Introduction to Kähler Geometry

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Preface

This is the note of a course given at Spring 2018 in East China Normal University.

Chapter 1

Kähler manifolds

1.1 Manifolds

Let M be an $2n$-dimensional manifold. Let TM be the tangent vector bundle over M. Let $\mathrm{End}(TM)$ be the real vector bundle over M such that the fibre $\mathrm{End}(TM)|_x$ for any $x \in M$ is canonically isomorphic to $\mathrm{End}(TM|_x)$. Let E be a vector bundle over M. Let $\mathscr{C}^\infty(M, E)$ be the space of smooth sections of E on M. Let $\Omega^r(M, E)$ be the smooth r-forms on M with values in E.

Definition 1.1.1. The manifold M is called a **almost complex manifold** if there exists $J \in \mathscr{C}^\infty(M, \mathrm{End}(TM))$ such that $J^2 = -\mathrm{Id}$. The endomorphism J is called the **almost complex structure** of TM.

For $x \in M$, the almost complex structure J induces a splitting of complex vector spaces,

$$T_xM \otimes_{\mathbb{R}} \mathbb{C} = T_x^{(1,0)}M \oplus T_x^{(0,1)}M, \tag{1.1.1}$$

where $T_x^{(1,0)}M$ and $T_x^{(0,1)}M$ are the eigenspaces of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. Since J is smooth, $T^{(1,0)}M = \{T_x^{(1,0)}M\}_{x\in M}$ and $T^{(0,1)}M = \{T_x^{(0,1)}M\}_{x\in M}$ are vector bundles.

A continuous map $\pi : E \to M$ between two Hausdorff spaces is called a complex vector bundle of rank r if for any $x \in M$, $E_x := \pi^{-1}(x)$ is a complex vector space of dimension r and there is a neighbourhood U of x and a homeomorphism

$$\psi : \pi^{-1}(U) \to U \times \mathbb{C}^r \tag{1.1.2}$$

such that for any $p \in U$, $\psi(E_p) = \{p\} \times \mathbb{C}^r$ and $\psi|_{E_p}$ is a complex linear space isomorphism. The pair (U, ψ) is called a local trivialization. For a complex vector bundle $\pi : E \to M$, E is called the total space and M the base space. We often say that E is a vector bundle over M. Notice that for two local trivializations (U_i, ψ_i) and (U_j, ψ_j), the map $\psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times \mathbb{C}^r \to (U_i \cap U_j) \times \mathbb{C}^r$ induces a transition map

$$\psi_{ij} : U_i \cap U_j \to GL(r, \mathbb{C}). \tag{1.1.3}$$

When $r = 1$, we will call E a complex line bundle.

It is easy to see that The eigenbundles $T^{(1,0)}M$ and $T^{(0,1)}M$ are complex vector bundles over M.

Let $T^{*(1,0)}M$ and $T^{*(0,1)}M$ be the dual bundles respectively. We denote by

$$\Omega^{p,q}(M) := \mathscr{C}^\infty(M, \Lambda^p(T^{*(1,0)}M) \otimes \Lambda^q(T^{*(0,1)}M)). \tag{1.1.4}$$

By (1.1.1), we have

$$\Omega^k(M, \mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(M). \tag{1.1.5}$$

If $\alpha \in \Omega^{p,q}(M)$, we say that α is a (p, q)-form. For $\alpha \in \Omega^{p,q}(M)$, from (1.1.5), we have $d\alpha = \sum_{j+k=p+q+1} (d\alpha)^{(j,k)}$, where $(d\alpha)^{(j,k)} \in \Omega^{j,k}(M)$. We define

$$\partial\alpha = (d\alpha)^{(p+1,q)}, \quad \bar{\partial}\alpha = (d\alpha)^{(p,q+1)}. \tag{1.1.6}$$

Let g be any Riemannian metric on TM compatible with J, i.e.,

$$g(JU, JV) = g(U, V) \tag{1.1.7}$$

for any $U, V \in T_xM$, $x \in M$.

Take $e_1 \in T_xM$. Then $g(Je_1, e_1) = 0$ by $J^2 = -\mathrm{Id}$. Take orthonormal vectors $e_1, \cdots, e_k \in T_xM$. If $e_{k+1} \notin \mathrm{span}\{e_1, Je_1, \cdots e_k, Je_k\}$, then so is Je_{k+1}. So we can construct an orthonormal basis of T_xM with the form $\{e_1, \cdots, e_{2n}\}$ such that $e_{n+i} = Je_i$, $1 \leq i \leq n$. Moreover,

$$\begin{aligned} T_x^{(1,0)}M &= \mathbb{C}\{e_1 - \sqrt{-1}e_{n+1}, \cdots, e_n - \sqrt{-1}e_{2n}\}, \\ T_x^{(0,1)}M &= \mathbb{C}\{e_1 + \sqrt{-1}e_{n+1}, \cdots, e_n + \sqrt{-1}e_{2n}\}. \end{aligned} \tag{1.1.8}$$

We also denote by g the $\mathbb{C}$-bilinear form on $TM \otimes_{\mathbb{R}} \mathbb{C}$ induced by g on TM. From (1.1.8), we could see that g vanishes on $T^{(1,0)}M \times T^{(1,0)}M$ and $T^{(0,1)}M \times T^{(0,1)}M$. That is, for any $Z, Z' \in T^{(1,0)}M$,

$$g(Z, Z') = g(\bar{Z}, \overline{Z'}) = 0. \tag{1.1.9}$$

Let

$$\theta_j = \frac{1}{\sqrt{2}}(e_j - \sqrt{-1}e_{n+j}), \quad \bar{\theta}_j = \frac{1}{\sqrt{2}}(e_j + \sqrt{-1}e_{n+j}), \quad 1 \le j \le n. \tag{1.1.10}$$

Then $\{\theta_j\}_{1\le j\le n}$ and $\{\bar{\theta}_j\}_{1\le j\le n}$ form orthonormal basis of complex vector spaces $T_x^{(1,0)}M$ and $T_x^{(0,1)}M$ respectively. Let $\{\theta^j\}_{1\le j\le n}$ be the dual frame of $\{\theta_j\}_{1\le j\le n}$. Let e^i be the dual of e_i. We have

$$\theta^j = \frac{1}{\sqrt{2}}(e^j + \sqrt{-1}e^{n+j}), \quad \bar{\theta}^j = \frac{1}{\sqrt{2}}(e^j - \sqrt{-1}e^{n+j}), \quad 1 \le j \le n. \tag{1.1.11}$$

From (1.1.11), we have

$$(\sqrt{-1})^n \theta^1 \wedge \cdots \wedge \theta^n \wedge \bar{\theta}^1 \wedge \cdots \wedge \bar{\theta}^n = e^1 \wedge \cdots \wedge e^{2n}. \tag{1.1.12}$$

Proposition 1.1.2. *The almost complex manifold is orientable.*

Proof. Let $\{e'_1, \cdots, e'_{2n}\}$ be another basis of TM. We may assume that $e'_{n+i} = Je'_i$, $1 \le i \le n$. Then there exists $\alpha \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ such that $\theta'^1 \wedge \cdots \wedge \theta'^n = \alpha \cdot \theta^1 \wedge \cdots \wedge \theta^n$. Since $\bar{\theta}'^1 \wedge \cdots \wedge \bar{\theta}'^n = \bar{\alpha} \cdot \bar{\theta}^1 \wedge \cdots \wedge \bar{\theta}^n$, from (1.1.12), we have $e'^1 \wedge \cdots \wedge e'^{2n} = |\alpha|^2 e^1 \wedge \cdots \wedge e^{2n}$.

So our proposition follows from $|\alpha|^2 > 0$. □

Let ω be the real 2-form defined by

$$\omega(X, Y) = g(JX, Y) \tag{1.1.13}$$

for vector fields X, Y. Note that $\omega(X,Y) = -\omega(Y,X)$ follows from $g(JX,Y) = g(J^2X, JY) = -g(JY, X)$. Since g is non-degenerate, by (1.1.9), ω is a non-degenerate real $(1,1)$-form. Since g is compatible with J, so is ω.

Conversely, we have the following lemma.

Lemma 1.1.3. *If there exists a non-degenerate real 2-form ω on M, then M is almost complex.*

Proof. Choose a metric g on TM. Since ω is real and non-degenerate, there exists invertible skew-symmetric $A \in \mathscr{C}^\infty(M, \mathrm{End}(TM))$ such that

$$\omega(X, Y) = g(AX, Y) \tag{1.1.14}$$

for any vector fields X, Y. Since $-A^2$ is positive definite, $(-A^2)^{1/2}$ is invertible. Then our lemma follows by defining $J = ((-A^2)^{1/2})^{-1}A$. □

Note that fixing a non-degenerate real $(1,1)$-form ω on almost complex manifold (M, J) compatible with J, we could construct a Riemannian metric compatible with J by

$$g(X, Y) = \omega(X, JY) \tag{1.1.15}$$

for vector fields X and Y.

Definition 1.1.4. A triple (g, J, ω) satisfying (1.1.7) and (1.1.13) is called a **compatible triple** of almost complex manifold M.

Definition 1.1.5. A **complex manifold** is a manifold with an atlas of charts to the open unit disk in $\mathbb{C}^n$, such that the transition maps are holomorphic.

Proposition 1.1.6. *The complex manifold is almost complex.*

Proof. Let $\{z^1, \cdots z^n\}$ be a local chart of a complex manifold M with complex dimension n. Denote by $z^k = x^k + \sqrt{-1}y^k$. Then $\{x^1, y^1, \cdots, x^n, y^n\}$ is a local chart of M as real manifold. So $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}\}$ is a real basis of TM. For $x \in M$, the linear transform $J_x : T_xM \to T_xM$ is defined by

$$J_x\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial y^k}, \quad J_x\left(\frac{\partial}{\partial y^k}\right) = -\frac{\partial}{\partial x^k}. \tag{1.1.16}$$

Obviously, $J_x^2 = -\operatorname{Id}$.

We claim that the definition of J_x does not depend on the coordinates. In fact, let $\{\theta^1, \cdots, \theta^n\}$ be another local complex chart of M. Then by the definition of complex manifold, z^j is holomorphic on θ^k for any $1 \leq j, k \leq n$. That is, for $\theta^k = u^k + \sqrt{-1}v^k$, we have $\frac{\partial x^j}{\partial u^k} = \frac{\partial y^j}{\partial v^k}$, $\frac{\partial x^j}{\partial v^k} = -\frac{\partial y^j}{\partial u^k}$ (Cauchy-Riemann equation). Therefore,

$$\begin{aligned}
J_x\left(\frac{\partial}{\partial u^k}\right) &= J_x\left(\frac{\partial x^j}{\partial u^k}\frac{\partial}{\partial x^j} + \frac{\partial y^j}{\partial u^k}\frac{\partial}{\partial y^j}\right) = \frac{\partial}{\partial v^k}, \\
J_x\left(\frac{\partial}{\partial v^k}\right) &= J_x\left(\frac{\partial x^j}{\partial v^k}\frac{\partial}{\partial x^j} + \frac{\partial y^j}{\partial v^k}\frac{\partial}{\partial y^j}\right) = -\frac{\partial}{\partial u^k}.
\end{aligned}$$

So the endomorphism J in (1.1.16) is global defined.

The proof of our proposition is completed. □

For a complex manifold M, the almost complex structure defined in (1.1.16) is called the canonical almost complex structure of M. Moreover,

$\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\}$ and $\{\frac{\partial}{\partial \bar{z}^1}, \cdots, \frac{\partial}{\partial \bar{z}^n}\}$ are basis of $T_x^{(1,0)}M$ and $T_x^{(0,1)}M$ respectively. Let dz^i, $d\bar{z}^i$ be the duals of $\frac{\partial}{\partial z^i}$, $\frac{\partial}{\partial \bar{z}^i}$ respectively. Then

$$dz^i = dx^i + \sqrt{-1}dy^i, \quad d\bar{z}^i = dx^i - \sqrt{-1}dy^i \tag{1.1.17}$$

and

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right), \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}\right). \tag{1.1.18}$$

From (1.1.9), locally for any $1 \le i, j \le n$,

$$g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = g\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}\right) = 0. \tag{1.1.19}$$

We write

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right), \quad g_{\bar{i}j} = g\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j}\right). \tag{1.1.20}$$

Then

$$g_{i\bar{j}} = \overline{g_{\bar{i}j}}. \tag{1.1.21}$$

From (1.1.13) and (1.1.18), we have

$$\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j. \tag{1.1.22}$$

We could easily check that the right hand side of (1.1.22) does not depend on the basis.

The Nijenhuis tensor $N^J : TM \times TM \to TM$ is given by

$$N^J(V, W) = [V, W] + J[JV, W] + J[V, JW] - [JV, JW] \tag{1.1.23}$$

for V, W vector fields on M.

Theorem 1.1.7 (Newlander-Nirenberg)**.** *Let (M, J) be a almost complex manifold. The following statements are equivalent:*

(1) M is a complex manifold and J is the canonical almost complex structure of M.

(2) $T^{(1,0)}M$ is formally integrable, that is, for any $X, Y \in \mathscr{C}^\infty(M, T^{(1,0)}M)$, $[X, Y] \in \mathscr{C}^\infty(M, T^{(1,0)}M)$.

(3) $T^{(0,1)}M$ is formally integrable.

(4) $N^J = 0$.

(5) On $\Omega^{(1,0)}(M)$, $d = \partial + \bar{\partial}$.

(6) On $\Omega^{(p,q)}(M)$, $d = \partial + \bar{\partial}$.

(7) $\bar{\partial}^2 = 0$.

If (M, J) satisfies one of the above statements, we say that the almost complex structure J is integrable.

Proof. (1)⇒(2): Write $X = X^i \frac{\partial}{\partial z^i}$ and $Y = Y^j \frac{\partial}{\partial z^j}$. Then

$$[X, Y] = X^i \frac{\partial Y^j}{\partial z^i} \frac{\partial}{\partial z^j} + Y^j \frac{\partial X^i}{\partial z^j} \frac{\partial}{\partial z^i} \in \mathscr{C}^\infty(M, T^{(1,0)}M).$$

(2)⇔(3) follows from $\overline{[X, Y]} = [\overline{X}, \overline{Y}]$.

(3)⇔(4): For $X, Y \in \mathscr{C}^\infty(M, TM)$, then $X + \sqrt{-1}JX$, $Y + \sqrt{-1}JY \in \mathscr{C}^\infty(M, T^{(0,1)}M)$. Let $Z = [X + \sqrt{-1}JX, Y + \sqrt{-1}JY]$. It is easy to calculate that $Z - \sqrt{-1}JZ = N^J(X, Y) - \sqrt{-1}JN^J(X, Y)$. So $Z \in \mathscr{C}^\infty(M, T^{(0,1)}M) \Leftrightarrow N^J(X, Y) = 0$.

(3)⇔(5): (5) is equivalent to that for any $\theta \in \Omega^{(1,0)}(M)$, $(d\theta)^{(0,2)} = 0$. For $X, Y \in \mathscr{C}^\infty(M, T^{(0,1)}M)$,

$$d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]) = -\theta([X, Y]).$$

So

$$\begin{aligned} d\theta(X, Y) = 0 \quad & \forall \theta \in \Omega^{(1,0)}(M), X, Y \in \mathscr{C}^\infty(M, T^{(0,1)}M) \\ \Leftrightarrow \theta([X, Y]) = 0 \quad & \forall \theta \in \Omega^{(1,0)}(M), X, Y \in \mathscr{C}^\infty(M, T^{(0,1)}M) \\ \Leftrightarrow [X, Y] \in \mathscr{C}^\infty(M, T^{(0,1)}M) \quad & \forall X, Y \in \mathscr{C}^\infty(M, T^{(0,1)}M) \end{aligned}$$

(5)⇔(6): Suppose (5) holds. By complex conjugation, on $\Omega^{(0,1)}(M)$, we have $d = \partial + \bar{\partial}$. Then (6) follows from the Leibniz rule.

(6)⇒(7) follows from $d^2 = 0$.

(7)⇒(5): Let $\{\theta^1, \cdots, \theta^n\}$ be a local frame of $T^{*(1,0)}M$. Let

$$d\theta^i = A^i_{jk}\theta^j \wedge \theta^k + B^i_{jk}\theta^j \wedge \bar{\theta}^k + C^i_{jk}\bar{\theta}^j \wedge \bar{\theta}^k.$$

Then (4) is equivalent to $C^i_{jk} = 0$, $\forall 1 \leq i, j, k \leq n$. Let $f : M \to \mathbb{C}$ be a smooth function. Then

$$\begin{aligned} 0 = \bar{\partial}^2 f = (d(\bar{\partial} f))^{(0,2)} = (d((\bar{\partial} - d)f))^{(0,2)} &= -(d(\partial f))^{(0,2)} \\ &= -\theta_i(f) C^i_{jk} \bar{\theta}^j \wedge \bar{\theta}^k. \end{aligned}$$

Since f is chosen arbitrarily, $C^i_{jk} = 0$, $\forall 1 \leq i, j, k \leq n$.

We will not prove (2)⇒(1) here. This part is very hard. We leave a reference to the reader. □

The only spheres which admit almost complex structures are S^2 and S^6 (Borel-Serre, 1953). In particular, S^4 cannot be given an almost complex structure (Ehresmann and Hopf). Whether the S^6 has a complex structure is an open question.

Definition 1.1.8. Let ω be a non-degenerate real valued 2-form on M. If $d\omega = 0$, ω is called a symplectic form on M. In this case, (M, ω) is called a **symplectic manifold**.

The following Proposition follows from Lemma 1.1.3.

Proposition 1.1.9. *The symplectic manifold is almost complex.*

Definition 1.1.10. Let (g, J, ω) be the compatible triple on almost complex manifold M defined in Definition 1.1.4. If one of the statements in Theorem 1.1.7 holds and $d\omega = 0$, M is called a **Kähler manifold**. In this case, ω is called the **Kähler form** and g is called the **Kähler metric**.

Example 1.1.11. Let $M = \mathbb{C}^n$. Then from (1.1.22),

$$\omega = \frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^i = \sum_{i=1}^{n} dx_i \wedge dy_i. \tag{1.1.24}$$

is a Kähler form of $\mathbb{C}^n$.

Example 1.1.12 (Projective space). The complex projective space $\mathbb{CP}^n$ is the set of complex lines in $\mathbb{C}^{n+1}$ or, equivalently,

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \backslash \{0\}) / \mathbb{C}^*, \tag{1.1.25}$$

where $\mathbb{C}^*$ acts by multiplication on $\mathbb{C}^{n+1}$. The topology of $\mathbb{CP}^n$ is induced by (1.1.25). The points of $\mathbb{CP}^n$ are written as $[z_0 : z_1 : \cdots : z_n]$ for $(z_0, \cdots, z_n) \neq (0, \cdots, 0)$, which means that for $\lambda \in \mathbb{C}^*$, $[\lambda z_0 : \lambda z_1 : \cdots : \lambda z_n]$ and $[z_0 : z_1 : \cdots : z_n]$ define the same point in $\mathbb{CP}^n$. The standard open covering of $\mathbb{CP}^n$ is given by

$$U_i = \{[z_0 : z_1 : \cdots : z_n] : z_i \neq 0\} \subset \mathbb{CP}^n. \tag{1.1.26}$$

It is open for the induced topology. Consider the bijective map $\varphi_i : U_i \to \mathbb{C}^n$ by

$$\varphi_i([z_0 : z_1 : \cdots : z_n]) = \left(\frac{z_0}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_n}{z_i} \right). \tag{1.1.27}$$

It is a homeomorphism. For the transition maps $\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$, for

$$(\theta_1, \cdots, \theta_n) = \left(\frac{z_0}{z_j}, \cdots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \cdots, \frac{z_n}{z_j} \right) \in \mathbb{C}^n, \tag{1.1.28}$$

we may assume $i < j$ and get

$$\begin{aligned}\varphi_i \circ \varphi_j^{-1}(\theta_1, \cdots, \theta_n) &= \left(\frac{z_0}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_n}{z_i}\right) \\ &= \left(\frac{\theta_1}{\theta_{i+1}}, \cdots, \frac{\theta_i}{\theta_{i+1}}, \frac{\theta_{i+2}}{\theta_{i+1}}, \cdots, \frac{\theta_j}{\theta_{i+1}}, \frac{1}{\theta_{i+1}}, \frac{\theta_{j+1}}{\theta_{i+1}} \cdots \frac{\theta_n}{\theta_{i+1}}\right). \end{aligned} \tag{1.1.29}$$

These maps are obviously bijective and holomorphic.

Similarly as in Example 1.3.13, consider the $(1,1)$-form

$$\omega = \sqrt{-1}\partial\bar{\partial}\log(|z|^2) = \sqrt{-1} \cdot \frac{|z|^2\delta_{ij} - \bar{z}_i z_j}{|z|^4} dz^i \wedge d\bar{z}^j \tag{1.1.30}$$

on $\mathbb{C}^{n+1}\backslash\{0\}$. Since the matrix $(g_{i\bar{j}}) = (|z|^2\delta_{ij} - \bar{z}_i z_j)$ is positive definite, we see that ω is a Kähler form on $\mathbb{C}^{n+1}\backslash\{0\}$. Observe that for $\lambda \in \mathbb{C}^*$,

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}\log(|\lambda z|^2) &= \sqrt{-1}\partial\bar{\partial}(\log|\lambda|^2 + \log|z|^2) \\ &= \sqrt{-1}\partial\bar{\partial}\log(|z|^2). \end{aligned} \tag{1.1.31}$$

So from (1.1.25), the $(1,1)$-form in (1.1.30) induces a $(1,1)$-form $\omega_{\mathbb{CP}}$ on $\mathbb{CP}^n$. We claim that it is a Kähler form on $\mathbb{CP}^n$. Restricted on U_i, from (1.1.28), (1.1.30) and (1.1.31),

$$\begin{aligned}\omega_{\mathbb{CP}}|_{U_i} &= \sqrt{-1}\partial\bar{\partial}\log(1 + |\theta|^2) \\ &= \sqrt{-1} \cdot \frac{(1+|\theta|^2)\delta_{kl} - \bar{\theta}_k\theta_l}{(1+|\theta|^2)^2} d\theta^k \wedge d\bar{\theta}^l. \end{aligned} \tag{1.1.32}$$

Since the matrix $((1+|\theta|^2)\delta_{kl} - \bar{\theta}_k\theta_l)$ is positive definite, we obtain that $\omega_{\mathbb{CP}}$ is a Kähler form and $(\mathbb{CP}^n, \omega_{\mathbb{CP}})$ is a Kähler manifold. The metric induced by (1.1.15), which we denote by g^{FS}, is called the Fubini-Study metric. By (1.1.32), on U_i,

$$g^{FS}_{k\bar{l}} = \frac{\partial^2}{\partial\theta_k\partial\bar{\theta}_l}\log(1+|\theta|^2) = \frac{(1+|\theta|^2)\delta_{kl} - \bar{\theta}_k\theta_l}{(1+|\theta|^2)^2}. \tag{1.1.33}$$

In the followings, we will also denote the Kähler form $\omega_{\mathbb{CP}}$ by ω_{FS}.

Remark that $\mathbb{CP}^n$ is simply connected. In fact, $\mathbb{CP}^n = S^{2n+1}/S^1$. From fibre exact sequence

$$\cdots \to \pi_1(S^{2n+1}) \to \pi_1(\mathbb{CP}^n) \to \pi_0(S^1) \to \cdots, \tag{1.1.34}$$

since $\pi_1(S^{2n+1}) = \pi_0(S^1) = \{1\}$, we have $\pi_1(\mathbb{CP}^n) = \{1\}$.

1.2 Vector bundles and connections

Definition 1.2.1. Let X and Y be complex manifolds. A continuous map $f : X \to Y$ is a **holomorphic** map if for any holomorphic charts (U, φ) and (U', φ') of X and Y, respectively, the map $\varphi' \circ f \circ \varphi : \varphi(f^{-1}(U') \cap U) \to \varphi'(U')$ is holomorphic.

Definition 1.2.2. Let M be a complex manifold and E be a complex vector bundle over M. We say that E is a **holomorphic** vector bundle if for any i, j such that $U_i \cap U_j \neq \emptyset$, ψ_{ij} in (1.1.3) is a holomorphic map.

Remark that the complex vector bundle could be defined over any manifolds, but the holomorphic vector bundle is only well-defined over complex manifolds.

It is easy to see that the total space of a holomorphic vector bundle is a complex manifold.

Proposition 1.2.3. *The complex vector bundle $T^{(1,0)}M$ over M is holomorphic.*

Proof. The proposition follows the fact that the transition map for $T^{(1,0)}M$ is the same as that of complex manifold M. □

Since $T^{(1,0)}M$ is locally spanned by $\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\}$, we will also regard it as the complex tangent bundle of M.

Example 1.2.4. Any canonical construction in linear algebra gives rise to a geometric version for complex (resp. holomorphic) vector bundles. Let E and F be complex (resp. holomorphic) vector bundles over M.

- The direct sum $E \oplus F$ is the complex (resp. holomorphic) vector bundle over M such that the fibre $(E \oplus F)|_x$ for any $x \in M$ is canonically isomorphic to $E|_x \oplus F|_x$ as complex vector spaces.

- The tensor product $E \otimes F$ is the complex (resp. holomorphic) vector bundle over M such that the fibre $(E \otimes F)|_x$ for any $x \in M$ is canonically isomorphic to $E|_x \otimes F|_x$ as complex vector spaces.

- The i-th exterior power $\Lambda^i E$ and the i-th symmetric power $S^i E$ are the complex (resp. holomorphic) vector bundle over M such that the fibres for any $x \in M$ are canonically isomorphic to $\Lambda^i(E|_x)$ and $S^i(E|_x)$ respectively.

- The dual bundle E^* is the complex (resp. holomorphic) vector bundle over M such that the fibre $E^*|_x$ for any $x \in M$ is canonically isomorphic to $(E|_x)^*$.

- The endomorphism bundle $\mathrm{End}(E)$ is the complex (resp. holomorphic) vector bundle over M such that the fibre $\mathrm{End}(E)|_x$ for any $x \in M$ is canonically isomorphic to $\mathrm{End}(E|_x)$.

Proposition 1.2.5. *The set $\gamma_n \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$ that consists of all pairs $(\ell, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}$ with $z \in \ell$ forms in a natural way a holomorphic line bundle over $\mathbb{CP}^n$.*

Proof. The projection $\pi : \gamma_n \to \mathbb{CP}^n$ is given by projecting to the first factor. Let $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$ be the standard open covering in (1.1.26). Let $\ell = [z_0 : \cdots : z_n]$. A canonical trivialization of γ_n over U_i is given by

$$\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}, \quad (\ell, z) \mapsto (\ell, z_i). \tag{1.2.1}$$

Then the transition maps $\ell \mapsto z_i/z_j$ is holomorphic. □

Let E be a complex vector bundle over a smooth manifold M. A linear map

$$\nabla^E : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, T^*M \otimes E) \tag{1.2.2}$$

is called a **connection** on E if for any $\varphi \in \mathscr{C}^\infty(M, \mathbb{C})$, $s \in \mathscr{C}^\infty(M, E)$ and vector field V, we have

$$\nabla^E_V(\varphi s) = V(\varphi)s + \varphi \nabla^E_V s. \tag{1.2.3}$$

Connections on E always exist. Indeed, let $\{U_k\}_{k\in I}$ be an open covering of M such that $E|_{U_k}$ is trivial for any $k \in I$. If $\{\xi_{kl}\}_{l=1,\cdots,r}$ is a local frame of $E|_{U_k}$, any section $s \in \mathscr{C}^\infty(U_k, E)$ has the form $s = \sum_{l=1}^r s_l \xi_{kl}$ with uniquely determined $s_l \in \mathscr{C}^\infty(U_k)$. We define a connection on $E|_{U_k}$ by $\nabla^E_k s := \sum_{l=1}^r ds_l \otimes \xi_{kl}$. Consider now a partition of unity $\{\psi_k\}_{k\in I}$ subordinated to $\{U_k\}_{k\in I}$. Then $\nabla^E s := \sum_k \nabla^E_k(\psi_k s)$, $s \in \mathscr{C}^\infty(M, E)$, defines a connection on E.

If $'\nabla^E$ is another connection on E, then by (1.2.3), $'\nabla^E - \nabla^E \in \Omega^1(M, \mathrm{End}(E))$.

If ∇^E is a connection on E, then there exists a unique extension $\nabla^E : \Omega^*(M, E) \to \Omega^{*+1}(M, E)$ verifying the Leibniz rule: for any $\alpha \in \Omega^k(M, \mathbb{C})$, $s \in \Omega^*(M, E)$, then

$$\nabla^E(\alpha \wedge s) = d\alpha \wedge s + (-1)^k \alpha \wedge \nabla^E s. \tag{1.2.4}$$

Proposition 1.2.6. *Let $(\nabla^E)^2 := \nabla^E \circ \nabla^E : \mathscr{C}^\infty(M, E) \to \Omega^2(M, E)$. For $s \in \mathscr{C}^\infty(M, E)$ and vector fields U, V on M, we have*

$$\left(\nabla^E\right)^2(U, V)s = \nabla^E_U \nabla^E_V s - \nabla^E_V \nabla^E_U s - \nabla^E_{[U,V]} s. \tag{1.2.5}$$

Proof. Let $\{e_i\}$ be a locally orthonormal frame of M and $\{e^i\}$ be its dual with respect to the metric. Then from (1.2.4),

$$\left(\nabla^E\right)^2 s = \nabla^E\left(e^i \otimes \nabla^E_{e_i} s\right) = de^i \otimes \nabla^E_{e_i} s - e^i \wedge e^j \otimes \nabla^E_{e_j} \nabla^E_{e_i} s. \tag{1.2.6}$$

Since

$$de^i(U,V) = U(e^i(V)) - V(e^i(U)) - e^i([U,V])$$

and

$$e^i \wedge e^j(U,V) = g(U,e_i)g(V,e_j) - g(U,e_j)g(V,e_i), \tag{1.2.7}$$

we have

$$\begin{aligned}\left(\nabla^E\right)^2(U,V)s = {} & U(g(V,e_i))\nabla^E_{e_i}s + g(V,e_i)\nabla^E_U\nabla^E_{e_i}s \\ & - V(g(U,e_i))\nabla^E_{e_i}s - g(U,e_i)\nabla^E_V\nabla^E_{e_i}s - \nabla^E_{[U,V]}s \\ = {} & \nabla^E_U\nabla^E_V s - \nabla^E_V\nabla^E_U s - \nabla^E_{[U,V]}s. \end{aligned} \tag{1.2.8}$$

The proof of this proposition is completed. □

Let R^E be the curvature of ∇^E. Then from Proposition 1.2.6, we have

$$(\nabla^E)^2 = R^E \in \Omega^2(M, \mathrm{End}(E)). \tag{1.2.9}$$

From the Leibniz's rule, the operator $(\nabla^E)^2$ and R^E could be extended to act on $\Omega^*(M,E)$. Moreover, they are also equal after the extension.

Proposition 1.2.7 (Bianchi Identity)**.** *The following identity holds,*

$$[\nabla^E, R^E] = 0. \tag{1.2.10}$$

Proof. Since $R^E = (\nabla^E)^2$,

$$[\nabla^E, R^E] = [\nabla^E, (\nabla^E)^2] = 0. \tag{1.2.11}$$

□

Let h^E be a **Hermitian metric** on E, i.e., a smooth family $\{h^E_x\}_{x\in M}$ of sesquilinear maps $h^E_x : E_x \times E_x \to \mathbb{C}$ such that $h^E_x(\xi,\xi) > 0$ for any $\xi \in E_x\backslash\{0\}$. We call (E, h^E) a Hermitian vector bundle on M. There always exist Hermitian metrics on E by using the partition of unity as above.

Example 1.2.8. By (1.1.21), for any $Z, Z' \in T^{(1,0)}M$,

$$h^{T^{(1,0)}M}(Z, Z') := g(Z, \overline{Z'}) \tag{1.2.12}$$

defines a Hermitian metric on $T^{(1,0)}M$. Let $h_{ij} = h^{T^{(1,0)}M}\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right)$. Then by (1.1.20),

$$h_{ij} = g_{i\bar{j}}. \tag{1.2.13}$$

Definition 1.2.9. A connection ∇^E is said to be a **Hermitian connection** on (E, h^E) if for any $s_1, s_2 \in \mathscr{C}^\infty(M, E)$,

$$dh^E(s_1, s_2) = h^E(\nabla^E s_1, s_2) + h^E(s_1, \nabla^E s_2). \tag{1.2.14}$$

There always exist Hermitian connections. Indeed, let ∇_0^E be a connection on E, then $h^E(\nabla_1^E s_1, s_2) = dh^E(s_1, s_2) - h^E(s_1, \nabla_0^E s_2)$ defines a connection ∇_1^E on E. Then $\nabla^E = \frac{1}{2}(\nabla_0^E + \nabla_1^E)$ is a Hermitian connection on (E, h^E).

In the rest of this section, we assume that E is a holomorphic vector bundle over a complex manifold M.

Let

$$\Omega^{p,q}(M, E) := \mathscr{C}^\infty(M, \Lambda^p(T^{*(1,0)}M) \otimes \Lambda^q(T^{*(0,1)}M) \otimes E). \tag{1.2.15}$$

Any section $s \in \mathscr{C}^\infty(M, E)$ has the local form $s = \sum_l \varphi_l \xi_l$, where $\{\xi_l\}$ is a holomorphic frame of E and φ_l are smooth functions. We set

$$\overline{\partial}^E s = \sum_l (\overline{\partial}\varphi_l)\xi_l, \tag{1.2.16}$$

where $\overline{\partial}\varphi_l = \sum_j d\bar{z}^j \frac{\partial}{\partial \bar{z}_j}\varphi_l$ in holomorphic coordinates $(z_1, \cdots, z_n)$. Then the operator

$$\overline{\partial}^E : \mathscr{C}^\infty(M, E) \to \Omega^{0,1}(M, E) \tag{1.2.17}$$

in (1.2.16) is well-defined.

Definition 1.2.10. A connection ∇^E on E is said to be a **holomorphic connection** if $\nabla_V^E s = i_V(\overline{\partial}^E s)$ for any $V \in T^{(0,1)}M$ and $s \in \mathscr{C}^\infty(M, E)$.

Let $\{\xi_l\}_{l=1,\cdots,r}$ be a local frame of E. Denote by $h = (h_{lk} = h^E(\xi_k, \xi_l))$ the matrix of h^E with respect to $\{\xi_l\}_{l=1,\cdots,r}$. Let $s_1 = \sum_k \varphi_{1k}\xi_k$, $s_2 = \sum_l \varphi_{2l}\xi_l$. Let $\varphi_i = (\varphi_{i1}, \cdots, \varphi_{ir})$ for $i = 0, 1$. Then

$$h^E(s_1, s_2) = \varphi_2 \cdot h \cdot \varphi_1^t = \langle h \cdot \varphi_1^t, \varphi_2^t \rangle. \tag{1.2.18}$$

The connection form $\Gamma = (\Gamma^l_k)$ of ∇^E with respect to $\{\xi_l\}_{l=1,\cdots,r}$ is defined by, with local 1-forms Γ^l_k,

$$\nabla^E \xi_k = \Gamma^l_k \xi_l. \tag{1.2.19}$$

For $s = \sum_k \varphi_k \xi_k$, denote by $\Gamma = (\Gamma_{lk}(:= \Gamma^l_k))$:

$$\Gamma s = (\xi_1, \cdots, \xi_r) \cdot \Gamma \cdot \varphi^t_1. \tag{1.2.20}$$

Recall that $R^E = d\Gamma + \Gamma \wedge \Gamma$.

Theorem 1.2.11. *There exists a unique holomorphic Hermitian connection ∇^E on (E, h^E), called the* ***Chern connection****. With respect to a local holomorphic frame, the connection matrix is given by*

$$\Gamma = h^{-1}\partial h. \tag{1.2.21}$$

Proof. From Definition 1.2.10, we only need to define ∇^E_U for $U \in T^{(1,0)}M$. Relation (1.2.14) implies for $V \in T^{(1,0)}M$, $s_1, s_2 \in \mathscr{C}^\infty(M, E)$,

$$V(h^E(s_1, s_2)) = h^E(\nabla^E_V s_1, s_2) + h^E(s_1, \nabla^E_{\overline{V}} s_2). \tag{1.2.22}$$

Since $\nabla^E_{\overline{V}} s = i_{\overline{V}}(\overline{\partial}^E s)$, the above equation defines ∇^E_V uniquely. Moreover, if $\{\xi_l\}_{l=1,\cdots,r}$ is a local holomorphic frame of E, by (1.2.18) and (1.2.20),

$$\langle \partial h \cdot \varphi^t_1, \varphi^t_2 \rangle = \langle h\Gamma \cdot \varphi^t_1, \varphi^t_2 \rangle. \tag{1.2.23}$$

Thus we get (1.2.21). □

Since E is holomorphic, similar to (1.2.4), the operator $\overline{\partial}^E$ extends naturally to $\overline{\partial}^E : \Omega^{*,*}(M, E) \to \Omega^{*,*+1}(M, E)$ and $(\overline{\partial}^E)^2 = 0$.

Let ∇^E be the Chern connection on (E, h^E). Then we have a decomposition

$$\nabla^E = (\nabla^E)^{1,0} + (\nabla^E)^{0,1} \tag{1.2.24}$$

such that

$$(\nabla^E)^{1,0} : \Omega^{*,*}(M, E) \to \Omega^{*+1,*}(M, E), \quad (\nabla^E)^{0,1} = \overline{\partial}^E. \tag{1.2.25}$$

From (1.2.22), $s_1, s_2 \in \mathscr{C}^\infty(M, E)$,

$$\begin{aligned} h^E\left(\left((\nabla^E)^{1,0}\right)^2 s_1, s_2\right) &= \partial h^E\left((\nabla^E)^{1,0} s_1, s_2\right) + h^E\left((\nabla^E)^{1,0} s_1, \overline{\partial}^E s_2\right) \\ &= \partial\left(\partial h^E(s_1, s_2) - h^E\left(s_1, \overline{\partial}^E s_2\right)\right) + \partial h^E\left(s_1, \overline{\partial}^E s_2\right) \\ &\qquad - h^E\left(s_1, (\overline{\partial}^E)^2 s_2\right) = 0. \end{aligned} \tag{1.2.26}$$

So $\left((\nabla^E)^{1,0}\right)^2 = 0$ and

$$(\nabla^E)^2 = \overline{\partial}^E \circ (\nabla^E)^{1,0} + (\nabla^E)^{1,0} \circ \overline{\partial}^E. \tag{1.2.27}$$

Then the curvature

$$R^E \in \Omega^{1,1}(M, \mathrm{End}(E)). \tag{1.2.28}$$

If $\mathrm{rank}(E) = 1$, $\mathrm{End}(E)$ is trivial. Since R^E is skew-adjoint, it is canonically identified as a (1,1)-form on M, such that $\sqrt{-1}R^E$ is real.

Example 1.2.12 (Tautological line bundle on $\mathbb{CP}^n$)**.** Recall that in Proposition 1.2.5, the point on γ_n is $(\ell, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}$. It is natural to define a Hermitian metric h on γ_n by $h(\ell, z) = |z|^2$. Now we study it in local coordinates. By (1.2.1), if h_i is the metric of h on U_i, then we have

$$h(\ell, z) = h_i(\ell) z_i \bar{z}_i = h_i(\ell)|z_i|^2. \tag{1.2.29}$$

So in the coordinates (U_i, θ), for $\ell = [z_0, \cdots, z_n]$,

$$h_i(\ell) = \frac{|z|^2}{|z_i|^2} = 1 + |\theta|^2. \tag{1.2.30}$$

By Theorem 1.2.11, the connection form of the Chern connection is

$$\Gamma = h_i^{-1} \frac{\partial h_i}{\partial \theta_k} d\theta^k = \frac{\bar{\theta}_k d\theta^k}{1 + |\theta|^2}. \tag{1.2.31}$$

The curvature

$$R^{\gamma_n} = d\Gamma = -\frac{(1 + |\theta|^2)\delta_{kl} - \bar{\theta}_k \theta_l}{(1 + |\theta|^2)^2} d\theta^k \wedge d\bar{\theta}^l. \tag{1.2.32}$$

By (1.1.32),

$$\omega_{FS} = -\sqrt{-1} R^{\gamma_n}. \tag{1.2.33}$$

Let ∇ be the Levi-Civita connection on (TM, g), which could be naturally extended linearly on $TX \otimes \mathbb{C}$.

Theorem 1.2.13. *Let M be a almost complex manifold with triple (g, J, ω). Then the following statements are equivalent.*

(1) (M, ω) is Kähler.

(2) the bundles $T^{(1,0)}M$ and $T^{(0,1)}M$ are preserved by ∇.

(3) $\nabla J = 0$.

Proof. (2)⇔(3) is obvious.

(3) $\Longrightarrow$ (1): From (1.1.23),

$$\begin{aligned} N^J(V,W) = \nabla_V W - \nabla_W V + J\nabla_{JV} W - J\nabla_W JV \\ + J\nabla_V JW - J\nabla_{JW} V - \nabla_{JV} JW + \nabla_{JW} JV \\ = J(\nabla_V J)W - J(\nabla_W J)V - (\nabla_{JV} J)W + (\nabla_{JW} J)V \end{aligned} \tag{1.2.34}$$

for vector fields V, W. So $\nabla J = 0$ implies $N^J = 0$. Since $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$, we have $\nabla\omega = 0$. From (A.1.6), we have $d\omega = 0$.

(1) $\Longrightarrow$ (3): Since $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$, for vector fields U, V, W, we have

$$\begin{aligned} d\omega(U,V,W) = U(\omega(V,W)) - V(\omega(U,W)) + W(\omega(U,V)) \\ - \omega([U,V],W) + \omega([U,W],V) - \omega([V,W],U) \\ = U(g(JV,W)) - V(g(JU,W)) + W(g(JU,V)) - g(J(\nabla_U V - \nabla_V U), W) \\ + g(J(\nabla_U W - \nabla_W U), V) - g(J(\nabla_V W - \nabla_W V), U) \\ = g((\nabla_U J)V, W) + g((\nabla_V J)W, U) + g((\nabla_W J)U, V). \end{aligned} \tag{1.2.35}$$

Since $g(JU,V) + g(U,JV) = 0$, by (1.2.34) and (1.2.35), we have

$$\begin{aligned} d\omega(JU,V,W) + d\omega(U,JV,W) = g((\nabla_{JU} J)V, W) + g((\nabla_V J)W, JU) \\ + g((\nabla_W J)JU, V) + g((\nabla_U J)JV, W) + g((\nabla_{JV} J)W, U) + g((\nabla_W J)U, JV) \\ = 2g((\nabla_W J)U, JV) - g(N^J(U,V), W). \end{aligned} \tag{1.2.36}$$

So $d\omega = 0$ and $N^J = 0$ imply $\nabla J = 0$.

Our theorem is completed. □

It is easy to verify that the restriction of ∇ on $T^{(1,0)}M$ is just the Chern connection on $(T^{(1,0)}M, h^{T^{(1,0)}M})$.

Theorem 1.2.14 (Normal coordinates)**.** *A complex manifold M with triple (g, J, ω) is Kähler if and only if around each point of M, there exist holomorphic coordinates in which $g_{i\bar{j}}(z) = \delta_{ij} + O(|z|^2)$.*

Proof. If $g_{i\bar{j}}(z) = \delta_{ij} + O(|z|^2)$, by (1.1.22),

$$d\omega = \sqrt{-1}\left(\frac{\partial g_{i\bar{j}}}{\partial x_k} dx^k + \frac{\partial g_{i\bar{j}}}{\partial y_k} dy^k\right) \wedge dz^i \wedge d\bar{z}^j = 0. \tag{1.2.37}$$

Conversely, let $(z_1, \cdots, z_n)$ be a holomorphic frame such that $g_{i\bar{j}}(0) = \delta_{ij}$. Then $g_{i\bar{j}} = \delta_{ij} + a_{ijk} z^k + a_{ij\bar{k}} \bar{z}^k + O(|z|^2)$. Since $g_{i\bar{j}} = \overline{g_{j\bar{i}}}$, we have $a_{ij\bar{k}} = \overline{a_{jik}}$.

Since $d\omega = 0$, we have $a_{ijk} = a_{kji}$. We choose another holomorphic frame $(\theta_1, \cdots, \theta_n)$ by

$$z_i = \theta_i - \frac{1}{2} a_{kij}\theta_j\theta_k. \tag{1.2.38}$$

This coordinate change is well-defined locally thanks to the holomorphic version of the local inversion theorem. Thus,

$$dz^i = d\theta^i - a_{kij}\theta_j d\theta^k. \tag{1.2.39}$$

So in this new coordinates,

$$\begin{aligned}
\omega &= \sqrt{-1}(\delta_{ij} + a_{ijk}z^k + a_{ij\bar{k}}\bar{z}^k + O(|z|^2))dz^i \wedge d\bar{z}^j \\
&= \sqrt{-1}(\delta_{ij} + a_{ijk}\theta^k + a_{ij\bar{k}}\bar{\theta}^k + O(|\theta|^2))(d\theta^i - a_{qil}\theta_l d\theta^q) \wedge (d\bar{\theta}^{\bar{j}} - \overline{a_{pjs}}\bar{\theta}_s d\bar{\theta}^{\bar{p}}) \\
&= \sqrt{-1}(\delta_{ij} + O(|\theta|^2))d\theta^i \wedge d\bar{\theta}^{\bar{j}}.
\end{aligned} \tag{1.2.40}$$

From (1.1.22), we have $g_{i\bar{j}}(\theta) = \delta_{ij} + O(|\theta|^2)$ □

In general, the normal coordinates in Riemannian geometry is different from that in Kähler geometry.

1.3 Curvatures

Let (M, ω) be a Kähler manifold. Let $R = \nabla^2$ be the curvature of the Levi-Civita connection. Then it is naturally extended as an endomorphism of $TM \otimes \mathbb{C}$ in a $\mathbb{C}$-linear way.

By Theorem 1.2.13, we see that $[R, J] = 0$. So for $U, V, W \in TM \otimes \mathbb{C}$,

$$R(U, V)JW = JR(U, V)W. \tag{1.3.1}$$

Recall that for $U, V, W, X \in TM \otimes \mathbb{C}$,

$$R(U, V, W, X) = g(R(U, V)X, W). \tag{1.3.2}$$

By (1.1.7) and (1.3.2), we have

$$R(U, V, JW, JX) = R(U, V, W, X). \tag{1.3.3}$$

So if $(W, X) \in T^{(1,0)}M \times T^{(1,0)}M$ or $T^{(0,1)}M \times T^{(0,1)}M$, $R(U, V, W, X) = 0$. Thus by (1.2.9), the curvatures are possibly non-vanishing only essentially for

$$(U, \overline{V}, W, \overline{X}) \in T^{(1,0)}M \times T^{(0,1)}M \times T^{(1,0)}M \times T^{(0,1)}M. \tag{1.3.4}$$

From Theorem 1.2.11 and (1.2.13),

$$R = d\Gamma + \Gamma \wedge \Gamma = -h^{-1}\partial\bar{\partial}h + h^{-1}\partial h \wedge h^{-1}\bar{\partial}h = -\partial\bar{\partial}\log(h), \tag{1.3.5}$$

where h is the matrix for $h^{T^{(1,0)}M}$. Here $\log(h)$ is defined by the power series expansion

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} \tag{1.3.6}$$

(or the inverse of the exponential map $\exp : \mathfrak{gl}(n,\mathbb{C}) \to GL(n,\mathbb{C})$). Take care that $\log(h)$ here, which depends on the frame, is not a global function on M. But $\partial\bar{\partial}\log(h)$ is. Locally, set $W = w^i\frac{\partial}{\partial z_i}$ and $X = x^j\frac{\partial}{\partial z_j}$. Let $w = (w_1, \cdots, w_n)$ and $x = (x_1, \cdots, x_n)$. Then by (1.2.18), (1.3.2) and (1.3.5),

$$R(U, \overline{V}, W, \overline{X}) = -\langle h\partial\bar{\partial}\log(h)(U, \overline{V})\bar{x}^t, w^t\rangle. \tag{1.3.7}$$

In local coordinates, from (1.3.5) and (1.3.7),

$$\begin{aligned} R_{i\bar{j}k\bar{l}} := R\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}\right) &= -\frac{\partial^2 h_{kl}}{\partial z_i \partial \bar{z}_j} + h^{st}\frac{\partial h_{sl}}{\partial z_i}\frac{\partial h_{kt}}{\partial \bar{z}_j} \\ &= -\frac{\partial^2 h_{ij}}{\partial z_k \partial \bar{z}_l} + h^{st}\frac{\partial h_{sj}}{\partial z_k}\frac{\partial h_{it}}{\partial \bar{z}_l}. \end{aligned} \tag{1.3.8}$$

Definition 1.3.1. Let Ric be the Ricci tensor in Riemannian geometry. For $X, Y \in TM \otimes \mathbb{C}$, we define the Ricci form $\mathrm{Ric}_\rho \in \Omega^{1,1}(M)$ by

$$\mathrm{Ric}_\rho(X, Y) = \mathrm{Ric}(JX, Y), \tag{1.3.9}$$

Proposition 1.3.2. *If the Kähler manifold (M, ω) is an Einstein manifold with Einstein constant k if and only if*

$$\mathrm{Ric}_\rho = k\omega. \tag{1.3.10}$$

Proof. Our corollary follows directly from Definition 1.3.1 and (1.1.13). □

Let $e_1, \cdots, e_{2n}$ be a locally orthonormal basis of TM such that $e_{n+i} = Je_i$ for $i = 1, \cdots, n$. Let $u_i = \frac{1}{\sqrt{2}}(e_i - \sqrt{-1}Je_i)$. Then $u_1, \cdots, u_n$ is a locally orthonormal basis of $T^{(1,0)}M$.

Proposition 1.3.3. *The Ricci form*

$$\mathrm{Ric}_\rho = \sqrt{-1}\,\mathrm{tr}^{T^{(1,0)}M}[R] = -\sqrt{-1}\partial\bar{\partial}(\log\det(h)). \tag{1.3.11}$$

Proof. By Definition 1.3.1,

$$\begin{aligned}\mathrm{Ric}_\rho(X,Y) = \mathrm{Ric}(JX,Y) &= \frac{1}{2}\sum_{i=1}^{2n}(R(Je_i,JX,Je_i,Y)+R(e_i,JX,e_i,Y))\\ = \frac{1}{2}\sum_{i=1}^{2n}(R(Y,Je_i,X,e_i)&+R(Je_i,X,Y,e_i)) = -\frac{1}{2}\sum_{i=1}^{2n}R(X,Y,Je_i,e_i)\\ = \sqrt{-1}\sum_{i=1}^{n}R(X,Y,\bar{u}_i,u_i) &= \sqrt{-1}\sum_{i=1}^{n}h(R(X,Y)u_i,u_i)\\ &= \sqrt{-1}\,\mathrm{tr}^{T^{(1,0)}M}[R(X,Y)]. \quad (1.3.12)\end{aligned}$$

From (1.3.5),

$$\mathrm{Ric}_\rho = -\sqrt{-1}\partial\bar{\partial}\,\mathrm{tr}^{T^{(1,0)}M}\log(h) = -\sqrt{-1}\partial\bar{\partial}\log\det(h). \quad (1.3.13)$$

The proof of our proposition is completed. □

Remark that in the last equality of (1.3.13), we use the matrix identity that

$$\mathrm{tr}\log(A) = \log\det(A) \quad (1.3.14)$$

holds for any complex non-degenerate matrix A.

Definition 1.3.4. Let M be a complex manifold with triple (g,J,ω). The metric g is called **Kähler-Einstein** if (M,ω) is Kähler and Einstein. In this case, we call (M,ω) a Kähler-Einstein manifold.

Proposition 1.3.5. *The Ricci form* $\mathrm{Ric}_\rho \in \Omega^{1,1}(M)$ *is closed, that is*

$$d\,\mathrm{Ric}_\rho = 0. \quad (1.3.15)$$

Proof. The proposition follows from the facts that the exterior differential d is local and

$$d\partial\bar{\partial} = \partial^2\bar{\partial} + \bar{\partial}\partial\bar{\partial} = -\partial\bar{\partial}^2 = 0. \quad (1.3.16)$$

□

Recall that if $X,Y \in T_xM$ such that $|X| = |Y| = 1$ and $g(X,Y) = 0$, then $R(X,Y,Y,X)$ is the sectional curvature of the plane P spanned by X,Y. As in the Riemannian geometry, we want to study the Kähler manifolds with constant curvature. Unfortunately, the space form of constant positive curvature, S^{2n}, is not Kähler unless $n = 1$. So we restrict us to only study the sectional curvature of the plane which is preserved by the almost complex structure.

Definition 1.3.6. Let P be the plan in T_xM invariant by J. Let X be a unit vector in P. Then

$$K(P) = R(X, JX, X, JX) \tag{1.3.17}$$

is called the **holomorphic sectional curvature** by P.

It is easy to see that the holomorphic sectional curvature by P does not depend on the choice of X in P.

Set $U = \frac{1}{\sqrt{2}}(X - \sqrt{-1}JX)$. Then

$$K(P) = -R(U, \overline{U}, U, \overline{U}). \tag{1.3.18}$$

Definition 1.3.7. If $K(P)$ is a constant for all planes P in T_xM invariant by J and for all points $x \in M$, then M is called a **space of constant holomorphic sectional curvature**.

Theorem 1.3.8. *The following identities are equivalent:*

(1) a Kähler manifold M is a space of constant holomorphic sectional curvature c;

(2) for any $A, B, C, D \in TM \otimes \mathbb{C}$,

$$\begin{aligned} R(A,B,C,D) = &-\frac{c}{4}\big(g(A,D)g(B,C) - g(A,C)g(B,D) \\ &+ g(A,JD)g(B,JC) - g(A,JC)g(B,JD) + 2g(A,JB)g(D,JC)\big); \end{aligned} \tag{1.3.19}$$

(3) for any $U, V, W, X \in T^{(1,0)}M$,

$$R(U, \overline{V}, W, \overline{X}) = -\frac{c}{2}\left(g(U,\overline{V})g(W,\overline{X}) + g(U,\overline{X})g(W,\overline{V})\right). \tag{1.3.20}$$

Proof. (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) are obvious. We only need to prove (1) $\Longrightarrow$ (2).

For $A, B, C, D \in TM \otimes \mathbb{C}$, let

$$\begin{aligned} R_0(A,B,C,D) = &\frac{1}{4}\big(g(A,D)g(B,C) - g(A,C)g(B,D) \\ &+ g(A,JD)g(B,JC) - g(A,JC)g(B,JD) + 2g(A,JB)g(D,JC)\big) \end{aligned} \tag{1.3.21}$$

It is easy to verify that

$$\begin{aligned} &R_0(A,B,C,D) = -R_0(B,A,C,D) = -R_0(A,B,D,C), \\ &R_0(A,B,C,D) = R_0(C,D,A,B), \\ &R_0(A,B,C,D) + R_0(B,C,A,D) + R_0(C,A,B,D) = 0, \\ &R_0(A,B,C,D) = R_0(JA,JB,C,D) = R_0(A,B,JC,JD). \end{aligned} \tag{1.3.22}$$

Recall that the curvature R also verifies (1.3.22). Since M is a space of constant holomorphic sectional curvature c,

$$R(A, JA, JA, A) = -cg(A, A)^2 = -cR_0(A, JA, JA, A). \qquad (1.3.23)$$

Set $T = R - cR_0$. From (1.3.22),

$$T(A, JB, JC, D) + T(A, JD, JC, B) + T(A, JC, JD, B) \qquad (1.3.24)$$

is symmetric in A, B, C, D. Since it vanishes for $A = B = C = D$ by (1.3.23), it must vanish identically.

Let $A = D$, $B = C$. We have

$$2T(A, JB, JB, A) + T(A, JA, JB, B) = 0. \qquad (1.3.25)$$

From (1.3.22),

$$\begin{aligned} 0 &= T(A, JA, JB, B) + T(JA, JB, A, B) + T(JB, A, JA, B) \\ &= T(A, JA, JB, B) - T(A, B, B, A) - T(A, JB, JB, A). \qquad (1.3.26) \end{aligned}$$

From (1.3.25) and (1.3.26),

$$3T(A, JB, JB, A) + T(A, B, B, A) = 0. \qquad (1.3.27)$$

Replacing B by JB,

$$3T(A, B, B, A) + T(A, JB, JB, A) = 0. \qquad (1.3.28)$$

Combining (1.3.27) and (1.3.28), we have

$$T(A, B, B, A) = 0 \qquad (1.3.29)$$

for any $A, B \in TM \otimes \mathbb{C}$. Thus

$$\begin{aligned} 0 = \frac{1}{2}T(A, B + C, B + C, A) &= \frac{1}{2}(T(A, B, C, A) + T(A, C, B, A)) \\ &= T(A, B, C, A). \qquad (1.3.30) \end{aligned}$$

By (1.3.30),

$$\begin{aligned} 0 = T(A + D, B, C, A + D) &= T(A, B, C, D) + T(D, B, C, A) \\ &= T(A, B, C, D) - T(C, A, B, D). \qquad (1.3.31) \end{aligned}$$

Replacing (A, B, C) by $(C, A < B)$ in (1.3.31),

$$T(C, A, B, D) = T(B, C, A, D). \tag{1.3.32}$$

So from (1.3.22),

$$T(A, B, C, D) = \frac{1}{3}(T(A, B, C, D) + T(C, A, B, D) + T(B, C, A, D)) = 0 \tag{1.3.33}$$

for any $A, B, C, D \in TM \otimes \mathbb{C}$. That means,

$$\begin{aligned} R(A, B, C, D) = -\frac{c}{4}\big(&g(A, D)g(B, C) - g(A, C)g(B, D) \\ &+ g(A, JD)g(B, JC) - g(A, JC)g(B, JD) + 2g(A, JB)g(D, JC)\big). \end{aligned} \tag{1.3.34}$$

The proof of our theorem is completed.

□

Corollary 1.3.9. *Let (M, ω) is a Kähler manifold, which is a space of constant holomorphic sectional curvature c. Then (M, ω) is Kähler-Einstein with Einstein constant $c(n + 1)/2$.*

Proof. Let $e_1, \cdots, e_{2n}$ be a locally orthonormal basis of TM such that $e_{n+i} = Je_i$ for $i = 1, \cdots, n$. By Theorem 1.3.8,

$$\begin{aligned} \mathrm{Ric}(X, Y) &= \sum_{i=1}^{n} R(e_i, X, e_i, Y) + \sum_{i=1}^{n} R(Je_i, X, Je_i, Y) \\ &= \frac{c}{4}\sum_{i=1}^{n}\big(g(X, Y) - g(X, e_i)g(Y, e_i) + 3g(X, Je_i)g(Y, Je_i)\big) \\ &+ \frac{c}{4}\sum_{i=1}^{n}\big(g(X, Y) - g(X, Je_i)g(Y, Je_i) + 3g(X, e_i)g(Y, e_i)\big) \\ &= \sum_{i=1}^{n}\frac{c}{2}\big(g(X, Y) + g(X, e_i)g(Y, e_i) + g(X, Je_i)g(Y, Je_i)\big) \\ &= \frac{(n+1)c}{2} g(X, Y). \end{aligned} \tag{1.3.35}$$

□

Corollary 1.3.10. *Let (M, g) is a Kähler manifold, which is a space of constant holomorphic sectional curvature c. If $c \geq 0$ (or $c \leq 0$), the sectional curvature of (M, g) is non-negative (or non-positive).*

Proof. By Theorem 1.3.8,

$$R(A,B,B,A)=\frac{c}{4}\left(|A|^2|B|^2-g(A,B)^2+3g(A,JB)^2\right). \tag{1.3.36}$$

□

Example 1.3.11 (Projective space)**.** Recall that in Example 1.1.12, we construct the Fubini-Study metric g^{FS} on $\mathbb{CP}^n$. In order to the curvature, we rescale the metric by

$$g_c=\frac{2}{c}g^{FS},\quad c>0. \tag{1.3.37}$$

Consider the unitary group $\mathrm{U}(n+1)$ on $\mathbb{C}^{n+1}$ (for any $A\in\mathrm{U}(n+1)$, $A\overline{A}^*=\mathrm{Id}$). Since $A\in\mathrm{U}(n+1)$ is linear, it induces an action on $\mathbb{CP}^n$ by

$$A([z])=[A(z)],\quad [z]\in\mathbb{CP}^n. \tag{1.3.38}$$

By definition, $\mathrm{U}(n+1)$ action preserves the Hermitian metric on $\mathbb{C}^{n+1}$. From (1.1.33), we see that g_c is $\mathrm{U}(n+1)$-invariant. On the other hand, the $\mathrm{U}(n+1)$-action on $\mathbb{CP}^n$ is holomorphic and transversal, i.e., for any $x,y\in\mathbb{CP}^n$, there exists $A\in\mathrm{U}(n+1)$ such that $y=Ax$. So the local structure of any two points on $\mathbb{CP}^n$ is the same up to the holomorphic isometry. Thus, in order to calculate the holomorphic sectional curvature, we only need to work on one point.

At the point $\theta=0$, we calculate from (1.1.33) that

$$g_{c,i\bar{j}}=\frac{2}{c}\delta_{ij},\quad g_c^{i\bar{j}}=\frac{c}{2}\delta_{ij},\quad \frac{\partial g_{c,i\bar{j}}}{\partial\theta_k}=\frac{\partial g_{c,i\bar{j}}}{\partial\bar{\theta}_k}=0. \tag{1.3.39}$$

Moreover,

$$\begin{aligned}\left.\frac{\partial^2 g_{c,i\bar{j}}}{\partial\theta_k\partial\bar{\theta}_l}\right|_{\theta=0} &= \frac{2}{c}\frac{\partial^4}{\partial\theta_i\partial\bar{\theta}_j\partial\theta_k\partial\bar{\theta}_l}\log(1+|\theta|^2)\Big|_{\theta=0}\\ &=\frac{2}{c}\left.\frac{\partial}{\partial\theta_i}\right|_{\theta=0}\frac{(\theta_j\delta_{kl}-\delta_{jk}\theta_l)(1+|\theta|^2)-2\theta_j(1+|\theta|^2)\delta_{kl}-\bar{\theta}_k\theta_l}{(1+|\theta|^2)^3}\\ &=-\frac{2}{c}(\delta_{ij}\delta_{kl}+\delta_{jk}\delta_{il}).\end{aligned} \tag{1.3.40}$$

By (1.3.8), we have

$$R_{i\bar{j}k\bar{l}}=\frac{c}{2}(g_{c,i\bar{j}}g_{c,k\bar{l}}+g_{c,j\bar{k}}g_{c,i\bar{l}}). \tag{1.3.41}$$

From Theorem 1.3.8, we see that $(\mathbb{CP}^n,g_c)$ is a space of constant holomorphic section curvature c for $c>0$. By Corollary 1.3.9, $\mathbb{CP}^n$ is a Kähler-Einstein manifold with Einstein constant $n+1$.

Example 1.3.12. Let $M = \mathbb{C}^n$ with trivial metric. Then the bisectional curvature vanishes.

Example 1.3.13 (Complex hyperbolic space). Let $M = B^n = \{z \in \mathbb{C}^n : |z| < 1\}$. Let

$$g_{i\bar{j}} = -\frac{\partial}{\partial z^i \partial \bar{z}^j} \log(1 - |z|^2) = \frac{(1 - |z|^2)\delta_{ij} + \bar{z}_i z_j}{(1 - |z|^2)^2}. \tag{1.3.42}$$

It is easy to see that the matrix $(g_{i\bar{j}})$ is positive definite. Thus it induces a metric on B^n. Then by (1.1.22),

$$\omega = -\sqrt{-1}\partial\bar{\partial}\log(1 - |z|^2) = \sqrt{-1} \cdot \frac{(1 - |z|^2)\delta_{ij} + \bar{z}_i z_j}{(1 - |z|^2)^2} dz^i \wedge d\bar{z}^j. \tag{1.3.43}$$

is a Kähler form of B^n.

Example 1.3.14 (Complex hyperbolic space)**.** Let $M = B^n = \{z \in \mathbb{C}^n : |z| < 1\}$. Let

$$g_c = -\frac{2}{c} g, \quad c < 0 \tag{1.3.44}$$

where g is the metric in (1.3.42). Then following the same process as in the study of projective space, we could calculate that

$$R_{i\bar{j}k\bar{l}} = \frac{c}{2}(g_{c,i\bar{j}} g_{c,k\bar{l}} + g_{c,j\bar{k}} g_{c,i\bar{l}}). \tag{1.3.45}$$

From Theorem 1.3.8, we see that $(\mathbb{CP}^n, g_c)$ is a space of constant holomorphic section curvature c for $c < 0$.

Theorem 1.3.15. *(Uniformization Theorem) For a complete Kähler manifold M of constant holomorphic sectional curvature c, its universal covering $\widetilde{M}$ is holomorphically isometric to one of the above examples.*

Proof. After rescaling, we only need to handle three cases: $c = -1, 0, 1$.

We prove $c \leq 0$ first. Let (M_c, g_c) be the Kähler manifold of constant holomorphic sectional curvature c in the above examples. Consider the exponential maps $\exp_0 : T_0 M_c \to M_c$ and $\exp_x : T_x\widetilde{M} \to \widetilde{M}$ respectively. By Corollary 1.3.10, the sectional curvatures of M_c and $\widetilde{M}$ are non-positive. By Cartan-Hadamard theorem, the exponential maps are diffeomorphisms. Here we use the complete property.

Identify both $T_0 M_c$ and $T_x\widetilde{M}$ with $\mathbb{R}^{2n}$ and define the map $\phi : \exp_x(\exp_0)^{-1}$. We only need to prove that ϕ is an isometry. By Cartan-Hadamard Theorem, for any $p \in M_c$ and $X \in T_pM_c$, there exist $v, w \in \mathbb{R}^{2n}$ such that $\exp_0(v) = p$ and $d\exp_0(v)(w) = X$. If $q = \phi(p)$, $\widetilde{X} = d\phi(X)$, then $\exp_x(v) = q$ and $d\exp_x(v)(w) = \widetilde{X}$. Set $\gamma(s,t) = \exp_0(s(v+tw))$. Let J be the corresponding Jacobi field. Then $J(1) = X$. By Jacobian equation,

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J - R(\dot\gamma, J)\dot\gamma = 0. \tag{1.3.46}$$

Take an orthonormal basis $e_1, \cdots, e_{2n}$ for T_0M_c, such that $e_1 = \dot\gamma/|\dot\gamma|$ and $e_{n+i} = Je_i$ for $i = 1, \cdots, n$. Parallel transport this basis along γ, then $\nabla_{\dot\gamma}e_i(s) = 0$ and $e_i(0) = e_i$. Write $J(s) = J^i(s)e_i(s)$, then (1.3.46) is

$$\frac{\partial^2 J^i(s)}{\partial^2 s} - |\dot\gamma|^2\langle R(e_1, e_j)e_1, e_i\rangle J^j(s) = 0. \tag{1.3.47}$$

By Theorem 1.3.8,

$$\langle R(e_1, e_j)e_1, e_i\rangle = \frac{c}{4}\left(\delta_{ij} - \delta_{1i}\delta_{1j} - \delta_{1,i-n}\delta_{j,n+1} + 2\delta_{1,i-n}\delta_{1,j-n}\right). \tag{1.3.48}$$

So X is uniquely determined by v, w and c. Since $\widetilde{X}$ satisfies the same equation with the same initial values, we have $|\widetilde{X}| = |X|$. That means ϕ is an isometry.

If $c > 0$, by Corollary 1.3.10, the Ricci curvature is positive. By Myers' Theorem, we know that $\widetilde{M}$ is compact. Let $U_0 \subset \mathbb{CP}^n$ be the open subset defined in (1.1.26). Then by the same argument, we can show that ϕ is an isometry from U_0 onto its image. Since U_0 is dense in $\mathbb{CP}^n$ and $\widetilde{M}$ is compact, we can extend ϕ to all of $\mathbb{CP}^n$ so that ϕ remains an isometry.

The proof of our theorem is completed. □

Definition 1.3.16. Given two J-invariant planes P and P' in T_xM, we define the **holomorphic bisectional curvature** $H(P, P')$ by

$$H(P, P') = R(X, JX, Y, JY), \tag{1.3.49}$$

where X is a unit vector in P and Y a unit vector in P'. It is a simple matter to verify that $R(X, JX, JY, Y)$ depends only on P and P'.

Set

$$U = \frac{1}{\sqrt{2}}(X - \sqrt{-1}JX), \quad V = \frac{1}{\sqrt{2}}(Y - \sqrt{-1}JY). \tag{1.3.50}$$

Then

$$\begin{aligned} H(P, P') = R(X, JX, Y, JY) &= R(U, \overline{U}, V, \overline{V}) \\ &= R(X, Y, Y, X) + R(X, JY, JY, X). \end{aligned} \tag{1.3.51}$$

If M is a space of constant holomorphic sectional curvature c, by Theorem 1.3.8 and (1.3.51),

$$\begin{aligned} H(P,P') &= R(X,Y,Y,X) + R(X,JY,JY,X) \\ &= \frac{c}{2}\left(1 + g(X,Y)^2 + g(X,JY)^2\right). \end{aligned} \tag{1.3.52}$$

It follows that, for a Kähler manifold of constant holomorphic sectional curva ture c, the holomorphic bisectional curvatures $H(P,P')$ lie between $c/2$ and c,

$$\frac{|c|}{2} \le |H(P,P')| \le |c|, \tag{1.3.53}$$

where the value $c/2$ is attained when P is perpendicular to P' and the value c is attained when $P = P'$.

We state an amazing theorem related to the bisectional curvature without proof to finish this introductory chapter.

A map $f : M \to N$ between two complex manifolds is called **biholomorphic** if f is a holomorphic homeomorphism.

Theorem 1.3.17 (Siu-Yau,Mori '80)**.** *Every compact Kähler manifold of positive bisectional curvature is biholomorphic to the complex projective space.*

Chapter 2

Topology of Kähler manifolds

2.1 Chern class

Definition 2.1.1. Let M be a smooth closed manifold with dimensional m. For $p \in \mathbb{Z}$, $0 \leq p \leq m$, the p-th **de Rham cohomology** of M (with complex coefficient) is defined by

$$H^p_{\mathrm{dR}}(M, \mathbb{C}) = \ker d|_{\Omega^p(M)} / d\Omega^{p-1}(M). \tag{2.1.1}$$

The (total) **de Rham cohomology** of M is defined as

$$H^*_{\mathrm{dR}}(M, \mathbb{C}) = \bigoplus_{p=0}^{m} H^p_{\mathrm{dR}}(M, \mathbb{C}). \tag{2.1.2}$$

From Definition 2.1.1, we see that any closed differential form ω on M, i.e., $d\omega = 0$, determines a cohomology class $[\omega] \in H^*_{\mathrm{dR}}(M, \mathbb{C})$. Moreover, two closed differential forms ω, ω' on M determine the same cohomology class if and only if there exists $\eta \in \Omega^*(M)$ such that $\omega - \omega' = d\eta$.

If ω, ω' are two closed differential forms on M and a is a constant function on M, then

$$[a\omega] = a[\omega], \quad [\omega + \omega'] = [\omega] + [\omega'] = [\omega'] + [\omega]. \tag{2.1.3}$$

Moreover, for $\eta, \eta' \in \Omega^*(M)$,

$$(\omega + d\eta) \wedge (\omega' + d\eta') = \omega \wedge \omega' + d(\eta \wedge \omega' + (-1)^{\deg \omega} \omega \wedge \eta' + \eta \wedge d\eta'). \tag{2.1.4}$$

Thus the cohomology class $[\omega \wedge \omega']$ depends only on $[\omega]$ and $[\omega']$. We denote it by $[\omega] \cdot [\omega']$. If ω'' is another closed differential forms on M, then

$$([\omega] + [\omega']) \cdot [\omega''] = [\omega] \cdot [\omega''] + [\omega'] \cdot [\omega'']. \tag{2.1.5}$$

From the above discussion, the de Rham cohomology of M carries a natural ring structure.

The importance of the de Rham cohomology lies in the de Rham theorem as follows.

Theorem 2.1.2. *Let M be a smooth closed oriented manifold with dimensional m. For $p \in \mathbb{Z}$, $0 \leq p \leq m$,*

(1) $\dim H^p_{\mathrm{dR}}(M, \mathbb{C}) < +\infty$*;*

(2) $H^p_{\mathrm{dR}}(M, \mathbb{C})$ *is canonically isomorphic to* $H^p_{\mathrm{Sing}}(M, \mathbb{C})$*, the p-th singular cohomology of M.*

By Theorem 2.1.2, we could see that $H^*_{\mathrm{dR}}(M, \mathbb{C})$ is a topological invariant, although we construct it from the differential structure and the differential forms. We usually simply denote it by $H^*(M, \mathbb{C})$.

Let E be a complex vector bundle over M. Recall that in (1.2.9), we interpret the curvature $R^E \in \Omega^2(M, \mathrm{End}(E))$ as the composition of connections. Furthermore, in view of the composition of the endomorphisms, for any $k \in \mathbb{N}$,

$$\left(R^E\right)^k = \overbrace{R^E \circ \cdots \circ R^E}^{k} : \mathscr{C}^\infty(M, E) \to \Omega^{2k}(M, E) \tag{2.1.6}$$

is a well-defined element lying in $\Omega^{2k}(M, \mathrm{End}(E))$.

For any $A \in \mathscr{C}^\infty(M, \mathrm{End}(E))$, the fiberwise trace of A forms a smooth function on M. We denote this function by $\mathrm{tr}[A]$. This further induces the map

$$\mathrm{tr} : \Omega^*(M, \mathrm{End}(E)) \to \Omega^*(M) \tag{2.1.7}$$

such that for any $\omega \in \Omega^*(M)$ and $A \in \Omega^*(M, \mathrm{End}(E))$,

$$\mathrm{tr}(\omega A) = \omega \, \mathrm{tr}[A]. \tag{2.1.8}$$

We also extend the Lie bracket operation on $\mathrm{End}(E)$ to $\Omega^*(M, \mathrm{End}(E))$ as follows: for any $\omega, \eta \in \Omega^*(M)$ and $A, B \in \mathscr{C}^\infty(M, \mathrm{End}(E))$,

$$[\omega A, \eta B] = (\omega A)(\eta B) - (-1)^{\deg \omega \cdot \deg \eta} (\eta B)(\omega A). \tag{2.1.9}$$

The following Proposition is obvious by (2.1.9).

Proposition 2.1.3. *For any $A, B \in \Omega^*(M, \mathrm{End}(E))$,*

$$\mathrm{tr}\left[[A, B]\right] = 0. \tag{2.1.10}$$

Proposition 2.1.4. *Let ∇^E be a connection on E. Then for any $A \in \Omega^*(M, \mathrm{End}(E))$,*

$$d\,\mathrm{tr}[A] = \mathrm{tr}\left[[\nabla^E, A]\right], \tag{2.1.11}$$

where

$$[\nabla^E, A] = \nabla^E \circ A - (-1)^{\deg A} A \circ \nabla^E \tag{2.1.12}$$

as in (2.1.9).

Proof. First of all, if $\widetilde{\nabla}^E$ is another connection on E, then by (1.2.3), $\nabla^E - \widetilde{\nabla}^E \in \Omega^1(M, \mathrm{End}(E))$. Thus by (2.1.3), we have $\mathrm{tr}\left[[\nabla^E - \widetilde{\nabla}^E, A]\right] = 0$. So the right hand side of (2.1.11) does not depend on the choice of ∇^E.

On the other hand, by (2.1.12), the right hand side of (2.1.11) is local. Thus for any $x \in M$, we could choose a sufficiently small open neighbourhood U_x of x such that $E|_{U_x}$ is trivial. Then we can take a trivial connection on $E|_{U_x}$ for which (2.1.11) holds obviously.

By combining the above independence and local properties, (2.1.11) holds on the whole manifold M.

The proof of our proposition is completed. □

Let

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots, \quad a_i \in \mathbb{C}, \tag{2.1.13}$$

be a power series in one variable. Since $R^E \in \Omega^2(M, \mathrm{End}(E))$,

$$\mathrm{tr}\left[f(R^E)\right] = a_0 + a_1\,\mathrm{tr}[R^E] + \cdots + a_n\,\mathrm{tr}\left[(R^E)^n\right] + \cdots \tag{2.1.14}$$

is an element in $\Omega^*(M, \mathbb{C})$, which only have finite terms.

We now state a form of the Chern-Weil theorem as follows.

Theorem 2.1.5. *(1) The form $\mathrm{tr}\left[f(R^E)\right]$ is closed. That is,*

$$d\,\mathrm{tr}\left[f(R^E)\right] = 0. \tag{2.1.15}$$

(2) If $\widetilde{\nabla}^E$ is another connection on E with curvature $\widetilde{R}^E$, then there is a differential form $\omega \in \Omega^*(M, \mathbb{C})$ such that

$$\mathrm{tr}\left[f(\widetilde{R}^E)\right] - \mathrm{tr}\left[f(R^E)\right] = d\omega. \tag{2.1.16}$$

Proof. (1) From Proposition 2.1.4,

$$d\,\mathrm{tr}\left[f(R^E)\right] = \mathrm{tr}\left[[\nabla^E, f(R^E)]\right] = \sum_i \mathrm{tr}\left[a_i\left[\nabla^E, (R^E)^i\right]\right] = 0 \qquad (2.1.17)$$

as we have the Bianchi Identity (cf. Proposition 1.2.7)

$$\left[\nabla^E, (R^E)^i\right] = \left[\nabla^E, (\nabla^E)^{2i}\right] = 0. \qquad (2.1.18)$$

(2) For any $t \in [0,1]$, let ∇_t^E be the deformed connection on E given by

$$\nabla_t^E = (1-t)\nabla^E + t\widetilde{\nabla}^E. \qquad (2.1.19)$$

Then ∇_t^E is a connection on E such that $\nabla_0^E = \nabla^E$ and $\nabla_1^E = \widetilde{\nabla}^E$. Moreover,

$$\frac{d\nabla_t^E}{dt} = \widetilde{\nabla}^E - \nabla^E \in \Omega^1(M, \mathrm{End}(E)). \qquad (2.1.20)$$

Let R_t^E be the curvature of ∇_t^E.

Let $f'(x)$ be the power series obtained from the derivative of $f(x)$. Then from Proposition 2.1.4 and (2.1.18),

$$\begin{aligned}\frac{d}{dt}\mathrm{tr}[f(R_t^E)] &= \mathrm{tr}\left[\frac{dR_t^E}{dt}f'(R_t^E)\right] = \mathrm{tr}\left[\frac{d(\nabla_t^E)^2}{dt}f'(R_t^E)\right] \\ &= \mathrm{tr}\left[\left[\nabla_t^E, \frac{d\nabla_t^E}{dt}\right]f'(R_t^E)\right] = \mathrm{tr}\left[\left[\nabla_t^E, \frac{d\nabla_t^E}{dt}f'(R_t^E)\right]\right] \\ &= d\,\mathrm{tr}\left[\frac{d\nabla_t^E}{dt}f'(R_t^E)\right]. \qquad (2.1.21)\end{aligned}$$

By (2.1.21), we have

$$\begin{aligned}\mathrm{tr}\left[f(\widetilde{R}^E)\right] - \mathrm{tr}\left[f(R^E)\right] &= \int_0^1 \frac{d}{dt}\mathrm{tr}[f(R_t^E)]dt \\ &= d\left(\int_0^1 \mathrm{tr}\left[\frac{d\nabla_t^E}{dt}f'(R_t^E)\right]dt\right). \qquad (2.1.22)\end{aligned}$$

The proof of our theorem is completed. □

Let

$$g(x) = b_0 + b_1x + \cdots + b_nx^n + \cdots, \quad b_i \in \mathbb{C}, \qquad (2.1.23)$$

be a power series in one variable.

Corollary 2.1.6. *(1) The form $g\left(\operatorname{tr}\left[f(R^E)\right]\right)$ is closed. That is,*

$$dg\left(\operatorname{tr}\left[f(R^E)\right]\right) = 0. \tag{2.1.24}$$

(2) If $\widetilde{\nabla}^E$ is another connection on E with curvature $\widetilde{R}^E$, letting $\nabla_t^E = (1-t)\nabla^E + t\widetilde{\nabla}^E$, we have

$$\begin{aligned} g\left(\operatorname{tr}\left[f(\widetilde{R}^E)\right]\right) &- g\left(\operatorname{tr}\left[f(R^E)\right]\right) \\ &= d\left(\int_0^1 g'\left(\operatorname{tr}\left[f(R_t^E)\right]\right) \cdot \operatorname{tr}\left[\frac{d\nabla_t^E}{dt} f'(R_t^E)\right] dt\right). \end{aligned} \tag{2.1.25}$$

By Theorem 2.1.5 (1), $g\left(\operatorname{tr}\left[f\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right)$ is a closed differential form which determines a cohomology class $\left[g\left(\operatorname{tr}\left[f\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right)\right]$ in $H^*(M,\mathbb{C})$. While Theorem 2.1.5 (2) says that this class does not depend on the choice of the connection ∇^E.

Definition 2.1.7. (1) The differential form $g\left(\operatorname{tr}\left[f\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right)$ is called the **Characteristic form** of E associated with ∇^E, f and g.

(2) The cohomology class $\left[g\left(\operatorname{tr}\left[f\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right)\right]$ is called the **Characteristic class** of E associated with f and g.

From (1.3.14), for $R^E \in \Omega^2(M, \operatorname{End}(E))$, we have

$$\det\left(I + \frac{\sqrt{-1}}{2\pi}R^E\right) = \exp\left(\operatorname{tr}\left[\log\left(I + \frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right) \tag{2.1.26}$$

in view of

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}, \quad \exp(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}. \tag{2.1.27}$$

Here I is the identity endomorphism of E.

Definition 2.1.8. The (total) **Chern form**, denoted by $c(E,\nabla^E)$, associated with ∇^E is defined by

$$c(E,\nabla^E) = \det\left(I + \frac{\sqrt{-1}}{2\pi}R^E\right). \tag{2.1.28}$$

We see that $c(E,\nabla^E)$ is a characteristic form in the sense of Definition 2.1.7. The associated characteristic class, denoted by $c(E)$, is called the (total)

Chern class of E. By (2.1.28), we have the decomposition of the (total) Chern form that

$$c(E,\nabla^E) = 1 + c_1(E,\nabla^E) + \cdots + c_k(E,\nabla^E) + \cdots \tag{2.1.29}$$

with

$$c_i(E,\nabla^E) \in \Omega^{2i}(M). \tag{2.1.30}$$

We call $c_i(E,\nabla^E)$ the i-th **Chern form** associated with ∇^E, and its associated cohomology class, denoted by $c_i(E)$, the i-th **Chern class** of E.

It is easy to see that if E is a trivial bundle, $c(E) = 1$.

Especially, by (2.1.26)-(2.1.29), the first Chern form

$$c_1(E,\nabla^E) = \frac{\sqrt{-1}}{2\pi}\,\mathrm{tr}[R^E] \in \Omega^2(M). \tag{2.1.31}$$

We rewrite (2.1.26) as

$$\log\left(\det\left(I + \frac{\sqrt{-1}}{2\pi}R^E\right)\right) = \mathrm{tr}\left[\log\left(I + \frac{\sqrt{-1}}{2\pi}R^E\right)\right]. \tag{2.1.32}$$

The from the power series expansion of $\log(1+x)$, we can deduce that for any integer $k \geq 0$, $\mathrm{tr}\left[(R^E)^k\right]$ can be written as a linear combination of various products of $c_i(E,\nabla^E)$'s.

Therefore, by Definition 2.1.7, any characteristic form (or characteristic class) could be written as a linear combination of various products of $c_i(E,\nabla^E)$'s (or $c_i(E)$'s). This establish the fundamental importance of the Chern class in the theory of characteristic classes of complex vector bundles.

Proposition 2.1.9. *Let E_1, E_2 be two vector bundles over M endowed with connections ∇^{E_1} and ∇^{E_2} respectively. Let R^{E_1} and R^{E_2} be the corresponding curvatures.*

(1) The curvature of the induced connection on the direct sum $E_1 \oplus E_2$ is given by

$$R^{E_1\oplus E_2} = R^{E_1} \oplus R^{E_2}. \tag{2.1.33}$$

(2) On the tensor product $E_1 \otimes E_2$, the induced curvature is given by

$$R^{E_1\otimes E_2} = R^{E_1} \otimes 1 \oplus 1 \otimes R^{E_2}. \tag{2.1.34}$$

(3) Let E^* be the dual of E, we have

$$R^{E^*} = -(R^E)^t. \tag{2.1.35}$$

(4) For a smooth map $f : N \to M$, we have

$$R^{f^*E} = f^*R^E. \tag{2.1.36}$$

Proposition 2.1.10. *Let E, E' be complex vector bundles over M endowed with connections ∇^E, $\nabla^{E'}$ respectively.*

(1) Let $\nabla^{E\oplus E'}$ be the induced connection on $E\oplus E'$,

$$c(E\oplus E', \nabla^{E\oplus E'}) = c(E,\nabla^E)\cdot c(E',\nabla^{E'}). \tag{2.1.37}$$

(2) Let $\nabla^{E\otimes E'}$ be the induced connection on $E\otimes E'$. If E' is a line bundle,

$$c_1(E\otimes E', \nabla^{E\otimes E'}) = c_1(E) + \mathrm{rank}(E)\cdot c_1(E'). \tag{2.1.38}$$

(3) Let ∇^{E^*} be the induced connection on the dual bundle E^*,

$$c_i(E^*, \nabla^{E^*}) = (-1)^i c_i(E,\nabla^E). \tag{2.1.39}$$

(4) Let $f: N\to M$ be a smooth map. Let $f^*\nabla^E$ be the induced connection on f^*E,

$$c_i(f^*E, f^*\nabla^E) = f^*c_i(E,\nabla^E). \tag{2.1.40}$$

(5) If $\mathrm{rank}(E)=k$, *then*

$$c_1(E,\nabla^E) = c_1(\Lambda^k E, \nabla^{\Lambda^k E}). \tag{2.1.41}$$

Proof. Note that (1), (3) and (4) follow directly from Proposition A.1.2. We only need to prove (2) and (5).

From Proposition A.1.2 (2), we have

$$\mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^{E\otimes E'}\right)\right] = \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^{E}\right)\right]\cdot(1+c_1(E',\nabla^{E'})). \tag{2.1.42}$$

Then (2) follows from taking the 2-form part of the two sides of (2.1.39).

Let $\nabla^{\Lambda^k E}$ be the connection on $\Lambda^k E$ induced from ∇^E. Let Γ and $\widetilde{\Gamma}$ be the connection forms of ∇^E and $\nabla^{\Lambda^k E}$. Let $\sigma_1,\cdots,\sigma_k$ be a local basis of sections of E. Then

$$\begin{aligned}\nabla^{\Lambda^k E}(\sigma_1\wedge\cdots\wedge\sigma_k) &= \sum_{i=1}^k \sigma_1\wedge\cdots\wedge\nabla^E\sigma_i\wedge\cdots\wedge\sigma_k\\ &= \sum_{i=1}^k \Gamma_{ii}\sigma_1\wedge\cdots\wedge\sigma_k.\end{aligned} \tag{2.1.43}$$

So we have

$$\widetilde{\Gamma} = \mathrm{tr}[\Gamma]. \tag{2.1.44}$$

Thus

$$\mathrm{tr}[R^E] = \mathrm{tr}[d\Gamma] + \mathrm{tr}[\Gamma \wedge \Gamma] = d\,\mathrm{tr}[\Gamma] = d\widetilde{\Gamma} = R^{\Lambda^k E}. \tag{2.1.45}$$

Therefore we get (5).

The proof of our proposition is completed. □

Example 2.1.11 (First Chern form of Chern connection)**.** Let M be a complex manifold and E be a holomorphic vector bundle over M with Hermitian metric h^E. Let ∇^E be the Chern connection on (E, h^E). By Theorem 1.2.11, the curvature of the Chern connection is

$$R^E = \bar{\partial}\partial \log(h^E). \tag{2.1.46}$$

Then the first Chern form

$$\begin{aligned} c_1(E, \nabla^E) = \frac{\sqrt{-1}}{2\pi}\,\mathrm{tr}[R^E] &= -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\,\mathrm{tr}\log(h^E) \\ &= -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\det(h^E) \in \Omega^{1,1}(M). \end{aligned} \tag{2.1.47}$$

Example 2.1.12 (γ_n on $\mathbb{CP}^n$)**.** By (1.2.32),

$$c_1(\gamma_n, \nabla^{\gamma_n}) = \frac{\sqrt{-1}}{2\pi}R^{\gamma_n} = -\frac{\sqrt{-1}}{2\pi}\frac{(1+|\theta|^2)d\theta_i \wedge d\bar{\theta}_i - \bar{\theta}_i\theta_j d\theta_i \wedge d\bar{\theta}_j}{(1+|\theta|^2)^2}. \tag{2.1.48}$$

From (1.2.33), we have

$$c_1(\gamma_n, \nabla^{\gamma_n}) = -\frac{1}{2\pi}\omega_{FS}, \tag{2.1.49}$$

where ω_{FS} is the Kähler form associated with the Fubini-Study metric on $\mathbb{CP}^n$.

If $n = 1$,

$$\int_{\mathbb{CP}^1}\omega_{FS} = \int_{\mathbb{C}}\frac{\sqrt{-1}d\theta \wedge d\bar{\theta}}{(1+|\theta|^2)^2} = \int_0^{2\pi}\int_0^{+\infty}\frac{2r}{(1+r^2)^2}dr d\varphi = 2\pi. \tag{2.1.50}$$

So from (2.1.49),

$$\int_{\mathbb{CP}^1} c_1(\gamma_1, \nabla^{\gamma_1}) = -1. \tag{2.1.51}$$

That means the first Chern class of the tautological bundle of $\mathbb{CP}^1$ is equal to -1 in $H^2(\mathbb{CP}^1, \mathbb{Z}) \simeq \mathbb{Z}$.

Definition 2.1.13. Let M be a complex manifold with complex dimension n. Then the holomorphic line bundle

$$K_M := T^{*(n,0)}M \tag{2.1.52}$$

is called the **canonical line bundle** of M.

Definition 2.1.14. Let M be a complex manifold. Let $T^{\mathbb{C}}M$ be the complex tangent bundle of M. We define

$$c_i(M) := c_i(T^{\mathbb{C}}M) = c_i(T^{(1,0)}M) \in H^{2i}(M, \mathbb{C}), \tag{2.1.53}$$

which is called the i-th **Chern class** of M.

From Proposition 2.1.10 (5), we have

$$c_1(M) = c_1(K_M^*) = -c_1(K_M). \tag{2.1.54}$$

If M is a complex manifold, by 1.1.7, $\bar{\partial}^2 = 0$. The following definition is well-defined.

Definition 2.1.15. Let M be a complex manifold. Then the (p, q)-Dolbeault cohomology is the vector space

$$H^{p,q}(M) := \frac{\operatorname{Ker}(\bar{\partial}|_{\Omega^{p,q}(M)})}{\operatorname{Im}(\bar{\partial}|_{\Omega^{p,q-1}(M)})}. \tag{2.1.55}$$

Note that if $\alpha \in \Omega^{p,q}$ is d-closed, it is $\bar{\partial}$-closed. It means that $[\alpha] \in H^{p,q}(M)$. So

- if E is a holomorphic vector bundle over M, $c_i(E) \in H^{i,i}(M)$;
- if (M, ω) is Kähler, $[\omega] \in H^{1,1}(M)$;
- if (M, ω) is Kähler, $[\operatorname{Ric}_\omega] \in H^{1,1}(M)$.

From Proposition 1.3.3, Definition 2.1.14 and (2.1.47), we have the following proposition.

Proposition 2.1.16. *Let (M, ω) be a Kähler manifold. Then the first Chern form of $T^{(1,0)}M$ associated with its Chern connection is*

$$c_1(T^{(1,0)}M, \nabla^{T^{(1,0)}M}) = \frac{\sqrt{-1}}{2\pi} R^{K_M^*} = -\frac{\sqrt{-1}}{2\pi} R^{K_M} = \frac{1}{2\pi} \operatorname{Ric}_\omega. \tag{2.1.56}$$

Here $\operatorname{Ric}_\omega$ *is the Ricci form in Definition 1.3.1. Moreover, for the Chern class,*

$$c_1(M) = \left[\frac{1}{2\pi} \operatorname{Ric}_\omega\right] \in H^{1,1}(M). \tag{2.1.57}$$

Theorem 2.1.17 (Calabi, Yau)**.** *Let (M, ω) be a Kähler manifold. For any $\rho \in [2\pi c_1(M)]$, there exists uniquely Kähler form ω' satisfying $[\omega'] = [\omega] \in H^{1,1}(M)$ such that $\mathrm{Ric}_{\omega'} = \rho$. In particular, if $c_1(M) = 0$, there exists Kähler form ω' such that $\mathrm{Ric}_{\omega'} = 0$, i.e., ω' is Ricci-flat or (M, ω') is a Kähler-Einstein manifold with Einstein constant 0.*

From (1.1.12) and (1.1.22), for Kähler form ω, we could calculate that

$$\begin{aligned}\omega^n &= (\sqrt{-1})^n g_{i_1,\bar{j}_1} \cdots g_{i_n,\bar{j}_n} dz^{i_1} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge dz^{i_n} \wedge d\bar{z}^{j_n} \\ &= (\sqrt{-1})^n g_{i_1,\bar{j}_1} \cdots g_{i_n,\bar{j}_n} \delta_{i_1,\cdots,i_n} \delta_{j_1,\cdots,j_n} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ &= (\sqrt{-1})^n g_{1,\bar{j}_{i_1}} \cdots g_{n,\bar{j}_{i_n}} \delta_{i_1,\cdots,i_n} \delta^{j_{i_1},\cdots,j_{i_n}}_{j_1,\cdots,j_n} \delta_{j_{i_1},\cdots,j_{i_n}} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ &= n!(\sqrt{-1})^n g_{1,\bar{j}_1} \cdots g_{n,\bar{j}_n} \delta_{j_1,\cdots,j_n} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ &= n!(\sqrt{-1})^n \det(g_{i\bar{j}}) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ &= 2^n n! \det(g_{i\bar{j}}) dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n. \end{aligned} \tag{2.1.58}$$

It means that ω^n is a volume form of M.

Definition 2.1.18. (1) A real (1,1)-form φ on a complex manifold M is called positive (resp. negative) if the symmetric tensor $\varphi(\cdot, J\cdot)$ is positive (resp. negative) definite. If $\varphi > 0$ (resp. $\varphi < 0$), as in (2.1.58), we have $\int_M \varphi^n > 0$ (resp. $\int_M \varphi^n < 0$).

(2)A cohomology class in $H^{1,1}(M) \cap H^2(M, \mathbb{R})$ is called positive (resp. negative) if it can be represented by a positive (resp. negative) (1,1)-form. (For the well-definedness of this definition, we need to figure out that if φ and φ' are two representatives of the cohomology, it is not possible that $\varphi > 0$ and $\varphi' < 0$. If not, we have $\int_M \varphi^n > 0$, $\int_M (\varphi')^n < 0$ and $(\varphi')^n - \varphi^n$ is d-exact. But by Stokes' formula, it is not possible.)

(3) A holomorphic line bundle L over a compact complex manifold is called positive (resp. negative) if there exists a Hermitian structure on L with Chern connection ∇^L and curvature $R^L = (\nabla^L)^2$ such that $\sqrt{-1}R^L$ is a positive (resp. negative) (1,1)-form.

From (2.1.31), it is easy to see that

$$L > 0 \Leftrightarrow c_1(L) > 0, \quad L < 0 \Leftrightarrow c_1(L) < 0. \tag{2.1.59}$$

Proposition 2.1.19. *If there exists a complex line bundle L over M such that $c_1(L) > 0$, then M is Kähler. Note that if $c_1(L) < 0$, $c_1(L^*) > 0$.*

Proof. From Definition 2.1.18, there exists a positive $(1,1)$-form φ such that $[\varphi] = c_1(L)$. Since $\varphi(\cdot, J\cdot)$ is positive definite, we take $g(\cdot,\cdot) := \varphi(\cdot, J\cdot)$ as the metric on M. Then φ is a closed Kähler form.

The proof of our proposition is completed. □

From (1.3.10) and (2.1.18) if (M,ω) is a Kähler-Einstein manifold with Einstein constant k, then

$$c_1(M) = k \cdot \left[\frac{1}{2\pi}\omega\right]. \tag{2.1.60}$$

Since $\omega > 0$, if $k > 0$ (resp. < 0), $c_1(M) > 0$ (resp. < 0).

Theorem 2.1.20 (Aubin, Yau)**.** *Let (M,ω) be a compact Kähler manifold. If $c_1(M) < 0$, there exists a unique Kähler-Einstein metric on M up to scalar factors.*

Comparing with Theorem 1.3.17, a recent result says that the negative holomorphic sectional curvature implies that the first Chern class is negative.

Theorem 2.1.21 (Wu-Yau, Tosatti-Yang, Diverio-Trapani)**.** *Let (M,ω) be a compact Kähler manifold with negative holomorphic sectional curvature. Then $c_1(M) < 0$.*

Definition 2.1.22. If M is a compact complex manifold with $c_1(M) > 0$, then it is called the **Fano manifold**.

By Proposition 2.1.19, the Fano manifold is Kähler.

Theorem 2.1.23 (Chen-Donaldson-Sun, Tian)**.** *Let (M,ω) be a compact Kähler manifold. If $c_1(M) > 0$, there exists a Kähler-Einstein metric on M if and only if M is K-stable.*

Note that in local coordinates,

$$c_1(T^{(1,0)}M, \nabla^{T^{(1,0)}M}) = -\frac{\sqrt{-1}}{2\pi}\frac{\partial^2 \log\det(g)}{\partial z_i \partial \bar{z}_j} dz^i \wedge d\bar{z}^j. \tag{2.1.61}$$

Example 2.1.24. We now compute the Chern class of $\mathbb{CP}^n$.

Let E be the orthogonal complement of γ_n using the standard Hermitian metric on $\mathbb{C}^{n+1}$, such that $\gamma_n \oplus E$ is a trivial complex vector bundle over $\mathbb{CP}^n$ with complex rank $n+1$. From the theory of vector bundles (cf. eg. Milnor "Characteristic class" Theorem 14.10), we could obtain that

$$T^{\mathbb{C}}\mathbb{CP}^n \simeq \mathrm{Hom}_{\mathbb{C}}(\gamma_n, E). \tag{2.1.62}$$

Observe that for complx line bundle γ_n, $\mathrm{Hom}_{\mathbb{C}}(\gamma_n,\gamma_n) \simeq \gamma_n^* \otimes \gamma_n$ is a trivial line bundle. (indeed, this result holds for all line bundles.) By adding the trivial line bundle on two sides of (2.1.62), we have

$$T^{\mathbb{C}}\mathbb{CP}^n \oplus \mathcal{E}^1 \simeq \mathrm{Hom}_{\mathbb{C}}(\gamma_n, \mathcal{E}^{n+1}). \tag{2.1.63}$$

Here $\mathcal{E}^k$ denotes the trivial complex vector bundle with rank k. Clearly the right hand side of (2.1.63) can be identified with the Whitney sum of $n+1$ copies of the dual bundle $\mathrm{Hom}_{\mathbb{C}}(\gamma_n, \mathcal{E}^1) \simeq \gamma_n^*$. Thus by Proposition 2.1.10 (1), (3), we have

$$c(\mathbb{CP}^n) = c(T^{\mathbb{C}}\mathbb{CP}^n \oplus \mathcal{E}^1) = c(\gamma_n^*)^{n+1} = (1 - c_1(\gamma_n))^{n+1}. \tag{2.1.64}$$

In particular,

$$c_1(\mathbb{CP}^n) = -(n+1)c_1(\gamma_n) = (n+1)\left[\frac{1}{2\pi}\omega^{FS}\right]. \tag{2.1.65}$$

From Proposition 2.1.16, we get the result again that $\mathbb{CP}^n$ is a Kähler-Einstein manifold with Einstein constant $n+1$.

From (2.1.64), $c_2(\mathbb{CP}^n) = \frac{n(n+1)}{2}c_1(\gamma_n)$. Combining with (2.1.65), we have

$$nc_1(\mathbb{CP}^n)^2 = 2(n+1)c_2(\mathbb{CP}^n). \tag{2.1.66}$$

For complex projective space, by (2.1.58),

$$\begin{aligned}\det\left(g^{FS}_{i\bar{j}}\right) &= \frac{\det\left((1+|\theta|^2)I - \bar{\theta}^t\theta\right)}{(1+|\theta|^2)^{2n}} \\ &= \frac{(1+|\theta|^2)^n \det(1 - |\theta|^2(1+|\theta|^2)^{-1})}{(1+|\theta|^2)^{2n}} = \frac{1}{(1+|\theta|^2)^{n+1}}.\end{aligned} \tag{2.1.67}$$

Here we use the identity of determinants:

$$\begin{aligned}\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} &= \det(A)\det(D - CA^{-1}B) \\ &= \det(D)\det(A - BD^{-1}C),\end{aligned} \tag{2.1.68}$$

for A, D invertible.

we could calculate that

$$\begin{aligned}&\int_{\mathbb{CP}^n}\left(\omega^{FS}\right)^n = 2^n n!\int_{U_0}\frac{1}{(1+|x|^2)^{n+1}}dx^1\wedge\cdots\wedge dx^{2n} \\ &= 2^n\pi^n n\int_0^{+\infty}\frac{2r^{2n-1}}{(1+r^2)^{n+1}}dr = 2^n\pi^n n\cdot n^{-1}\left(\frac{r^2}{1+r^2}\right)\Bigg|_0^{\infty} = (2\pi)^n.\end{aligned} \tag{2.1.69}$$

Here we use the formula

$$\int_{\mathbb{R}^n} f(|x|)dx^1\wedge\cdots\wedge dx^{2n} = \frac{2\pi^n}{(n-1)!}\int_0^{+\infty} r^{2n-1}f(r)dr. \tag{2.1.70}$$

Therefore, we have

$$\int_{\mathbb{CP}^n} c_1(\mathbb{CP}^n)^n = (n+1)^n. \tag{2.1.71}$$

Theorem 2.1.25 (Miyaoka-Yau Inequality)**.** *Let M^n be a compact Kähler manifold.*

(1) If M is Kähler-Einstein with $k > 0$, then

$$n \int_M c_1(M)^n \leq 2(n+1) \int_M c_1(M)^{n-2} c_2(M), \tag{2.1.72}$$

with equality if and only if $M = \mathbb{CP}^n$.

(2) If M is Kähler-Einstein with $k < 0$, then

$$n(-1)^{n-2} \int_M c_1(M)^n \leq 2(n+1)(-1)^{n-2} \int_M c_1(M)^{n-2} c_2(M), \tag{2.1.73}$$

with equality if and only if $M = \mathbb{H}_c^n / \Gamma$.

Theorem 2.1.26 (Fujita '18)**.** *If M^n is a Fano manifold with Kähler-Einstein metric, then*

$$\int_M c_1(M)^n \leq (n+1)^n, \tag{2.1.74}$$

with equality if and only if M is biholomorphic to $\mathbb{CP}^n$.

Remark 2.1.27. For the line bundles, the first Chern class is a complete invariant. It means that for any element in $H^2(M, \mathbb{Z})$, there exists a line bundle such that this element is the first Chern class of this bundle and if two line bundles are not isomorphic, then the first Chern classes of them are not equal. This is not right for vector bundles with higher rank.

We now list some other common characteristic classes here.

- The **Chern character form** associated with ∇^E is defined by

  $$\mathrm{ch}(E, \nabla^E) = \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi} R^E\right)\right] \in \Omega^{\mathrm{even}}(M). \tag{2.1.75}$$

  The associated cohomology class, denoted by $\mathrm{ch}(E)$, is called the **Chern character** of E. For complex vector bundles E_1, E_2,

  $$\mathrm{ch}(E_1 \oplus E_2) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2), \tag{2.1.76}$$
  $$\mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1)\,\mathrm{ch}(E_2). \tag{2.1.77}$$

  The Chern character is a polynomial with respect to the Chern classes:

  $$\mathrm{ch} = r + c_1 + \frac{1}{2}(-2c_2 + c_1^2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots . \tag{2.1.78}$$

  Here $r = \mathrm{rank}\, E$.

- The **Todd form** associated with ∇^E is defined by

$$\mathrm{Td}(E, \nabla^E) = \det\left(\frac{\frac{\sqrt{-1}}{2\pi}R^E}{1-\exp\left(-\frac{\sqrt{-1}}{2\pi}R^E\right)}\right) \in \Omega^{\mathrm{even}}(M). \quad (2.1.79)$$

  The associated cohomology class, denoted by $\mathrm{Td}(E)$, is called the **Todd class** of E. The Todd class is a polynomial with respect to the Chern classes:

$$\mathrm{Td} = \frac{1}{2}c_1 + \frac{1}{12}(c_2 + c_1^2) + \frac{1}{24}c_1c_2 + \cdots . \quad (2.1.80)$$

Recall that in (1.2.17), for holomorphic vector bundle E, $\bar{\partial}^E : \Omega^{p,q}(M,E) \to \Omega^{p,q+1}(M,E)$ is well-defined and $\left(\bar{\partial}^E\right)^2 = 0$.

Definition 2.1.28. Let M be a complex manifold and E be a holomorphic vector bundle. Then the Dolbeault cohomology $H^q(M,E)$ is the vector space

$$H^q(M,E) := \frac{\mathrm{Ker}(\bar{\partial}^E|_{\Omega^{0,q}(M,E)})}{\mathrm{Im}(\bar{\partial}^E|_{\Omega^{0,q-1}(M,E)})}. \quad (2.1.81)$$

We also denote by

$$H^{p,q}(M,E) := H^q(M, \Lambda^p T^{*(1,0)}M \otimes E). \quad (2.1.82)$$

Theorem 2.1.29 (Hirzebruch-Riemann-Roch Theorem)**.** *Let M be a complex manifold and E be a holomorphic vector bundle. Then*

$$\sum_{i=0}^{n}(-1)^i \dim_{\mathbb{C}} H^i(M,E) = \int_M \mathrm{Td}(T^{1,0}M)\,\mathrm{ch}(E). \quad (2.1.83)$$

Remark 2.1.30 (Characteristic class for real bundle)**.** Let now E be a real vector bundle over M, and ∇^E be a connection on E. Let R^E be the curvature of E. Proceeding in exactly the same way as (2.1.6)-Definition 2.1.7 for real vector bundles with connections, we could also get Chern-Weil theory for real vector bundles. In the following examples, we assume that E is a real bundle.

- The **Pontrjagin form** associated with ∇^E is defined by

$$p(E, \nabla^E) = \det\left(\left(I - \left(\frac{R^E}{2\pi}\right)^2\right)^{1/2}\right) \in \Omega^{4k}(M). \quad (2.1.84)$$

The associated cohomology class, denoted by $p(E)$, is called the **Pontrjagin class** of E. As the Chern form, $p(E,\nabla^E)$ admits a decomposition

$$p(E,\nabla^E) = 1 + p_1(E,\nabla^E) + \cdots + p_k(E,\nabla^E) + \cdots \tag{2.1.85}$$

with $p_i(E,\nabla^E)\Omega^{4i}(M)$. We call $p_i(E,\nabla^E)$ the i-th Pontrjagin form associated with ∇^E and the associated class $p_i(E)$ the i-th Pontrjagin class of E. For $i \geq 0$,

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}). \tag{2.1.86}$$

- The **Hirzebruch's L-form** associated with ∇^E is defined by

$$L(E,\nabla^E) = \det\left(\left(\frac{\frac{\sqrt{-1}}{2\pi}R^E}{\tanh\left(\frac{\sqrt{-1}}{2\pi}R^E\right)}\right)^{1/2}\right) \in \Omega^{4k}(M). \tag{2.1.87}$$

The associated cohomology class, denoted by $L(E)$, is called the **L-class** of E. The L-class is a polynomial with respect to the Pontrjagin classes:

$$L = \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2) + \frac{1}{3^3\cdot 5\cdot 7}(62p_3 - 13p_1p_2 + 2p_1^3) + \cdots . \tag{2.1.88}$$

- The **Â-form** associated with ∇^E is defined by

$$\hat{A}(E,\nabla^E) = \det\left(\left(\frac{\frac{\sqrt{-1}}{4\pi}R^E}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^E\right)}\right)^{1/2}\right) \in \Omega^{4k}(M). \tag{2.1.89}$$

The associated cohomology class, denoted by $\hat{A}(E)$, is called the **Â-class** of E. The Â-class is a polynomial with respect to the Pontrjagin classes:

$$\hat{A} = -\frac{1}{24}p_1 + \frac{1}{2^7\cdot 3^2\cdot 5}(-4p_2 + 7p_1^2) - \frac{1}{2^{10}\cdot 3^3\cdot 5\cdot 7}(16p_3 - 44p_1p_2 + 31p_1^3) + \cdots . \tag{2.1.90}$$

If E is oriented,

$$\mathrm{Td}(E\otimes\mathbb{C}) = \hat{A}(E)^2. \tag{2.1.91}$$

2.2 Hodge theory

Let M be a closed oriented Riemannian manifold. For $x \in M$, the metric on M induces a metric on T_x^*M, thus a metric g^Λ on $\Lambda^k T_x^* M$ for $k \le n := \dim M$. Explicitly, let $e^1, \cdots, e^n$ be an orthonormal basis of T^*M. Then $e_I \in \Lambda^k T^*M$ with $I = \{i_1 < \cdots < i_k\}$ forms an orthonormal basis of $\Lambda^k T_x^* M$. The volume form is locally given by

$$\mathrm{vol} := e^1 \wedge \cdots \wedge e^n. \tag{2.2.1}$$

Definition 2.2.1. The Hodge $*$-operator

$$* : \Lambda^k T^*M \to \Lambda T^{n-*} M \tag{2.2.2}$$

is defined by

$$*(e^{i_1} \wedge \cdots \wedge e^{i_k}) = \delta_{i_1,\cdots,i_k,j_1,\cdots,j_{n-k}} e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}, \tag{2.2.3}$$

for $\{i_1, \cdots, i_k, j_1, \cdots, j_{n-k}\} = \{1, \cdots, n\}$. In particular, we have $*1 = \mathrm{vol}$ and $*\,\mathrm{vol} = 1$.

Proposition 2.2.2. *(1) for any $\alpha, \beta \in \Lambda^k T^*M$, we have*

$$\alpha \wedge *\beta = g^\Lambda(\alpha, \beta)\,\mathrm{vol} \tag{2.2.4}$$

(2) On $\Lambda^k T^*M$,

$$*^2 = (-1)^{k(n-k)}. \tag{2.2.5}$$

(3) The $*$-operator is an isometry:

$$g^\Lambda(*\alpha, *\beta) = g^\Lambda(\alpha, \beta). \tag{2.2.6}$$

(4) For $\alpha \in \Lambda^k T^*M$, we have

$$g^\Lambda(\alpha, *\beta) = (-1)^{k(n-k)} g^\Lambda(*\alpha, \beta). \tag{2.2.7}$$

Proof. For (1), by (2.2.3),

$$\begin{aligned} &e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge *(e^{i_1} \wedge \cdots \wedge e^{i_k}) \\ &\qquad = \delta_{i_1,\cdots,i_k,j_1,\cdots,j_{n-k}} e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge e^{j_1} \wedge \cdots \wedge e^{j_{n-k}} = \mathrm{vol}\,. \end{aligned} \tag{2.2.8}$$

For (2), by (2.2.3), we have

$$\begin{aligned} *^2\,(e^{i_1} \wedge \cdots \wedge e^{i_k}) &= \delta_{i_1,\cdots,i_k,j_1,\cdots,j_{n-k}} * e^{j_1} \wedge \cdots \wedge e^{j_{n-k}} \\ &= \delta_{i_1,\cdots,i_k,j_1,\cdots,j_{n-k}} \delta_{j_1,\cdots,j_{n-k},i_1,\cdots,i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \\ &= (-1)^{k(n-k)} e^{i_1} \wedge \cdots \wedge e^{i_k}. \end{aligned} \tag{2.2.9}$$

For (3), by (2.2.5),

$$\begin{aligned} g^{\Lambda}(*\alpha, *\beta)\,\mathrm{vol} = *\alpha \wedge *^2\beta = (-1)^{k(n-k)} * \alpha \wedge \beta = \beta \wedge *\alpha \\ = g^{\Lambda}(\beta, \alpha)\,\mathrm{vol} = g^{\Lambda}(\alpha, \beta)\,\mathrm{vol} \end{aligned} \tag{2.2.10}$$

For (4), by (2.2.5) and (2.2.6),

$$g^{\Lambda}(\alpha, *\beta) = g^{\Lambda}(*\alpha, *^2\beta) = (-1)^{k(n-k)} g^{\Lambda}(*\alpha, \beta) \tag{2.2.11}$$

The proof of our proposition is completed. □

Definition 2.2.3. We define an inner product on forms $\langle \cdot, \cdot \rangle_{\mathbb{R}} : \Omega^k(M, \mathbb{R}) \times \Omega^k(M, \mathbb{R}) \to \mathbb{R}$ by

$$\langle \alpha, \beta \rangle_{\mathbb{R}} := \int_M g^{\Lambda}(\alpha, \beta) dv = \int_M \alpha \wedge *\beta. \tag{2.2.12}$$

We denote by $d^* : \Omega^*(M, \mathbb{R}) \to \Omega^{*-1}(M, \mathbb{R})$ the formal adjoint of d with respect to $\langle \cdot, \cdot \rangle_{\mathbb{R}}$, i.e., for any $\alpha, \beta \in \Omega^*(M, \mathbb{R})$,

$$\langle d\alpha, \beta \rangle_{\mathbb{R}} = \langle \alpha, d^*\beta \rangle_{\mathbb{R}}. \tag{2.2.13}$$

Proposition 2.2.4. *On $\Omega^k(M)$,*

$$d^* = (-1)^{n(k-1)+1} * d * . \tag{2.2.14}$$

Proof. By stokes' formula and Proposition 2.2.2, for $\alpha \in \Omega^{k-1}(M, \mathbb{R}), \beta \in \Omega^k(M, \mathbb{R})$, we have

$$\begin{aligned} \langle d\alpha, \beta \rangle_{\mathbb{R}} &= \int_M d\alpha \wedge *\beta = -(-1)^{k-1} \int_M \alpha \wedge d * \beta \\ &= (-1)^{k+(k-1)(n-k+1)} \int_M \alpha \wedge *^2 d * \beta = (-1)^{n(k-1)+1} \langle \alpha, *d * \beta \rangle_{\mathbb{R}}. \end{aligned} \tag{2.2.15}$$

The proof of our proposition is completed. □

Since $d^2 = 0$, by (2.2.13), we have

$$(d^*)^2 = 0. \tag{2.2.16}$$

We define the Laplace-Beltrami operator $\Delta_{\mathbb{R}}$ by

$$\Delta_{\mathbb{R}} := (d + d^*)^2 = dd^* + d^*d. \tag{2.2.17}$$

Proposition 2.2.5. *We have*

$$\ker(\Delta_{\mathbb{R}}) = \ker(d) \cap \ker(d^*). \tag{2.2.18}$$

Proof. The proposition follows from

$$\langle \Delta_{\mathbb{R}}\alpha, \alpha \rangle = \|d\alpha\|^2 + \|d^*\alpha\|^2. \tag{2.2.19}$$

The proof is completed. □

Theorem 2.2.6 (Hodge Theorem, real version)**.** *For any $k \in \mathbb{N}$, we have the orthogonal decomposition, called the Hodge decomposition*

$$\begin{aligned} \Omega^k(M) &= \ker(\Delta_{\mathbb{R}}|_{\Omega^k}) \oplus \operatorname{Im}(\Delta_{\mathbb{R}}|_{\Omega^k}) && (2.2.20)\\ &= \ker(\Delta_{\mathbb{R}}|_{\Omega^k}) \oplus \operatorname{Im}(d|_{\Omega^{k-1}}) \oplus \operatorname{Im}(d^*|_{\Omega^{k+1}}) && (2.2.21) \end{aligned}$$

and the canonical isomorphism

$$\ker(\Delta_{\mathbb{R}}|_{\Omega^k}) \simeq H^k(M, \mathbb{R}). \tag{2.2.22}$$

Especially, the space $\ker(\Delta_{\mathbb{R}}|_{\Omega^k})$ *is finite-dimensional.*

Corollary 2.2.7 (Poincaré duality)**.** *The bilinear form $\int_M \alpha \wedge \beta$ induces a non-degenerate pairing*

$$H^k(M, \mathbb{R}) \times H^{n-k}(M, \mathbb{R}) \to \mathbb{R}. \tag{2.2.23}$$

In other words, we get

$$H^k(M, \mathbb{R}) \simeq (H^{n-k}(M, \mathbb{R}))^*. \tag{2.2.24}$$

Proof. Take $[\alpha] \in H^k(M, \mathbb{R})$. Then by Hodge theorem, there exists $\alpha \in [\alpha]$ such that $\alpha \in \ker(\Delta_{\mathbb{R}}|_{\Omega^k})$. Thus by Proposition 2.2.5, $d^*\alpha = 0$. By Proposition 2.2.4, we have $d{*}\alpha = 0$. If $\int_M \alpha \wedge \beta = 0$ for any $\beta \in H^{n-k}(M, \mathbb{R})$, then $\int_M |\alpha|^2 dv = \int_M \alpha \wedge *\alpha = 0$. Thus $[\alpha] = 0$.

The proof of the corollary is completed. □

Now we assume that M is a closed complex manifold with $\dim_{\mathbb{C}} M = n$. As usual, let g be a Riemannian metric on TM. Then it could be $\mathbb{C}$-linearly extended on $TM \otimes \mathbb{C}$. We denote by

$$T^{*(p,q)}M = \Lambda^p(T^{*(1,0)}M) \otimes \Lambda^q(T^{*(0,1)}M). \tag{2.2.25}$$

Then by (1.1.1)

$$\Lambda^k(T^*M \otimes \mathbb{C}) = \bigoplus_{p+q=k} T^{*(p,q)}M. \tag{2.2.26}$$

From (1.2.12), the Riemannian metric g on TM induces a Hermitian metric h on $T^{(1,0)}M$, thus a Hermitian metric h^Λ on $T^{*(p,q)}M$. As in (1.2.12), for $\alpha, \beta \in \Omega^{p,q}(M)$, we have

$$h^\Lambda(\alpha, \beta) = g^\Lambda(\alpha, \bar{\beta}). \tag{2.2.27}$$

We extend the Hodge $*$-operator $\mathbb{C}$-linearly to

$$* : \Lambda^k(T^*M \otimes \mathbb{C}) \to \Lambda^{2n-k}(T^*M \otimes \mathbb{C}). \tag{2.2.28}$$

By Definition 2.2.3, we have

$$* : T^{*(p,q)}M \to T^{*(n-q,n-p)}M. \tag{2.2.29}$$

As in Definition 2.2.3, we define the Hermitian inner product $\langle \cdot, \cdot \rangle : \Omega^{p,q}(M) \times \Omega^{p,q}(M) \to \mathbb{C}$ by

$$\langle \alpha, \beta \rangle_{\mathbb{C}} := \int_M h^\Lambda(\alpha, \beta) dv = \int_M \alpha \wedge * \bar{\beta}. \tag{2.2.30}$$

By Definition 2.2.3 and Proposition 2.2.4, since $\dim_{\mathbb{R}} M$ is even, we have the following proposition.

Proposition 2.2.8. *Let ∂^* and $\bar{\partial}^*$ be the formal adjoint of ∂ and $\bar{\partial}$ respectively. Then we have*

$$d^* = \partial^* + \bar{\partial}^*, \quad (\partial^*)^2 = (\bar{\partial}^*)^2 = 0. \tag{2.2.31}$$

and

$$\partial^* = - * \bar{\partial} *, \quad \bar{\partial}^* = - * \partial * \tag{2.2.32}$$

Definition 2.2.9. The Laplacians associated with ∂ and $\bar{\partial}$ are defined as

$$\Delta_\partial = (\partial + \partial^*)^2 = \partial\partial^* + \partial^*\partial, \quad \Delta_{\bar{\partial}} = (\bar{\partial} + \bar{\partial}^*)^2 = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}. \tag{2.2.33}$$

Clearly,

$$\Delta_\partial, \Delta_{\bar{\partial}} : \Omega^{p,q}(M) \to \Omega^{p,q}(M). \tag{2.2.34}$$

The following proposition is an analogue of Proposition 2.2.5. The proof is the same.

Proposition 2.2.10. *We have*

$$\ker(\Delta_\partial) = \ker(\partial) \cap \ker(\partial^*), \quad \ker(\Delta_{\bar{\partial}}) = \ker(\bar{\partial}) \cap \ker(\bar{\partial}^*). \tag{2.2.35}$$

Theorem 2.2.11 (Hodge Theorem, complex version)**.** *Let M be a closed complex manifold. Then we have two natural orthogonal decompositions*

$$\Omega^{p,q}(M) = \ker(\Delta_\partial|_{\Omega^{p,q}}) \oplus \mathrm{Im}(\partial|_{\Omega^{p-1,q}}) \oplus \mathrm{Im}(\partial^*|_{\Omega^{p+1,q}}) \tag{2.2.36}$$

and

$$\Omega^{p,q}(M) = \ker(\Delta_{\bar\partial}|_{\Omega^{p,q}}) \oplus \mathrm{Im}(\bar\partial|_{\Omega^{p,q-1}}) \oplus \mathrm{Im}(\bar\partial^*|_{\Omega^{p,q+1}}). \tag{2.2.37}$$

The spaces $\ker(\Delta_\partial|_{\Omega^{p,q}})$ *and* $\ker(\Delta_{\bar\partial}|_{\Omega^{p,q}})$ *are finite dimensional. And*

$$\ker(\Delta_{\bar\partial}|_{\Omega^{p,q}}) \simeq H^{p,q}(M), \tag{2.2.38}$$

the (p,q)-Dolbeault cohomology.

Let E be a holomorphic vector bundle over M. In Definition 2.1.28, the operator $\bar\partial^E$ induces the Dolbeault cohomology group $H^*(M,E)$. Let h^E be a Hermitian metric on E. As in Definition 2.2.3, we define an inner product on forms $\langle\cdot,\cdot\rangle_E : \Omega^{0,q}(M,E) \times \Omega^{0,q}(M,E) \to \mathbb{C}$ by

$$\langle s,t\rangle_E := \int_M h^{\Lambda\otimes E}(s,t)dv. \tag{2.2.39}$$

Here $h^{\Lambda\otimes E}$ denotes by the Hermitian metric on $\Lambda^*(T^*M\otimes\mathbb{C})\otimes E$ induced by h^Λ and h^E. We denote by $\bar\partial^{E,*} : \Omega^{0,*}(M,E) \to \Omega^{0,*-1}(M,E)$ the formal adjoint of $\bar\partial^E$ with respect to $\langle\cdot,\cdot\rangle_E$, i.e., for any $s,t\in\Omega^*(M,E)$,

$$\langle\bar\partial^E s,t\rangle_E = \langle s,\bar\partial^{E,*}t\rangle_E. \tag{2.2.40}$$

As in (2.2.16), we have

$$\left(\bar\partial^{E,*}\right)^2 = 0. \tag{2.2.41}$$

Definition 2.2.12. The Hermitian metric h^E on E induces a $\mathbb{C}$-anti-linear isomorphism $h : E \simeq E^*$. The map

$$\bar{*}_E : T^{*(p,q)}M\otimes E \to T^{*(n-p,n-q)}M\otimes E^* \tag{2.2.42}$$

is defined by $\bar{*}_E(\alpha\otimes A) = *(\bar\alpha)\otimes h^E(A)$.

With this notation, for $s,t\in T^{*(p,q)}M\otimes E$,

$$h^{\Lambda\otimes E}(s,t) = s\wedge\bar{*}_E(t), \tag{2.2.43}$$

where "$\wedge$" is the exterior product in the form part and the evaluation map $E\otimes E^*\to\mathbb{C}$ in the bundle part. From Proposition 2.2.2 (2), on $T^{*(p,q)}M\otimes E$,

$$\bar{*}_{E^*}\circ\bar{*}_E = (-1)^{p+q}. \tag{2.2.44}$$

Proposition 2.2.13. *The formal adjoint operator*

$$\bar{\partial}^{E,*} = -\bar{*}_{E^*} \circ \bar{\partial}^{E^*} \circ \bar{*}_E. \tag{2.2.45}$$

Proof. For any holomorphic sections $s = \alpha \otimes A \in \Omega^{p,q}(M, E)$ and $t = \beta \otimes A' \in \Omega^{p,q+1}(M, E)$,

$$\begin{aligned}
\langle s, \bar{\partial}^{E,*} t\rangle_E = \langle \bar{\partial}^E s, t\rangle_E = \int_M \bar{\partial}^E s \wedge \bar{*}_E t = \int_M \bar{\partial}\alpha \wedge *\bar{\beta} \otimes A \otimes h(A') \\
\int_M (\bar{\partial}(\alpha \wedge *\bar{\beta} \otimes A \otimes h(A')) - (-1)^{p+q+1} \alpha \wedge \bar{\partial}(*\bar{\beta} \otimes A \otimes h(A'))) \\
= \int_M d(\alpha \wedge *\bar{\beta} \otimes A \otimes h(A')) - (-1)^{p+q+1} \int_M \alpha \wedge \bar{\partial}(*\bar{\beta} \otimes A \otimes h(A')) \\
= -(-1)^{p+q+1} \int_M s \wedge \bar{\partial}^{E^*}(\bar{*}_E t) = -\int_M s \wedge \bar{*}_{E^*} \circ \bar{*}_{E^*} \bar{\partial}^E (\bar{*}_E t) \\
= -\langle s, \bar{*}_{E^*} \circ \bar{\partial}^{E^*} \circ \bar{*}_E t\rangle_E.
\end{aligned} \tag{2.2.46}$$

The proof of our proposition is completed. □

Definition 2.2.14. The Laplacian associated with $\bar{\partial}^E$, which is called the Kodaira-Laplacian, is defined as

$$\Box^E = (\bar{\partial}^E + \bar{\partial}^{E,*})^2 = \bar{\partial}^E \bar{\partial}^{E,*} + \bar{\partial}^{E,*} \bar{\partial}^E = \left[\bar{\partial}^E, \bar{\partial}^{E,*}\right]. \tag{2.2.47}$$

Theorem 2.2.15 (Hodge Theorem, holomorphic bundle version)**.** *Let M be a closed complex manifold and E be a holomorphic vector bundle over M. Then we have the orthogonal decomposition*

$$\Omega^{0,q}(M, E) = \ker(\Box^E|_{\Omega^{0,q}}) \oplus \operatorname{Im}(\bar{\partial}^E|_{\Omega^{0,q-1}}) \oplus \operatorname{Im}(\bar{\partial}^{E,*}|_{\Omega^{0,q+1}}). \tag{2.2.48}$$

The spaces $\ker(\Box^E|_{\Omega^{0,q}})$ *is finite dimensional. And*

$$\ker(\Box^E|_{\Omega^{0,q}}) \simeq H^{0,q}(M, E). \tag{2.2.49}$$

Theorem 2.2.16 (Serre duality)**.** *Let M be a closed connected complex manifold. For $s \in \Omega^{0,q}(M, E)$, $t \in \Omega^{0,n-q}(M, K_M \otimes E^*) = \Omega^{n,n-q}(M, E^*)$, the bilinear form $\int_M s \wedge t$ induces a non-degenerate pairing*

$$H^q(M, E) \times H^{n-q}(M, K_M \otimes E^*) \to \mathbb{C}. \tag{2.2.50}$$

In other words, we get

$$H^q(M, E) \simeq (H^{n-q}(M, K_M \otimes E^*))^*. \tag{2.2.51}$$

Proof. Take $[\alpha] \in H^q(M, E)$. Then by Hodge theorem, there exists $\alpha \in [\alpha]$ such that $\alpha \in \ker(\square^E|_{\Omega^{0,q}})$. Thus by Proposition 2.2.13, we have $\bar{\partial}^{E^*}\bar{*}_E\alpha = 0$. If $\int_M \alpha \wedge \beta = 0$ for any $\beta \in H^{n-q}(M, K_M \otimes E^*)$, then $\int_M |\alpha|^2 dv = \int_M \alpha \wedge \bar{*}_E\alpha = 0$. Thus $[\alpha] = 0$.

The proof of the theorem is completed. □

By taking $E = T^{*(p,0)}M$, we have

Corollary 2.2.17 (Serre duality)**.** *Let M be a closed connected complex manifold. The bilinear form $\int_M \alpha \wedge \beta$ induces a non-degenerate pairing*

$$H^{p,q}(M) \times H^{n-p,n-q}(M) \to \mathbb{C}. \tag{2.2.52}$$

In other words, we get

$$H^{p,q}(M) \simeq (H^{n-p,n-q}(M))^*. \tag{2.2.53}$$

Remark that (2.2.51) is $\mathbb{C}$-linear and does not depend on the metrics on M and E.

Let ∇^E be the Chern connection on E. Recall that $(\nabla^E)^{1,0}$ is the $(1,0)$-part of ∇^E defined in (1.2.24). We denote by $(\nabla^E)^*$ and $(\nabla^E)^{1,0*}$ the formal adjoints of ∇^E and $(\nabla^E)^{1,0}$ with respect to $\langle\cdot,\cdot\rangle_E$ in (2.2.39) respectively.

Recall that $\widetilde{\nabla}$ is the connection defined in Proposition **??**. It is a connection on $TM \otimes \mathbb{C}$ and it preserves TM. We still denote by $\widetilde{\nabla}$ the induced connection on TM. Then it preserves the metric on M. Let T be the torsion of $\overline{\nabla}$. Then $T \in \Lambda^2(T^*M) \otimes TM$ is defined by

$$T(U, V) = \widetilde{\nabla}_U V - \widetilde{\nabla}_V U - [U, V], \tag{2.2.54}$$

for vector fields U, V. Then T maps $T^{(1,0)}M \otimes T^{(1,0)}M$ (resp. $T^{(0,1)}M \otimes T^{(0,1)}M$) into $T^{(1,0)}M$ (resp. $T^{(1,0)}M$) and vanish on $T^{(1,0)}M \otimes T^{(0,1)}M$. Indeed, for $U = U_i \frac{\partial}{\partial z_i} \in T^{1,0}M$, $V = V_j \frac{\partial}{\partial \bar{z}_j} \in T^{0,1}M$, we have

$$\widetilde{\nabla}_V U = \nabla_V^{T^{(1,0)}M} U = i_V \bar{\partial}^{T^{(1,0)}M} U = V_j \frac{\partial U_i}{\partial \bar{z}_j} \frac{\partial}{\partial z_i}, \tag{2.2.55}$$

and

$$\widetilde{\nabla}_U V = \overline{\nabla_{\overline{U}}^{T^{(1,0)}M} \overline{V}} = U_i \frac{\partial V_j}{\partial z_i} \frac{\partial}{\partial \bar{z}_j}. \tag{2.2.56}$$

Thus we have

$$\widetilde{\nabla}_U V - \widetilde{\nabla}_V U = [U, V]. \tag{2.2.57}$$

Let

$$\widetilde{\nabla}^E = \widetilde{\nabla} \otimes 1 + 1 \otimes \nabla^E. \tag{2.2.58}$$

Lemma 2.2.18. *Let $\{e_j\}$ be a locally orthonormal basis of TM and $\{e^j\}$ be the duals. We have*

$$\nabla^E = e^j \wedge \widetilde{\nabla}^E_{e_j} + \frac{1}{2} g(T(e_j, e_k), e_l) e^j \wedge e^k i_{e_l}, \tag{2.2.59}$$

$$(\nabla^E)^* = -i_{e_j} \wedge \widetilde{\nabla}^E_{e_j} - g(T(e_j, e_k), e_k) i_{e_j} + \frac{1}{2} g(T(e_j, e_k), e_l) e^l \wedge i_{e_k} i_{e_j}. \tag{2.2.60}$$

Especially, if $E = \mathbb{C}$, we have

$$d = e^j \wedge \widetilde{\nabla}_{e_j} + \frac{1}{2} g(T(e_j, e_k), e_l) e^j \wedge e^k i_{e_l}, \tag{2.2.61}$$

$$d^* = -i_{e_j} \wedge \widetilde{\nabla}_{e_j} - g(T(e_j, e_k), e_k) i_{e_j} + \frac{1}{2} g(T(e_j, e_k), e_l) e^l \wedge i_{e_k} i_{e_j}. \tag{2.2.62}$$

Proof. We prove (2.2.61) first. We denote by $\mathbf{d}$ the right hand side of (2.2.61). It is easy to see that for any homogeneous differential forms α, β, we have

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \mathbf{d}\beta. \tag{2.2.63}$$

So we only need to show that $\mathbf{d}$ agrees with d on functions, which is clear, and 1-forms. For any $f \in \mathscr{C}^\infty(M)$,

$$\begin{aligned} e^j \wedge \widetilde{\nabla}_{e_j} df &= e^j \wedge e^k \langle \widetilde{\nabla}_{e_j} df, e_k \rangle = e^j \wedge e^k \big(e_j(e_k(f)) - \langle df, \widetilde{\nabla}_{e_j} e_k \rangle \big) \\ &= \frac{1}{2} e^j \wedge e^k \big(e_j(e_k(f)) - \langle df, \widetilde{\nabla}_{e_j} e_k \rangle - e_k(e_j(f)) - \langle df, \widetilde{\nabla}_{e_k} e_j \rangle \big) \\ &= -\frac{1}{2} e^j \wedge e^k \langle T(e_j, e_k), df \rangle. \end{aligned} \tag{2.2.64}$$

Thus $\mathbf{d}$ coincides with d on 1-forms. Thus we get (2.2.61).

For (2.2.59), let $s = \alpha \otimes A \in \Omega^*(M, E)$. Then by (2.2.61),

$$\begin{aligned} \widetilde{\nabla}^E(\alpha \otimes A) &= d\alpha \otimes A + (-1)^{\deg \alpha} \alpha \wedge \nabla^E A \\ &= e^j \wedge \widetilde{\nabla}^E_{e_j} + \frac{1}{2} g(T(e_j, e_k), e_l) e^j \wedge e^k i_{e_l}. \end{aligned} \tag{2.2.65}$$

Now we prove (2.2.62). From the knowledge of differential geometry, for any $\theta \in \Omega^1(M)$, the function $\mathrm{tr}(\nabla\theta)$ is given by

$$\mathrm{tr}(\nabla\theta) = e_j(\alpha(e_j)) - \theta(\nabla_{e_j} e_j). \tag{2.2.66}$$

Then we have

$$\int_M \operatorname{tr}(\nabla\theta)dv = 0. \tag{2.2.67}$$

For $\alpha, \beta \in \Omega^*(M)$, take $\theta = -g^\Lambda(i.\alpha, \beta)$. We have

$$\operatorname{tr}(\nabla\alpha) = -e_j\big(g^\Lambda(i_{e_j}\alpha, \beta)\big) + g^\Lambda(i_{\nabla_{e_j}e_j}\alpha, \beta). \tag{2.2.68}$$

Since $i_{e_j}\widetilde{\nabla}_{e_j}\alpha = \widetilde{\nabla}_{e_j}i_{e_j}\alpha - i_{\widetilde{\nabla}_{e_j}e_j}\alpha$, we have

$$\begin{aligned} g^\Lambda(e^j \wedge \widetilde{\nabla}_{e_j}\alpha, \beta) = g^\Lambda(\widetilde{\nabla}_{e_j}\alpha, i_{e_j}\beta) &= e_j\big(g^\Lambda(\alpha, i_{e_j}\beta)\big) - g^\Lambda(\alpha, \widetilde{\nabla}_{e_j}i_{e_j}\beta) \\ &= e_j\big(g^\Lambda(\alpha, i_{e_j}\beta)\big) - g^\Lambda(\alpha, i_{e_j}\widetilde{\nabla}_{e_j}\beta) + g^\Lambda(\alpha, i_{\widetilde{\nabla}_{e_j}e_j}\beta) \\ &= -g^\Lambda(\alpha, i_{e_j}\widetilde{\nabla}_{e_j}\beta) - \operatorname{tr}(\nabla\theta) - g(T(e_k, e_j), e_j)g^\Lambda(\alpha, i_{e_k}\beta) \end{aligned} \tag{2.2.69}$$

Thus

$$\big(e^j \wedge \widetilde{\nabla}_{e_j}\big)^* = -i_{e_j}\widetilde{\nabla}_{e_j} - g(T(e_j, e_k), e_k)i_{e_j}. \tag{2.2.70}$$

We get (2.2.62).

Using the same argument in (2.2.65), we get (2.2.60).

The proof of our lemma is completed. □

Let $\mathcal{A}$ be a ring and $f, g : TM \otimes T^*M \otimes \mathbb{C} \to \mathcal{A}$ be two linear maps. Then from (1.1.18), we have

$$\begin{aligned} \sum_{i=1}^{2n} f(e^i)g(e_i) &= \sum_{i=1}^{n}\left(f(\theta^i)g(\theta_i) + f(\bar\theta^i)g(\bar\theta_i)\right), \\ \sum_{i=1}^{2n} f(e_i)g(e_i) &= \sum_{i=1}^{n}\left(f(\theta_i)g(\bar\theta_i) + f(\bar\theta_i)g(\theta_i)\right). \end{aligned} \tag{2.2.71}$$

By taking the $(1,0)$-part and the $(0,1)$-part of (2.2.59) and (2.2.60) and using (2.2.71), we have the following lemma.

Lemma 2.2.19. *Let $\{\theta_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}M$. Then we have*

$$\bar\partial^E = \bar\theta^j \wedge \widetilde{\nabla}^E_{\bar\theta_j} + \frac{1}{2}g(T(\bar\theta_j, \bar\theta_k), \theta_l)\bar\theta^j \wedge \bar\theta^k i_{\bar\theta_l}, \tag{2.2.72}$$

$$\bar\partial^{E,*} = -i_{\bar\theta_j} \wedge \widetilde{\nabla}^E_{\theta_j} - g(T(\theta_j, \theta_k), \bar\theta_k)i_{\bar\theta_j} + \frac{1}{2}g(T(\theta_j, \theta_k), \bar\theta_l)\bar\theta^l \wedge i_{\bar\theta_k}i_{\bar\theta_j}, \tag{2.2.73}$$

$$(\nabla^E)^{1,0} = \theta^j \wedge \widetilde{\nabla}^E_{\theta_j} + \frac{1}{2} g(T(\theta_j, \theta_k), \bar{\theta}_l)\theta^j \wedge \theta^k i_{\theta_l}, \tag{2.2.74}$$

$$(\nabla^E)^* = -i_{\theta_j} \wedge \widetilde{\nabla}^E_{\bar{\theta}_j} - g(T(\bar{\theta}_j, \bar{\theta}_k), \theta_k) i_{\theta_j} + \frac{1}{2} g(T(\bar{\theta}_j, \bar{\theta}_k), \theta_l)\theta^l \wedge i_{\theta_k} i_{\theta_j}. \tag{2.2.75}$$

Let ω be the real $(1,1)$-form associated with g in (1.1.13).

Definition 2.2.20. We define the **Lefschetz operator** $L = (\omega\wedge) \otimes 1$ on $\Lambda^{\cdot,\cdot}(T^*M) \otimes E$ and its adjoint $\Lambda = i(\omega)$ with respect to $h^{\Lambda \otimes E}$.

For $\{\theta_j\}_{j=1}^n$ a local orthonormal frame of $T^{(1,0)}M$, by (1.1.13), we have

$$L = \sqrt{-1}\theta^j \wedge \bar{\theta}^j \wedge, \quad \Lambda = -\sqrt{-1} i_{\bar{\theta}_j} i_{\theta_j}. \tag{2.2.76}$$

It is easy to see that

$$\Lambda = *^{-1} \circ L \circ *. \tag{2.2.77}$$

Definition 2.2.21. The holomorphic Kodaira Laplacian is defined by

$$\overline{\square}^E = \left[(\nabla^E)^{1,0}, (\nabla^E)^{1,0*}\right] = (\nabla^E)^{1,0}(\nabla^E)^{1,0*} + (\nabla^E)^{1,0*}(\nabla^E)^{1,0}. \tag{2.2.78}$$

The Hermitian torsion operator is defined by

$$\mathcal{T} := [\Lambda, \partial\omega] = [i(\omega), \partial\omega]. \tag{2.2.79}$$

Theorem 2.2.22 (Generalized Kähler identities)**.**

$$\left[\bar{\partial}^{E,*}, L\right] = \sqrt{-1}\left((\nabla^E)^{1,0} + \mathcal{T}\right), \tag{2.2.80}$$

$$\left[(\nabla^E)^{1,0*}, L\right] = -\sqrt{-1}\left(\bar{\partial}^E + \overline{\mathcal{T}}\right), \tag{2.2.81}$$

$$\left[\Lambda, \bar{\partial}^E\right] = -\sqrt{-1}\left((\nabla^E)^{1,0*} + \mathcal{T}^*\right), \tag{2.2.82}$$

$$\left[\Lambda, (\nabla^E)^{1,0}\right] = \sqrt{-1}\left(\bar{\partial}^{E,*} + \overline{\mathcal{T}}^*\right), \tag{2.2.83}$$

$$\left[\bar{\partial}^E, L\right] = \left[(\nabla^E)^{1,0}, L\right] = \left[\Lambda, \bar{\partial}^{E,*}\right] = \left[\Lambda, (\nabla^E)^{1,0*}\right] = 0. \tag{2.2.84}$$

Proof. Note that (2.2.82) and (2.2.83) are the adjoints of (2.2.80) and (2.2.81). We only need to prove the first two formulas.

From (2.2.73),

$$\left[\bar{\partial}^{E,*}, L\right] = -\left[i_{\bar{\theta}_j} \wedge \widetilde{\nabla}^E_{\theta_j}, L\right] - g(T(\theta_j, \theta_k), \bar{\theta}_k)\left[i_{\bar{\theta}_j}, L\right] + \frac{1}{2} g(T(\theta_j, \theta_k), \bar{\theta}_l)\left[\bar{\theta}^l \wedge i_{\bar{\theta}_k} i_{\bar{\theta}_j}, L\right]. \quad (2.2.85)$$

By (2.2.76),

$$\left[i_{\bar{\theta}_j}, L\right] = -\sqrt{-1}\theta^j \wedge . \quad (2.2.86)$$

Also by (2.2.76),

$$[\widetilde{\nabla}^E_{\theta_j}, L] = \sqrt{-1}(\widetilde{\nabla}_{\theta_j}\theta^k) \wedge \bar{\theta}^k \wedge + \sqrt{-1}\theta^k \wedge (\widetilde{\nabla}_{\theta_j}\bar{\theta}^k)\wedge = \sqrt{-1}\big(-g(\widetilde{\nabla}_{\theta_j}\theta_l, \bar{\theta}_k) - g(\theta_l, \widetilde{\nabla}_{\theta_j}\bar{\theta}_k)\big)\theta^l \wedge \bar{\theta}^k \wedge = 0. \quad (2.2.87)$$

Thus by (2.2.86) and (2.2.87),

$$-\left[i_{\bar{\theta}_j} \wedge \widetilde{\nabla}^E_{\theta_j}, L\right] = -\left[i_{\bar{\theta}_j}, L\right] \wedge \widetilde{\nabla}^E_{\theta_j} = \sqrt{-1}\theta^j \wedge \widetilde{\nabla}^E_{\theta_j}. \quad (2.2.88)$$

From (2.2.86), we have

$$\left[\bar{\theta}^l \wedge i_{\bar{\theta}_k} i_{\bar{\theta}_j}, L\right] = \bar{\theta}^l \wedge \left(\left[i_{\bar{\theta}_k}, L\right] i_{\bar{\theta}_j} + i_{\bar{\theta}_k}\left[i_{\bar{\theta}_j}, L\right]\right) = -\sqrt{-1}\bar{\theta}^l \left(\theta^k \wedge i_{\bar{\theta}_j} + i_{\bar{\theta}_k}\theta^j\right). \quad (2.2.89)$$

Thus,

$$\left[\bar{\partial}^{E,*}, L\right] = \sqrt{-1}\theta^j \wedge \widetilde{\nabla}^E_{\theta_j} + \sqrt{-1} g(T(\theta_j, \theta_k), \bar{\theta}_k)\theta^j + \sqrt{-1} g(T(\theta_j, \theta_k), \bar{\theta}_l)\theta^j \wedge \bar{\theta}^l \wedge i_{\theta_j}. \quad (2.2.90)$$

From (2.2.87), we see that $\widetilde{\nabla}\omega = 0$. By (2.2.74), we have

$$\partial\omega = \frac{1}{2} g(T(\theta_j, \theta_k), \bar{\theta}_l)\theta^j \wedge \theta^k i_{\theta_l}\omega = \frac{\sqrt{-1}}{2} g(T(\theta_j, \theta_k), \bar{\theta}_l)\theta^j \wedge \theta^k \wedge \bar{\theta}^l. \quad (2.2.91)$$

So from (2.2.90), since

$$[\Lambda, \theta^j] = -\sqrt{-1} i_{\bar{\theta}_j}, \quad [\Lambda, \bar{\theta}^j] = -\sqrt{-1} i_{\theta_j}, \quad (2.2.92)$$

we have

$$\mathcal{T} = \frac{1}{2} g(T(\theta_j, \theta_k), \bar{\theta}_l) \left(2\theta^k \wedge \bar{\theta}^l \wedge i_{\bar{\theta}_j} - 2\delta_{jl}\theta^k - \theta^j \wedge \theta^k \wedge i_{\theta_l}\right). \tag{2.2.93}$$

By (2.2.74), (2.2.90) and (2.2.93), we get (2.2.80).

As the computation is local, we can choose a locally holomorphic frame of E to reduce the proof of (2.2.81) to the case that E is a trivial bundle. Then (2.2.81) follows from (2.2.80) by conjugation.

The formula (2.2.84) follows directly from the Leibniz's rule.

The proofs of the generalized Kähler identities are completed. □

For super-commutator

$$[B, C] = BC - (-1)^{|B||C|} CB, \tag{2.2.94}$$

where $|\cdot|$ is the degree, the Jacobi identity reads

$$(-1)^{|C||A|}[A, [B, C]] + (-1)^{|A||B|}[B, [C, A]] + (-1)^{|B||C|}[C, [A, B]] = 0. \tag{2.2.95}$$

Theorem 2.2.23 (Bochner-Kodaira-Nakano formula)**.**

$$\square^E = \overline{\square}^E + \sqrt{-1}[R^E, \Lambda] + [(\nabla^E)^{1,0}, \mathcal{T}^*] - [\bar{\partial}^E, \overline{\mathcal{T}}^*]. \tag{2.2.96}$$

Proof. From Theorem 2.2.22, (1.2.27), (2.2.47), (2.2.78) and (2.2.95), we have

$$\begin{aligned}\square^E &= \left[\bar{\partial}^E, \bar{\partial}^{E,*}\right] = -\sqrt{-1}\left[\bar{\partial}^E, \left[\Lambda, (\nabla^E)^{1,0}\right]\right] - \left[\bar{\partial}^E, \overline{\mathcal{T}}^*\right] \\ &= -\sqrt{-1}\left[\Lambda, \left[(\nabla^E)^{1,0}, \bar{\partial}^E\right]\right] - \sqrt{-1}\left[(\nabla^E)^{1,0}, \left[\bar{\partial}^E, \Lambda\right]\right] - \left[\bar{\partial}^E, \overline{\mathcal{T}}^*\right] \\ &= -\sqrt{-1}\left[\Lambda, R^E\right] + \left[(\nabla^E)^{1,0}, (\nabla^E)^{1,0*}\right] + [(\nabla^E)^{1,0}, \mathcal{T}^*] - \left[\bar{\partial}^E, \overline{\mathcal{T}}^*\right].\end{aligned} \tag{2.2.97}$$

The proof of our theorem is complete. □

Now we assume that (M, ω) is Kähler.

Theorem 2.2.24. *Assume that (M, ω) is Kähler. Then*

$$\begin{gathered}[\bar{\partial}^*, L] = \sqrt{-1}\partial, \quad [\partial^*, L] = -\sqrt{-1}\bar{\partial}, \quad [\Lambda, \bar{\partial}] = -\sqrt{-1}\partial^*, \\ [\Lambda, \partial] = \sqrt{-1}\bar{\partial}^*, \quad \left[\bar{\partial}, L\right] = [\partial, L] = \left[\Lambda, \bar{\partial}^*\right] = [\Lambda, \partial^*] = 0, \\ \square^E = \overline{\square}^E + \sqrt{-1}[R^E, \Lambda], \quad \Delta_{\mathbb{R}} = 2\Delta_{\partial} = 2\Delta_{\bar{\partial}}.\end{gathered} \tag{2.2.98}$$

Proof. By Proposition **??** and (2.2.12), if (M,ω) is Kähler, $\mathcal{T}=0$. Thus we only need to prove the last formula.

By Theorem 2.2.22, we have

$$[\partial,\bar{\partial}^*]=-\sqrt{-1}[\partial,[\Lambda,\partial]]=\partial\Lambda\partial-\partial^2\Lambda+\Lambda\partial^2-\partial\Lambda\partial=0. \tag{2.2.99}$$

Thus

$$\begin{aligned}\Delta_{\mathbb{R}}=-[d,d^*]=[\partial+\bar{\partial},\partial^*+\bar{\partial}^*]=\Delta_\partial+\Delta_{\bar{\partial}}+[\partial,\bar{\partial}^*]+[\bar{\partial},\partial^*]\\ =2\Delta_\partial=2\Delta_{\bar{\partial}}.\end{aligned} \tag{2.2.100}$$

The proof of our theorem is completed. □

The following theorem is the direct consequence of Theorem 2.2.22 and 2.2.24.

Theorem 2.2.25. *Assume that (M,ω) is Kähler. We denote by $\Delta:=\Delta_\partial=2\Delta_{\bar{\partial}}$. Then*

(1) $H^k(M,\mathbb{C})\simeq\bigoplus_{p+q=k}H^{p,q}(M)$;

(2) $H^{p,q}(M)\simeq\overline{H^{q,p}(M)}$ and Serre duality yields $H^{p,q}(M)\simeq H^{n-p,n-q}(M)^*$;

(3) Δ commutes with $*,\partial,\bar{\partial},\partial^*,\bar{\partial}^*,L$ and Λ.

Since $\Delta\circ *=*\circ\Delta$ and $*^2=(-1)^{p(n-q)}$, the Hodge $*$-map induces an isomorphism

$$*:H^{p,q}(M)\simeq H^{n-q,n-p}(M). \tag{2.2.101}$$

Theorem 2.2.26 ($\partial\bar{\partial}$-lemma)**.** *Let M be a compact Kähler manifold. Then for a d-closed form α of type (p,q), the following conditions are equivalent:*

(1) The form α is d-exact, i.e., $\alpha=d\beta$ for some $\beta\in\Omega^{p+q+1}(M,\mathbb{C})$.

(2) The form α is ∂-exact, i.e., $\alpha=\partial\beta$ for some $\beta\in\Omega^{p-1,q}(M)$.

(3) The form α is $\bar{\partial}$-exact, i.e., $\alpha=\bar{\partial}\beta$ for some $\beta\in\Omega^{p,q-1}(M)$.

(4) The form α is $\partial\bar{\partial}$-exact, i.e., $\alpha=\partial\bar{\partial}\beta$ for some $\beta\in\Omega^{p-1,q-1}(M)$.

Proof. It is obvious that (4) implies (1), (2) and (3). By Hodge theory, if any of (1), (2) and (3) holds, we see that α is orthogonal to $\ker(\Delta)$. Since α is d-closed, it is ∂-closed and $\bar{\partial}$-closed. Since $\alpha\perp\mathrm{Im}\partial^*$, we have $\alpha=\partial\gamma$. Now we use the Hodge decomposition with respect to $\bar{\partial}$ to the form γ. Then $\gamma=\bar{\partial}\beta+\bar{\partial}^*\beta'+\beta''$ for some harmonic form β''. Thus, $\alpha=\partial\bar{\partial}\beta+\partial\bar{\partial}^*\beta'$. By (2.2.99), we have $\partial\bar{\partial}^*=-\bar{\partial}^*\partial$. Thus

$$\|\partial\bar{\partial}^*\beta'\|^2=\|\bar{\partial}^*\partial\beta'\|^2=(\bar{\partial}\bar{\partial}^*\partial\beta',\partial\beta')=(\bar{\partial}\partial\bar{\partial}\beta-\bar{\partial}\alpha,\partial\beta')=0. \tag{2.2.102}$$

We have $\alpha=\partial\bar{\partial}\beta$.

The proof is completed. □

2.3 Relations of Hodge numbers

Let (M,ω) be a compact Kähler manifold. Since $[L,\Delta_{\mathbb{R}}] = [\Lambda,\Delta_{\mathbb{R}}] = 0$, the Lefschetz operator L and its dual Λ induce maps between cohomology groups:

$$L : H^k(M,\mathbb{R}) \to H^{k+2}(M,\mathbb{R}), \quad \Lambda : H^k(M,\mathbb{R}) \to H^{k-2}(M,\mathbb{R}). \tag{2.3.1}$$

Definition 2.3.1. Let (M,ω) be a compact Kähler manifold. Then the primitive cohomology is defined by

$$\begin{aligned} H^k(M,\mathbb{R})_{\text{prim}} &:= \operatorname{Ker}\left(\Lambda : H^k(M,\mathbb{R}) \to H^{k-2}(M,\mathbb{R})\right), \\ H^{p,q}(M)_{\text{prim}} &:= \operatorname{Ker}\left(\Lambda : H^{p,q}(M) \to H^{p-1,q-1}(M)\right). \end{aligned} \tag{2.3.2}$$

Note that the primitive cohomology does not depend on the chosen Kähler structure and only on the cohomology class of the Kähler form $[\omega] \in H^{1,1}(M)$.

Theorem 2.3.2 (Hard Lefschetz Theorem)**.** *Let (M,ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} M = n$. Then for $k \le n$,*

$$L^{n-k} : H^k(M,\mathbb{R}) \simeq H^{2n-k}(M,\mathbb{R}) \tag{2.3.3}$$

and for any k,

$$H^k(M,\mathbb{R}) = \bigoplus_{i \ge (k-n)_+} L^i H^{k-2i}(X,\mathbb{R})_{\text{prim}}, \tag{2.3.4}$$

where $a_+ = \max\{a,0\}$. Moreover, these two isomorphisms are compatible with the bidegree decomposition. It means that for $k \le n$,

$$L^{n-k} : H^{p,k-p}(M) \simeq H^{n+p-k,n-p}(M), \tag{2.3.5}$$

and for any k,

$$H^{p,q}(M) = \bigoplus_{i \ge (p+q-n)_+} L^i H^{p-i,q-i}(X)_{\text{prim}}. \tag{2.3.6}$$

In particular,

$$H^k(M,\mathbb{R})_{\text{prim}} = \bigoplus_{p+q=k} H^{p,q}(M)_{\text{prim}}. \tag{2.3.7}$$

In order to prove the Hard Lefschetz theorem, we need the following lemma.

Lemma 2.3.3. *For $\alpha \in \Lambda^k T^*M$, we have*

$$[L, \Lambda]\alpha = (k - n)\alpha \tag{2.3.8}$$

and

$$[L^i, \Lambda]\alpha = i(k - n + i - 1)L^{i-1}\alpha. \tag{2.3.9}$$

Proof. We prove (2.3.8) by induction. If $\dim_{\mathbb{C}} M = 1$, for $\alpha \in \Lambda^0 T^*M$, $[L, \Lambda]\alpha = -\Lambda L\alpha = -\alpha$; for $\alpha \in \Lambda^1 T^*M$, $[L, \Lambda]\alpha = 0$; for $\alpha \in \Lambda^2 T^*M$, $[L, \Lambda]\alpha = L\Lambda\alpha = \alpha$. (2.3.8) holds.

Assume that (2.3.8) holds for $\dim_{\mathbb{C}} M = m$. If $\dim_{\mathbb{C}} M = m + 1$, for $x \in M$, we split T_xM by $T_xM = U \oplus V$ such that $\dim_{\mathbb{R}} V = 2$. Then $\Lambda^k T^*M = \oplus_{i=0}^2 \Lambda^{k-i} U \otimes \Lambda^i V$. For $\alpha \in \Lambda^k T^*M$, $\alpha = \beta_0 \otimes \beta_0' + \beta_1 \otimes \beta_1' + \beta_2 \otimes \beta_2'$. Thus, for $j = 0, 1, 2$,

$$\begin{aligned}[L, \Lambda]\beta_j \otimes \beta_j' &= L\left(\Lambda(\beta_j) \otimes \beta_j' + \beta_j \otimes \Lambda(\beta_j')\right) - \Lambda\left(L(\beta_j) \otimes \beta_j' + \beta_j \otimes L(\beta_j')\right) \\ &= [L, \Lambda](\beta_j) \otimes \beta_j' + \beta_j \otimes [L, \Lambda](\beta_j') = (k - j - m)\beta_j \otimes \beta_j' + (j - 1)\beta_j \otimes \beta_j' \\ &= (k - m - 1)\beta_j \otimes \beta_j'. \end{aligned} \tag{2.3.10}$$

Therefore, we get (2.3.8).

We also prove (2.3.9) by induction. By (2.3.8), (2.3.9) holds for $i = 1$. Assume that (2.3.9) holds for $i = m$. For $i = m + 1$,

$$\begin{aligned}[L^{m+1}, \Lambda]\alpha &= L^{m+1}\Lambda\alpha - \Lambda L^{m+1}\alpha = L[L^m, \Lambda]\alpha + [L, \Lambda]L^m\alpha \\ &= m(k - n + m - 1)L^m\alpha + (2m + k - n)L^m\alpha \\ &= (m + 1)(k - n + m)L^m\alpha. \end{aligned} \tag{2.3.11}$$

Therefore, we get (2.3.9).

The proof of our lemma is completed. □

The Hard Lefschetz theorem Theorem 2.3.2 follows directly from $[L, \Delta_{\mathbb{R}}] = 0$ and the following proposition.

Proposition 2.3.4. *Let $P^k = \{\alpha \in \Lambda^k T^*M : \Lambda\alpha = 0\}$.*

(i) If $u \in P^k$, then $L^s u = 0$ for $s \geq (n - k + 1)_+$.

(ii) If $k > n$, then $P^k = 0$.

(iii) The map $L^{n-k} : \Lambda^k T^*M \to \Lambda^{2n-k} T^*M$ is bijective.

(iv) If $k \leq n$, then $P^k = \{\alpha \in \Lambda^k T^*M : L^{n-k+1}\alpha = 0\}$.

(v) There exists orthogonal decomposition $\Lambda^k T^*M = \oplus_{i \geq (k-n)_+} L^i(P^{k-2i})$.

Proof. For (i), by (2.3.9), for $u \in P^k$,

$$\Lambda^s L^r u = \Lambda^{s-1}(\Lambda L^r - L^r \Lambda)u = r(n-k-r+1)\Lambda^{s-1}L^{r-1}u. \tag{2.3.12}$$

By induction, for $r \geq s$, we have

$$\Lambda^s L^r u = r(r-1)\cdots(r-s+1)\cdot(n-k-r+1)\cdots(n-k-r+s)L^{r-s}u. \tag{2.3.13}$$

Take $r = n+1$. Then $L^r u = 0$. Thus

$$(n+1)\cdots(n-s+2)\cdot(-k)\cdots(-k+s-1)L^{r-s}u = 0. \tag{2.3.14}$$

So if $s \leq k$, we have $L^{n+1-s}u = 0$, which is equivalent to (i).

Take $s = 0$ in (i). We get (ii).

(iii) Since $\mathrm{rk}(\Lambda^k T^*M) = \mathrm{rk}(\Lambda^{2n-k}T^*M)$, we only need to prove the injectivity. We prove it by induction on k. For $k = 0$, L^n is injective. We assume that the injectivity holds for $k \leq m-1$. For $k = m$, $r \leq n-k$, we can assume that L^{r-1} is injective on $\Lambda^m T^*M$. For $\alpha \in \Lambda^m T^*M$, if $L^r\alpha = 0$, then by Lemma 2.3.3,

$$\begin{aligned} L^{r-1}&(L\Lambda - r(m-n+r-1)\,\mathrm{Id})\alpha \\ &= [L^r, \Lambda]\alpha - r(m-n+r-1)L^{r-1}\alpha = 0. \end{aligned} \tag{2.3.15}$$

Thus $(L\Lambda - r(m-n+r-1)\,\mathrm{Id})\alpha = 0$. Since $r \leq n-m$, $r(m-n+r-1) \neq 0$. Thus there exists $\beta \in \Lambda^{m-2}T^*M$, such that $\alpha = L\beta$ and $L^{r+1}\beta = 0$. Since L^{r+1} is injective, $\beta = 0$. So $\alpha = 0$ and L^r is injective. By induction, we get L^m is injective. So (iii) holds for any $k \leq n$.

(iv) If $\alpha \in P^k$, from (ii), we have $L^{n-k+1}\alpha = 0$. If $L^{n-k+1}\alpha = 0$, we have $L^{n-k+1}\Lambda\alpha = 0$. Since L^{n-k+2} is bijective, we have $\Lambda\alpha = 0$.

(v) is equivalent to the statement that for any $\alpha \in \Lambda T^*M$, there exists unique decomposition

$$\alpha = \sum_{r \geq (k-n)_+} L^r u^r, \quad u_r \in P^{k-2r}. \tag{2.3.16}$$

We first study the uniqueness. Assume that $\alpha = 0$ and there exists r such that $u_r \neq 0$. Let s be the largest integer such that $u_s \neq 0$. Then

$$\Lambda^s \alpha = 0 = \sum_{(k-n)_+ \leq r \leq s} \Lambda^s L^r u_r = \sum_{(k-n)_+ \leq r \leq s} \Lambda^{s-r}\Lambda^r L^r u_r = \sum_{(k-n)_+ \leq r \leq s} c_{k,r}\Lambda^{s-r}u_r, \tag{2.3.17}$$

where $c_{k,r} = r!(n-k+r+1)\cdots(n-k+2r)$ by (2.3.13). Since $\Lambda u_r = 0$ for any r, we have $u_s = 0$, a contradiction.

For the existence, by (iii), we can assume that $k \le n$ and prove it by induction on k. It is obvious for $k = 1$. We assume that (2.3.16) exists for any $k < m$. By (iii), for $\alpha \in \Lambda^m T^*M$ there exists $\beta \in \Lambda^{m-2}T^*M$, such that $L^{n-m+2}\beta = L^{n-m+1}\alpha$. Let $\alpha_0 = \alpha - L\beta$. Then $L^{n-m+1}\alpha_0 = 0$. From (iv), $\alpha_0 \in P^m$.

The proof of our proposition is completed. □

Let $b^k(M) := \dim_{\mathbb{C}} H^k(M,\mathbb{C})$ be the usual **Betti number** of M and let $h^{p,q}(M) := \dim_{\mathbb{C}} H^{p,q}(M)$ be the so-called **Hodge number** of M when M is a complex manifold. Remark that b^k is a topological invariant but $h^{p,q}$ might be not, which depends on the complex structure.

In this section, we assume that (M,ω) is a compact Kähler manifold.

By Theorem 2.2.25 and (2.2.101), we have

$$\begin{aligned}
&b^k = \sum_{p+q=k} h^{p,q}; \quad \Leftarrow H^k(M,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(M);\\
&h^{p,q} = h^{q,p}; \quad \Leftarrow H^{p,q}(M) \simeq \overline{H^{q,p}(M)};\\
&h^{p,q} = h^{n-p,n-q}; \quad \Leftarrow \text{Serre duality:} H^{p,q}(M) \simeq H^{n-p,n-q}(M)^*;\\
&h^{p,q} = h^{n-q,n-p}; \quad \Leftarrow \text{Hodge}* : H^{p,q}(M) \simeq H^{n-q,n-p}(M);\\
&h^{p,k-p} = h^{n+p-k,n-p}, \forall p \le k \le n, \Leftarrow \text{Hard Lefschetz} : H^{p,k-p}(M) \simeq H^{n+p-k,n-p}(M).
\end{aligned} \tag{2.3.18}$$

A popular picture to describe the relations is the **Hodge diamond**:

$$\begin{array}{ccccccccc}
b^0 & & & & h^{0,0} & & & & \\
b^1 & & & h^{1,0} & & h^{0,1} & & & \\
b^2 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & & & & \cdots & & & & \\
b^n & h^{n,0} & & & \textbf{Serre} \searrow & & & h^{0,n} & \updownarrow \textbf{Hodge} \\
 & & & & \cdots & & & & \\
b^{2n-2} & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\
b^{2n-1} & & & h^{n,n-1} & & h^{n-1,n} & & & \\
b^{2n} & & & & h^{n,n} & & & & \\
 & & & & \overset{\longleftrightarrow}{\textbf{Conj}}. & & & &
\end{array}$$

Theorem 2.3.5. *On compact Kähler manifolds, we have*

(1) the odd Betti numbers b^{2k+1} are even;

(2) $h^{1,0} = \frac{1}{2}b^1$ is a topological invariant;

(3) the even Betti numbers b^{2k} are positive;

(4) $h^{p,p}$ are positive.

(5) if $k = p + q \leq n$, then $h^{p,q} \geq h^{p-1,q-1}$, $b_k \geq b_{k-2}$; If $k = p + q \geq n$, then $h^{p,q} \geq h^{p+1,q+1}$, $b_k \geq b_{k+2}$.

Proof. The first statement follows from

$$b^{2k+1} = \sum_{p=0}^{2k+1} h^{p,2k+1-p} = 2\sum_{p=0}^{k} h^{p,2k+1-p} \tag{2.3.19}$$

(2) is obvious.

For (3), if $\omega^k = d\alpha$, by Stokes' theorem, $\int_M \omega^n = \int_M d(\alpha \wedge \omega^{n-k}) = 0$. It will not happen since by (2.1.58), ω^n is a volume form. So ω^k is d-closed and not d-exact.

For (4), we observe that $\omega^p \in \Omega^{p,p}(M)$ and is $\bar\partial$-exact. If $\omega^p = \bar\partial\beta$, then $\omega^n = \bar\partial(\beta \wedge \omega^{n-p})$ is $\bar\partial$-exact. But $[\omega]^n \in H^{2n}(M,\mathbb{C}) \simeq H^{n,n}(M)$ is not equal to 0 since it is a volume form. So ω^p is not $\bar\partial$-exact.

For (5), let $h^{p,q}_{\text{prim}} = \dim H^{p,q}(M)_{\text{prim}}$. Then Theorem 2.3.2 says that if $p+q \leq n$,

$$h^{p,q} = h^{p,q}_{\text{prim}} + h^{p-1,q-1}_{\text{prim}} + \cdots \tag{2.3.20}$$

and if $p + q \geq n$,

$$h^{p,q} = h^{n-q,n-p}_{\text{prim}} + h^{n-q-1,n-p-1}_{\text{prim}} + \cdots . \tag{2.3.21}$$

So we get (5).

The proof of our theorem is completed. □

Corollary 2.3.6. *The only sphere that admits a Kähler structure is S^2.*

Let $P^{p,q}_{\mathbb{C}} = \{\alpha \in \Lambda^p T^{(1,0)*}M \otimes \Lambda^q T^{(0,1)*}M : \Lambda\alpha = 0\}$.

Lemma 2.3.7. *For $\alpha \in P^{p,q}_{\mathbb{C}}$, $p + q = k$, we have*

$$*\alpha = (-1)^{\frac{k(k+1)}{2}} \sqrt{-1}^{p-q} \frac{L^{n-k}\alpha}{(n-k)!}. \tag{2.3.22}$$

Proof. We only need to prove it at one point of M. In this proof we regard $T^{(1,0)*}M$ as $\mathbb{C}^n$. Let $dz_1, \cdots dz_n$ be a basis. For $S = \{i_1, \cdots, i_s\}$, we denote by $\omega_S = \left(\frac{\sqrt{-1}}{2}\right)^s dz_{i_1} \wedge d\bar z_{i_1} \wedge \cdots \wedge dz_{i_s} \wedge d\bar z_{i_s}$. We can write

$$\alpha = \sum_{A,B,S} \gamma_{A,B,S} dz_A \wedge dz_M \wedge \omega_S, \tag{2.3.23}$$

where A, B, S are disjoint subsets of $\{1, \cdots, n\}$. Let $\alpha_{A,B} = \sum_S \gamma_{A,B,S} dz_A \wedge d\bar{z}_B \wedge \omega_S$. Thus $\Lambda\alpha = 0$ implies that $\Lambda\alpha_{A,B} = 0$. So we only need to prove the lemma for

$$\alpha = dz_A \wedge d\bar{z}_B \wedge \sum_S \gamma_S \omega_S. \tag{2.3.24}$$

In this sum, we only need to consider the subsets $S \subset K := \{1, \cdots, n\} - (A \cup B)$ and the cardinal $m = |S| = (k - |A| - |B|)/2$. Since $\Lambda\alpha = 0$, for any $N \subset K$ with $|N| = m - 1$, we have

$$\sum_{i \in K - N} \gamma_{N \cup \{i\}} = 0. \tag{2.3.25}$$

Let ${}^cS$ be the complement of S in K. Then by (2.2.30), we have

$$\begin{aligned}(d\bar{z}_A \wedge dz_B \wedge \omega_S) \wedge *(dz_A \wedge d\bar{z}_B \wedge \omega_S) &= \mathrm{vol} \\ &= \left(\frac{\sqrt{-1}}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n. \end{aligned} \tag{2.3.26}$$

After a careful calculation, we have

$$*(dz_A \wedge d\bar{z}_B \wedge \omega_S) = (-1)^{m + \frac{k(k+1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^{n-k} \sqrt{-1}^{p-q} dz_A \wedge d\bar{z}_B \wedge \omega_{{}^cS} \tag{2.3.27}$$

So

$$*\alpha = \sum_{S \subset K} (-1)^{m + \frac{k(k+1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^{n-k} \sqrt{-1}^{p-q} \gamma_S dz_A \wedge d\bar{z}_B \wedge \omega_{{}^cS}. \tag{2.3.28}$$

On the other hand,

$$\begin{aligned}&(-1)^{\frac{k(k+1)}{2}} \sqrt{-1}^{p-q} \frac{L^{n-k}\alpha}{(n-k)!} \\ &\qquad = (-1)^{\frac{k(k+1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^{n-k} \sqrt{-1}^{p-q} \sum_{S,N} \gamma_S dz_A \wedge d\bar{z}_B \wedge \omega_{S \cup N} \\ &= \sum_{J \subset K} (-1)^{\frac{k(k+1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^{n-k} \sqrt{-1}^{p-q} \left(\sum_{S \subset J} \gamma_S\right) dz_A \wedge d\bar{z}_B \wedge \omega_J, \end{aligned} \tag{2.3.29}$$

where N runs through the subsets of cardinal $n - k$ contained in K and disjoint from S.

For every $r \leq m$, let $S_r = \sum_{|N \cap J| = r} \gamma_N$. Then $S_0 = \gamma_{^cJ}$ and $S_m = \sum_{N \subset J} \gamma_N$. Then (2.3.25) implies that $(r+1)S_{r+1} = -(m-r)S_r$. So $S_m = (-1)^m S_0$. It means that

$$\sum_{S \subset J} \gamma_S = (-1)^m \gamma_{^cJ}. \tag{2.3.30}$$

From (2.3.28), (2.3.29) and (2.3.30), we get (2.3.22).

The proof of our lemma is completed. □

Let (M, ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} M = n$. The Poincaré duality implies a non-degenerate pairing

$$\langle\ ,\ \rangle : H^k(M, \mathbb{R}) \times H^{n-k}(M, \mathbb{R}) \to \mathbb{R}. \tag{2.3.31}$$

We define the intersection form Q on $H^k(M, \mathbb{R})$, $k \leq n$ by

$$Q(\alpha, \beta) = \langle L^{n-k}\alpha, \beta \rangle = \int_M \omega^{n-k} \wedge \alpha \wedge \beta. \tag{2.3.32}$$

Clearly, it is symmetric for k even and antisymmetric for k odd. Thus on $H^k(M, \mathbb{C})$, the sesquilinear form

$$H_k(\alpha, \beta) = (\sqrt{-1})^k Q(\alpha, \bar{\beta}) \tag{2.3.33}$$

is a Hermitian form.

Lemma 2.3.8. *For $k \leq n$, the Lefschetz decomposition*

$$H^k(M, \mathbb{C}) = \bigoplus_{i \geq 0} L^i H^{k-2i}(X, \mathbb{C})_{\text{prim}}. \tag{2.3.34}$$

is orthogonal for H^k. Moreover, on each primitive component $L^i H^{k-2i}(X, \mathbb{C})_{\text{prim}}$, H_k induces the form $(-1)^i H_{k-2i}$.

Proof. For $\alpha = L^r\alpha'$, $\beta = L^s\beta'$, with α', β' primitive and $r < s$, we have $L^{n-k}\alpha \wedge \beta = L^{n-k+r+s}\alpha' \wedge \beta'$. By Proposition 2.3.4 (iv), we have $L^{n-k+r+s}\alpha' = 0$. Thus $H_k(\alpha, \beta) = 0$. The second statement is obvious.

The proof of our lemma is completed. □

The curve case of the following theorem is due to Riemann.

Theorem 2.3.9 (Hodge-Riemann bilinear relation)**.** *Let (M, ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} M = n$. The decomposition $H^k(M, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(M)$ is orthogonal for H_k. Moreover, the form $(\sqrt{-1})^{p-q-k}(-1)^{\frac{k(k+1)}{2}} H_k$ is positive definite on $H^{p,q}(M)_{\text{prim}}$.*

Proof. The first statement follows directly by counting the degrees.

For the second statement, for $\alpha \in H^{p,q}(M)_{\text{prim}}$ the harmonic form, by Lemma 2.3.7,

$$\begin{aligned} H_k(\alpha,\alpha) &= (\sqrt{-1})^k \int_M \alpha \wedge L^{n-k}\bar{\alpha} \\ &= (\sqrt{-1})^k (n-k)!(-1)^{\frac{k(k-1)}{2}} \sqrt{-1}^{q-p} \int_M \alpha \wedge *\bar{\alpha} \\ &= (n-k)!(-1)^{\frac{k(k-1)}{2}} \sqrt{-1}^{q-p+k} \|\alpha\|^2_{L_2}. \end{aligned} \quad (2.3.35)$$

The proof of our theorem is completed. □

Theorem 2.3.10 (Hodge index theorem)**.** *Let (M,ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} M = n$ even. Let $\operatorname{sign}(Q)$ be the signature of the intersection form $Q(\alpha,\beta) = \int_M \alpha \wedge \beta$ on $H^n(M,\mathbb{R})$. Then*

$$\operatorname{sign}(Q) = \sum_{p,q} (-1)^p h^{p,q}. \quad (2.3.36)$$

In particular, the number $\sum_{p,q}(-1)^p h^{p,q}$ is a topological invariant.

Proof. For $n = 2k$, $\alpha \in H^{p,q}(M)_{\text{prim}}$, $p+q = n-2r$, we have

$$\operatorname{sign}(Q) = (-1)^k \operatorname{sign}(H). \quad (2.3.37)$$

By (2.3.35),

$$\begin{aligned} H(L^r\alpha) = (-1)^r H_{n-2r}(\alpha) &= (-1)^r(-1)^{k-r-p}(2r)!\|\alpha\|^2_{L^2} \\ &= (2r)!(-1)^{k+p}\|\alpha\|^2_{L^2}. \end{aligned} \quad (2.3.38)$$

So

$$\operatorname{sign}(Q) = \sum_{r\geq 0, p+q=n-2r} (-1)^p h^{p,q}_{\text{prim}} = \sum_{p+q=n} (-1)^p \sum_{j\geq 0} (-1)^j h^{p-j,q-j}_{\text{prim}}. \quad (2.3.39)$$

By (2.3.20), we have $h^{p,q}_{\text{prim}} = h^{p,q} - h^{p-1,q-1}$. So

$$\begin{aligned} \operatorname{sign}(Q) &= \sum_{p+q=n} (-1)^p \left(h^{p,q} + 2\sum_{j>0} (-1)^j h^{p-j,q-j} \right) \\ &\overset{(1)}{=} \sum_{p+q=n} (-1)^p \left(h^{p,q} + \sum_{j\neq 0} (-1)^j h^{p-j,q-j} \right) \\ &= \sum_{p+q \text{ even}} (-1)^p h^{p,q} \overset{(2)}{=} \sum_{p,q} (-1)^p h^{p,q}. \end{aligned} \quad (2.3.40)$$

Here (1) uses the Serre duality and (2) follows from $(-1)^p h^{p,q} + (-1)^q h^{q,p} = 0$ if $p+q$ is odd.

The proof of our theorem is completed. □

Definition 2.3.11. Let M be a compact complex manifold of dimension n. The Hirzebruch χ_y-genus is the polynomial

$$\chi_y := \sum_{p,q=0}^{n} (-1)^q h^{p,q} y^p. \tag{2.3.41}$$

It is a special case of the elliptic genus, a mathematical analogue of the partition function in physics. The following theorem is the corollary of the Hirzebruch-Riemann-Roch Theorem 2.1.29.

Theorem 2.3.12. *In local terms,*

$$\chi_y = \int_M \mathrm{Td}(T^{1,0}M) \left(\sum_{p=0}^{n} y^p \, \mathrm{ch}(T^{(p,0)*}M) \right). \tag{2.3.42}$$

If $y = 0$, $\chi_0 = \sum_{q=0}^{n} (-1)^q h^{0,q}$ and $\mathrm{Td}(M) := \int_M \mathrm{Td}(T^{1,0}M)$ are two definitions of the arithmetic genus in the history.

If $y = 1$, and if M is Kähler with even complex dimension, then $\chi_1 = \mathrm{sign}(Q)$ in Theorem 2.3.10. In this case, (2.3.42) reads

$$\mathrm{sign}(Q) = \int_M L(M), \tag{2.3.43}$$

where L is defined in (2.1.87). This is the Hirzebruch signature theorem, which also holds for compact $4k$-dimensional manifolds.

If $y = -1$, and if M is Kähler,

$$\chi_{-1} = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q} = \sum_{k=0}^{n} (-1)^k b_k = e(M), \tag{2.3.44}$$

the Euler number. In this case, Theorem 2.3.12 means that

$$e(M) = \int_M c_n(M) = c_n(M). \tag{2.3.45}$$

This is the Gauss-Bonnet-Chern Theorem for complex manifolds. Note that (2.3.45) also holds on compact complex manifolds.

We finish this chapter by the famous Hodge conjecture.

Definition 2.3.13. The fundamental class $[Z] \in H^{p,p}(M)$ of a complex submanifold $Z \subset M$ of codimension p in M is defined by

$$\int_M \alpha \wedge [Z] = \int_Z \alpha|_Z \tag{2.3.46}$$

for any $\alpha \in H^{2n-2p}(M)$.

Definition 2.3.14. If M is a complex submanifold of a complex projective space, then M is called a projective manifold.

Now we could state a version of the Hodge conjecture.

Conjecture 2.3.15 (Hodge conjecture)**.** Let M be a projective manifold. For any $\alpha \in H^{p,p}(M) \cap H^{2p}(M, \mathbb{Q})$, it could be generated linearly by the fundamental classes with coefficients in $\mathbb{Q}$.

Remark that the Hodge conjecture is false for Kähler manifolds. And there exists $\alpha \in H^{p,p}(M) \cap H^{2p}(M, \mathbb{Z})$ such that it could not be generated linearly by the fundamental classes with coefficients in $\mathbb{Z}$.

Here we summarize the supercommutative relations of $\partial, \bar{\partial}, \partial^*, \bar{\partial}^*, L$ and Λ for compact Kähler manifold, which contains the Kähler identity.

Let $[A, B] = AB - (-1)^{|A||B|}BA$.

B \ A	∂	$\bar{\partial}$	∂^*	$\bar{\partial}^*$	L	Λ
∂	∂^2	0	Δ	0	0	$\sqrt{-1}\bar{\partial}^*$
$\bar{\partial}$	0	$\bar{\partial}^2$	0	Δ	0	$-\sqrt{-1}\partial^*$
∂^*	Δ	0	$\partial^{*,2}$	0	$\sqrt{-1}\bar{\partial}$	0
$\bar{\partial}^*$	0	Δ	0	$\bar{\partial}^{*,2}$	$-\sqrt{-1}\partial$	0
L	0	0	$-\sqrt{-1}\bar{\partial}$	$\sqrt{-1}\partial$	0	$n-k$
Λ	$-\sqrt{-1}\bar{\partial}^*$	$\sqrt{-1}\partial^*$	0	0	$k-n$	0

Chapter 3

Positive vector bundles and vanishing theorems

3.1 Bochner methods and vanishing theorem

For a vector bundle E over a Riemannian manifold M with a connection ∇^E, by taking a locally orthonormal basis, the usual Bochner Laplacian Δ^E is defined by

$$\Delta^E = -\sum_{j=1}^{\dim_{\mathbb{R}} M} \left((\nabla^E_{e_j})^2 - \nabla^E_{\nabla^{TX}_{e_j} e_j} \right). \tag{3.1.1}$$

We assume that the vector bundle E admits a Euclidean metric if it is real or a Hermitian metric if it is complex. We denote the corresponding metric by $\langle \cdot, \cdot \rangle$. We assume that the connection ∇^E preserves the metric on E.

For $s_1, s_2 \in \mathscr{C}^\infty(M, E)$ with compact support, we have

$$\begin{aligned}\int_M \langle \Delta^E s_1, s_2 \rangle dv &= \sum_{j=1}^{\dim_{\mathbb{R}} M} \int_M \langle \nabla^E_{e_j} s_1, \nabla^E_{e_j} s_2 \rangle dv - \int_M \operatorname{tr}(\nabla \alpha) dv \\ &= \sum_{j=1}^{\dim_{\mathbb{R}} M} \int_M \langle \nabla^E_{e_j} s_1, \nabla^E_{e_j} s_2 \rangle dv = \int_M \langle s_1, \Delta^E s_2 \rangle dv, \end{aligned} \tag{3.1.2}$$

where $\alpha(Y) = \langle \nabla^E_Y s_1, s_2 \rangle$.

Lemma 3.1.1. *Let V be a real vector space with basis e_i. For any $A \in \operatorname{End}(V)$, there exists a unique endomorphism $\lambda(A)$, which is called the* ***derivation****, on ΛV, such that it coincides with A on V and satisfies the Leibniz's* *rules:*

$$\lambda(A)(a \wedge b) = A(a) \wedge b + a \wedge A(b), \tag{3.1.3}$$

where $a, b \in \Lambda V$*. Explicitly, it is given by*

$$\lambda(A) = \langle e^j, Ae_k \rangle e_j \wedge i_{e_k}. \tag{3.1.4}$$

Proof. The uniqueness is obvious. We only need to prove that (3.1.4) is a derivation. Firstly, for $e_k \in V$, we have $\lambda(A)e_k = \langle e^j, Ae_k\rangle e_j = Ae_k$. Secondly, the operator $e_j \wedge i_{e_k}$ satisfies the Leibniz's rule (3.1.3).

The proof of our lemma is completed. □

Theorem 3.1.2 (Weitzenböck's formula)**.** *Let* R *be the curvature of the Levi-Civita connection on* TM*. Then*

$$(d + d^*)^2 = \Delta^{\Lambda T^*M} - \sum_{ijkl} R_{ijkl} e^k \wedge i_{e_l} e^i \wedge i_{e_j}. \tag{3.1.5}$$

In particular, on the space of one forms, we have

$$\Delta_{\mathbb{R}} = (d + d^*)^2 = \Delta^{\Lambda T^*M} + \mathrm{Ric}(e_i, e_j) e^i \wedge i_{e_j}. \tag{3.1.6}$$

Proof. Let $\nabla^{\Lambda T^*M}$ be the connection on ΛT^*M induced by the Levi-Civita connection ∇. Let $R^{\Lambda T^*M}$ be the curvature of $\nabla^{\Lambda T^*M}$. From (2.2.61) and (2.2.62), we have

$$d = e^j \wedge \nabla^{\Lambda T^*M}_{e_j}, \quad d^* = -i_{e_j} \nabla^{\Lambda T^*M}_{e_j}. \tag{3.1.7}$$

Since the formulas (3.1.5) and (3.1.6) do not depend on the choice of the locally orthonormal coordinates. We choose the normal coordinates. Notice that

$$e^i \wedge i_{e_j} + i_{e_j} e^i \wedge = \delta_{ij} \,\mathrm{Id}\,. \tag{3.1.8}$$

We have

$$\begin{aligned} dd^* + d^*d = -e^i \wedge i_{e_j} \nabla^{\Lambda T^*M}_{e_i} \nabla^{\Lambda T^*M}_{e_j} - i_{e_j} e^i \wedge \nabla^{\Lambda T^*M}_{e_j} \nabla^{\Lambda T^*M}_{e_i} \\ = -\nabla^{\Lambda T^*M}_{e_i} \nabla^{\Lambda T^*M}_{e_i} - R^{\Lambda T^*M}(e_i, e_j) e^i \wedge i_{e_j}. \end{aligned} \tag{3.1.9}$$

Let R^{TM} be the curvature of the Levi-Civita connection ∇. It is easy to see that $R^{\Lambda T^*M}$ is the derivation of R^{T^*M}. By (3.1.4), we have

$$R^{\Lambda T^*M} = \langle e_k, R^{T^*M} e^l \rangle e^k \wedge i_{e_l} = \langle R^{TM} e_l, e_k \rangle e^k \wedge i_{e_l}. \tag{3.1.10}$$

Combining (3.1.9) and (3.1.10), we have

$$dd^* + d^*d = \Delta^{\Lambda T^*M} - R_{ijkl} e^k \wedge i_{e_l} e^i \wedge i_{e_j}. \tag{3.1.11}$$

From (3.1.8), we have

$$\begin{aligned} R_{ijkl} e^k \wedge i_{e_l} e^i \wedge i_{e_j} &= -R_{ijkl} e^k \wedge e^i \wedge i_{e_l} i_{e_j} + R_{ijki} e^k \wedge i_{e_j} \\ &= -R_{ijkl} e^k \wedge e^i \wedge i_{e_l} i_{e_j} - \mathrm{Ric}(e_i, e_j) e^i \wedge i_{e_j}. \end{aligned} \tag{3.1.12}$$

Notice that the first term on the right-hand side vanishes on one forms. Then we get (3.1.6).

The proof of our theorem is completed. □

Definition 3.1.3. A function (resp. a twofold symmetric covariant tensor, etc) on a manifold is **quasi-positive** if it is everywhere nonnegative (resp. positive semi-definite, etc) and is positive (resp. positive definite, etc) at a point. **Quasi-negativity** is dually defined.

Theorem 3.1.4 (Bochner 1946)**.** *For a compact orientable Riemannian manifold M of nonnegative Ricci curvature, its first Betti number $b_1 \leq \dim M$, with the upper bound attained by the flat torus. If the Ricci curvature is quasi-positive, then $b_1 = 0$.*

Proof. From (3.1.2), for any $\alpha \in \Omega^1(M)$, then

$$\int_M \langle \Delta^{\Lambda T^*M} \alpha, \alpha \rangle dv = \sum_{j=1}^{\dim_{\mathbb{R}} M} \|\nabla_{e_j}^{\Lambda T^*M} \alpha\|_{L_2}^2 \geq 0. \tag{3.1.13}$$

If the Ricci curvature is quasi-positive, there exists $x \in M$ such that $\alpha = 0$ on a neighbourhood of x. Since

$$\int_M \langle \mathrm{Ric}(e_i, e_j) e^i \wedge i_{e_j} \alpha, \alpha \rangle dv \geq 0, \tag{3.1.14}$$

by (3.1.6) and (3.1.13), we have $\nabla^{\Lambda T^*M} \alpha = 0$. So $\alpha \equiv 0$. Thus $\ker \Delta_{\mathbb{R}} = 0$. From the Hodge theorem 2.2.6, we have $b_1 = 0$.

If the Ricci curvature is nonnegative, we have

$$\int_M \langle \mathrm{Ric}(e_i, e_j) e^i \wedge i_{e_j} \alpha, \alpha \rangle dv \geq 0. \tag{3.1.15}$$

If $\alpha \in \ker \Delta_{\mathbb{R}}$, from (3.1.6), (3.1.13) and (3.1.15), we have $\nabla^{\Lambda T^*M} \alpha = 0$. For any $x \in M$, we have

$$b_1 \leq \dim_{\mathbb{R}} \{\alpha_x : \nabla^{\Lambda T^*M} \alpha = 0\} = \dim_{\mathbb{R}} M. \tag{3.1.16}$$

Notice that for torus T^n, $H^n(T^n, \mathbb{R}) = H^1(S^1, \mathbb{R})^{\otimes n} = \mathbb{R}^n$. Thus the proof of our theorem is completed. □

Now we consider the Kähler case.

Let (M,ω) be a compact orientable Kähler manifold. Let E be a Hermitian holomorphic vector bundle over M with Hermitian connection ∇^E. We simply denote by $\Delta^{0,\cdot}$ the Laplacian with respect to the connection $\nabla^{\Lambda T^{0,1}M\otimes E}$ induced by the connections $\nabla^{T^{(0,1)}M}$ and ∇^E. Recall that $K_M^* = \Lambda^n(T^{1,0}M)$ and

$$\operatorname{tr}\left[R^{T^{1,0}M}\right] = R^{K_M^*} = -\sqrt{-1}\,\mathrm{Ric}_\omega\,. \tag{3.1.17}$$

Theorem 3.1.5 (Bochner-Kodaira). *Let E be a Hermitian holomorphic vector bundle over the Kähler manifold M. In a local holomorphic coordinate system,*

$$\begin{aligned}\square^E = (\bar\partial^E + \bar\partial^{E,*})^2 = \frac{1}{2}\Delta^{0,\cdot} - \frac{1}{2}R^E(\theta_i,\bar\theta_i) \\ + \left(R^E + \frac{1}{2}\operatorname{tr}\left[R^{T^{1,0}M}\right]\right)(\theta_j,\bar\theta_k)\bar\theta^k\wedge i_{\bar\theta_j}.\end{aligned} \tag{3.1.18}$$

Proof. By Theorem 1.2.14, we could choose the normal holomorphic coordinates. In this coordinates around $x\in M$, we have $[\nabla, i_{\bar\theta_k}] = [\nabla, \bar\theta^k\wedge] = 0$ and $[\bar\theta_j,\theta_k] = \nabla_{\bar\theta_j}\theta_k - \nabla_{\theta_k}\bar\theta_j = 0$ at x.

By (2.2.72) and (2.2.73), $\bar\partial^E = \bar\theta^j\wedge\nabla^{\Lambda T^{(0,1)}M\otimes E}_{\bar\theta_j}$ and $\bar\partial^{E,*} = -i_{\bar\theta_j}\nabla^{\Lambda T^{(0,1)}M\otimes E}_{\theta_j}$. We simply denote by $\nabla^{0,\cdot} := \nabla^{\Lambda T^{(0,1)}M\otimes E}$ Thus

$$\begin{aligned}\bar\partial^E\bar\partial^{E,*} + \bar\partial^{E,*}\bar\partial^E &= -\bar\theta^j\wedge i_{\bar\theta_k}\nabla^{0,\cdot}_{\bar\theta_j}\nabla^{0,\cdot}_{\theta_k} - i_{\bar\theta_k}\bar\theta^j\wedge\nabla^{0,\cdot}_{\theta_k}\nabla^{0,\cdot}_{\bar\theta_j} \\ &= -(\bar\theta^j\wedge i_{\bar\theta_k} + i_{\bar\theta_k}\bar\theta^j\wedge)\nabla^{0,\cdot}_{\theta_k}\nabla^{0,\cdot}_{\bar\theta_j} - \bar\theta^j\wedge i_{\bar\theta_k}\left(\nabla^{0,\cdot}_{\bar\theta_j}\nabla^{0,\cdot}_{\theta_k} - \nabla^{0,\cdot}_{\theta_k}\nabla^{0,\cdot}_{\bar\theta_j}\right) \\ &= -\nabla^{0,\cdot}_{\theta_j}\nabla^{0,\cdot}_{\bar\theta_j} + R^E(\theta_k,\bar\theta_j)\bar\theta^j i_{\bar\theta_k} + R^{T^{0,\cdot *}M}(\theta_k,\bar\theta_j)\bar\theta^j i_{\bar\theta_k}.\end{aligned} \tag{3.1.19}$$

By (2.2.71),

$$\sum_{i=1}^{2n}\nabla^{0,\cdot}_{e_i}\nabla^{0,\cdot}_{e_i} = \sum_{i=1}^{n}\left(\nabla^{0,\cdot}_{\theta_i}\nabla^{0,\cdot}_{\bar\theta_i} + \nabla^{0,\cdot}_{\bar\theta_i}\nabla^{0,\cdot}_{\theta_i}\right) = 2\sum_{i=1}^{n}\nabla^{0,\cdot}_{\theta_i}\nabla^{0,\cdot}_{\bar\theta_i} - \sum_{i=1}^{n}R^{0,\cdot}(\theta_i,\bar\theta_i). \tag{3.1.20}$$

Since we choose the normal coordinates for Kähler manifold, by (2.2.71), $\sum_{i=1}^{2n}\nabla^{TX}_{e_i}e_i = \sum_{i=1}^{n}\nabla^{TX}_{\theta_i}\bar\theta_i + \sum_{i=1}^{n}\nabla^{TX}_{\bar\theta_i}\theta_i = 0$. So

$$-\nabla^{0,\cdot}_{\theta_j}\nabla^{0,\cdot}_{\bar\theta_j} = \frac{1}{2}\Delta^{0,\cdot} - \frac{1}{2}R^E(\theta_i,\bar\theta_i) - \frac{1}{2}R^{T^{0,*}M}(\theta_i,\bar\theta_i). \tag{3.1.21}$$

From Lemma 3.1.1,

$$R^{\Lambda T^{0,1*}M} = \langle\bar{\theta}_l, R^{T^{0,1*}M}\bar{\theta}^s\rangle\bar{\theta}^l \wedge i_{\bar{\theta}_s} = g(R\theta_s, \bar{\theta}_l)\bar{\theta}^l \wedge i_{\bar{\theta}_s}. \quad (3.1.22)$$

Thus

$$R^{T^{0,\cdot *}M}(\theta_k, \bar{\theta}_j)\bar{\theta}^j i_{\bar{\theta}_k} - \frac{1}{2}R^{T^{0,*}M}(\theta_i, \bar{\theta}_i) = -R_{k\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_s}\bar{\theta}^j \wedge i_{\bar{\theta}_k} + \frac{1}{2}R_{j\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_s} \quad (3.1.23)$$

By Bianchi Identity, $R_{k\bar{j}s\bar{l}} + R_{sk\bar{j}\bar{l}} + R_{\bar{j}sk\bar{l}} = 0$. Since $R_{sk\bar{j}\bar{l}} = 0$, we have

$$R_{k\bar{j}s\bar{l}} = R_{s\bar{j}k\bar{l}}. \quad (3.1.24)$$

As in (3.1.8), we have

$$\bar{\theta}^i \wedge i_{\bar{\theta}_j} + i_{\bar{\theta}_j}\bar{\theta}^i \wedge = \delta_{ij}\,\mathrm{Id}\,. \quad (3.1.25)$$

So

$$R_{k\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_s}\bar{\theta}^j \wedge i_{\bar{\theta}_k} = R_{k\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_k}\bar{\theta}^j \wedge i_{\bar{\theta}_s} = -R_{k\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_s}\bar{\theta}^j \wedge i_{\bar{\theta}_k} + R_{k\bar{j}j\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_k} + R_{j\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_s} \quad (3.1.26)$$

It implies

$$-R_{k\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_s}\bar{\theta}^j \wedge i_{\bar{\theta}_k} = -R_{j\bar{j}s\bar{l}}\bar{\theta}^l \wedge i_{\bar{\theta}_s}. \quad (3.1.27)$$

Recall that in (2.1.56), we get

$$\mathrm{tr}[R^{T^{1,0}M}] = R^{K_M^*} = \mathrm{Ric}_\omega\,. \quad (3.1.28)$$

Since

$$-R_{j\bar{j}s\bar{l}} = -R_{s\bar{l}j\bar{j}} = g(R(\theta_s, \bar{\theta}_l)\theta_j, \bar{\theta}_j) = \mathrm{tr}[R^{T^{1,0}M}](\theta_s, \bar{\theta}_l), \quad (3.1.29)$$

We obtain the theorem.

Our proof of the theorem is completed. □

Theorem 3.1.6. *On a compact Kähler manifold M with quasi-positive bisectional curvature, we have $h^{1,1} = 1$.*

Proof. In this case, $E = \Lambda(T^{*(1,0)}M)$. We have

$$R^{\Lambda(T^{*(1,0)}M)}(\theta_i, \bar{\theta}_i) = R_{i\bar{i}l\bar{s}}\theta^l \wedge i_{\theta_s} \tag{3.1.30}$$

and

$$R^{\Lambda(T^{*(1,0)}M)}(\theta_j, \bar{\theta}_k)\bar{\theta}^k \wedge i_{\bar{\theta}_j} = R_{j\bar{k}l\bar{s}}\theta^l \wedge i_{\theta_s}\bar{\theta}^k \wedge i_{\bar{\theta}_j}. \tag{3.1.31}$$

Thus by (3.1.18),

$$\Box^E - \frac{1}{2}\Delta^{0,\cdot} = -\frac{1}{2}R_{i\bar{i}l\bar{s}}\theta^l \wedge i_{\theta_s} + R_{j\bar{k}l\bar{s}}\theta^l \wedge i_{\theta_s}\bar{\theta}^k \wedge i_{\bar{\theta}_j} - \frac{1}{2}R_{j\bar{k}i\bar{i}}\bar{\theta}^k \wedge i_{\bar{\theta}_j}. \tag{3.1.32}$$

For harmonic real $(1,1)$-form α, if we write $\alpha = \sum_{i,j} \alpha_{ij}\theta^i \wedge \bar{\theta}^j$, we have

$$\sum_{i,j} \alpha_{ij}\theta^i \wedge \bar{\theta}^j = \alpha = \bar{\alpha} = \overline{\alpha_{ij}}\bar{\theta}^i \wedge \theta^j = -\sum_{i,j} \overline{\alpha_{ji}}\theta^i \wedge \bar{\theta}^j. \tag{3.1.33}$$

Thus after an orthogonal transform, we could assume that α could be written as $\alpha = \sum_i \sqrt{-1}\alpha_i\theta^i \wedge \bar{\theta}^i$ where α_i is a real-valued function. From (3.1.32), we have

$$\frac{1}{2}\Delta^{0,\cdot}\alpha = \frac{\sqrt{-1}}{2}R_{i\bar{i}l\bar{k}}\alpha_k\theta^l \wedge \bar{\theta}^k - \sqrt{-1}R_{i\bar{k}l\bar{i}}\alpha_i\theta^l \wedge \bar{\theta}^k + \frac{\sqrt{-1}}{2}R_{l\bar{k}i\bar{i}}\alpha_l\theta^l \wedge \bar{\theta}^k. \tag{3.1.34}$$

Taking the conjuation,

$$\begin{aligned}\sqrt{-1}R_{l\bar{k}i\bar{i}}\alpha_l\theta^l \wedge \bar{\theta}^k &= \sqrt{-1}R_{k\bar{l}i\bar{i}}\alpha_k\theta^k \wedge \bar{\theta}^l = -\sqrt{-1}R_{k\bar{l}i\bar{i}}\alpha_k\bar{\theta}^l \wedge \theta^k \\ &= \sqrt{-1}R_{\bar{k}l\bar{i}i}\alpha_k\theta^l \wedge \bar{\theta}^k = \sqrt{-1}R_{l\bar{k}i\bar{i}}\alpha_k\theta^l \wedge \bar{\theta}^k.\end{aligned} \tag{3.1.35}$$

So we have

$$\frac{1}{2}\Delta^{0,\cdot}\alpha = \sqrt{-1}R_{i\bar{i}l\bar{k}}\alpha_k\theta^l \wedge \bar{\theta}^k - \sqrt{-1}R_{i\bar{k}l\bar{i}}\alpha_i\theta^l \wedge \bar{\theta}^k. \tag{3.1.36}$$

From (3.1.2) and (3.1.24), for harmonic real $(1,1)$-form α, we have

$$\begin{aligned}\sum_i \|\nabla^{0,\cdot}_{e_i}\alpha\|^2_{L_2} &= -\int_M (2R_{i\bar{i}k\bar{k}}\alpha_k^2 + 2R_{i\bar{k}k\bar{i}}\alpha_i\alpha_k)dv \\ &= -\int_M R_{i\bar{i}k\bar{k}}(\alpha_i - \alpha_k)^2 dv.\end{aligned} \tag{3.1.37}$$

If the bisectional curvature is quasi-positive, we have $\alpha_i = \alpha_k$ for any i, k. Thus $\alpha = \phi \cdot \omega$, where ϕ is a real-valued function. Since $\nabla^{0,\cdot}_{e_i}\alpha = 0$, we see that ϕ is a constant. Thus $h^{1,1} = 1$.

The proof of our theorem is completed. □

In general, a markable extension of Theorem 1.3.17 (Siu-Yau, Mori) exists.

Theorem 3.1.7 (Mok 1988)**.** *A compact Kähler manifold with quasi-positive bisectional curvature is biholomorphic to complex projective space.*

Theorem 3.1.8. *For negative holomorphic line bundle L over complex manifold M, we have $H^0(M,L)=0$ for $p>0$.*

Proof. Take $E=L$ in (3.1.18). If L is negative, by Definition 2.1.18, we have $R^L(\theta_i,\bar\theta_i)=\sqrt{-1}R^L(\theta_i,J\bar\theta_i)<0$. Following the same arguments, we get our theorem. □

Theorem 3.1.9. *Let (M,ω) be a compact Kähler manifold such that* Ric_ω *is quasi-positive. Then $h^{p,0}=0$ for any $p>0$.*

Proof. Let α be a harmonic $(p,0)$-form. Then by Theorem 3.1.5and (3.1.22),

$$\Delta^{0,\cdot}\alpha = R^{\Lambda(T^{*(1,0)}M)}(\theta_i,\bar\theta_i)\alpha = R_{i\bar i l\bar s}\theta^l\wedge i_{\theta_s}\alpha \tag{3.1.38}$$

From Definition 2.1.18, if Ric_ω is quasi-positive, then $\mathrm{Ric}_\omega(\cdot,J\cdot)$ is quasi-positive. From (1.3.12),

$$\mathrm{Ric}_\omega(\theta_l,J\bar\theta_s) = -\sqrt{-1}\,\mathrm{Ric}_\omega(\theta_l,\bar\theta_s) = R_{l\bar s i\bar i} = -R_{i\bar i l\bar s}. \tag{3.1.39}$$

So for any l,s, $R_{i\bar i l\bar s}$ is quasi-negative. So $\int_M\langle R_{i\bar i l\bar s}\theta^l\wedge i_{\theta_s}\alpha,\alpha\rangle<0$. Since $\int_M\langle\Delta^{0,\cdot}\alpha,\alpha\rangle\geq 0$, we see that $\alpha=0$.

The proof of our theorem is completed. □

Corollary 3.1.10 (Kobayashi)**.** *A compact connected Kähler manifold with positive Ricci curvature is simply connected.*

Proof. Since $h^{p,q}=h^{q,p}$, we see that for any $p>0$, $h^{0,p}=0$. Notice that the only holomorphic functions on connected compact complex manifold are constants. Thus $h^{0,0}=1$. So $\chi_0(M)=\sum_{p=0}^n(-1)^p h^{0,p}=1$.

From the Myer's theorem, since M is compact and the Ricci tensor has the positive lower bound, the fundamental group $\pi_1(M)$ is finite. Let $\tilde M$ be the universal cover of M. Then $\tilde M$ is compact with positive Ricci curvature. It implies that $\chi_0(\tilde M)=1$. We lift the geometric structure of M onto $\tilde M$. Then we have

$$\int_{\tilde M}\mathrm{Td}(T^{(1,0)}\tilde M) = |\pi_1(M)|\int_M \mathrm{Td}(T^{(1,0)}M). \tag{3.1.40}$$

From the Hirzebruch-Riemann-Roch theorem,

$$\int_{\tilde{M}} \mathrm{Td}(T^{(1,0)}\tilde{M}) = \chi_0(\tilde{M}) = 1 = \chi_0(M) = \int_M \mathrm{Td}(T^{(1,0)}M). \tag{3.1.41}$$

So we get $\pi_1(M) = 1$.

The proof of our corollary is completed. □

Corollary 3.1.11. *Fano manifolds are simply connected.*

Proof. Let M be a Fano manifold. Then $c_1(M) > 0$. From the Calabi-Yau theorem 2.1.17, there exists a Kähler form ω such that $\mathrm{Ric}_\omega > 0$.

The proof is completed. □

Theorem 3.1.12 (Nakano's inequality)**.** *For holomorphic vector bundle E over a compact Kähler manifold M, and any $s \in \Omega^{\cdot,\cdot}(M,E)$,*

$$\langle \square^E s, s\rangle_E \geq \langle [\sqrt{-1}R^E, \Lambda]s, s\rangle_E. \tag{3.1.42}$$

Proof. By Bochner-Kodaira-Nakano formula Theorem 2.2.23,

$$\begin{aligned} \langle \square^E s, s\rangle_E &= \|\bar\partial^E s\|^2_{L^2} + \|\bar\partial^{E,*} s\|^2_{L^2} \\ &= \|(\nabla^E)^{1,0} s\|^2_{L^2} + \|(\nabla^E)^{1,0*} s\|^2_{L^2} + \langle \sqrt{-1}[R^E, \Lambda]s, s\rangle_E. \end{aligned} \tag{3.1.43}$$

The proof of our theorem is completed. □

Theorem 3.1.13. *Let M be a compact complex manifold of complex dimension n and L be a positive holomorphic line bundle over M. Then*

(a) (Kodaira vanishing theorem) if $q > 0$

$$H^q(M, L \otimes K_M) = 0; \tag{3.1.44}$$

(b) (Nakano vanishing theorem) if $p + q > n$,

$$H^{p,q}(M, L) = 0. \tag{3.1.45}$$

Proof. Since L is positive, $\omega = \frac{\sqrt{-1}}{2\pi}R^L$ is a positive $(1,1)$-form. Let g^{TX} be the associated Kähler metric on TM. As $\omega = \sqrt{-1}\theta^i \wedge \bar\theta^i$, by (2.2.92), we have

$$[\omega, \Lambda] = \theta^i \wedge i_{\theta_i} - i_{\bar\theta_i}\bar\theta^i \wedge . \tag{3.1.46}$$

Thus for $s \in \Omega^{p,q}(M, L)$, we have

$$[\omega, \Lambda]s = (p + q - n)s. \tag{3.1.47}$$

Then the Nakano's inequality Theorem 3.1.12 implies that if $\square^L s = 0$, it follows that $s = 0$ whenever $p + q > n$. By Hodge theorem for holomorphic vector bundle $\Lambda^p(T^{*(1,0)}M) \otimes L$, we get (b). (a) is a case of (b) for $p = n$.

The proof of our theorem is completed. □

Theorem 3.1.14 (Kodaira-Serre vanishing theorem)**.** *Let L be a positive holomorphic line bundle and E be a holomorphic vector bundle. Then there exists $p_0 > 0$ such that for any $p \geq p_0$,*

$$H^q(M, L^p \otimes E) = 0 \quad \text{for any } q > 0. \tag{3.1.48}$$

Proof. From (3.1.19), for any $s \in \oplus_{p\geq 1}\Omega^{0,q}(M, L^p \otimes E)$,

$$\begin{aligned} \langle \Box^{L^p\otimes E} s, s\rangle &= \|\bar\partial^{L^p\otimes E} s\|^2_{L_2} + \|\bar\partial^{L^p\otimes E,*} s\|^2_{L_2} = \sum_{i=1}^n \|\nabla^{0,\cdot}_{\bar\theta_i} s\|^2_{L_2} \\ &\quad + \langle R^{L^p\otimes E}(\theta_j, \bar\theta_k)\bar\theta^k \wedge i_{\bar\theta_j} s, s\rangle + \langle R^{\Lambda T^{*(0,1)}M}(\theta_j, \bar\theta_k)\bar\theta^k \wedge i_{\bar\theta_j} s, s\rangle \\ &\geq p\langle R^L(\theta_j, \bar\theta_k)\bar\theta^k \wedge i_{\bar\theta_j} s, s\rangle + \langle R^{\Lambda T^{*(0,1)}M\otimes E}(\theta_j, \bar\theta_k)\bar\theta^k \wedge i_{\bar\theta_j} s, s\rangle. \end{aligned} \tag{3.1.49}$$

We identify the two form R^L with the Hermitian matrix $\dot R^L \in \mathrm{End}(T^{(1,0)}M)$ such that for $X, Y \in T^{(1,0)}M$,

$$R^L(X, \overline{Y}) = \langle \dot R^L X, \overline{Y}\rangle. \tag{3.1.50}$$

After an orthogonal transform, we could assume that

$$\dot R^L(x) = \mathrm{diag}(a_1(x), \cdots, a_n(x)) \in \mathrm{End}(T^{(1,0)}_x M). \tag{3.1.51}$$

Since L is positive, for any $x \in M$ and $1 \leq j \leq n$, $a_j(x) > 0$. So there exists $C_0 > 0$ such that

$$\langle R^L(\theta_j, \bar\theta_k)\bar\theta^k \wedge i_{\bar\theta_j} s, s\rangle = \langle \sum_j a_j(x)\bar\theta^j \wedge i_{\bar\theta_j} s, s\rangle \geq C_0 \|s\|^2_{L_2} \tag{3.1.52}$$

Thus from (3.1.49) and (3.1.52), there exists $C_1 > 0$ such that

$$\langle \Box^{L^p\otimes E} s, s\rangle \geq (C_0 p - C_1)\|s\|^2_{L_2}. \tag{3.1.53}$$

If p is taken large enough such that $C_0 p - C_1 > 0$, we have $\ker \Box^{L^p\otimes E} = 0$. From the Hodge theory, we obtain the Kodaira-Serre vanishing theorem. □

For complex manifold, we also have the corresponding Bochner-Kodaira type formula. We only state here without proof.

Let M be a compact complex manifold and E be a holomorphic vector bundle over M. There are two natural connections: Levi-Civita connection ∇ and Chern connection $\widetilde\nabla$. If the manifold is Kähler, they are equal.

Set

$$S := \widetilde\nabla - \nabla. \tag{3.1.54}$$

Take $S^B \in \Omega^1(M, \mathrm{End}(TM))$ such that

$$g(S^B(U)V, W) = \frac{\sqrt{-1}}{2}\big((\partial - \bar{\partial})\omega\big)(U, V, W) \tag{3.1.55}$$

for any $U, V, W \in TM$. The **Bismut connection** ∇^B on TM is defined by

$$\nabla^B := \nabla + S^B = \widetilde{\nabla} + S^B - S. \tag{3.1.56}$$

Remark that the Bismut connection preserves the complex structure. Thus it induces a natural connection ∇^B on $\Lambda(T^{*(0,1)}M)$. Let $\nabla^{B,\Lambda^{0,\cdot}}$, $\nabla^{B,\Lambda^{0,\cdot}\otimes E}$ be the connections on $\Lambda(T^{*(0,1)}M)$, $\Lambda(T^{*(0,1)}M) \otimes E$ defined by

$$\nabla^{B,\Lambda^{0,\cdot}} = \nabla^B + \langle S(\cdot)\theta_j, \bar{\theta}_j\rangle, \tag{3.1.57}$$

$$\nabla^{B,\Lambda^{0,\cdot}\otimes E} = \nabla^{B,\Lambda^{0,\cdot}} \otimes 1 + 1 \otimes \nabla^E. \tag{3.1.58}$$

For any $v \in TM$ with the decomposition $v = v^{1,0} + v^{0,1} \in T^{(1,0)}M \oplus T^{(0,1)}M$, let $\bar{v}^{1,0,*}$ be the metric dual of $v^{1,0}$. Then we set

$$c(v) := \sqrt{2}(\bar{v}^{1,0,*} \wedge - i_{v^{0,1}}) \in \mathrm{End}(\Lambda(T^{*(0,1)}M)). \tag{3.1.59}$$

We verify easily that for $U, V \in TM$,

$$c(U)c(V) + c(V)c(U) = -2g(U, V). \tag{3.1.60}$$

For a skew-adjoint endomorphism A of TM, from (3.1.59), we could calculate that

$$\begin{aligned}\frac{1}{4}g(Ae_i, e_j)c(e_i)c(e_j) = &-\frac{1}{2}g(A\theta_j, \bar{\theta}_j) + g(A\theta_l, \bar{\theta}_m)\bar{\theta}^m \wedge i_{\bar{\theta}_l} \\ &+ \frac{1}{2}g(A\theta_l, \theta_m)i_{\bar{\theta}_l}i_{\bar{\theta}_m} + \frac{1}{2}g(A\bar{\theta}_l, \bar{\theta}_m)i_{\bar{\theta}_l}i_{\bar{\theta}_m}\bar{\theta}^l \wedge \bar{\theta}^m \wedge .\end{aligned} \tag{3.1.61}$$

For $i_1 < \cdots < i_j$, we define

$$^c(e^{i_1} \wedge \cdots \wedge e^{i_j}) = c(e_{i_1}) \cdots c(e_{i_j}). \tag{3.1.62}$$

Then by extending $\mathbb{C}$-linearly, ^cA is defined for any $A \in \Lambda(T^*M \otimes_{\mathbb{R}} \mathbb{C})$.

Theorem 3.1.15 (Bismut's Lichnerowicz formula)**.** *Let $\Delta^{B,\Lambda^{0,\cdot}\otimes E}$ be the Laplacian of $\nabla^{B,\Lambda^{0,\cdot}\otimes E}$ as in (3.1.1). Let r^M be the scalar curvature of M. We have*

$$\begin{aligned}2\square^E = 2(\bar{\partial}^E + \bar{\partial}^{E,*})^2 = \Delta^{B,\Lambda^{0,\cdot}\otimes E} + \frac{r^M}{4} + {}^c\left(R^E + \frac{1}{2}\operatorname{tr}\left[R^{T^{1,0}M}\right]\right) \\ + \frac{\sqrt{-1}}{2}\,{}^c(\partial\bar{\partial}\omega) - \frac{1}{8}|(\partial - \bar{\partial})\omega|^2.\end{aligned} \tag{3.1.63}$$

Remark that it generalises the Bochner-Kodaira for Kähler case.

3.2 Proof of the Hodge theory

Definition 3.2.1. Let M be a compact Riemannian (resp. complex) manifold. Let (E, h^E) be a complex (resp. holomorphic) Hermitian vector bundle over M. If there exists Hermitian connection ∇^E and Hermian endomorphism $Q \in \mathscr{C}^\infty(M, \mathrm{End}(E))$, i.e., $Q_x^* = Q_x$ associated with h^E, such that

$$H = \Delta^E + Q, \tag{3.2.1}$$

we say H is a **generalised Laplacian**.

For example, from the Weitzenböck's formula Theorem 3.1.2 and Bismut's Lichnerowicz formula Theorem 3.1.15, we see that $(d + d^*)^2$ and $(\bar{\partial}^E + \bar{\partial}^{E,*})^2$ are all generalised Laplacian. In this section, we prove the Hodge theorem for the generalised Laplacian.

Let $L^2(M, E)$ be the completion of $\mathscr{C}^\infty(M, E)$ with respect to the norm $\|\cdot\|_{L^2}^2$. Let ∇^{E*} be the adjoint of ∇^E, which means that for any $u \in \mathscr{C}^\infty(M, E)$, $s \in \mathscr{C}^\infty(M, T^*M \otimes E)$, we have $\langle \nabla^E u, s\rangle = \langle u, \nabla^{E*} s\rangle$.

For $u \in L^2(M, E)$, if there exists $C > 0$ such that for any $s \in \mathscr{C}^\infty(M, T^*M \otimes E)$,

$$|\langle u, \nabla^{E*} s\rangle| \leq C\|s\|_{L^2}, \tag{3.2.2}$$

from the Riesz representation theorem, there exists $v \in L^2(M, E)$, such that for any $s \in \mathscr{C}^\infty(M, T^*M \otimes E)$,

$$\langle u, \nabla^{E*} s\rangle = \langle v, s\rangle. \tag{3.2.3}$$

We define $\nabla^E u = v$.

Definition 3.2.2. We define the Sobolev spaces by

$$\mathcal{H}^0(M, E) = L^2(M, E), \tag{3.2.4}$$

and

$$\mathcal{H}^k(M, E) = \{u \in L^2(M, E) : (\nabla^E)^k u \in L^2(M, (T^*M)^{\otimes k} \otimes E)\}. \tag{3.2.5}$$

In this case, for $s \in \mathcal{H}^k(M, E)$, $k \geq 0$, the Sobolev norm is defined by

$$\|s\|_k^2 := \|s\|_{L^2}^2 + \sum_{l=1}^{k} \|(\nabla^E)^l s\|_{L^2}^2. \tag{3.2.6}$$

Set

$$\mathcal{H}^{-k}(M,E) = \mathcal{H}^k(M,E)^*. \tag{3.2.7}$$

In this case, for $u \in \mathcal{H}^{-k}(M,E)$, the Sobolev norm is defined by

$$\|u\|_{-k} := \sup_{0 \neq s \in \mathcal{H}^k(M,E)} \frac{\langle u, s\rangle}{\|s\|_k}. \tag{3.2.8}$$

Remark that the condition in (3.2.5) means that for any $s \in \mathscr{C}^\infty(M, (T^*M)^{\otimes k} \otimes E)$,

$$|\langle u, (\nabla^{E*})^k s\rangle| \le C\|s\|_{L_2}. \tag{3.2.9}$$

Notice that the definition of the Sobolev space and the Sobolev norm here depend on the connection ∇^E. However, it does not matter. The Sobolev spaces with respect to different connections are equivalent. In other words, for $k \ge 0$, if $\|\cdot\|'_k$ be the Sobolev norm with respect to $\nabla^{E'}$, there exist $c, C > 0$ such that for any $s \in \mathscr{C}^\infty(M,E)$, we have

$$c\|s\|_k \le \|s\|'_k \le C\|s\|_k. \tag{3.2.10}$$

The proof of it is left to the interesting reader. (Forgive me. I don't want to write this sentence. I believe that it appears the first time in this note. The proof is tedious and not useful in other places.)

Now we introduce some basic estimates associated with the Sobolev spaces. The proof of them are not hard and can be found in almost all introductory PDE books. We don't state the proof here because it is out the scope of this note.

Theorem 3.2.3 (Elliptic estimates and regularity). *If $u \in L^2(M,E)$ and $Hu \in \mathcal{H}^{k-1}(M,E)$ for $k \ge 0$. Then $u \in \mathcal{H}^{k+1}(M,E)$ and there exists $C > 0$ such that for any $s \in \mathcal{H}^{k+1}(M,E)$, we have*

$$\|s\|^2_{k+1} \le C\|Hs\|^2_{k-1} + C\|s\|^2_{L^2}. \tag{3.2.11}$$

Theorem 3.2.4 (Sobolev embedding). *If $m > k + \frac{n}{2}$, then there exists $C > 0$ such that if $u \in \mathcal{H}^m(M,E)$, we have $u \in \mathscr{C}^k(M,E)$ and*

$$|u|_{\mathscr{C}^k} \le C\|u\|_m, \tag{3.2.12}$$

where

$$|u|_{\mathscr{C}^k} := \sup_{0 \le l \le k, x \in M} \left|(\nabla^E)^l u(x)\right|. \tag{3.2.13}$$

Theorem 3.2.5 (Rellich's lemma)**.** *For any $k \geq 0$, the embedding*

$$j : \mathcal{H}^{k+1}(M,E) \to \mathcal{H}^{k}(M,E) \tag{3.2.14}$$

is a compact operator, which means that the closure of the image of the bounded set is compact, i.e., for any bounded sequence $\{x_n\} \subset \mathcal{H}^{k+1}(M,E)$, the set $\{j(x_n)\}$ has a convergent subsequence.

Theorem 3.2.6 (Spectral theorem for compact operators)**.** *Let $\mathcal{H}$ be a infinite dimensional Hilbert space. Let B be a compact self-adjoint operator on $\mathcal{H}$. Then there exists a complete orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ of $\mathcal{H}$ and a real sequence $\{0, \lambda_1, \cdots, \lambda_n, \cdots\}$ such that $B\varphi_j = \lambda_j\varphi_j$, $\dim \mathrm{Ker}(B - \lambda_j) < +\infty$ and $\lim_{j\to+\infty} \lambda_j = 0$.*

Theorem 3.2.7. *Let H be a generalised Laplacian. Then there exists a complete orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ of $L^2(M,E)$ such that*

(1) $\varphi_j \in \mathscr{C}^{\infty}(M,E)$;

(2) $H\varphi_j = \lambda_j\varphi$, where $-\infty < \lambda_1 \leq \lambda_2 \leq \cdots + \infty$;

(3) $\dim \mathrm{Ker}(H|_{\mathcal{H}^2(M,E)} - \lambda_j) < +\infty$.

Proof. Note that if the theorem holds for H, it also holds for $H + C$, where $C \in \mathbb{R}$ is a constant. So we could assume that there exists $C_0 > 0$ such that $Q - C_0 \,\mathrm{Id}$ is positive-definite.

We claim that the map $H : \mathcal{H}^1(M,E) \to \mathcal{H}^{-1}(M,E)$ is bijective and there exists $C > 0$ such that $\|Hs\|_{-1} \geq C\|s\|_1$ for any $s \in \mathcal{H}^1(M,E)$. Indeed, from (3.2.6) and (3.2.8), there exists $C > 0$ such that

$$\|Hs\|_{-1}\|s\|_1 \geq \langle Hs, s\rangle = \langle \nabla^E s, \nabla^E s\rangle + \langle Qs, s\rangle \geq C\|s\|_1^2. \tag{3.2.15}$$

Thus we have $\|Hs\|_{-1} \geq C\|s\|_1$. It also implies that H is injective and $\mathrm{Im}H$ is closed, which means that if $Hs_j \to v$ in $\mathcal{H}^{-1}$, then there exists $s \in \mathcal{H}^1$, $s_j \to s$ and $Hs = v$. If $\mathrm{Im}H \neq \mathcal{H}^{-1}(M,E)$, then there exists $u_0 \in (\mathcal{H}^{-1}(M,E))^* = \mathcal{H}^1(M,E)$ such that $u_0 \perp \mathrm{Im}H$, i.e., for any $u \in \mathcal{H}^1(M,E)$, $\langle Hu, u_0\rangle = 0$. Take $u = u_0$. Since $\langle Hu_0, u_0\rangle \geq C\|u_0\|_0^2$ where $C > 0$, we have $u_0 = 0$. Thus H is surjective.

From Rellich's lemma Theorem 3.2.5, the embedding $\mathcal{H}^1(M,E) \to L^2(M,E)$ is compact. Since $L^2(M,E) \subset \mathcal{H}^{-1}(M,E)$, the operator

$$H^{-1} : L^2(M,E) \hookrightarrow \mathcal{H}^{-1}(M,E) \to \mathcal{H}^1(M,E) \hookrightarrow L^2(M,E) \tag{3.2.16}$$

is compact. From Theorem 3.2.6, since $\ker H^{-1} = 0$, there exists a complete orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ of $L^2(M,E)$ and a real sequence $\{\lambda_1^{-1}, \cdots, \lambda_n^{-1}, \cdots\}$ such that $H^{-1}\varphi_j = \lambda_j^{-1}\varphi_j$, $\dim \mathrm{Ker}(H^{-1} - \lambda_j^{-1}) < +\infty$ and $\lim_{j\to+\infty} \lambda_j^{-1} = 0$. Thus we only need to prove that φ_j is smooth for any j.

Since $\varphi_j \in L^2(M,E)$, $H\varphi_j = \lambda_j\varphi_j \in L^2(M,E)$. From Theorem 3.2.3, we have $\varphi_j \in \mathcal{H}^2(M,E)$. Since $H\varphi_j = \lambda_j\varphi_j \in \mathcal{H}^2(M,E)$, $\varphi_j \in \mathcal{H}^4(M,E)$. By induction, for any $k > 0$, $\varphi_j \in \mathcal{H}^{2k}(M,E)$. By Sobolev embedding theorem (Theorem 3.2.4), we have $\varphi_j \in \mathscr{C}^\infty(M,E)$.

The proof of our theorem is completed. □

Theorem 3.2.8 (Hodge decomposition theorem)**.** *(a) For any $k \geq 1$, $H : \mathcal{H}^k(M,E) \to \mathcal{H}^{k-2}(M,E)$ is well-defined. We have $\ker(H|_{\mathcal{H}^k(M,E)}) \subset \mathscr{C}^\infty(M,E)$, which does not depend on the choice of $k \in \mathbb{N}$. We denote it by $\ker H$.*

(b) The image $\mathrm{Im}(H|_{\mathcal{H}^{k+2}(M,E)})$ is closed in $\mathcal{H}^k(M,E)$. And we have orthogonal decompositions

$$H^k(M,E) = \ker H \oplus \mathrm{Im}(H|_{\mathcal{H}^{k+2}(M,E)}), \tag{3.2.17}$$

$$\mathscr{C}^\infty(M,E) = \ker H \oplus \mathrm{Im}(H|_{\mathscr{C}^\infty(M,E)}). \tag{3.2.18}$$

Proof. For (a), the well-definedness of $H : \mathcal{H}^k(M,E) \to \mathcal{H}^{k-2}(M,E)$ follows directly from elliptic estimate (3.2.11). If $s \in L^2(M,E)$, $Hs = 0$, also by elliptic estimate (3.2.11), for any $k \geq 0$, there exists $C_k > 0$, such that $\|s\|_k \leq C_k\|s\|_0$. From the Sobolev embedding Theorem 3.2.4, we have $s \in \mathscr{C}^\infty(M,E)$. Thus, for any $k \geq 0$, $\ker(H|_{\mathcal{H}^k(M,E)}) = \ker(H|_{\mathscr{C}^\infty(M,E)})$.

For (b), we have $L^2(M,E) = \ker H \oplus (\ker H)^\perp$. We only need to prove that $(\ker H)^\perp = \mathrm{Im}(H|_{\mathcal{H}^2(M,E)})$.

If $u \in \ker H \cap \mathrm{Im}(H|_{\mathcal{H}^2(M,E)})$, then there exists $v \in \mathcal{H}^2(M,E)$ such that $u = Hv$. Since u is smooth and H is formally self-adjoint, from (3.2.3), we have

$$\langle u,u\rangle = \langle u,Hv\rangle = \langle Hu,v\rangle = 0. \tag{3.2.19}$$

Thus $u = 0$. It means that

$$\ker H \cap \mathrm{Im}(H|_{\mathcal{H}^2(M,E)}) = \{0\}. \tag{3.2.20}$$

If $u \in (\ker H)^\perp$, from Theorem 3.2.16, we have the fourier expansion $u = \sum_i' a_i\varphi_i$, where the sum $\sum_i'$ runs over $i \in \mathbb{N}$ such that $\lambda_i \neq 0$. Set $v = \sum_i' \lambda_i^{-1}a_i\varphi_i$. Take $c = \inf_i\{\lambda_i^2 : \lambda_i \neq 0\}$. Then

$$|v|^2 = \sum_i{}' \lambda_i^{-2}|a_i|^2 < c^{-1}\sum_i{}' |a_i|^2 = c^{-1}\|u\|^2_{L^2} < +\infty \tag{3.2.21}$$

and $Hv = u$. From the regularity Theorem 3.2.3, we have $v \in \mathcal{H}^2(M,E)$. It means that

$$(\ker H)^\perp \subset \mathrm{Im}(H|_{\mathcal{H}^2(M,E)}). \tag{3.2.22}$$

From (3.2.20) and (3.2.22), we have

$$L^2(M,E) = \ker H \oplus \mathrm{Im}(H|_{\mathcal{H}^2(M,E)}). \tag{3.2.23}$$

If $u \in \mathscr{C}^\infty(M,E)$, we have the decomposition $u = u_0 + u_1$ associated with (3.2.23). From (a), $u_0 \in \mathscr{C}^\infty(M,E)$. Thus $u_1 \in \mathscr{C}^\infty(M,E)$. Since there exists $v \in \mathcal{H}^2(M,E)$ such that $u = Hv$, from the regularity property, we have $v \in \mathscr{C}^\infty(M,E)$. Following the same argument, if $u \in \mathcal{H}^k(M,E)$, we have $u_1 \in \mathcal{H}^k(M,E)$ and $v \in \mathcal{H}^{k+2}(M,E)$.

The proof of our theorem is completed. □

Now we use the Hodge decomposition theorem to prove the real version of the Hodge theorem (Theorem 2.2.6).

From Weitzenböck formula Theorem 3.1.2, $\Delta_{\mathbb{R}}$ is a generalised Laplacian. From the Hodge decomposition theorem by taking $E = \Lambda T^*M$, we have

$$\Omega^*(M) = \ker \Delta_{\mathbb{R}} \oplus \mathrm{Im}\Delta_{\mathbb{R}}. \tag{3.2.24}$$

Since $\Delta_{\mathbb{R}} = dd^* + d^*d$, $\mathrm{Im}\Delta_{\mathbb{R}} = \mathrm{Im}d + \mathrm{Im}d^*$. Since $\langle d\beta, d^*\beta' \rangle = \langle d^2\beta, \beta' \rangle = 0$, we have $\mathrm{Im}d + \mathrm{Im}d^* = \mathrm{Im}d \oplus \mathrm{Im}d^*$.

If $u \in \mathrm{Im}d^*$, $u = d^*v'$ and $v \in \ker d$, we have $\langle u, v \rangle = \langle d^*v', v \rangle = \langle v' dv \rangle = 0$. It means that $\ker d \perp \mathrm{Im}d^*$. Since

$$\Omega^*(M) = \ker \Delta_{\mathbb{R}} \oplus \mathrm{Im}d \oplus \mathrm{Im}d^* \tag{3.2.25}$$

and $\ker \Delta_{\mathbb{R}} \subset \ker d$, $\mathrm{Im}d \subset \ker d$, we have $\ker d = \ker \Delta_{\mathbb{R}} \oplus \mathrm{Im}d$. It implies that the de Rham cohomology

$$H^*(M,E) \simeq \ker \Delta_{\mathbb{R}}. \tag{3.2.26}$$

For complex case, note that from Bismut's Lichnerowicz formula, $2\Box^E$ is a generalised Laplacian. Following the same argument above, we get Theorem 2.2.15.

3.3 Kodaira embedding theorem

Theorem 3.3.1 (Kodaira embedding theorem I '54). *A compact complex manifold M is projective if and only if M could be equipped with a positive line bundle.*

Let $\{s_i^p\}_{i=1}^{d_p}$ ($d_p := \dim H^0(M, L^p)$) be any orthonormal basis of $H^0(M, L^p)$ with respect to the usual L^2-norm (2.2.39). By Hodge theory, s_i^p's are holomorphic sections of L^p.

Definition 3.3.2. Let

$$\mathrm{Bl}_p := \{x \in M : s(x) = 0, \forall s \in H^0(M, L^p)\}, \tag{3.3.1}$$

which is called the base locus. The Kodaira map Φ_p is defined by

$$\Phi_p : M \backslash \mathrm{Bl}_p \to \mathbb{CP}^{d_p - 1}, \quad x \mapsto (s_1^p(x) : \cdots : s_{d_p}^p(x)). \tag{3.3.2}$$

Definition 3.3.3. Let L be a holomorphic line bundle.
It is called **semi-ample** if there exists p_0 such that for all $p \geq p_0$, $\mathrm{Bl}_p = \emptyset$.
It is called **ample** if it is semi-ample and Φ_p is an embedding.
It is called **very ample** if $\mathrm{Bl}_1 = \emptyset$ and Φ_1 is an embedding.

It is obvious that L is ample if and only if there exists p_0 such that for all $p \geq p_0$, L^p is very ample.

Theorem 3.3.4 (Kodaira embedding theorem II '54). *The holomorphic line bundle L is ample if and only if it is positive.*

For any $s \in H^0(M, L^p)$, we could write

$$s = \sum_{i=1}^{d_p} a_i s_i^p, \quad a_i \in \mathbb{C}. \tag{3.3.3}$$

Let γ be the tautological line bundle over $\mathbb{CP}^{d_p-1}$. For $([l], z) \in \gamma$, $z \in l \subset \mathbb{C}^{d_p}$. we define $\sigma_s \in \gamma^*$ such that for any $([l], z) \in \gamma$,

$$\langle \sigma_s([l]), ([l], z) \rangle := \sum_{i=1}^{d_p} a_i z_i. \tag{3.3.4}$$

Easy to see that for any $\zeta \in \gamma^*$, there exists $s \in H^0(M, L^p)$ such that $\zeta = \sigma_s$.

Proposition 3.3.5. *Let L be a semi-simple line bundle. Then for $p \geq p_0$, $\Phi_p : M \to \mathbb{CP}^{d_p-1}$ is holomorphic and*

$$\Psi_p : \Phi_p^* \gamma^* \to L^p, \quad \Phi^* \sigma_s \mapsto s \tag{3.3.5}$$

defines a canonical isomorphism from $\Phi_p \gamma^*$ to L^p over M.

Proof. Since s_i^p's are holomorphic sections on L^p, from (3.3.2), Φ_p is holomorphic.

Let

$$S = (s_1^p(x), \cdots, s_{d_p}^p(x)) \in \mathbb{C}^{d_p}. \tag{3.3.6}$$

Then

$$\begin{aligned}\langle \Phi_p^* \sigma_s(x), \Phi_p^*(\Phi_p(x), S(x)) \rangle &= \langle \sigma(\Phi_p(x)), (\Phi_p(x), S(x)) \rangle \\ &= \sum_{i=1}^{d_p} a_i s_i^p(x) = s(x).\end{aligned} \tag{3.3.7}$$

Thus $\Phi_p^* \sigma_s(x) = 0$ if and only if $s(x) = 0$.

Since s and $\Phi_p^* \sigma_s$ are holomorphic sections of L^p and $\Phi_p^* \gamma^*$, Ψ_p is holomorphic. Thus it is continuous and the inverse of it is continuous.

The proof of this proposition is completed. □

Corollary 3.3.6. *If L is ample, then it is positive.*

Proof. If L is ample, then Φ_p is an embedding. Since γ^* is positive, $\Phi_p^* \gamma^*$ is positive. By Proposition 3.3.5, L^p is positive. So is L.

The proof of the corollary is completed. □

From now on, we assume that L is positive. From the Kodaira vanising theorem (Theorem 3.1.14),

$$H^q(M, L^p) = 0 \quad \text{for any } q \geq 0, p \gg 1. \tag{3.3.8}$$

Let P_p be the orthogonal projection from $\Omega^{0,*}(M, L^p)$ on to $\ker(\bar{\partial}^{L^p} + \bar{\partial}^{L^p,*}) = H^*(M, L^p)$. From the Kodaira vanishing theorem, if we only consider the case for p large enough,

$$P_p : \Omega^{0,*}(M, L^p) \to H^0(M, L^p). \tag{3.3.9}$$

Let

$$P_p(x, x') := \sum_{i=1}^{d_p} s_i^p(x) \otimes s_i^p(x)^* \in L_x^p \otimes (L^p)_{x'}^*. \tag{3.3.10}$$

Proposition 3.3.7. *For any* $s \in \Omega^{0,*}(M, L^p)$*,*

$$(P_p s)(x) = \int_{x' \in M} P_p(x, x') s(x') dx'. \tag{3.3.11}$$

Here $P_p(x, x')$ *is called the* ***Bergman kernel*** *associated with* L^p*.*

Proof. For any $s \in \Omega^{0,*}(M, L^p)$,

$$\begin{aligned}(P_p s)(x) &= \sum_{i=1}^{d_p} \left(\int_M s_i^p(x')^* \cdot s(x') dx' \right) \cdot s_i^p(x) \\ &= \int_M \left(\sum_{i=1}^{d_p} s_i^p(x) \otimes s_i^p(x')^* \right) \cdot s(x') dx' = \int_{x' \in M} P_p(x, x') s(x') dx'. \end{aligned} \tag{3.3.12}$$

The proof of this proposition is completed. □

Observe that $L^p \otimes (L^p)^*$ is a trivial line bundle. From (**??**), $P_p(x, x)$ is a complex valued function on M. If we take the adjoint with respect to h^{L^p}, we have

$$P_p(x, x) = \sum_{i=1}^{d_p} |s_i^p(x)|^2_{h^{L^p}}. \tag{3.3.13}$$

Proposition 3.3.8. *For any* $x \in M$*,*

$$h^{\Phi_p^* \gamma^*}(x) = P_p(x, x)^{-1} h^{L^p}(x). \tag{3.3.14}$$

Proof. Under the isomorphism (3.3.5), for any holomorphic section s on L^p, from (3.3.7) and (3.3.13),

$$\begin{aligned}\left|\Phi_p^* \sigma_s(x)\right|^2_{h^{\Phi_p^* \gamma^*}} = |\sigma_s(\Phi_p(x))|^2_{h^{\gamma^*}} &= \frac{|\langle \sigma_s(\Phi_p(x)), (\Phi_p(x), S(x)) \rangle|^2_{h^{L^p}}}{|(\Phi_p(x), S(x))|^2_{h^\gamma}} \\ &= \frac{|s(x)|^2_{L^p}}{\sum_{i=1}^{d_p} |s_i^p(x)|^2_{h^{L^p}}} = P_p(x, x)^{-1} |s(x)|^2_{L^p} \end{aligned} \tag{3.3.15}$$

The proof of this proposition is completed. □

The following theorem started from Tian '90 (also Bouche '90, Ruan '98) following the suggestion of Yau '87 was first established by Catlin '97, Zelditch '98.

Theorem 3.3.9. *For any $k, k' \in \mathbb{N}$, there exist $C_{k,k'} > 0$ and $b_r \in \mathscr{C}^\infty(M, \mathbb{C})$, $0 \le r \le k$ such that for any $p \in \mathbb{N}$,*

$$\left| P_p(x,x) - \sum_{r=0}^{k} b_r(x) p^{n-r} \right|_{\mathscr{C}^{k'}(M)} \le C_{k,k'} p^{n-k-1} \tag{3.3.16}$$

and

$$b_0 = \det \left(\frac{\dot{R}^L}{2\pi} \right). \tag{3.3.17}$$

Remark that Lu '00 and Lu-Tian '04 calculated b_1, b_2, b_3 used by Donaldson in his work on the existence of Kähler metrics with constant scalar curvature.

Proposition 3.3.10. *If L is positive, then it is semi-ample.*

Proof. If $R^L > 0$, by (3.1.50), $b_0 = \det \left(\dot{R}^L / 2\pi \right) > 0$. From Theorem 3.3.9, for p large enough, $P_p(x,x) > 0$. Thus our proposition follows from (3.3.1) and (3.3.13).

The proof of our proposition is completed. □

Theorem 3.3.11 (Tian '90-Ruan '98)**.** *Assume that (L, h^L) is positive. Then the induced Fubini-Study metric $\frac{1}{p}\Phi_p^*(\omega_{FS})$ converges in $\mathscr{C}^\infty$-topology to $\omega = \sqrt{-1}R^L$. For any $l \ge 0$, there exists $C_l > 0$ such that*

$$\left| \frac{1}{p}\Phi_p^*(\omega_{FS}) - \omega \right|_{\mathscr{C}^l(M)} \le \frac{C_l}{p}. \tag{3.3.18}$$

Proof. From (1.2.33) and (2.1.46), we have

$$\omega_{FS} = \sqrt{-1}R^{\gamma^*} = \sqrt{-1}\bar{\partial}\partial \log |\sigma_s|^2_{h^{\gamma^*}}. \tag{3.3.19}$$

Thus from Proposition 3.3.8, (2.1.46) and (3.3.2),

$$\begin{aligned} \Phi_p^* \omega_{FS} &= \sqrt{-1}\bar{\partial}\partial \log |\Phi_p^* \sigma_s|^2_{h^{\gamma^*}} \\ &= \sqrt{-1}\bar{\partial}\partial \log |s(x)|^2_{h^{L^p}} - \sqrt{-1}\bar{\partial}\partial \log P_p(x,x) \\ &= \sqrt{-1}R^{L^p} - \sqrt{-1}\bar{\partial}\partial \log P_p(x,x) \\ &= p\omega - \sqrt{-1}\bar{\partial}\partial \log P_p(x,x) \end{aligned} \tag{3.3.20}$$

From Theorem 3.3.9, we have

$$\bar{\partial}\partial \log P_p(x,x) = \bar{\partial}\partial \log(p^n P_p(x,x)) = \bar{\partial}\partial \log b_0(x) + O(p^{-1}). \tag{3.3.21}$$

Thus our theorem follows directly from Theorem 3.3.9, (3.3.20) and (3.3.21). □

Proposition 3.3.12. *The Kodaira map* Φ_p*is an immersion for* $p \gg 1$.

Proof. From Theorem 3.3.11, for any $l \geq 0$, there exists $C_l > 0$ such that

$$\left| \frac{1}{p} \Phi_p^*(g_{FS}) - g^{TM} \right|_{\mathscr{C}^l(M)} \leq \frac{C_l}{p}. \tag{3.3.22}$$

For $v \in T_x M$, $v \neq 0$, we have $g^{TM}(v, v) > 0$. From (3.3.22), for p large, we have $\Phi_p^*(g_{FS})(v, v) > 0$. It means that $g_{FS}(\Phi_* v, \Phi_* v) > 0$, which implies that $\Phi_* v \neq 0$.

The proof of our proposition is completed. □

Chapter 4

Kähler-Einstein metric

Appendix A

Riemann Geometry revisited

A.1 Connection and curvature

In this book, we use the **Einstein Summation Convention** as follows: A basis of V is denoted by $v_1, \cdots, v_n$. For $v \in V$,

$$v = \sum_i \alpha^i v_i = \alpha^i v_i = (v_1, \cdots, v_n) \cdot \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{pmatrix}. \tag{A.1.1}$$

Let (M, g) be a Riemannian manifold with Levi-civita connection ∇. Then the Levi-civita connection is uniquely determined by **Koszul's formula**:

$$\begin{aligned} 2g(\nabla_Y X, Z) = & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ & - g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y). \end{aligned} \tag{A.1.2}$$

For $\alpha \in \Omega^k(M)$, $\beta \in \Omega^r(M)$, $v_1, \cdots, v_{k+r} \in TM$,

$$\begin{aligned} & \alpha \wedge \beta(v_1, \cdots, v_{k+r}) \\ & = \frac{1}{k!r!} \sum_{\sigma \in S_{k+r}} (-1)^{|\sigma|} \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \cdot \beta((v_{\sigma(k+1)}, \cdots, v_{\sigma(k+r)})). \end{aligned} \tag{A.1.3}$$

Let $d : \Omega^*(M) \to \Omega^{*+1}(M)$ be the exterior differential. It is characterized by

(1) $d^2 = 0$;

(2) for $\varphi \in \mathscr{C}^\infty(M)$, $d\varphi$ is the one form such that $(d\varphi)(U) = U(\varphi)$ for any vector field U;

(3) for any $\alpha \in \Omega^k(M)$, $\beta \in \Omega^*(M)$, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \tag{A.1.4}$$

We denote by $\varepsilon : T^*M \otimes \Lambda^*(T^*M) \to \Lambda^*(T^{*+1}M)$ the exterior product.

Proposition A.1.1. *For the exterior differentiation operator d,*

$$d = \varepsilon \circ \nabla. \tag{A.1.5}$$

For $\alpha \in \Omega^k(M)$ and vector fields $X_0, \cdots, X_k$, we have

$$\begin{aligned} d\alpha(X_0, \cdots, X_k) &= \sum_{i=0}^{k} (-1)^i X_i(\alpha(X_0, \cdots, \widehat{X_i}, \cdots, X_k)) \\ &\quad + \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_k) \\ &= \sum_{i=0}^{k} (-1)^i (\nabla_{X_i} \alpha)(X_0, \cdots, \widehat{X_i}, \cdots, X_k). \end{aligned} \tag{A.1.6}$$

In fact, in (A.1.6), we only need ∇ is a torsion-free connection.

Let E be a vector bundle over M. Let ∇^E be a connection on E. Let R^E be the curvature of ∇^E defined by

$$R^E(U, V) = \nabla^E_U \nabla^E_V - \nabla^E_V \nabla^E_U - \nabla^E_{[U,V]}. \tag{A.1.7}$$

Proposition A.1.2. *Let E_1, E_2 be two vector bundles over M endowed with connections ∇^{E_1} and ∇^{E_2} respectively. Let R^{E_1} and R^{E_2} be the corresponding curvatures.*

(1) The curvature of the induced connection on the direct sum $E_1 \oplus E_2$ is given by

$$R^{E_1 \oplus E_2} = R^{E_1} \oplus R^{E_2}. \tag{A.1.8}$$

(2) On the tensor product $E_1 \otimes E_2$, the induced curvature is given by

$$R^{E_1 \otimes E_2} = R^{E_1} \otimes 1 \oplus 1 \otimes R^{E_2}. \tag{A.1.9}$$

(3) Let E^* be the dual of E, we have

$$R^{E^*} = -(R^E)^t. \tag{A.1.10}$$

(4) For a smooth map $f : N \to M$, we have

$$R^{f^*E} = f^* R^E. \tag{A.1.11}$$

The **gradient** of $f : M \to \mathbb{R}$, denoted by $\text{grad}f = \nabla f$, is defined as the vector field satisfying

$$g(v, \nabla f) = df(v) \tag{A.1.12}$$

for any $v \in TM$. In local coordinates, we have

$$\nabla f = g^{ij}\partial_i(f)\partial_j. \tag{A.1.13}$$

The **Hessian** $\text{Hess}f$ is defined by

$$\text{Hess}f = \frac{1}{2}L_{\nabla f}g. \tag{A.1.14}$$

The **divergence** of a vector field

$$\text{div}X = \text{tr}(\nabla X) = dx^i(\nabla_{\partial_i}X) = \sum_i g(\nabla_{e_i}X, e_i). \tag{A.1.15}$$

The **Laplacian**

$$\Delta f = \text{tr}(\nabla(\nabla f)) = \text{div}(\nabla f). \tag{A.1.16}$$

We define the **Curvature** of Levi-civita connection by:

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{A.1.17}$$

In local coordinates,

$$R(\partial_i, \partial_j)\partial_k = R^l_{ijk}\partial_l. \tag{A.1.18}$$

Then

$$R^l_{ijk} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^s_{jk}\Gamma^l_{is} - \Gamma^s_{ik}\Gamma^l_{js}. \tag{A.1.19}$$

We denote by

$$R(X, Y, Z, W) = g(R(X, Y)W, Z). \tag{A.1.20}$$

Then the curvature has the following properties:

- **Skew-symmetric**:

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = R(Y, X, W, Z). \tag{A.1.21}$$

- **Symmetric**:

$$R(X,Y,Z,W) = R(Z,W,X,Y). \tag{A.1.22}$$

- **Bianchi's first identity**:

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0. \tag{A.1.23}$$

- **Bianchi's second identity**:

$$(\nabla_Z R)(X,Y)W + (\nabla_Y R)(Z,X)W + (\nabla_X R)(Y,Z)W = 0. \tag{A.1.24}$$

The **curvature operator** $\mathfrak{R} : \Lambda^2 M \to \Lambda^2 M$ is a self-adjoint operator such that

$$g(\mathfrak{R}(X \wedge Y), Z \wedge W) = R(X,Y,W,Z), \tag{A.1.25}$$

where

$$\begin{aligned} g(X \wedge Y, Z \wedge W) &= g(X,Z)g(Y,W) - g(X,W)g(Y,Z) \\ &= \det \begin{pmatrix} g(X,Z) & g(X,W) \\ g(Y,Z) & g(Y,W) \end{pmatrix}. \end{aligned} \tag{A.1.26}$$

The **sectional curvature** of (v,w) is defined by

$$\sec(v,w) = \frac{R(v,w,v,w)}{g(v \wedge w, v \wedge w)} = \frac{g(\mathfrak{R}(w \wedge v), w \wedge v)}{g(v \wedge w, v \wedge w)}. \tag{A.1.27}$$

It only depends on the plane $\pi = \mathrm{span}\{v,w\}$.

A Riemann manifold has **constant curvature** k if $\sec(\pi) = k$ for all 2-planes in T_pM. It is equivalent to $\mathfrak{R}(\omega) = k\omega$ for all $\omega \in \Lambda_p^2 M$. If $n \geq 3$, the sphere $S^{n-1}(r)$ has constant curvature r^{-2}.

The **Ricci curvature** of (v,w) is defined by

$$\begin{aligned} \mathrm{Ric}(v,w) = \sum_{i=1}^n R(e_i,v,e_i,w) &= \sum_{i=1}^n R(v,e_i,w,e_i) \\ &= \sum_{i=1}^n R(e_i,w,e_i,v). \end{aligned} \tag{A.1.28}$$

Thus Ric is a symmetric bilinear form. We adopt the language that $\mathrm{Ric} \geq k$ if all eigenvalues of Ric are $\geq k$. That is, $Ric(v,v) \geq kg(v,v)$ for all v.

If $\mathrm{Ric}(v,w) = kg(v,w)$ for all v, w, then (M,g) is said to be an **Einstein manifold** with **Einstein constant** k. If (M,g) has constant curvature k, then (M,g) is also Einstein with Einstein constant $(n-1)k$.

The **scalar curvature** is defined by

$$\mathrm{scal} = \mathrm{tr}(\mathrm{Ric}) = 2\mathrm{tr}\mathfrak{R} = 2\sum_{i<j}\sec(e_i,e_j). \tag{A.1.29}$$

It is easy to calculate that

$$d\mathrm{tr}(\mathrm{Ric}) = 2\mathrm{div}(\mathrm{Ric}). \tag{A.1.30}$$

Let

$$\mathrm{Ric}(v) = \sum_{i=1}^{n} R(v,e_i)e_i. \tag{A.1.31}$$

Lemma A.1.3 (Schur 1886 Ch2 Lemma 3)**.** *Assume $n \geq 3$ and one of the following conditions hold:*

(a) $\sec(\pi) = f(p)$ *for all 2-plane π in T_pM and $p \in M$.*

(b) $\mathrm{Ric}(v) = (n-1)f(p)v$ *for all $v \in T_pM$ and $p \in M$.*

Then in either case f must be constant. In other words, the metric has constant curvature or is Einstein, respectively.

Corollary A.1.4. *For $n \geq 3$, (M,g) is Einstein iff*

$$\mathrm{Ric} = \frac{\mathrm{scal}}{n}g. \tag{A.1.32}$$

Proposition A.1.5. *Let $\tilde{g} = e^{2\psi}g$. Then*

(a)

$$\widetilde{\nabla}_X Y = \nabla_X Y + X(\psi)Y + Y(\psi)X - g(X,Y)\nabla\psi. \tag{A.1.33}$$

(b) If X, Y orthonormal with respect to g,

$$\begin{aligned} e^{2\psi}\widetilde{\sec}(X,Y) = \sec(X,Y) &- \mathrm{Hess}\,\psi(X,X) - \mathrm{Hess}\,\psi(Y,Y) \\ &- |\nabla\psi|^2 + X(\psi)^2 + Y(\psi)^2. \end{aligned} \tag{A.1.34}$$

A.2 Curvature and topology

Theorem A.2.1 (Hopf-Rinow 1931 Ch5 Thm 16)**.** *The following statements are equivalent:*

(a) M is geodesically complete, i.e., all geodesics are defined for all time.

(b) M is geodesically complete at p, i.e., all geodesics through p are defined for all time.

(c) M satisfies the Heine-Borel property, i.e., every closed bounded set is compact.

(d) M is metrically complete.

From now on, all manifolds are assumed to be connected and complete.

Theorem A.2.2. *If M is a closed simply connected manifold with constant curvature k, then $k > 0$ and $M = S^n$.*

Theorem A.2.3. *If M is geodesically complete and noncompact with constant curvature k, then $k \leq 0$ and the universal cover is diffeomorphic to $\mathbb{R}^n$.*

Theorem A.2.4 (Killing 1893, Hopf 1926)**.** *If (M, g) is a connected, geodesically complete Riemannian manifold with constant curvature k, then the universal cover is isometric to S^n_k, $\mathbb{R}^n$ and $\mathbb{H}^n_k$.*

Theorem A.2.5 (Myers-Steenrod 1939 Ch5 Thm18)**.** *Let (M, g) and (N, g') be Riemannian manifolds and $F : M \to N$ a bijection. If F is distance-preserving, i.e., $d_{g'}(F(p), F(q)) = d_g(p, q)$ for all $p, q \in M$, then F is a Riemannian isometry.*

Let $\bar{\gamma} : (-\varepsilon, \varepsilon) \times [a, b]$ be a smooth variation of a smooth curve $\gamma(t) = \bar{\gamma}(0, t)$. Consider the **Energy functional**:

$$E(\gamma) = \frac{1}{2}\int_0^1 |\dot{\gamma}|^2 dt. \tag{A.2.1}$$

Lemma A.2.6 (The first variation formula Ch5 Lemma 11)**.**

$$\frac{dE(\gamma_s)}{ds} = -\int_a^b g\left(\frac{\partial^2\bar{\gamma}}{\partial t^2}, \frac{\partial\bar{\gamma}}{\partial s}\right) dt + g\left(\frac{\partial\bar{\gamma}}{\partial t}, \frac{\partial\bar{\gamma}}{\partial s}\right)\Bigg|_{(s,a)}^{(s,b)}. \tag{A.2.2}$$

Theorem A.2.7 (Ch5 Thm 13)**.** *If γ is a local minimum for E, then γ is a smooth geodesic.*

Theorem A.2.8 (Synge's second variation formula, 1926 Ch6 Thm21)**.** *If γ is a geodesic, then*

$$\begin{aligned}\frac{d^2E(\gamma_s)}{ds^2}\bigg|_{s=0} = \int_a^b \left|\frac{\partial^2\bar{\gamma}}{\partial t\partial s}\right|^2 dt - \int_a^b g\left(R\left(\frac{\partial\bar{\gamma}}{\partial s}, \frac{\partial\bar{\gamma}}{\partial t}\right)\frac{\partial\bar{\gamma}}{\partial t}, \frac{\partial\bar{\gamma}}{\partial s}\right) dt \\ + g\left(\frac{\partial\bar{\gamma}}{\partial t}, \frac{\partial^2\bar{\gamma}}{\partial s^2}\right)\bigg|_a^b. \end{aligned} \tag{A.2.3}$$

Jacobi fields: we define two vector fields T, U along $\bar{\gamma}$:

$$T(\bar{\gamma}(s,t)) = d\bar{\gamma}\left(\left.\frac{\partial}{\partial t}\right|_{(s,t)}\right), \quad U(\bar{\gamma}(s,t)) = d\bar{\gamma}\left(\left.\frac{\partial}{\partial s}\right|_{(s,t)}\right). \tag{A.2.4}$$

We usually write $T = \partial_t$, $U = \partial_s$. Assume that for any $s \in (-\varepsilon, \varepsilon)$, $\bar{\gamma}(s,t)$ is geodesic. Then we have

$$[T, U] = d\bar{\gamma}\left(\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right]\right) = 0 \tag{A.2.5}$$

and

$$\nabla_T T = 0. \tag{A.2.6}$$

So

$$\nabla_T \nabla_T U = \nabla_T \nabla_U T = \nabla_T \nabla_U T - \nabla_U \nabla_T T - \nabla_{[T,U]} T = -R(U,T)T. \tag{A.2.7}$$

Let $J(t) = U(\bar{\gamma}(0,t))$. Then we have the **Jacobi Equation**:

$$\ddot{J} + R(J, \dot{\gamma})\dot{\gamma} = 0. \tag{A.2.8}$$

The field J is called the **Jacobi field**. In case $J(0) = 0$, it can be constructed via the geodesic variation

$$\bar{\gamma}(s,t) = \exp_p\left(t\left(\dot{\gamma}(0) + s\dot{J}(0)\right)\right). \tag{A.2.9}$$

For $w \in T_v T_p M$ such that $w = \dot{J}(0)$, we have

$$d(\exp_p(v))(w) = J(1). \tag{A.2.10}$$

Theorem A.2.9 (Mangoldt 1881, Hadamard 1889, Cartan 1925 Thm22)**.** *If (M, g) is complete, connected, and has* $\sec \le 0$*, then the universal cover is diffeomorphic to $\mathbb{R}^n$.*

Proof.

$$\frac{d}{dt}\left(\frac{1}{2}|J(t)|^2\right) = g(\dot{J}, J). \tag{A.2.11}$$

$$\begin{aligned} \frac{d^2}{dt^2}\left(\frac{1}{2}|J(t)|^2\right) &= \frac{d}{dt} g(\dot{J}, J) = g(\ddot{J}, J) + g(\dot{J}, \dot{J}) \\ &= -g(R(J,\dot{\gamma})\dot{\gamma}, J) + |\dot{J}|^2 \ge |\dot{J}|^2. \end{aligned} \tag{A.2.12}$$

Integrating,

$$g(\dot{J}, J) \geq \int_0^t |\dot{J}|^2 dt > 0. \tag{A.2.13}$$

Integrating,

$$\frac{1}{2}|J(t)|^2 > 0. \tag{A.2.14}$$

□

Theorem A.2.10 (Hopf-Rinow 1931, Myers 1932 Cor 12). *Suppose (M, g) is complete and satisfies $\sec \geq k > 0$. Then M is compact and satisfies $\operatorname{diam}(M, g) \leq \pi/\sqrt{k} = \operatorname{diam} S^n_k$. In particular, M has finite fundamental group.*

Theorem A.2.11 (Myers 1941 Thm 25). *Suppose (M, g) is complete and satisfies $\operatorname{Ric} \geq (n-1)k > 0$. Then $\operatorname{diam}(M, g) \leq \pi/\sqrt{k}$. Furthermore, M has finite fundamental group.*

Theorem A.2.12 (Synge 1936 Thm 26). *Let M be a compact manifold with* $\sec > 0$.

(1) If M is even-dimensional and orientable, then M is simply connected.

(2) If M is odd-dimensional, then M is orientable.

Theorem A.2.13 (Thm 28). *If (M, g) has $\sec \leq K$, $K > 0$, then*

$$\exp_p : B(0, \pi/\sqrt{K}) \to M \tag{A.2.15}$$

has no critical points.

Theorem A.2.14 (Rauch-Berger-Klingenberg 1951-61 cor13). *Let M be a closed simply connected n-manifold with $4 > \sec \geq 1$, then M is $(n-1)$-connected and hence a homotopy sphere.*

Theorem A.2.15 (thm 35). *The set of Killing fields $\mathfrak{iso}(M, g)$ is a Lie algebra of dimension $\leq n(n+1)/2$. Furthermore, if M is compact (or complete), then $\mathfrak{iso}(M, g)$ is the Lie algebra of $\operatorname{Iso}(M, g)$. If $\dim \operatorname{Iso}(M, g) n(n+1)/2$, then (M, g) has constant curvature.*

Theorem A.2.16. *Suppose (M, g) is compact, oriented, and has $\operatorname{Ric} \leq 0$. We have*

$$\dim(\operatorname{Iso}(M, g)) \leq \dim M \tag{A.2.16}$$

and $\operatorname{Iso}(M, g)$ is finite if $\operatorname{Ric} < 0$.

Theorem A.2.17. *Suppose (M, g) is compact, oriented, and has* $\mathrm{Ric} \leq 0$*. Let $p = \dim(\mathrm{Iso}(M, g))$. We have that the universal cover splits isometrically as $\widetilde{M} = \mathbb{R}^p \times N$.*

Theorem A.2.18. *If (M, g) is a compact, oriented, and has* $\mathrm{Ric} \geq 0$*, then $b_1(M) \leq n = \dim M$, with equality holding iff (M, g) is a flat torus.*

Proposition A.2.19 (cor20)**.** *Suppose M is orientable. If $\mathfrak{R} \geq 0$, then*

$$b_k(M) \leq b_k(T^n) = C_n^k. \tag{A.2.17}$$

And if $\mathfrak{R} > 0$ somewhere, then $b_k(M) = 0$ for $k \leq n - 1$.

Theorem A.2.20 (SY Cheng 75 Thm 62)**.** *If (M, g) is a complete Riemannian manifold with* $\mathrm{Ric} \geq (n-1)k > 0$ *and* $\mathrm{diam} = \pi/\sqrt{k}$*, then (M, g) is isometric to S_k^n.*

Theorem A.2.21 (Gallot-Gromov 80 thm63)**.** *If M is a Riemannian manifold of dimension n such that* $\mathrm{Ric} \geq (n-1)k$ *and* $\mathrm{diam}(M) \leq D$*, then there is a function $C(n, kD^2)$ such that*

$$b_1(M) \leq C(n, kD^2). \tag{A.2.18}$$

Moreover, $\lim_{\varepsilon \to 0} C(n, \varepsilon) = n$. In particular, there is $\varepsilon(n) > 0$ such that if $kD^2 \geq -\varepsilon(n)$, then $b_1(M) \leq n$.

Let

$$\begin{aligned} \mathfrak{M}(n, k, v, D) = \{&\text{compact } (M^n, g) : \\ &\mathrm{Ric} \geq (n-1)k,\ \mathrm{vol} \geq v,\ \mathrm{diam} \leq D\}. \end{aligned} \tag{A.2.19}$$

Theorem A.2.22 (Anderson 90 thm 64)**.** *There are only finitely many fundamental groups among the manifolds in $\mathfrak{M}(n, k, v, D)$ for fixed n, k, v, D.*

Theorem A.2.23 (Cheeger-Gromoll 71 thm 68)**.** *If (M, g) contains a line and has* $\mathrm{Ric} \geq 0$*, then (M, g) is isometric to a product $(H \times \mathbb{R}, g_0 + dt^2)$.*

Theorem A.2.24 (Cheeger-Gromoll 71 thm69)**.** *Suppose (M, g) is a compact Riemannian manifold with* $\mathrm{Ric} \geq 0$*. Then the universal cover $(\tilde{M}, \tilde{g})$ splits isometrically as a product $N \times \mathbb{R}$, where N is a compact manifold.*

Corollary A.2.25. *Suppose (M, g) is a compact Riemannian manifold with* $\mathrm{Ric} \geq 0$*. If M is $K(\pi, 1)$, i.e., the universal cover is contractible, then the universal cover is Euclidean space and (M, g) is a flat manifold.*

Corollary A.2.26. *Suppose* (M, g) *is a compact Riemannian manifold with* $\mathrm{Ric} \geq 0$. *If* $\mathrm{Ric} > 0$ *on some* T_pM, *then* $\pi_1(M)$ *is finite.*

Corollary A.2.27. *Suppose* (M, g) *is a compact Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Then* $b_1(M) \leq \dim M$, *with equality holding iff* (M, g) *is a flat torus.*

Proposition A.2.28 (Cor43)**.** *Suppose* K *is a compact submanifold of a complete Riemannian manifold* (M, g) *and suppose the distance function* $r = d(\cdot, K)$ *is regular everywhere on* $M - K$. *Then* M *is diffeomorphic to the normal bundle of* K *in* M. *In particular, if* $K = \{p\}$, *then* M *is diffeomorphic to* $\mathbb{R}^n$.

Theorem A.2.29 (thm80)**.** *If* M *is a simply connected closed Riemannian manifold with* $1 \leq \sec \leq 4 - \delta$, *then* M *is homeomorphic to a sphere.*

Theorem A.2.30 (Berger 62, Grove-Schiohama 77 thm 81)**.** *If* (M, g) *is a closed Riemannian manifold with* $\sec \geq 1$ *and* $\mathrm{diam} > \pi/2$, *then* M *is homeomorphic to a sphere.*

Theorem A.2.31 (thm 82)**.** *Suppose* (M, g) *is simply connected of dimension* n *with* $1 \leq \sec \leq 4 + \varepsilon$.

(1) (Berger 83) If n *is even, then there is* $\varepsilon(n) > 0$ *such that* M *must be homeomorphic to a sphere or diffeomorphic to one of the spaces* $\mathbb{C}P^{n/2}$, $\mathbb{H}P^{n/4}$, $\mathbb{O}P^2$.

(2) (Abresch-Meyer 94) If n *is odd, then there is an* $\varepsilon > 0$, *which can be chosen independently of* n, *such that* M *is homeomorphic to a sphere.*

Theorem A.2.32 (Grove-Gromoll 87, Wilking 01 thm 83)**.** *Suppose* (M, g) *is closed and satisfies* $\sec \geq 1$, $\mathrm{diam} \geq \pi/2$. *Then one of the following cases holds:*

(1) M *is homeomorphic to a sphere.*

(2) M *is isometric to a finite quotient* $S^n(1)/\Gamma$, *where the action of* Γ *is reducible (has an invariant subspace).*

(3) M *is isometric to one of* $\mathbb{C}P^{n/2}$, $\mathbb{H}P^{n/4}$, $\mathbb{C}P^{n/2}/\mathbb{Z}_2$ *for* $n = 2 \mod 4$.

(4) M *is isometric to* $\mathbb{O}P^2$.

Theorem A.2.33 (Cheeger-Gromoll-Meyer 69,72 thm84)**.** *If* (M, g) *is a complete non-compact Riemannian manifold with* $\sec \geq 0$, *then* M *contains a soul* $S \subset M$, *which is a closed totally convex submanifold, such that* M *is diffeomorphic to the normal bundle of* S. *Moreover, when* $\sec > 0$, *the soul is a point and* M *is diffeomorphic to* $\mathbb{R}^n$.

Theorem A.2.34 (Gromov 78,81 thm 86)**.** *There is a constant $C(n)$ such that any complete manifold (M, g) with* $\sec \geq 0$ *satisfies:*

(1) $\pi_1(M)$ can be generated by $\leq C(n)$ generators.

(2) For any field F of coefficients the Betti numbers are bounded:

$$\sum_{i=0}^{n} b_i(M, F) = \sum_{i=0}^{n} \dim H_i(M, F) \leq C(n). \tag{A.2.20}$$

Theorem A.2.35 (Grove-Peterson 88 thm 87)**.** *Given an integer $n > 1$ and numbers $v, D, k \in (0, \infty)$, the class of Riemannian n-manifolds with*

$$\operatorname{diam} \leq D, \ \operatorname{vol} \geq v, \ \sec \geq -k^2 \tag{A.2.21}$$

contains only finite many homotopy types.