bar{T})\mathcal{D}_{ij}(\mathbf{w})] - 2\sum_{j=1}^{d}\frac{\partial}{\partial x_j}[\mu'(\bar{T})\mathcal{T}\mathcal{D}_{ij}(\bar{\mathbf{u}})] \\
&\quad + \sum_{j=1}^{d}\bar{u}_j\frac{\partial w_i}{\partial x_j} + \sum_{j=1}^{d} w_j\frac{\partial \bar{u}_i}{\partial x_j} + 2(\omega\times\mathbf{w})_i + \frac{\partial q}{\partial x_i} - \beta\mathcal{T}\frac{\partial\phi}{\partial x_i} = \mathbf{f}_i^* \\
&\quad 1\le i\le d \qquad \text{in } \Omega
\end{aligned}
\tag{3.8a}
$$

$$
\nabla\cdot\mathbf{w} = f^* \qquad \text{in } \Omega \tag{3.8b}
$$

$$
-\triangle\mathcal{T} + \lambda\mathbf{w}\cdot\nabla\bar{T} + \lambda\bar{\mathbf{u}}\cdot\ \nabla\mathcal{T} = g^* \qquad \text{in } \Omega \tag{3.8c}
$$

$$
\mathbf{w} = 0,\ \mathcal{T} = 0 \qquad \text{on } \Gamma \tag{3.8d}
$$

is uniquely solvable for any $(\mathbf{f}^*, f^*, g^*) \in [H^{-1}]^d \times L^2 \times H^{-1}$.

Theorem 4. Let $(\mathbf{u}^h, T^h)$ strongly converge to one solution $(\bar{\mathbf{u}}, \bar{T})$ of the system (1.1) in $\mathcal{X}_0 \times \mathcal{Y}$ as $h \to 0$. An optimal prior error estimate

$$
\|\bar{\mathbf{u}} - \mathbf{u}^h\|_{L^2} + \|\bar{T} - T^h\|_{L^2} + h(\|\bar{\mathbf{u}} - \mathbf{u}^h\|_{H^1} + \|\bar{T} - T^h\|_{H^1}) \le Ch^{r+1} \tag{3.9}
$$

holds if the solution $(\bar{\mathbf{u}}, \bar{p}, \bar{T})$ of the system (1.1) is a nonsingular solution and satisfies regular conditions such that $\bar{\mathbf{u}} \in [H^{r+1}]^d$ and $\bar{T} \in H^{r+1}$. Here the constant C is dependent on the above-mentioned norms of the solution $(\bar{\mathbf{u}}, \bar{p}, \bar{T})$ of the system (1.1) but independent of mesh size h.

Proof. By (3.5), we have the estimate

$$
\begin{aligned}
\|\nabla(\bar{\mathbf{u}} - \mathbf{u}^h)\|_{L^2} \le C\{&(\|\bar{T} - T^h\|_{H^1} + \|\bar{\mathbf{u}} - \mathbf{u}^h\|_{L^2})\|\bar{\mathbf{u}}\|_{H^2} \\
&+ \|\bar{T} - T^h\|_{L^2} + \inf_{\mathbf{v}^h\in\mathcal{X}_0^h}\|\bar{\mathbf{u}} - \mathbf{v}^h\|_{H^1}\}
\end{aligned}
\tag{3.10a}
$$

$$
\|\nabla(\bar{T} - T^h)\|_{L^2} \le C\{\|\bar{\mathbf{u}} - \mathbf{u}^h\|_{L^2}\|\bar{T}\|_{H^2} + \inf_{z^h\in\mathcal{Z}^h}\|\bar{T} - z^h\|_{H^1}\} \tag{3.10b}
$$

Define the linear bounded operator $\mathcal{Q}$ from $\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}$ onto its dual space by

$$
\begin{aligned}
&\{\mathcal{Q}[(\mathbf{w}, q, \mathcal{T})], (\mathbf{v}, e, z)\} \\
&\quad = a(T; \mathbf{w}, \mathbf{v}) + 2[\mu'(\bar{T})\mathcal{T}\mathcal{D}(\bar{\mathbf{u}}), \mathcal{D}(\mathbf{v})] + b(\bar{\mathbf{u}}; \mathbf{w}, \mathbf{v}) \\
&\qquad + b(\mathbf{w}; \bar{\mathbf{u}}, \mathbf{v}) + 2(\omega\times\mathbf{w}, \mathbf{v}) - (\beta\mathcal{T}\nabla\phi, \mathbf{v}) - (q, \nabla\cdot\mathbf{v}) + (\nabla\cdot\mathbf{w}, e) \\
&\qquad + d(\mathcal{T}, z) + \lambda c(\mathbf{w}; \nabla\bar{T}, z) + \lambda c(\bar{\mathbf{u}}; \nabla\mathcal{T}, z) \qquad \forall\,(\mathbf{v}, e, z) \in \mathcal{X}\times\mathcal{Y}\times\mathcal{Z}
\end{aligned}
\tag{3.11}
$$

The first-order differential approximation (3.8) is equivalent to the following variational problem:

Seek $(\mathbf{w}, q, \mathcal{T}) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ such that

$$\{\mathcal{Q}[(\mathbf{w}, q, \mathcal{T})], (\mathbf{v}, e, z)\} = (\mathbf{f}^*, \mathbf{v}) + (f^*, e) + (g^*,\ z) \qquad \forall (\mathbf{v}, e, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \tag{3.12}$$

Define the adjoin operator $\mathcal{Q}^*$ of the operator $\mathcal{Q}$ by

$$\{\mathcal{Q}[(\mathbf{w}, q, \mathcal{T})], (\mathbf{v}, e, z)\} = \{(\mathbf{w}, q, \mathcal{T}), \mathcal{Q}^*[(\mathbf{v}, e, z)]\} \qquad \forall (\mathbf{w}, q, \mathcal{T}),\ (\mathbf{v}, e, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \tag{3.13}$$

Operator $\mathcal{Q}$ is an isomorphism from $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ onto $[H^{-1}]^d \times L^2 \times H^{-1}$. Therefore, its adjoin operator $\mathcal{Q}^*$ is also an isomorphism from $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ onto $[H^{-1}]^d \times L^2 \times H^{-1}$. The adjoin operator $\mathcal{Q}^*$ is equivalent to the adjoin problem of the system (3.8):

$$\begin{aligned} &-2\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left[\mu(\bar{T})\mathcal{D}_{ij}(\mathbf{v})\right] - \sum_{j=1}^{d} \bar{u}_j \frac{\partial v_i}{\partial x_j} + \sum_{i=1}^{d} v_i \frac{\partial \bar{u}_i}{\partial x_j} \\ &\quad - 2(\omega \times \mathbf{v})_i - \frac{\partial e}{\partial x_i} + \lambda z \frac{\partial \bar{T}}{\partial x_i} = \mathbf{f}^*_{1i}, \quad 1 \le i \le d \qquad \text{in } \Omega \end{aligned} \tag{3.14a}$$

$$\nabla \cdot \mathbf{v} = f_1^* \qquad \text{in } \Omega \tag{3.14b}$$

$$-\triangle z - \lambda \bar{\mathbf{u}} \cdot \nabla z + \mu'(\bar{T}) \sum_{i,j=1}^{d} \mathcal{D}_{ij}(\bar{\mathbf{u}})\mathcal{D}_{ij}(\mathbf{v}) - \beta \nabla \phi \cdot \mathbf{v} = g_1^* \qquad \text{in } \Omega \tag{3.14c}$$

$$\mathbf{v} = 0,\ z = 0 \qquad \text{on } \Gamma \tag{3.14d}$$

Introduce the auxiliary function $(\mathbf{w}, q, \mathcal{T}) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ such that

$$\begin{aligned} &-2\sum_{j=1}^{d} \frac{\partial}{\partial x_j}(\mu(\bar{T})\mathcal{D}_{ij}(\mathbf{w})) - \sum_{j=1}^{d} \bar{u}_j \frac{\partial w_i}{\partial x_j} + \sum_{i=1}^{d} w_i \frac{\partial \bar{u}_i}{\partial x_j} \\ &\quad - 2(\omega \times \mathbf{w})_i - \frac{\partial q}{\partial x_i} + \lambda \mathcal{T} \frac{\partial \bar{T}}{\partial x_i} \\ &= \bar{u}_i - u_i^h \qquad 1 \le i \le d \qquad \text{in } \Omega \end{aligned} \tag{3.15a}$$

$$\nabla \cdot \mathbf{w} = 0 \qquad \text{in } \Omega \tag{3.15b}$$

$$\begin{aligned} &-\triangle \mathcal{T} - \lambda \bar{\mathbf{u}} \cdot \nabla \mathcal{T} + 2\mu'(\bar{T}) \sum_{i,j=1}^{d} \mathcal{D}_{ij}(\bar{\mathbf{u}})\mathcal{D}_{ij}(\mathbf{w}) - \beta \nabla \phi \cdot \mathbf{w} \\ &= \bar{T} - T^h \qquad \text{in } \Omega \end{aligned} \tag{3.15c}$$

$$\mathbf{w} = 0,\ \mathcal{T} = 0 \qquad \text{on } \Gamma \tag{3.15d}$$

Since the operator $\mathcal{Q}^*$ is an isomorphism from $\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}$ onto its dual space, the system (3.15) has one unique solution $(\mathbf{w}, q, \mathcal{T})$ in $\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}$ and there exists a constant C such that

$$\|\mathbf{w}\|_{H^1} + \|q\|_{L^2} + \|\mathcal{T}\|_{H^1} \le C\big\{\|\mathbf{u}^h - \bar{\mathbf{u}}\|_{H^{-1}} + \|T^h - \bar{T}\|_{H^{-1}}\big\} \tag{3.16}$$

Rewrite the system (3.15) as

$$\begin{aligned} &-2\sum_{j=1}^{d}\frac{\partial}{\partial x_j}(\mu(\bar{T})\mathcal{D}_{ij}(\mathbf{w})) - \frac{\partial q}{\partial x_i} \\ &\quad = \sum_{j=1}^{d}\bar{u}_j\frac{\partial w_i}{\partial x_j} - \sum_{i=1}^{d} w_i\frac{\partial \bar{u}_i}{\partial x_j} + 2(\omega\times\mathbf{w})_i - \lambda\mathcal{T}\frac{\partial \bar{T}}{\partial x_i} \\ &\qquad + \bar{u}_i - u_i^h \qquad 1\le i\le d \qquad \text{in } \Omega \end{aligned} \tag{3.17a}$$

$$\nabla\cdot\mathbf{w} = 0 \qquad \text{in } \Omega \tag{3.17b}$$

$$\begin{aligned} &-\triangle\mathcal{T} - \lambda\bar{\mathbf{u}}\cdot\ \nabla\mathcal{T} \\ &\quad = -2\mu'(\bar{T})\sum_{i,j=1}^{d}\mathcal{D}_{ij}(\bar{\mathbf{u}})\mathcal{D}_{ij}(\mathbf{w}) + \beta\nabla\phi\cdot\mathbf{w} + \bar{T} - T^h \qquad \text{in } \Omega \end{aligned} \tag{3.17c}$$

$$\mathbf{w} = 0,\ \mathcal{T} = 0 \qquad \text{on } \Gamma \tag{3.17d}$$

Note that the domains Ω is smooth or convex. Therefore $(\mathbf{w}, q, \mathcal{T}) \in [H^2]^d \times H^1 \times H^2$ (see [14, 33]) and satisfies a prior estimate

$$\|\mathbf{w}\|_{H^2} + \|q\|_{H^1} + \|\mathcal{T}\|_{H^2} \le C\{\|\mathbf{u}^h - \bar{\mathbf{u}}\|_{L^2} + \|T^h - \bar{T}\|_{L^2}\} \tag{3.18}$$

Let $a'(T^h - \bar{T}, \mathbf{w}) = 2[\mu'(\bar{T})(T^h - \bar{T})\mathcal{D}(\bar{\mathbf{u}}), \mathcal{D}(\mathbf{w})]$. By using the auxiliary function $(\mathbf{w}, \mathcal{T})$, we have

$$\begin{aligned} &(\mathbf{u}^h - \bar{\mathbf{u}},\ \mathbf{u}^h - \bar{\mathbf{u}}) + (T^h - \bar{T}, T^h - \bar{T}) \\ &\quad = a(\bar{T}; \mathbf{u}^h - \bar{\mathbf{u}}, \mathbf{w}) + a'(T^h - \bar{T}, \mathbf{w}) + b(\bar{\mathbf{u}}; \mathbf{u}^h - \bar{\mathbf{u}}, \mathbf{w}) \\ &\qquad + b(\mathbf{u}^h - \bar{\mathbf{u}}; \bar{\mathbf{u}}, \mathbf{w}) + 2\big[\omega\times(\mathbf{u}^h - \bar{\mathbf{u}}), \mathbf{w}\big] - \big[\beta(T^h - \bar{T})\nabla\phi, \mathbf{w}\big] \\ &\qquad + d(T^h - \bar{T}, \mathcal{T}) + \lambda c(\bar{\mathbf{u}}, T^h - \bar{T}, \mathcal{T}) + \lambda c(\mathbf{u}^h - \bar{\mathbf{u}}; \bar{T}, \mathcal{T}) \\ &\quad = a(T^h; \mathbf{u}^h, \mathbf{w}) - a(\bar{T}; \bar{\mathbf{u}}, \mathbf{w}) + b(\mathbf{u}^h; \mathbf{u}^h, \mathbf{w}) - b(\bar{\mathbf{u}}; \bar{\mathbf{u}}, \mathbf{w}) \\ &\qquad + 2\big[\omega\times(\mathbf{u}^h - \bar{\mathbf{u}}), \mathbf{w}\big] - \big[\beta(T^h - \bar{T})\nabla\phi, \mathbf{w}\big] \\ &\qquad + d(T^h - \bar{T}, \mathcal{T}) + \lambda c(\mathbf{u}^h; T^h, \mathcal{T}) - \lambda c(\bar{\mathbf{u}}; \bar{T}, \mathcal{T}) \\ &\qquad + a(\bar{T}; \mathbf{u}^h, \mathbf{w}) - a(T^h; \mathbf{u}^h, \mathbf{w}) + a'(T^h - \bar{T}, \mathbf{w}) \\ &\qquad + b(\bar{\mathbf{u}} - \bar{\mathbf{u}}^h; \mathbf{u}^h - \bar{\mathbf{u}}, \mathbf{w}) + \lambda c(\mathbf{u}^h - \bar{\mathbf{u}}; \bar{T} - T^h, \mathcal{T}) \end{aligned} \tag{3.19}$$

To estimate the terms on the right-hand side of (3.19), we have

$$\begin{aligned}
&a(\bar{T};\mathbf{u}^h,\mathbf{w}) - a(T^h;\mathbf{u}^h,\mathbf{w}) + a'(T^h - \bar{T},\mathbf{w}) \\
&\quad = a(\bar{T};\bar{\mathbf{u}},\mathbf{w}) - a(T^h;\bar{\mathbf{u}},\mathbf{w}) + a'(T^h - \bar{T},\mathbf{w}) \\
&\qquad + a(\bar{T};\mathbf{u}^h - \bar{\mathbf{u}},\mathbf{w}) - a(T^h;\mathbf{u}^h - \bar{\mathbf{u}},\mathbf{w}) \\
&\quad \le C[\|\bar{T} - T^h\|_{H^1}^2 + \|\bar{T} - T^h\|_{H^1}\|\bar{\mathbf{u}} - \mathbf{u}^h\|_{H^1}]\|\mathbf{w}\|_{H^2}
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&b(\bar{\mathbf{u}} - \bar{\mathbf{u}}^h;\mathbf{u}^h - \bar{\mathbf{u}},\mathbf{w}) + \lambda c(\mathbf{u}^h - \bar{\mathbf{u}};\bar{T} - T^h,\mathcal{T}) \\
&\quad \le C\Big(\|\bar{\mathbf{u}} - \bar{\mathbf{u}}^h\|_{H^1}^2\|\mathbf{w}\|_{H^1} + \|\bar{\mathbf{u}} - \bar{\mathbf{u}}^h\|_{H^1}\|\bar{T} - T^h\|_{H^1}\|\mathcal{T}\|_{H^1}\Big)
\end{aligned} \tag{3.21}$$

By (3.5),

$$\begin{aligned}
&a(T^h;\mathbf{u}^h,\mathbf{w}) - a(\bar{T};\bar{\mathbf{u}},\mathbf{w}) + b(\bar{\mathbf{u}}^h;\mathbf{u}^h,\mathbf{w}) - b(\bar{\mathbf{u}};\bar{\mathbf{u}},\mathbf{w}) \\
&\qquad + 2\big[\omega \times (\mathbf{u}^h - \bar{\mathbf{u}}),\mathbf{w}\big] - \big[\beta(T^h - \bar{T})\nabla\phi,\mathbf{w}\big] \\
&\qquad + d(T^h - \bar{T},\mathcal{T}) + \lambda c(\mathbf{u}^h,T^h,\mathcal{T}) - \lambda c(\bar{\mathbf{u}};\bar{T},\mathcal{T}) \\
&\quad = \inf_{\mathbf{v}^h \in \mathcal{X}_0^h} \big\{a(T^h;\mathbf{u}^h,\mathbf{w} - \mathbf{v}^h) - a(\bar{T};\bar{\mathbf{u}},\mathbf{w} - \mathbf{v}^h) \\
&\qquad\qquad + b(\bar{\mathbf{u}}^h;\mathbf{u}^h,\mathbf{w} - \mathbf{v}^h) - b(\bar{\mathbf{u}};\bar{\mathbf{u}},\mathbf{w} - \mathbf{v}^h) \\
&\qquad\qquad + 2\big[\omega \times (\mathbf{u}^h - \bar{\mathbf{u}}),\mathbf{w} - \mathbf{v}^h\big] - (\beta(T^h - \bar{T})\nabla\phi,\mathbf{w} - \mathbf{v}^h)\big\} \\
&\qquad + \inf_{z^h \in \mathcal{Z}^h}[d(T^h - \bar{T},\mathcal{T} - z^h) + \lambda c(\mathbf{u}^h,T^h,\mathcal{T} - z^h) - \lambda c(\bar{\mathbf{u}};\bar{T},\mathcal{T} - z^h)] \\
&\quad \le Ch\big(\|\bar{T} - T^h\|_{H^1} + \|\bar{\mathbf{u}} - \mathbf{u}^h\|_{H^1}\big)(\|\mathbf{w}\|_{H^2} + \|\mathcal{T}\|_{H^2})
\end{aligned} \tag{3.22}$$

Substituting (3.20)–(3.22) into (3.19) leads to

$$\begin{aligned}
&\|\bar{\mathbf{u}} - \mathbf{u}^h\|_{L^2} + \|\bar{T} - T^h\|_{L^2} \\
&\quad \le C\{h(\|\bar{T} - T^h\|_{H^1} + \|\bar{\mathbf{u}} - \mathbf{u}^h\|_{H^1}) + (\|\bar{T} - T^h\|_{H^1} + \|\bar{\mathbf{u}} - \mathbf{u}^h\|_{H^1})^2\}
\end{aligned} \tag{3.23}$$

Substituting (3.23) into (3.10) yields

$$\begin{aligned}
&\|\bar{\mathbf{u}} - \mathbf{u}^h\|_{H^1} + \|\bar{T} - T^h\|_{H^1} \\
&\quad \le C\{h(\|\bar{T} - T^h\|_{H^1} + \|\bar{\mathbf{u}} - \mathbf{u}^h\|_{H^1}) + (\|\bar{T} - T^h\|_{H^1} + \|\bar{\mathbf{u}} - \mathbf{u}^h\|_{H^1})^2 + h^r\}
\end{aligned} \tag{3.24}$$

Note that $\|\bar{T} - T^h\|_{H^1} + \|\bar{\mathbf{u}} - \mathbf{u}^h\|_{H^1} \to 0$ as $h \to 0$. Therefore, we have the error estimate from (3.24) that

$$\|\bar{\mathbf{u}} - \mathbf{u}^h\|_{H^1} + \|\bar{T} - T^h\|_{H^1} \le Ch^r \tag{3.25}$$

for sufficiently small h. Here the constant C is dependent on the solution $(\bar{\mathbf{u}},\bar{T})$ of (1.1) but independent of h. Equations (3.23) and (3.25) lead to the prior error estimate (3.9).

4. CONVERGENCE ANALYSIS FOR MIXED FINITE-ELEMENT SCHEME

In this section, we analyze the convergence rate of the mixed finite-element scheme.

Theorem 5. If the system (1.1) has multiple solutions, the solution sequence $(\mathbf{u}^h, p^h, T^h)$ of the mixed finite-element scheme (2.7) can be divided into several subsequences which weakly converge to different solutions $(\mathbf{u}, p, T)$ of the system (1.1) in $[H^1]^d \times L^2 \times H^1$ as $h \to 0$, while $(\mathbf{u}^h, T^h)$ is strongly convergent in $[H^s]^d \times H^1$ for any s in $0 \le s < 1$. If $\mathbf{u} \in [W^{1,q}]^d$ for some $q > d$, $(\mathbf{u}^h, p^h)$ is strongly convergent in $[H^1]^d \times L^2$.

Proof. By Theorem 2, the solution sequence $(\mathbf{u}^h, p^h, T^h)$ of the system (2.7) is bounded in $[H^1]^d \times L^2 \times H^1$. The compact imbedding theory concludes, therefore, that the sequence $(\mathbf{u}^h, p^h, T^h)$ may be divided into several subsequences which are weakly convergent in $[H^1]^d \times L^2 \times H^1$ and $(\mathbf{u}^h, T^h)$ strongly convergent in $[H^s]^d \times H^s$ for any s in $0 \le s < 1$. Let $\lim_{h\to 0}(\mathbf{u}^h, p^h, T^h) = (\bar{\mathbf{u}}, \bar{p}, \bar{T})$ weakly in $[H^1]^d \times L^2 \times H^1$. It is clear that $\lim_{h\to 0}(\mathbf{u}^h, T^h) = (\bar{\mathbf{u}}, \bar{T})$ strongly in $[H^s]^d \times H^s$ for any s in $0 \le s < 1$. Similarly to the argument of theorem 3, we can prove that $(\bar{\mathbf{u}}, \bar{p}, \bar{T})$ satisfies

$$a(\bar{T}; \bar{\mathbf{u}}, \mathbf{v}) + b(\bar{\mathbf{u}}; \bar{\mathbf{u}}, \mathbf{v}) - (\bar{p}, \nabla \cdot \mathbf{v}) + 2(\omega \times \bar{\mathbf{u}}, \mathbf{v}) = (\mathbf{f} + \beta \bar{T} \nabla \phi, \mathbf{v}), \qquad \forall\, \mathbf{v} \in \mathcal{X} \tag{4.1a}$$

$$(\nabla \cdot \bar{\mathbf{u}}, q) = 0 \qquad \forall\, q \in \mathcal{Y} \tag{4.1b}$$

$$d(\bar{T}, z) + \lambda c(\bar{\mathbf{u}}; \bar{T}, z) = (g, z) \qquad \forall\, z \in \mathcal{Z} \tag{4.1c}$$

Hence, $(\bar{\mathbf{u}}, \bar{p}, \bar{T})$ is one solution of the system (1.1).

Corresponding to the solution $(\bar{\mathbf{u}}, \bar{p}, \bar{T})$, we have, by (2.7) and (1.1),

$$\begin{aligned} a(T^h; \mathbf{u}^h - \bar{\mathbf{u}}, \mathbf{v}^h) &+ b(\mathbf{u}^h; \mathbf{u}^h - \bar{\mathbf{u}}, \mathbf{v}^h) + 2(\omega \times (\mathbf{u}^h - \bar{\mathbf{u}}), \mathbf{v}^h) - (p^h - \bar{p}, \nabla \cdot \mathbf{v}^h) \\ &= a(\bar{T}; \bar{\mathbf{u}}, \mathbf{v}^{\mathbf{h}}) - a(T^h; \bar{\mathbf{u}}, \mathbf{v}^{\mathbf{h}}) + b(\bar{\mathbf{u}} - \mathbf{u}^h; \bar{\mathbf{u}}, \mathbf{v}^h) \\ &\quad + (\beta(T^h - \bar{T}) \nabla \phi, \mathbf{v}^h) \qquad \forall\, \mathbf{v}^h \in \mathcal{X}^h \end{aligned} \tag{4.2a}$$

$$\left[\nabla \cdot (\mathbf{u}^h - \bar{\mathbf{u}}), q^h\right] = 0 \qquad \forall\, q^h \in \mathcal{Y}^h \tag{4.2b}$$

$$d(T^h - \bar{T}, z^h) + \lambda c(\mathbf{u}^h; T^h - \bar{T}, z^h) = \lambda c(\bar{\mathbf{u}} - \mathbf{u}^h; \bar{T}, z^h) \qquad \forall\, z^h \in \mathcal{Z}^h \tag{4.2c}$$

A similar analysis to that of Theorem 3 concludes that $T^h \to \bar{T}$ strongly in H^1 as $h \to 0$ and $\mathbf{u}^h \to \bar{\mathbf{u}}$ strongly in $[H^1]^d$ $h \to 0$ if $\bar{\mathbf{u}} \in [W^{1,q}]^d$ for some $q > d$.

Note that

$$\begin{aligned}(p^h - \bar{p}, \nabla \cdot \mathbf{v}^h) &= a(T^h; \mathbf{u}^h, \mathbf{v}^h) - a(\bar{T}; \bar{\mathbf{u}}, \mathbf{v}^h) + b(\mathbf{u}^h; \mathbf{u}^h, \mathbf{v}^h) - b(\bar{\mathbf{u}}; \bar{\mathbf{u}}, \mathbf{v}^h) \\ &\quad + 2(\omega \times (\mathbf{u}^h - \bar{\mathbf{u}}), \mathbf{v}^h) - (\beta(T^h - \bar{T})\nabla\phi, \mathbf{v}^h) \\ &\le C[\|\bar{\mathbf{u}} - \mathbf{u}^h\|_{H^1} + \|\bar{T} - T^h\|_{H^1}]\|\mathbf{v}^h\|_{H^1} \end{aligned} \tag{4.3}$$

Therefore,

$$\begin{aligned}\|p^h - q^h\|_{L^2} &\le \frac{1}{\beta_0} \sup_{\mathbf{v}^h \in \mathcal{X}^h} \frac{(p^h - q^h, \nabla \cdot \mathbf{v}^h)}{\|\mathbf{v}^h\|_{H^1}} \\ &\le C[\|\bar{\mathbf{u}} - \mathbf{u}^h\|_{H^1} + \|\bar{T} - T^h\|_{H^1}\ ] + \|\bar{p} - q^h\|_{L^2} \qquad \forall\, q^h \in \mathcal{Y}^h \end{aligned} \tag{4.4}$$

which implies that $p^h \to \bar{p}$ strongly in L^2.

Theorem 6. Let $(\mathbf{u}^h, p^h, T^h)$ strongly converge to one solution $(\bar{\mathbf{u}}, \bar{p}, \bar{T})$ of the system (1.1) in $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ as $h \to 0$. An optimal prior error estimate

$$\|\bar{\mathbf{u}} - \mathbf{u}^h\|_{L^2} + \|\bar{T} - T^h\|_{L^2} + h(\|\bar{\mathbf{u}} - \mathbf{u}^h\|_{H^1} + \|\bar{T} - T^h\|_{H^1} + \|\bar{p} - p^h\|_{L^2}) \le Ch^{r+1} \tag{4.5}$$

holds if the solution $(\bar{\mathbf{u}}, \bar{p}, \bar{T})$ of the system (1.1) is nonsingular and satisfies $\bar{\mathbf{u}} \in [H^{r+1}]^d$, $\bar{T} \in H^{r+1}$ and $\bar{p} \in H^r$. Here the constant C is dependent on the above-mentioned norms on the solution $(\bar{\mathbf{u}}, \bar{p}, \bar{T})$ of the system (1.1) but independent of mesh size h.

Proof. Under the condition of Theorem 6, we have, by (4.2), the estimate

$$\begin{aligned}&\|\nabla(\mathbf{u}^h - \bar{\mathbf{u}})\|_{L^2}^2 + \|\nabla(T^h - \bar{T})\|_{L^2}^2 \\ &\quad \le C\{\|T^h - \bar{T}\|_{L^2}^2 + \|\mathbf{u}^h - \bar{\mathbf{u}}\|_{L^2}^2 + \inf_{\mathbf{v}^h \in \mathcal{X}^h} \|\bar{\mathbf{u}} - \mathbf{v}^h\|_{H^1}^2 + \inf_{z^h \in \mathcal{Z}^h} \|\bar{T} - z^h\|_{H^1}^2\} \\ &\qquad + \varepsilon\|p^h - \bar{p}\|_{L^2}^2 \end{aligned} \tag{4.6}$$

and

$$\|p^h - \bar{p}\|_{L^2} \le C[\|\bar{\mathbf{u}} - \mathbf{u}^h\|_{H^1} + \|\bar{T} - T^h\|_{H^1} + \inf_{q^h \in \mathcal{Y}^h} \|\bar{p} - q^h\|_{L^2}] \tag{4.7}$$

(4.6) and (4.7) lead to

$$\begin{aligned}&\|\mathbf{u}^h - \bar{\mathbf{u}}\|_{H^1} + \|T^h - \bar{T}\|_{H^1} + \|p^h - \bar{p}\|_{L^2} \\ &\quad \le C\{\|T^h - \bar{T}\|_{L^2} + \|\mathbf{u}^h - \bar{\mathbf{u}}\|_{L^2} + \inf_{\mathbf{v}^h \in \mathcal{X}^h} \|\bar{\mathbf{u}} - \mathbf{v}^h\|_{H^1} \\ &\qquad + \inf_{q^h \in \mathcal{Y}^h} \|\bar{p} - q^h\|_{L^2} + \inf_{z^h \in \mathcal{Z}^h} \|\bar{T} - z^h\|_{H^1}\} \end{aligned} \tag{4.8}$$

Introduce the auxiliary function $(\mathbf{w}, q, \mathcal{T}) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ to satisfy (3.15) with all right-hand terms replaced by the solution of the mixed finite-element scheme (2.7).

By using the auxiliary function $(\mathbf{w}, q, \mathcal{T})$, we have

$$
\begin{aligned}
&(\mathbf{u}^h - \bar{\mathbf{u}},\ \mathbf{u}^h - \bar{\mathbf{u}}) + (T^h - \bar{T}, T^h - \bar{T}) \\
&\quad = a(\bar{T}; \mathbf{u}^h - \bar{\mathbf{u}}, \mathbf{w}) + a'(T^h - \bar{T}, \mathbf{w}) + b(\bar{\mathbf{u}}; \mathbf{u}^h - \bar{\mathbf{u}}, \mathbf{w}) + b(\mathbf{u}^h - \bar{\mathbf{u}}; \bar{\mathbf{u}}, \mathbf{w}) \\
&\qquad - (p^h - \bar{p}, \nabla \cdot \mathbf{w}) + 2\big[\omega \times (\mathbf{u}^h - \bar{\mathbf{u}}), \mathbf{w}\big] - \big[\beta(T^h - \bar{T})\nabla\phi, \mathbf{w}\big] \\
&\qquad + d(T^h - \bar{T}, \mathcal{T}) + \lambda c(\bar{\mathbf{u}}, T^h - \bar{T}, \mathcal{T}) + \lambda c(\mathbf{u}^h - \bar{\mathbf{u}}; \bar{T}, \mathcal{T}) \\
&\quad = a(T^h; \mathbf{u}^h, \mathbf{w}) - a(\bar{T}; \bar{\mathbf{u}}, \mathbf{w}) + b(\mathbf{u}^h; \mathbf{u}^h, \mathbf{w}) - b(\bar{\mathbf{u}}; \bar{\mathbf{u}}, \mathbf{w}) \\
&\qquad - (p^h - \bar{p}, \nabla \cdot \mathbf{w}) + 2\big[\omega \times (\mathbf{u}^h - \bar{\mathbf{u}}), \mathbf{w}\big] - \big[\beta(T^h - \bar{T})\nabla\phi, \mathbf{w}\big] \\
&\qquad + d(T^h - \bar{T}, \mathcal{T}) + \lambda c(\mathbf{u}^h, T^h, \mathcal{T}) - \lambda c(\bar{\mathbf{u}}; \bar{T}, \mathcal{T}) \\
&\qquad + a(\bar{T}; \mathbf{u}^h, \mathbf{w}) - a(T^h; \mathbf{u}^h, \mathbf{w}) + a'(T^h - \bar{T}, \mathbf{w}) \\
&\qquad + b(\bar{\mathbf{u}} - \bar{\mathbf{u}}^h; \mathbf{u}^h - \bar{\mathbf{u}}, \mathbf{w}) + \lambda c(\mathbf{u}^h - \bar{\mathbf{u}}; \bar{T} - T^h, \mathcal{T}) \qquad (4.9)
\end{aligned}
$$

which yields, by a similar method to that used in proving Theorem 4, the estimate

$$
\begin{aligned}
&\|\mathbf{u}^h - \bar{\mathbf{u}}\|_{L^2} + \|T^h - \bar{T}\|_{L^2} \\
&\quad \le C\big[h(\|T^h - \bar{T}\|_{H^1} + \|\mathbf{u}^h - \bar{\mathbf{u}}\|_{H^1} + \|p^h - \bar{p}\|_{L^2}) \\
&\qquad + (\|T^h - \bar{T}\|_{H^1} + \|\mathbf{u}^h - \bar{\mathbf{u}}\|_{H^1})^2\big] \qquad (4.10)
\end{aligned}
$$

The error estimate (4.5) can thus be obtained by (4.8) and (4.10).

5. CONCLUDING REMARKS

A standard finite-element scheme and a mixed finite-element scheme are developed for convective heat transfer with system rotation and variable thermal properties. The existence and convergence of solutions are established for the two schemes. These schemes are capable of capturing not only the nonsingular solutions but also the singular solutions. For the nonsingular solutions, their convergence rate is optimal in H^1-norm and L^2-norm.

In particular, we define two nonlinear operators by (2.13) and (2.14). This leads to two iterative procedures: one based on the standard finite-element method and one based on the mixed finite-element method. The former is to find $(\mathbf{u}_k^h, T_k^h) = \mathcal{L}\big[(\mathbf{u}_{k-1}^h, T_{k-1}^h)\big]$ such that

$$
\begin{aligned}
&a(T_{k-1}^h; \mathbf{u}_k^h, \mathbf{v}^h) + b(\mathbf{u}_{k-1}^h; \hat{\mathbf{u}}_k^h, \mathbf{v}^h) + 2(\omega \times \hat{\mathbf{u}}_k^h, \mathbf{v}^h) \\
&\quad = (\mathbf{f} + \beta T_k^h \nabla\phi, \mathbf{v}^h) \qquad \forall \mathbf{v}^h \in \mathcal{X}_0^h \qquad (5.1a)
\end{aligned}
$$

$$
d(T_k^h, \psi^h) + \lambda c(\mathbf{u}_{k-1}^h; T_k^h, \psi^h) = (g, \psi^h) \qquad \forall \psi^h \in \mathcal{Z}^h \qquad (5.1b)
$$

starting from an initial approximation $(\mathbf{u}_0^h, T_0^h)$ in $\mathcal{M}^h(K_1, K_2)$ for $k = 1,\ 2, \ldots$. The latter is to find $(\mathbf{u}_k^h, p_k^h, T_k^h) = \mathcal{L}\big[(\mathbf{u}_{k-1}^h, p_{k-1}^h, T_{k-1}^h)\big]$ such that

$$
\begin{aligned}
&a(T_{k-1}^h; \mathbf{u}_k^h, \mathbf{v}^h) + b(\mathbf{u}_{k-1}^h; \mathbf{u}_k^h, \mathbf{v}^h) + 2(\omega \times \mathbf{u}_k^h, \mathbf{v}^h) - (q_k^h, \nabla \cdot \mathbf{v}^h) \\
&\quad = (\mathbf{f} + \beta T_{k-1}^h \nabla\phi, \mathbf{v}^h) \qquad \forall \mathbf{v}^h \in \mathcal{X}^h \qquad (5.2a)
\end{aligned}
$$

$$(\nabla \cdot \hat{\mathbf{u}}_k^h, \varphi^h) = 0 \qquad \forall\, \varphi^h \in \mathcal{Y}^h \tag{5.2b}$$

$$d(T_k^h, \psi^h) + \lambda c(\mathbf{u}_{k-1}^h; T_k^h, \psi^h) = (g, \psi^h) \qquad \forall\ \psi^h \in \mathcal{Z}^h \tag{5.2c}$$

starting from an initial approximation $(\mathbf{u}_0^h, p_0^h, T_0^h)$ in $\mathcal{N}^h(K_1, K_2, K_3)$ for $k = 1,\ 2, \ldots$.

The convergence analyses show that the solution sequence $\left[(\mathbf{u}_k^h, T_k^h)\right]_{k=0}^{\infty}$ defined by (5.1) and $\left[(\mathbf{u}_k^h, p_k^h, T_k^h)\right]_{k=0}^{\infty}$ defined by (5.2) can be divided into several subsequences which strongly converge to different solutions of systems (2.4) and (2.7), respectively. Therefore, all solutions of systems (2.4) and (2.7) can be obtained by stating iterative procedures (5.1) and (5.2) with different initial functions. While the convergent rates of iterative procedures (5.1) and (5.2) may be lower in general, they are globally convergent. Therefore, the iterative procedures (5.1) and (5.2) can be used to find better initial approximations for other schemes of higher convergent rate, such as Newton-type and conjugate-gradient-type schemes.

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