Note of Analysis on Manifolds

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Preface

This is the note of a course given at Spring 2019 in East China Normal University.

Chapter 1

Differential and pseudodifferential operators on vector bundles

1.1 Differential operators on manifolds

1.1.1 What is manifold?

Etymologically, manifold means a collection of maps. In fact, it really is.

For thousands of years, people have never stopped exploring the land we live on.

They explore the earth and draw the picture to note down everything they see, which is called the map. The idea of drawing the map to record the roads is not hard, even an old horse could do it in its brain. But usually one map is not enough. For example, if we want to drive the car from Pudong airport to ECNU, we would be disappointed to see that, not like our neighborhood, we could not find ECNU in the screen map containing the airport.

So we have to glue another map together to find the road we need.

In order to glue the maps together, the only necessary condition is that **in the intersection area, a continuous road in one map must continuous in another one.** Otherwise we will get lost. (Of course, due to the modern technologies, AI could do this for us even with a sweet voice.)

On the other hand, the fact that earth is a round ball obviously also gives the fundamental obstruction to describe everything in one map. We need to draw the map piece by piece and glue them together.

However, it is unbelievable and a miracle that we can get the converse: from the maps with scales and the gluing methods we can see that our earth is a ball, not a torus, by the Gauss-Bonnet theorem. (If the reader knows the Gauss-Bonnet theorem before, which we will not mention in the following sections, please try to explain it.)

Let us explain a bit the importance of this genius idea. For the earth, the earth is a round ball is not news. We could work hard and earn enough money to take the spacecraft to the moon to confirm that the earth is really round, does not has a hole and not like a cup.

But for the space-age now, we want to explore the universe. It is not easy to decide that the universe has a hole or not. To be honest, we are not the God. We cannot see the universe outside it like the people on the moon. The only method is to search it, draw the 3-dimensional map everywhere, glue them together and try to obtain the global properties by the local explorations.

For our mathematicians, we are used to abstract the ideas and put all dimensions together. We define the manifold as the topological spaces which can be studied in this process. Let us end the story and start to study the math.

Definition 1.1.1. Let M be a Hausdorff and second countable topological space. We say that M is a **topological manifold** of dimension n if every point $x \in M$ has an open neighborhood, which is homeomorphic to an open set in $\mathbb{R}^n$.

For topological manifold M, there exists a open cover $\{U_i\}_i$ of M such that each U_i has a map $\phi_i : U_i \to \mathbb{R}^n$ and $\phi_i : U_i \to \phi(U_i) \subset \mathbb{R}^n$ is a homeomorphism. The pair (U_i, ϕ_i) is called a chart and the collection $\{(U_i, \phi_i)\}_i$ is called an atlas.

1.1.2 How to do analysis on manifold?

Since M is a topological space, we could study a continuous function $f : M \to \mathbb{R}$. But if we want to apply the achievements of human after Newton on manifolds, we have to find a way to take the derivative of f. **We don't know how to do analysis on manifold, but we know how to do it on $\mathbb{R}^n$.**

The most natural idea is to take the derivative on each chart (U_i, ϕ_i). If

$$f \circ \phi_i^{-1} : \phi_i(U_i) \subset \mathbb{R}^n \to \mathbb{R} \tag{1.1.1}$$

is smooth, we could take the derivative of $f \circ \phi_i^{-1}$ as the derivative of f. However, if $U_i \cap U_j \neq \emptyset$, for $x \in U_i \cap U_j$, even if $f \circ \phi_i^{-1}$ is smooth at x, $f \circ \phi_j^{-1}$ may be not differentiable at x. If we want to make

$$f \circ \phi_j^{-1} = (f \circ \phi_i^{-1}) \circ (\phi_i \circ \phi_j^{-1}) \tag{1.1.2}$$

smooth at x, We need to assume further that

$$\phi_{ij} := \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \subset \mathbb{R}^n \to \phi_i(U_i \cap U_j) \subset \mathbb{R}^n \tag{1.1.3}$$

is smooth. If for any i, j such that $U_i \cap U_j \neq \emptyset$, (1.1.3) is smooth, the statement "f is smooth at $x \in M$" is meaningful.

Remark that $f(x) \equiv 0$ is a smooth map.

Definition 1.1.2. A smooth atlas on a topological manifold is an atlas $\{(U_i, \phi_i)\}_i$ such that for any i, j such that $U_i \cap U_j \neq \emptyset$, (1.1.3) is smooth. We say two smooth atlases are equivalent if they determine the same collection of smooth functions on M. The equivalent class of smooth atlases is called a smooth structure. A topological manifold with a smooth structure is called a smooth manifold.

The main purpose of this definition is to establish a home for the smooth function on manifold.

Furthermore, we could define a smooth map between two smooth manifolds.

Definition 1.1.3. Let M and N be two smooth manifolds with smooth atlases $\{(U_i, \phi_i)\}_i$ and $\{(V_j, \varphi_j)\}_j$. Let $f : M \to N$ be a continuous map. For $m \in U_i \subset M$, we say that f is smooth at m if for any V_j containing $f(m)$, $\varphi_j \circ f \circ \phi_i^{-1}$ is smooth at $\phi_i(m)$. If f is smooth at any $m \in M$, we say that f is a smooth map.

If the smooth map f has an inverse, which is also smooth, we say f is a diffeomorphism. For example, ϕ_{ij} in (1.1.3) is a diffeomorphism.

In this note, we will work in the $\mathscr{C}^\infty$ category. In the following sections, if we say "a manifold", we mean "a smooth manifold"; if we say "a function", we mean "a smooth function"...

Once we understand how to define the smooth structure on M, the next natural question is

1.1.3 How to define the partial differential of a smooth function?

The naive idea is to take $\frac{\partial}{\partial x_k^{(j)}}(f \circ \phi_j^{-1})$ in the chart U_j as the partial differential $\frac{\partial}{\partial x_k} f$. In order to distinguish the partial differential in different charts, we use the notation $\frac{\partial}{\partial x_k^{(j)}}$ to represent the partial differential on the coordinate on $\phi_j(U_j)$. After all we know how to take partial differential on $\mathbb{R}^n$. We could do anything locally on $\mathbb{R}^n$. But the same problem appear. For another chart (U_i, ϕ_i) such that $U_i \cap U_j \neq \emptyset$, for $x \in \phi_j(U) \subset \mathbb{R}^n$, if $\frac{\partial}{\partial x_k^{(i)}}(f \circ \phi_i^{-1})$ and $\frac{\partial}{\partial x_k^{(j)}}(f \circ \phi_j^{-1})$ represent the same function $\frac{\partial}{\partial x_k} f$, from

(1.1.2), we need $\frac{\partial(f\circ\phi_j^{-1})}{\partial x_k^{(j)}}(x) = \frac{\partial(f\circ\phi_i^{-1})}{\partial x_k^{(i)}}(\phi_{ij}(x))$. However, by the chain's rule,

$$\frac{\partial(f\circ\phi_j^{-1})}{\partial x_k^{(j)}}(x) = \sum_{l=1}^{n} \frac{\partial(f\circ\phi_i^{-1})}{\partial x_l^{(i)}}(\phi_{ij}(x)) \cdot \frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x), \tag{1.1.4}$$

where ϕ_{ij}^l is the l-th component of the image of ϕ_{ij} in $\mathbb{R}^n$. This means that our naive idea is not compatible with our construction of the smooth structure. By (1.1.2) and (1.1.4), $\frac{\partial}{\partial x_k^{(j)}}(f\circ\phi_j^{-1})$ and $\sum_{l=1}^n \frac{\partial\phi_{ij}^l}{\partial x_k^{(j)}}(x)\frac{\partial}{\partial x_l^{(i)}}(f\circ\phi_i^{-1})$ represent the same function. Thus the partial differential

$$\frac{\partial}{\partial x_k^{(j)}} \text{ in } U_j \text{ corresponds to } \sum_{l=1}^{n} \frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\frac{\partial}{\partial x_l^{(i)}} \text{ in } U_i. \tag{1.1.5}$$

For the convenience, we could write (1.1.5) by matrix.

$$\begin{pmatrix} \frac{\partial}{\partial x_1^{(j)}} \\ \vdots \\ \frac{\partial}{\partial x_n^{(j)}} \end{pmatrix} \sim \left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)} \cdot \begin{pmatrix} \frac{\partial}{\partial x_1^{(i)}} \\ \vdots \\ \frac{\partial}{\partial x_n^{(i)}} \end{pmatrix}. \tag{1.1.6}$$

From this relation, we could glue the partial differentials in each chart together to get a global partial differential, which we call a vector field.

Definition 1.1.4. Let M be a manifold with smooth structure $\{(U_i,\phi_i)\}_i$. A vector field X on M is a collection of partial differentials $\sum_{k=1}^n a_k^{(j)}\frac{\partial}{\partial x_k^{(j)}}$, $a_k^{(j)}(x) \in \mathscr{C}^\infty(\phi_j(U_j)\subset\mathbb{R}^n,\mathbb{R})$, on each $\phi_j(U_j)\subset\mathbb{R}^n$ such that if $m\in U_i\cap U_j$,

$$a_l^{(i)}(\phi_i(m)) = \sum_{k=1}^{n} a_k^{(j)}(\phi_j(m))\frac{\partial\phi_{ij}^l}{\partial x_k^{(j)}}(\phi_j(m)). \tag{1.1.7}$$

We often write $X|_{U_j} = \sum_{k=1}^n a_k^{(j)}\frac{\partial}{\partial x_k^{(j)}}$. Thus for a smooth function $f\in\mathscr{C}^\infty(M,\mathbb{R})$ and a vector field X, we could define Xf as a smooth function on M.

By definition, a vector field X is a map

$$X : \mathscr{C}^\infty(M,\mathbb{R}) \to \mathscr{C}^\infty(M,\mathbb{R}). \tag{1.1.8}$$

We must check the well-definedness of this definition. If $m\in U_i\cap U_j\cap U_s$, from (1.1.3), we have

$$\phi_{si}\circ\phi_{ij} = \phi_{sj}. \tag{1.1.9}$$

By chain's rule,

$$\frac{\partial \phi_{sj}^l}{\partial x_k^{(j)}}(\phi_j(m)) = \sum_{t=1}^n \frac{\partial \phi_{si}^l}{\partial x_t^{(i)}}(\phi_i(m)) \cdot \frac{\partial \phi_{ij}^t}{\partial x_k^{(j)}}(\phi_j(m)). \tag{1.1.10}$$

Thus from (1.1.7) and (1.1.10),

$$\begin{aligned} a_s^{(l)}(\phi_s(m)) &= \sum_{t=1}^n a_t^{(i)}(\phi_i(m)) \frac{\partial \phi_{si}^l}{\partial x_t^{(i)}}(\phi_i(m)) \\ &= \sum_{t=1}^n \sum_{k=1}^n a_k^{(j)}(\phi_j(m)) \frac{\partial \phi_{ij}^t}{\partial x_k^{(j)}}(\phi_j(m)) \frac{\partial \phi_{si}^l}{\partial x_t^{(i)}}(\phi_i(m)) \\ &= \sum_{k=1}^n a_k^{(j)}(\phi_j(m)) \frac{\partial \phi_{sj}^l}{\partial x_k^{(j)}}(\phi_j(m)). \end{aligned} \tag{1.1.11}$$

Thus our glue is compatible with the smooth structure on M.

In order to simplify the notations and perfect the theory, we want to find a home for the vector field, which we call the tangent bundle.

Let $\sqcup_i(U_i \times \mathbb{R}^n)$ be the disjoint union of $U_i \times \mathbb{R}^n$. We define an equivalent relation "$\sim$" such that $(x, (a_1^{(i)}, \cdots, a_n^{(i)})) \in U_i \times \mathbb{R}^n \sim (y, (a_1^{(j)}, \cdots, a_n^{(j)})) \in U_j \times \mathbb{R}^n$ if and only if $x = y$ and for any $1 \le l \le n$, $a_l^{(i)} = \sum_{k=1}^n a_k^{(j)} \frac{\partial \phi_{ij}^l}{\partial x_k}(x)$, i.e.,

$$(a_1^{(i)}, \cdots, a_n^{(i)}) = (a_1^{(j)}, \cdots, a_n^{(j)}) \cdot \left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x) \right)_{(k,l)} \tag{1.1.12}$$

From (1.1.9) and (1.1.11), we see that this relation "$\sim$" is really an equivalent relation.

Definition 1.1.5. The tangent bundle is defined as the quotient space of the equivalent relation with the quotient topology, i.e., $TM := \sqcup_i(U_i \times \mathbb{R}^n)/\sim$.

Proposition 1.1.6. *The tangent bundle TM is a manifold.*

The proof is left to an exercise.

Let $\pi : TM \to M$ be the natural projection from $(x, v) \in TM$ to x.

Proposition 1.1.7. *A vector field is a smooth map $X : M \to TM$ such that $\pi \circ X = \mathrm{Id}$.*

The proof is left to an exercise.

A vector field is also called a section of TM. We denote by $\mathscr{C}^\infty(M, TM)$ the set of sections of TM.

1.1.4 What about the exterior differential?

Our natural idea is to take

$$d^{(j)}(f \circ \phi_j^{-1}) = \frac{\partial}{\partial x_k^{(j)}}(f \circ \phi_j^{-1}) \cdot dx_k^{(j)} \tag{1.1.13}$$

in the chart U_j as the exterior differential df. From the coordinate transformation formula in multivariable calculus, as in (1.1.6), we have

$$\left(dx_1^{(i)}, \cdots, dx_n^{(i)}\right) \sim \left(dx_1^{(j)}, \cdots, dx_n^{(j)}\right) \cdot \left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)}. \tag{1.1.14}$$

Equivalently, we have

$$\left(dx_1^{(j)}, \cdots, dx_n^{(j)}\right) \sim \left(dx_1^{(i)}, \cdots, dx_n^{(i)}\right) \cdot \left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)}^{-1}. \tag{1.1.15}$$

Thus from (1.1.6) and (1.1.15), we have

$$\begin{aligned} d^{(j)}(f \circ \phi_j^{-1})(x) &= \left(dx_1^{(j)}, \cdots, dx_n^{(j)}\right) \begin{pmatrix} \frac{\partial}{\partial x_1^{(j)}} \\ \vdots \\ \frac{\partial}{\partial x_n^{(j)}} \end{pmatrix} (f \circ \phi_j^{-1})(x) \\ &\sim \left(dx_1^{(i)}, \cdots, dx_n^{(i)}\right) \cdot \left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)}^{-1} \cdot \left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)} \cdot \begin{pmatrix} \frac{\partial}{\partial x_1^{(i)}} \\ \vdots \\ \frac{\partial}{\partial x_n^{(i)}} \end{pmatrix} (f \circ \phi_i^{-1})(\phi_{ij}(x)) \\ &= \left(dx_1^{(i)}, \cdots, dx_n^{(i)}\right) \cdot \begin{pmatrix} \frac{\partial}{\partial x_1^{(i)}} \\ \vdots \\ \frac{\partial}{\partial x_n^{(i)}} \end{pmatrix} (f \circ \phi_i^{-1})(\phi_{ij}(x)) = d^{(i)}(f \circ \phi_i^{-1})(\phi_{ij}(x)). \end{aligned} \tag{1.1.16}$$

Therefore, not like the partial differential, for the exterior differential, our naive idea is right. What we obtain is the following proposition.

Proposition 1.1.8. *The exterior differential defined d in (1.1.13) is globally defined.*

Like the vector field, we need to construct a home for df. We assume that on U_j, $df|_{U_j} = \sum_{k=1}^n b_k^{(j)} dx_k^{(j)}$. From (1.1.15),

$$df|_{U_i\cap U_j} = \left(dx_1^{(i)}, \cdots, dx_n^{(i)}\right) \begin{pmatrix} b_1^{(i)} \\ \vdots \\ b_n^{(i)} \end{pmatrix} \sim \left(dx_1^{(j)}, \cdots, dx_n^{(j)}\right) \begin{pmatrix} b_1^{(j)} \\ \vdots \\ b_n^{(j)} \end{pmatrix}$$
$$\sim \left(dx_1^{(i)}, \cdots, dx_n^{(i)}\right) \cdot \left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)}^{-1} \cdot \begin{pmatrix} b_1^{(j)} \\ \vdots \\ b_n^{(j)} \end{pmatrix}. \quad (1.1.17)$$

Thus

$$(b_1^{(i)}, \cdots, b_n^{(i)}) = (b_1^{(j)}, \cdots, b_n^{(j)}) \cdot \left(\left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)}^{-1}\right)^T. \quad (1.1.18)$$

Definition 1.1.9. We define an equivalent relation "$\sim$" such that $(x, (b_1^{(i)}, \cdots, b_n^{(i)})) \in U_i \times \mathbb{R}^n \sim (y, (b_1^{(j)}, \cdots, b_n^{(j)})) \in U_j \times \mathbb{R}^n$ if and only if $x = y$ and (1.1.18) holds. The cotangent bundle is defined as the quotient space of this equivalent relation with the quotient topology, i.e., $T^*M := \sqcup_i (U_i \times \mathbb{R}^n)/\sim$.

From (1.1.5), the relation "$\sim$" is really an equivalent relation. Let $\pi' : T^*M \to M$ be the natural projection. As in the tangent bundle case, we denote by $\mathscr{C}^\infty(M, T^*M)$ the set of smooth maps $s : M \to T^*M$ such that $\pi' \circ s = \mathrm{Id}$. From the construction above, we have $df \in \mathscr{C}^\infty(M, T^*M)$. An element in $\mathscr{C}^\infty(M, T^*M)$ is also called a 1-form. Conversely, for any $\alpha \in \mathscr{C}^\infty(M, T^*M)$, there exists $b_k^i : \phi(U_i) \to \mathbb{R}$, $1 \le k \le n$ such that $\alpha|_{U_i} = \sum_{k=1}^n b_k^{(i)} dx_k^{(i)}$. From the knowledge of the multivariable calculus, there exists $f \in \mathscr{C}^\infty(\phi(U_i), \mathbb{R})$ such that $\alpha|_{U_i} = df$. Remark that this converse process is local. In general, for $\alpha \in \mathscr{C}^\infty(M, T^*M)$, we can not find $f \in \mathscr{C}^\infty(M, \mathbb{R})$ such that $\alpha = df$.

Furthermore, for $\alpha \in \mathscr{C}^\infty(M, T^*M)$, we can also define the exterior derivative of α: $d\alpha$. For

$$\alpha|_{U_i} = \sum_{k=1}^n b_k^{(i)} dx_k^{(i)}, \quad (1.1.19)$$

the natural idea to define $d\alpha$ by $d\alpha|_{U_i} = \sum_{l=1}^n \sum_{k=1}^n \frac{\partial b_k^{(i)}}{\partial x_l^{(i)}} dx_l^{(i)} dx_k^{(i)}$. As usual, we need to check the coordinate transformation. We simply denote by Φ the matrix $\left(\frac{\partial\phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)}$. From (1.1.6), (1.1.14) and (1.1.12), we have

$$b_k^{(i)}=\sum_{t=1}^n b_t^{(j)}\Phi_{kt}^{-1},\quad \frac{\partial}{\partial x_l^{(i)}}\sim\sum_{m=1}^n\Phi_{lm}^{-1}\frac{\partial}{\partial x_m^{(j)}},\quad dx_l^{(i)}\sim\sum_{q=1}^n dx_q^{(j)}\Phi_{ql}. \quad (1.1.20)$$

Thus

$$\begin{aligned}\sum_{l=1}^n\sum_{k=1}^n\frac{\partial b_k^{(i)}}{\partial x_l^{(i)}}dx_l^{(i)}dx_k^{(i)}&=\sum_{k,l,t,m,p,q=1}^n\Phi_{lm}^{-1}\frac{\partial(b_t^{(j)}\Phi_{kt}^{-1})}{\partial x_m^{(j)}}dx_q^{(j)}\Phi_{ql}dx_p^{(j)}\Phi_{pk}\\&=\sum_{t,m=1}^n\frac{\partial b_t^{(j)}}{\partial x_m^{(j)}}dx_m^{(j)}dx_t^{(j)}+\sum_{t,m,p,k=1}^n b_t^{(j)}\frac{\partial(\Phi_{kt}^{-1})}{\partial x_m^{(j)}}\Phi_{pk}dx_m^{(j)}dx_p^{(j)}\\&=\sum_{t,m=1}^n\frac{\partial b_t^{(j)}}{\partial x_m^{(j)}}dx_m^{(j)}dx_t^{(j)}-\sum_{t,m,p,k=1}^n b_t^{(j)}\Phi_{kt}^{-1}\frac{\partial^2\phi_{ij}^k}{\partial x_m^{(j)}\partial x_p^{(j)}}dx_m^{(j)}dx_p^{(j)}. \quad (1.1.21)\end{aligned}$$

The annoying term

$$\sum_{t,k=1}^n b_t^{(j)}\Phi_{kt}^{-1}\left(\sum_{m,p=1}^n\frac{\partial^2\phi_{ij}^k}{\partial x_m^{(j)}\partial x_p^{(j)}}dx_m^{(j)}dx_p^{(j)}\right) \quad (1.1.22)$$

appear. Note that $\frac{\partial^2\phi_{ij}^k}{\partial x_m^{(j)}\partial x_p^{(j)}}=\frac{\partial^2\phi_{ij}^k}{\partial x_p^{(j)}\partial x_m^{(j)}}$. A genius idea is that we define $dx_m^{(j)}dx_p^{(j)}=-dx_p^{(j)}dx_m^{(j)}$ to let (1.1.22) vanish. In order to avoid the ambiguity, for this anti-commutation property, we introduce a new notation: wedge product $\wedge$. And we use the notation $dx_m\wedge dx_p$ to replace $dx_m dx_p$ in the image of $d\alpha$. Here $dx_m\wedge dx_p$ means

$$dx_m\wedge dx_p=-dx_p\wedge dx_m. \quad (1.1.23)$$

Now we could define

$$d\alpha|_{U_i}=\sum_{l=1}^n\sum_{k=1}^n\frac{\partial b_k^{(i)}}{\partial x_l^{(i)}}dx_l^{(i)}\wedge dx_k^{(i)}. \quad (1.1.24)$$

From the arguments above, the definition in (1.1.24) does not depend on the choice of the coordinate. The next thing is to construct a home for the image of $d\alpha$. Now we do it in general. I'm tired to construct them one by one.

As in (1.1.23), we introduce the notation $dx_{p_1}\wedge\cdots\wedge dx_{p_k}$ satisfying

$$dx_{p_1}\wedge\cdots\wedge dx_{p_k}=\delta_{p_1,\cdots,p_k}^{q_1,\cdots,q_k}dx_{q_1}\wedge\cdots\wedge dx_{q_k}. \quad (1.1.25)$$

For $\alpha^{(i)}_{p_1,\cdots,p_k} \in \mathscr{C}^\infty(\phi(U_i),\mathbb{R})$, $1 \le p_1,\cdots,p_k \le n$, (1.1.19) is generalized to a k-form

$$\sum_{1\le p_1,\cdots,p_k\le n} \alpha^{(i)}_{p_1,\cdots,p_k} dx^{(i)}_{p_1} \wedge \cdots \wedge dx^{(i)}_{p_k} \tag{1.1.26}$$

on U_i. From (1.1.25), we arrange and restate (1.1.26) as

$$\sum_{1\le p_1<\cdots<p_k\le n} \beta^{(i)}_{p_1,\cdots,p_k} dx^{(i)}_{p_1} \wedge \cdots \wedge dx^{(i)}_{p_k}, \tag{1.1.27}$$

where $\beta^{(i)}_{p_1,\cdots,p_k} \in \mathscr{C}^\infty(\phi(U_i),\mathbb{R})$ and anti-commutes on $p_1,\cdots,p_k$. From (1.1.14), if $\sum_{1\le p_1<\cdots<p_k\le n} \beta^{(i)}_{p_1,\cdots,p_k} dx^{(i)}_{p_1}\wedge\cdots\wedge dx^{(i)}_{p_k}$ and $\sum_{1\le q_1<\cdots<q_k\le n} \beta^{(j)}_{q_1,\cdots,q_k} dx^{(j)}_{q_1}\wedge\cdots\wedge dx^{(j)}_{q_k}$ represent the same object on $U_i\cap U_j$, we can construct $(\Lambda^k\Phi)^{q_1,\cdots,q_k}_{p_1,\cdots,p_k} \in \mathscr{C}^\infty(\phi(U_j),\mathbb{R})$ such that

$$\beta^{(i)}_{p_1,\cdots,p_k} = \sum_{1\le q_1<\cdots<q_k\le n} (\Lambda^k\Phi)^{q_1,\cdots,q_k}_{p_1,\cdots,p_k} \cdot \beta^{(j)}_{q_1,\cdots,q_k}. \tag{1.1.28}$$

As in Definition 1.1.9, we define the bundle of exterior differentials.

Definition 1.1.10. We define an equivalent relation "$\sim_\Lambda$" on $\sqcup_i(U_i \times \mathbb{R}^{C^k_n})$ such that $(x, (\beta^{(i)}_{p_1,\cdots,p_k})_{1\le p_1<\cdots<p_k\le n}) \in U_i\times\mathbb{R}^{C^k_n} \sim (y, (\beta^{(j)}_{p_1,\cdots,p_k})_{1\le p_1<\cdots<p_k\le n}) \in U_j \times \mathbb{R}^{C^k_n}$ if and only if $x = y$ and (1.1.28) holds. The bundle of k-th exterior differentials is defined as the quotient space of this equivalent relation with the quotient topology, i.e., $\Lambda^k T^*M := \sqcup_i(U_i \times \mathbb{R}^{C^k_n})/\sim_\Lambda$.

Remark that if $k = n$, we have

$$\Lambda^n\Phi = (\det\Phi)^{-1}. \tag{1.1.29}$$

We could define $\mathscr{C}^\infty(M,\Lambda^kT^*M)$ as before. An element in $\mathscr{C}^\infty(M,\Lambda^kT^*M)$ is called a k-form on M. For $\alpha \in \mathscr{C}^\infty(M,\Lambda^kT^*M)$, $\alpha|_{U_i}$ could be written as (1.1.27). We define

$$d\alpha|_U = \sum_{1\le p_1<\cdots<p_k\le n} \sum_{t=1}^n \frac{\partial\beta^{(i)}_{p_1,\cdots,p_k}}{\partial x^{(i)}_t} dx^{(i)}_t \wedge dx^{(i)}_{p_1} \wedge\cdots\wedge dx^{(i)}_{p_k}. \tag{1.1.30}$$

As the same process in (1.1.20)-(1.1.24), we could obtain that $d\alpha$ is globally defined, which does not depend on the choice of the coordinate (try too fix it). Thus $d\alpha \in \mathscr{C}^\infty(M,\Lambda^{k+1}T^*M)$. Now we get a well-defined exterior differential

$$d : \mathscr{C}^\infty(M,\Lambda^kT^*M) \to \mathscr{C}^\infty(M,\Lambda^{k+1}T^*M). \tag{1.1.31}$$

Proposition 1.1.11. *(1)* $d^2 = 0$*;*

(2) for $\varphi \in \mathscr{C}^\infty(M)$*,* $d\varphi$ *is the one form such that* $\langle d\varphi, U\rangle = U(\varphi)$ *for any vector field* U*;*

(3) for any $\alpha \in \mathscr{C}^\infty(M, \Lambda^k T^*M)$*,* $\beta \in \mathscr{C}^\infty(M, \Lambda^l T^*M)$*, then*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \tag{1.1.32}$$

The proof is left to an exercise. In fact, the exterior differential d is characterized by the properties in Proposition 1.1.11.

The differential forms and the exterior is very useful.

Theorem 1.1.12 (de Rham Theorem)**.** *If* M *is compact, the* k*-th cohomology of* M *with real coefficients*

$$H^k(M, \mathbb{R}) \simeq \frac{\mathrm{Ker}(d : \mathscr{C}^\infty(M, \Lambda^k T^*M) \to \mathscr{C}^\infty(M, \Lambda^{k+1} T^*M))}{\mathrm{Im}(d : \mathscr{C}^\infty(M, \Lambda^{k-1} T^*M) \to \mathscr{C}^\infty(M, \Lambda^k T^*M))}. \tag{1.1.33}$$

This theorem verifies our naive philosophy to get the global property from the local charts.

1.1.5 Differential operator on vector bundles

Now we generalize the tangent bundle, cotangent bundle, bundle of exterior differential in Definition 1.1.5, 1.1.9 and 1.1.10 to the vector bundle.

Definition 1.1.13. Let $\{U_i\}$ be an open covering of M. Let $m \in \mathbb{N}$. The transition function is a group of maps $\{\Psi_{ij} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{R})\}$ such that if $U_i \cap U_j \cap U_k \neq \emptyset$,

$$\Psi_{ki} \circ \Psi_{ij} = \Psi_{kj}. \tag{1.1.34}$$

We define an equivalent relation "$\sim$" on $\sqcup_i (U_i \times \mathbb{R}^m)$ such that $(x, (b_1^{(i)}, \cdots, b_m^{(i)})) \in U_i \times \mathbb{R}^m \sim (y, (b_1^{(j)}, \cdots, b_m^{(j)})) \in U_j \times \mathbb{R}^m$ if and only if $x = y$ and

$$(b_1^{(i)}, \cdots, b_m^{(i)}) = (b_1^{(j)}, \cdots, b_m^{(j)}) \cdot \Psi_{ij}(x). \tag{1.1.35}$$

A **vector bundle** with rank m is defined as the quotient space of this equivalent relation with the quotient topology, i.e., $E := \sqcup_i (U_i \times \mathbb{R}^m)/\sim$. As a manifold, E is called the total space and M is called the base space. We also have the natural projection (smooth) map $\pi : E \to M$. A (smooth) section of E is a (smooth) map $s : M \to E$ such that $\pi \circ s = \mathrm{Id}_M$. We denote by $\mathscr{C}^\infty(M, E)$ the set of sections.

If we replace $\mathbb{R}$ by $\mathbb{C}$ in Definition 1.1.13, we get a complex vector bundle.

Since the closure of U_i could be covered by coordinate charts, we always assume that U_i here is a coordinate chart.

Obviously, TM, T^*M and $\Lambda^k T^*M$ are vector bundles.

We could also construct new vector bundles from the old one. Let E and F be two vector bundles. We may assume that they are defined on the same covering $\{U_i\}$ (if not, take the common refinement). Let $\{\Psi_{ij}^E : U_i \cap U_j \to \mathrm{GL}(m, \mathbb{R}/\mathbb{C})\}$ and $\{\Psi_{ij}^F : U_i \cap U_j \to \mathrm{GL}(m', \mathbb{R}/\mathbb{C})\}$ be transition functions of E and F. Then we could construct

(1) $E \oplus F$, with transition function $\Psi_{ij}^E \oplus \Psi_{ij}^F$,

(2) $E \otimes F$, with transition function $\Psi_{ij}^E \otimes \Psi_{ij}^F$,

(3) E^*, with transition function $\left(\left(\Psi_{ij}^E\right)^{-1}\right)^T$,

(4) $\Lambda^k E$, with transition function $\Lambda^k \Psi_{ij}^E$ as in (1.1.28).

We usually denote by $\Lambda^\bullet E := \oplus_{k=1}^m \Lambda^k E$.

Obviously, $T^*M = (TM)^*$.

Now we generalize (1.1.30) and (1.1.31) to the differential operator.

We begin by fixing notation. For an n-tuple of nonnegative integers $\alpha = (\alpha_1, \cdots, \alpha_n)$, we set $|\alpha| = \sum_{i=1}^n \alpha_i$, and for each $\xi \in \mathbb{R}^n$, we set $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. In local coordinates $(x_1, \cdots, x_n)$, we denote by $\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$.

Definition 1.1.14. Let E and F be two complex vector bundles over M with $\operatorname{rank} E = p$ and $\operatorname{rank} F = q$. A **differential operator** of order m on M is a linear map $P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, F)$ such that on each U_i,

$$P|_{U_i} = \sum_{|\alpha| \le m} A_\alpha^{(i)}(x) \frac{\partial^{|\alpha|}}{\partial x^{(i),\alpha}}, \tag{1.1.36}$$

where each $A_\alpha^{(i)}(x)$ is a $q \times p$-matrix of smooth functions and where $A_\alpha^{(i)} \neq 0$ for some α with $|\alpha| = m$.

By the knowledge of linear algebra, the complex matrix is easier to be handled than the real one. In the followings, we always assume that the differential operator acts on a complex vector bundle, except otherwise stated. For a real vector bundle, we first tensor it by $\mathbb{C}$, and then do the analysis.

We need to explain a bit about the right hand side of (1.1.36). For $s \in \mathscr{C}^\infty(M, E)$, on U_i, we could write $s|_{U_i} = \sum_{k=1}^p f_p s_p$, where $f_p \in \mathscr{C}^\infty(\phi(U_i) \subset \mathbb{R}^n, \mathbb{C})$ and $(s_1, \cdots, s_p)$ is a basis of $\mathbb{C}^p$. Consider a system of partial differential equations

$$\begin{cases} \sum_\alpha a_{11}^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_1 + \cdots \sum_\alpha a_{1p}^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_p = g_1, \\ \quad \cdots \quad \cdots \quad \cdots \\ \sum_\alpha a_{q1}^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_1 + \cdots \sum_\alpha a_{qp}^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_p = g_q, \end{cases} \tag{1.1.37}$$

where $g_i \in \mathscr{C}^\infty(\phi(U_i) \subset \mathbb{R}^n, \mathbb{C})$. Note any system of PDEs could be written in the form of (1.1.37). Let $A_\alpha = (a_{ij}^\alpha)_{q\times p}$. We write (1.1.37) in the sense of matrix:

$$\sum_\alpha A_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} \begin{pmatrix} f_1 \\ \vdots \\ f_p \end{pmatrix} = \begin{pmatrix} g_1 \\ \vdots \\ g_q \end{pmatrix}, \tag{1.1.38}$$

which explains the right hand side of (1.1.36).

Easy to see that $d : \mathscr{C}^\infty(M, \Lambda^k T^*M \otimes \mathbb{C}) \to \mathscr{C}^\infty(M, \Lambda^{k+1} T^*M \otimes \mathbb{C})$ is a differential operator.

1.1.6 How to take derivative on the section of a vector bundle?

In Section 1.1.4, we define the exterior differential on a smooth function on M. In Definition 1.1.14, we see that it is a differential operator. If there is a vector bundle E over the manifold, the section is the natural generalization of the function. Is it possible to generalize the exterior differential d on the sections?

We repeat the process in (1.1.16) and (1.1.17) to see what is the same and what is not.

Let $s_k^{(i)} \in \mathscr{C}^\infty(U_i, \mathbb{C}^n)$ be the constant function on U_i such that for any $x \in U_i$, $s_k^{(i)}(x) = e_k$, $1 \le k \le m$. For any $s \in \mathscr{C}^\infty(M, E)$, there exist $f_k \in \mathscr{C}^\infty(U_i, \mathbb{C})$, $1 \le k \le m$, such that

$$s|_{U_i} = \sum_{k=1}^m f_k^{(i)} s_k^{(i)}. \tag{1.1.39}$$

From the definition of the vector bundle, on $U_i \cap U_j$,

$$(f_1^{(i)}, \cdots, f_m^{(i)}) = (f_1^{(j)}, \cdots, f_m^{(j)}) \cdot \Psi_{ij}(x). \tag{1.1.40}$$

Thus

$$\begin{pmatrix} s_1^{(j)} \\ \vdots \\ s_m^{(j)} \end{pmatrix} \sim \Psi_{ij}(x) \begin{pmatrix} s_1^{(i)} \\ \vdots \\ s_m^{(i)} \end{pmatrix}. \tag{1.1.41}$$

Naively, we try to define the exterior differential by $ds|_{U_i} = \sum_{k=1}^m (df_k^{(i)}) s_k^{(i)}$. The same problem appear.

$$\sum_{k=1}^m (df_k^{(i)}) s_k^{(i)} = (df_1^{(i)}, \cdots, df_m^{(i)}) \begin{pmatrix} s_1^{(i)} \\ \vdots \\ s_m^{(i)} \end{pmatrix}$$
$$\sim (df_1^{(j)}, \cdots, df_m^{(j)}) \begin{pmatrix} s_1^{(j)} \\ \vdots \\ s_m^{(j)} \end{pmatrix} + (f_1^{(j)}, \cdots, f_m^{(j)})(d\Psi_{ij}(x))\Psi_{ij}(x)^{-1} \begin{pmatrix} s_1^{(j)} \\ \vdots \\ s_m^{(j)} \end{pmatrix}. \tag{1.1.42}$$

The annoying term is $(d\Psi_{ij}(x))\Psi_{ij}(x)^{-1}$, which is a matrix of 1-form. But in this case, we don't have any idea kill it. Thus the exterior differential

$$d \quad \text{in } U_i \text{ corresponds to} \quad d + (d\Psi_{ij}(x))\Psi_{ij}(x)^{-1} \quad \text{in } U_j. \tag{1.1.43}$$

Remark that the matrix of 1-form $(d\Psi_{ij}(x))\Psi_{ij}(x)^{-1}$ acts on $s|_{U_j} = \sum_{k=1}^m (df_k^{(j)}) s_k^{(j)}$ by

$$(f_1^{(j)}, \cdots, f_m^{(j)})(d\Psi_{ij}(x))\Psi_{ij}(x)^{-1} \begin{pmatrix} s_1^{(j)} \\ \vdots \\ s_m^{(j)} \end{pmatrix}. \tag{1.1.44}$$

We will handle it by the same method as the vector field. We glue the exterior differentials by the transformation (1.1.43) together and give it a name: connection.

Definition 1.1.15. A (affine) connection ∇^E on E is a collection of $d + A^{(i)}$, where $A^{(i)}$ is a matrix of 1-forms, on each U_i such that on $U_i \cap U_j$,

$$A^{(j)} = \Psi_{ij}(x) A^{(i)} \Psi_{ij}(x)^{-1} + (d\Psi_{ij}(x))\Psi_{ij}(x)^{-1}. \tag{1.1.45}$$

It is easy to see that ∇^E is a map

$$\nabla^E : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, T^*M \otimes E). \tag{1.1.46}$$

(Why? try to fix it.)

Recall that T^*M are the dual bundle of TM. Then for a vector field X and a 1-form α, we can define $\alpha(X) \in \mathscr{C}^\infty(M, \mathbb{R})$. In fact we've already used it in Proposition 1.1.11. Now we define

$$\iota_X : \mathscr{C}^\infty(M, T^*M) \to \mathscr{C}^\infty(M, \mathbb{R}), \quad \iota_X(\alpha) := \alpha(X). \tag{1.1.47}$$

Then by Proposition 1.1.11, we have

$$\iota_X \circ d = X : \mathscr{C}^\infty(M,\mathbb{R}) \to \mathscr{C}^\infty(M,\mathbb{R}). \tag{1.1.48}$$

Note that we can naturally define ι_X on $\mathscr{C}^\infty(M, T^*M \otimes E)$ by acting only on the T^*M part. Then we could define

$$\nabla^E_X := \iota_X \circ \nabla^E : \mathscr{C}^\infty(M,E) \to \mathscr{C}^\infty(M,E). \tag{1.1.49}$$

The operator ∇^E_X is usually regarded as taking the partial derivative on a section of a vector bundle along the direction X.

Exercise: Please check that ∇^E and ∇^E_X are all differential operators.

Proposition 1.1.16. *(1) For any $s_1, s_2 \in \mathscr{C}^\infty(M,E)$, we have*

$$\nabla^E(s_1+s_2) = \nabla^E s_1 + \nabla^E s_2. \tag{1.1.50}$$

(2) For any $s \in \mathscr{C}^\infty(M,E)$ and $f \in \mathscr{C}^\infty(M,\mathbb{C})$, we have

$$\nabla^E(fs) = (df)s + f\nabla^E s. \tag{1.1.51}$$

Proof. By definition. □

In many books, Proposition 1.1.16 is taken as the definition of the connection.

Remark that not like the exterior differential, the connection on the vector bundle is not uniquely defined. From Definition 1.1.15, locally, the difference of two connections is a matrix of 1-forms. Globally, the difference of two connections is a section of $\mathscr{C}^\infty(M, T^*M \otimes \mathrm{End}(E))$, where $\mathrm{End}(E) = E \otimes E^*$.

For a connection ∇^E on E and any $k \in \mathbb{N}$, there exists a unique extension $\nabla^E : \mathscr{C}^\infty(M, \Lambda^k T^*M \otimes E) \to \mathscr{C}^\infty(M, \Lambda^{k+1} T^*M \otimes E)$ verifying the Leibniz rule: for $\alpha \in \mathscr{C}^\infty(M, \Lambda^q T^*M)$, $s \in \mathscr{C}^\infty(M, \Lambda^{k-q} T^*M \otimes E)$, we have

$$\nabla^E(\alpha \wedge s) = d\alpha \wedge s + (-1)^q \alpha \wedge \nabla^E s. \tag{1.1.52}$$

1.2 Sobolev space

1.2.1 Integral, metric and partition of unity

The main purpose of this note is to study the differential operator. In Definition 1.1.14, a differential operator is a linear map

$$P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, F). \tag{1.2.1}$$

Once we see the linear map, the want to use the linear algebra, the matrix theory to study it. Unfortunately, in general, $\mathscr{C}^\infty(M, E)$ and $\mathscr{C}^\infty(M, F)$ are infinite dimensional vector space. The tool of studying the infinite dimensional vector spaces is the functional analysis. So naturally we plan to use the functional analysis to study the differential operator. In order to use the functional analysis, we firstly need to define an inner product on $\mathscr{C}^\infty(M, E)$ and extend it to the Hilbert space. After all, the theory of functional analysis we know for the undergraduates are based on the Hilbert space.

How to define a Hermitian product on $\mathscr{C}^\infty(M, E)$?

We first study it for $M = \mathbb{R}^n$, $E = \mathbb{C}$. For $f, g \in \mathscr{C}^0(\mathbb{R}^n, \mathbb{C})$, the classical Hermitian product is defined by

$$\langle f, g\rangle := \int_{\mathbb{R}^n} f(x) \cdot \overline{g(x)} dv_x. \tag{1.2.2}$$

Note that the right hand side of (1.2.2) might be infinity. We denote by

$$\|f\|^2_{L^2} := \langle f, f\rangle. \tag{1.2.3}$$

Let $L^2(\mathbb{R}^n)$ be the completion of the set $\{f \in \mathscr{C}^0(\mathbb{R}^n, \mathbb{C}) : \|f\|_{L^2} < +\infty\}$ with respect to the norm $\| \cdot \|_{L^2}$. It is a Hilbert space.

Let the support of f, $\mathrm{supp}(f)$, be the closure of

$$\{x \in \mathbb{R}^n : f(x) \neq 0\}. \tag{1.2.4}$$

We denote by $\mathscr{C}^\infty_0(M, \mathbb{C})$ be the set of smooth functions with compact support. It is easy to see that $\mathscr{C}^\infty_0(M, \mathbb{C}) \subset L^2(\mathbb{R}^n)$. Moreover, we all know that the completion of $\mathscr{C}^\infty_0(M, \mathbb{C})$ with respect to the norm $\| \cdot \|_{L^2}$ is $L^2(\mathbb{R}^n)$. Similarly, we denote by $\mathscr{C}^\infty_0(M, E)$ be the set of smooth sections with compact support.

The next step is to define the Hermitian product on $\mathscr{C}^\infty(M, \mathbb{C})$ and $L^2(M)$. Naturally, we want to follow the definition in (1.2.2). The problem is

How to define the integration on a manifold?

As in (1.2.4), for $f \in \mathscr{C}^\infty(M, \mathbb{C})$, we define

$$\operatorname{supp}(f) := \overline{\{x \in M : f(x) \neq 0\}}. \tag{1.2.5}$$

For a chart U_j of M, if $\operatorname{supp}(f) \subset U_j$, naturally, we can define

$$\int_M f dv := \int_{\mathbb{R}^n} f \circ \phi_j^{-1}(x) dv_x^{(j)}. \tag{1.2.6}$$

As usual, we need to check it for another chart. If $\operatorname{supp}(f) \subset U_i \cap U_j$, in chart U_i, by coordinate transformation formula,

$$\begin{aligned} \int_{\mathbb{R}^n} f \circ \phi_i^{-1}(x) dv_x^{(i)} &= \int_{\mathbb{R}^n} f \circ \phi_i^{-1}(\phi_{ij}(x)) |\det(\Phi_{ij})| dv_x^{(j)} \\ &= \int_{\mathbb{R}^n} f \circ \phi_j^{-1}(x) |\det(\Phi_{ij})| dv_x^{(j)}. \end{aligned} \tag{1.2.7}$$

The annoying term $|\det(\Phi_{ij})|$ prevents us defining the integral over a manifold in a natural way. We overcome it with the same method as the vector field: glue them by a transformation relation together to get a vector bundle, then take the section to do things.

Definition 1.2.1. We define the density bundle $|\Lambda|$ over M by the transition function $\Psi_{ij} = |\det(\Phi_{ij})|^{-1}$. It is a 1-dimensional real vector bundle.

Then the integral over M could be defined as a linear form

$$\int_M : \mathscr{C}_0^\infty(M, |\Lambda|) \to \mathbb{R}. \tag{1.2.8}$$

This idea is reasonable. But it is a little abstract. So we'll not go this way.

From (1.1.29), if all $\det(\Phi_{ij}) > 0$, we see that $|\Lambda| = \Lambda^n T^* M$. We are familiar with $\Lambda^n T^* M$. So we are shamed to assume in this note that all $\det(\Phi_{ij}) > 0$. Now we give it a name.

Definition 1.2.2. We say M is oriented if there exists an atlas such that for any $U_i \cap U_j \neq \emptyset$, $\det(\Phi_{ij}) > 0$.

In this note, we always assume that M is oriented.

From this point of view, we see that the integral is more natural defined on n-forms than the smooth functions.

For $\alpha \in \mathscr{C}^\infty(M, \Lambda^n T^* M)$ such that $\operatorname{supp}(\alpha) \subset U_i$, from the argument above, if on U_i, $\alpha = f \cdot dx_1^{(i)} \wedge \cdots \wedge dx_n^{(i)}$, we see that the definition

$$\int_M \alpha := \int_{\mathbb{R}^n} f \cdot dx_1^{(i)} \cdots dx_n^{(i)} \tag{1.2.9}$$

is meaningful.

As in (1.1.24), in multivariable calculus, the term $dx_1 \cdots dx_n$ could be explained as then n-form $dx_1 \wedge \cdots \wedge dx_n$.

Now we want to integrate the n-form on the whole manifold. We introduce the trick of partition of unity.

Partition of unity

In the definition of manifold, we assume that M is second countable. From the knowledge of the topology, it implies that M is paracompact, that is, each open covering of M admits a locally finite refinement. Thus in the followings, we always assume that our covering of the atlas is locally finite, i.e., each point only lives in finite charts.

Theorem 1.2.3 (Partition of unity). *There exists a family of smooth functions* $\{\varphi_i\}$ *such that* $\mathrm{supp}(\varphi_i) \subset U_i$ *and*

$$\sum_i \varphi_i(x) = 1. \tag{1.2.10}$$

Remark that since we assume that $\{U_i\}$ is locally finite, for each $x \in M$, the sum in (1.2.10) is a finite sum.

Proof. We could choose an open covering $\{V_i\}$ such that $\overline{V_i} \subset U_i$. Then we could construct functions $g_i \in \mathscr{C}^\infty(M, \mathbb{R})$ such that $V_i \subset \mathrm{supp}(g_i) \subset U_i$. Thus $g(x) := \sum_i g_i(x) > 0$ for any $x \in M$. Then we could take $\varphi_i := g_i/g$. □

For a n-form α, we have $\mathrm{supp}(\varphi_i \cdot \alpha) \subset U_i$. Thus from (1.2.9), we could define

$$\int_M \alpha = \int_M (\sum_i \varphi_i(x))\alpha = \sum_i \int_M \varphi_i \cdot \alpha. \tag{1.2.11}$$

Note that our functions of partition of unity are not unique. We need to check our definition in (1.2.11) does not depend on the choice of the partition of unity. It is left to the reader.

Until now, we obtain the definition of the integration of a n-form. The definition is naturally extended to the integration of any differential form by taking $\int_M \beta = 0$ for any $\beta \in \mathscr{C}^\infty(M, \Lambda^k T^*M)$ for $k < n$.

Since we want to extend (1.2.2) to the manifold, we also need to define the integration of a function.

From the construction of $\Lambda^n T^*M$, there exists nowhere vanishing n-form (not unique) on M. Such a nowhere vanishing n-form is called a volume form, usually denoted by dv_x.

Remark that the existence of the nowhere vanishing n-form implies that $\Lambda^n T^*M$ is a 1-dimensional trivial vector bundle (Since M is oriented).

After taking a volume form, we could define the integration of a function f by taking the integration of $f\,dv_x \in \mathscr{C}^\infty(M, \Lambda^n T^*M)$.

Now for $f, g \in \mathscr{C}^\infty(M, \mathbb{C})$, the classical Hermitian product is defined by

$$\langle f, g\rangle := \int_M f(x) \cdot \overline{g(x)} dv_x. \tag{1.2.12}$$

We denote the norm by

$$\|f\|_{L^2}^2 := \langle f, f\rangle. \tag{1.2.13}$$

Let $L^2(M)$ be the completion of the set $\{f \in \mathscr{C}^\infty(M, \mathbb{C}) : \|f\|_{L^2} < +\infty\}$ with respect to the norm $\|\cdot\|_{L^2}$. It is also a Hilbert space.

The next question is how to do these things for sections of a vector bundle?

The key point is how to do $f(x) \cdot \overline{g(x)}$ for sections.

For vector bundle, the fiber is a vector space. Let $\pi : E \to M$ be the projection. For each $x \in M$, $E_x := \pi^{-1}(x)$ is a complex vector space and $f(x), g(x) \in E_x$ are vectors. From the knowledge of linear algebra, if there is a Hermitian inner product $\langle \cdot, \cdot\rangle_x$ on E_x, we could replace $f(x) \cdot \overline{g(x)}$ by $\langle f(x), g(x)\rangle_x$. We also need the inner product $\langle \cdot, \cdot\rangle_x$ depends smoothly on x. Usually, we also denote by

$$h_x^E(f(x), g(x)) := \langle f(x), g(x)\rangle_x. \tag{1.2.14}$$

Note that $h_x^E(\cdot, \cdot)$ is linear on the first variable and conjugate linear on the second one. Such map is called the sesquilinear map.

Definition 1.2.4. The Hermitian metric is a smooth family $\{h_x^E\}_{x\in M}$ of sesquilinear maps $h_x^E : E_x \times E_x \to \mathbb{C}$ such that $h_x^E(\xi, \xi) > 0$ for any $\xi \in E_x \backslash \{0\}$.

For the real vector bundle F, the corresponding metric is usually called the Euclidean metric.

Definition 1.2.5. The Euclidean metric is a smooth family $\{g_x^F\}_{x\in M}$ of bilinear maps $g_x^F : F_x \times F_x \to \mathbb{R}$ such that $g_x^F(\xi, \xi) > 0$ for any $\xi \in F_x \backslash \{0\}$.

Proposition 1.2.6. *There always exist Hermitian metrics on E.*

Proof. For any $U_i \times \mathbb{C}^m$, we could easily construct a smooth family of Hermitian products h_i^E on each fiber, e.g., taking the classical Hermitian product on $\mathbb{C}^m$. Let $\{\varphi_i\}$ be a partition of unity with respect to $\{U_i\}$. Then $\sum_i \varphi_i h_i^E$ is a Hermitian metric. □

Similarly, there always exist Euclidean metric on real vector bundles.

Definition 1.2.7. A Euclidean metric on TM is called a Riemannian metric. A manifold with a Riemannian metric is called a Riemannian manifold.

Remark that the Hermitian (Euclidean) metric is far from unique.

Let h^E be a Hermitian metric on E. Now for $f, g \in \mathscr{C}^\infty(M, E)$, the Hermitian product is defined by

$$\langle f, g \rangle := \int_M h_x^E(f(x), g(x)) dv_x. \tag{1.2.15}$$

We denote the norm by

$$\|f\|_{L^2}^2 := \langle f, f \rangle. \tag{1.2.16}$$

Let $L^2(M, E)$ be the completion of the set $\{f \in \mathscr{C}^\infty(M, E) : \|f\|_{L^2} < +\infty\}$ with respect to the norm $\|\cdot\|_{L^2}$. It is also a Hilbert space. Similarly, we could denote the set of sections with compact support by $\mathscr{C}_0^\infty(M, E)$ and $\mathscr{C}_0^\infty(M, E)$ is dense in $L^2(M, E)$ with respect to the norm $\|\cdot\|_{L^2}$.

Once we extend the set of sections to a Hilbert space, naively, we want to extend the differential operator to

$$P : L^2(M, E) \to L^2(M, F) \tag{1.2.17}$$

If P is bounded, we could use a whole theory of functional analysis we learned to study the differential operator.

Unfortunately, the world is not as good as we think.

We will see this from the easiest differential operator: The derivative $\frac{d}{dt}$ on $\mathscr{C}_0^1(\mathbb{R})$.

Proposition 1.2.8. *The derivative $\frac{d}{dt}$ is unbounded with respect to the norm $\|\cdot\|_{L^2}$.*

Proof. Let $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$ such that $\|\varphi\|_{L^2} = 1$. Let $\|\frac{d\varphi}{dt}\|_{L^2} = C$. Then

$$\|n^{1/2}\varphi(nt)\|_{L^2}^2 = \int_{-\infty}^{+\infty} n\varphi^2(nt)dt = \int_{-\infty}^{+\infty} \varphi^2(t)dt = 1. \tag{1.2.18}$$

But

$$\begin{aligned} \left\| n^{1/2} \frac{d}{dt}\varphi(nt) \right\|_{L^2}^2 &= \int_{-\infty}^{+\infty} n^3 \left(\frac{d\varphi}{dt}(nt) \right)^2 dt \\ &= \int_{-\infty}^{+\infty} n^2 \left(\frac{d\varphi}{dt}(t) \right)^2 dt = n^2 C. \end{aligned} \tag{1.2.19}$$

Thus $\frac{d}{dt}$ is unbounded. □

Definition 1.2.9. Let W, W' be two Banach spaces. Let $P : W \to W'$ be a linear operator with domain $D(P)$. We say P is a closed operator if for $x_n \in D(P)$, $x_n \to x$, $Px_n \to y$, we have $x \in D(P)$ and $y = Px$.

Proposition 1.2.10. *The derivative* $\frac{d}{dt}$ *is a closed operator.*

Proof. The domain of $\frac{d}{dt}$ is $\mathscr{C}_0^1(\mathbb{R})$. We take $x_n(t) \in \mathscr{C}_0^1(\mathbb{R})$ and $x_n(t) \to x(t) \in L^2(\mathbb{R})$, $\frac{dx_n(t)}{dt} \to y(t) \in L^2(\mathbb{R})$, then we have $x_n(t) \to \int_{-\infty}^t y(s)ds$. Thus $x(t) = \int_{-\infty}^t y(s)ds$. So $x(t) \in \mathscr{C}_0^1(\mathbb{R})$ and $\frac{dx(t)}{dt} = y(t)$. □

Theorem 1.2.11 (Closed graph theorem)**.** *Let* W, W' *be two Banach spaces. Let* $P : W \to W'$ *be a closed operator with domain* $D(P)$. *If* $D(P)$ *is closed, then* P *is bounded.*

This is my most hate theorem. It prevents us to extend the domain of the derivative operator to the whole $L^2(\mathbb{R})$. (If we could extend, closed graph theorem implies $\frac{d}{dt}$ is bounded, which is a contradiction with Proposition 1.2.8).

For a large class of differential operator, we will meet the same obstruction coming from the functional analysis.

In the history of the differential operator, there are two ways to overcome this obstruction, any of them is not easy:

(1) reduce the Hilbert spaces to smaller Banach spaces, called the Sobolev spaces, such that the differential operator is bounded on these Hilbert spaces with respect to the new norms, which is the main purpose of this section;

(2) define the differential operator on a dense subset of the Hilbert space, which is the main idea of our next chapter.

1.2.2 Sobolev space

In 19th century, Gauss studied the electrostatic field and posed a famous problem, called the Dirichlet problem, that for a domain $\Omega \subset \mathbb{R}^2$, find a solution $u(x, y) \in \mathscr{C}^2(\Omega) \cap \mathscr{C}^0(\overline{\Omega})$ of

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u|_{\partial\Omega} = f, & f \in \mathscr{C}^0(\partial\Omega). \end{cases} \tag{1.2.20}$$

Later, Riemann discussed this problem and stated the Dirichlet Principle. For

$$\begin{aligned} &I(u) = \int_\Omega |\nabla u|^2 dxdy, \\ &u \in A = \{u \in \mathscr{C}^1(\Omega) : u_x, u_y \in L^2, u|_{\partial\Omega} = f\}, \end{aligned} \tag{1.2.21}$$

$I(u) \geq 0$, thus $\inf_A I(u)$ exists. Riemann said there exists $u_0 \in A$ such that $u_0 = \inf_A I(u)$. Then u_0 is the solution of (1.2.20).

In 1870, Weierstrass posed a counter example to explain that $\min_A I(u)$ may not exist. Sometimes, we cannot take $u_0 \in A$ such that $u_0 = \inf_A I(u)$. Later, people recognized that even $\min_A I(u)$ exists, it may not have the enough regularity.

In 1900, Hilbert confirmed the Dirichlet Principle for the smooth boundary. In his point of view, this is a very important problem, so that he posed three problems about this in his famous 23 problems.

Later, Sobolev stated a strategy to handle this problem. Firstly, we complete A into a complete space $\overline{A}$ with respect to some norm. In the complete space $\overline{A}$, obviously $\min_{\overline{A}} = \inf_{\overline{A}}$. Thus we could get a minimum element $u_0 \in \overline{A}$. Then we could use other method to study the regularity of it. This complete space is called the Sobolev space.

As usual, we first discuss the Sobolev space on $\mathbb{R}^n$. Since our main purpose is to study the differential operator on manifold without boundary, we will not discuss the boundary condition.

Let $\mathscr{S}$ be the set of $\mathbb{C}^m$-valued smooth function u on $\mathbb{R}^n$ such that for any n-tuple α and $k \in \mathbb{N}$, there exists $C_{\alpha,k} > 0$, such that

$$\left|(1+|x|)^k \frac{\partial^{|\alpha|}u}{\partial x^\alpha}(x)\right| \leq C_{\alpha,k}. \tag{1.2.22}$$

It is called the Schwartz space or space of rapidly decreasing functions.

For $u \in \mathscr{S}$, we define a norm $\|u\|_k$ on this space by

$$\|u\|_k^2 = \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} \left|\frac{\partial^{|\alpha|}u}{\partial x^\alpha}(x)\right|^2 dx. \tag{1.2.23}$$

In this part, we assume that all functions are $\mathbb{C}^m$-valued.

Definition 1.2.12. The completion of $\mathscr{S}$ relative to the norm $\|\cdot\|_k$ is the Sobolev space $\mathbf{H}^k$.

In order to do more things, we recall and summarize the knowledge of Fourier analysis on $\mathbb{R}^n$.

Let $u \in L^1(\mathbb{R}^n)$. The Fourier transform $\hat{u}$ of u is defined by

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} u(x)dx. \tag{1.2.24}$$

We denote by

$$D^\alpha = i^{-|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha}. \tag{1.2.25}$$

Proposition 1.2.13. *For $u \in \mathscr{S}$, we have*

$$\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi), \tag{1.2.26}$$

$$\widehat{x^\alpha u}(\xi) = (-1)^{|\alpha|} D_\xi^\alpha \hat{u}(\xi), \tag{1.2.27}$$

and the Plancherel's formula

$$(u, v)_{L^2} = (\hat{u}, \hat{v})_{L^2}. \tag{1.2.28}$$

Recall that the convolution product $*$ is defined by

$$u * v(x) = \int_{\mathbb{R}^n} u(x-y)v(y)dy, \tag{1.2.29}$$

for any $u, v \in \mathscr{S}$.

Proposition 1.2.14. *For $u, v \in \mathscr{S}$, we have*

$$\widehat{u \cdot v} = \hat{u} * \hat{v}, \quad \widehat{u * v} = \hat{u} \cdot \hat{v}. \tag{1.2.30}$$

Proposition 1.2.15. *The Fourier transform defines an isomorphism $:\mathscr{S} \to \mathscr{S}$. The inverse is given by*

$$u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} \hat{u}(\xi) d\xi. \tag{1.2.31}$$

From (1.2.23), (1.2.26) and (1.2.28), we have

$$\begin{aligned} \|u\|_k^2 = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^2}^2 = \sum_{|\alpha| \le k} \|\widehat{D^\alpha u}\|_{L^2}^2 = \sum_{|\alpha| \le k} \|\xi^\alpha \hat{u}\|_{L^2}^2 \\ = \int_{\mathbb{R}^n} \left(\sum_{|\alpha| \le k} \xi^{2\alpha} \right) |\hat{u}(\xi)|^2 d\xi. \end{aligned} \tag{1.2.32}$$

Since there exist $c_1, c_2 > 0$ such that

$$c_1(1+|\xi|)^{2k} \le \sum_{|\alpha| \le k} \xi^{2\alpha} \le c_2(1+|\xi|)^{2k}, \tag{1.2.33}$$

$\| \cdot \|_k$ is equivalent to the norm $\|u\|_k'^2$ given by

$$\|u\|_k'^2 = \int_{\mathbb{R}^n} (1+|\xi|)^{2k} |\hat{u}(\xi)|^2 d\xi. \tag{1.2.34}$$

In some literatures, the weight part is $(1+|\xi|^2)^k$. The norm defined by this weight is also equivalent to $\|\cdot\|_k$.

In Functional analysis, the Hilbert spaces with the equivalent norms are topological isomorphism. Thus they could be regarded as the same space.

Following this way, we could define the Sobolev norm and the Sobolev space of any order s, $s \in \mathbb{R}$.

Definition 1.2.16. For $s \in \mathbb{R}$ and $u \in \mathscr{S}$, we define the s-th Sobolev norm $\|\cdot\|_s$ by

$$\|u\|_s^2 = \int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\hat{u}(\xi)|^2 d\xi. \tag{1.2.35}$$

The completion of $\mathscr{S}$ with respect to this norm is the Sobolev space $\mathbf{H}^s$.

For $k \in \mathbb{N}$, the uniform $\mathscr{C}^k$-norm of $u \in \mathscr{C}^k$ is defined by

$$\|u\|_{\mathscr{C}^k}^2 := \sup_{\mathbb{R}^n} \sum_{|\alpha|\le k} |D^\alpha u|^2. \tag{1.2.36}$$

It is well-known that this norm is complete on any bounded domain.

Theorem 1.2.17 (Sobolev Embedding Theorem)**.** *For each $k \in \mathbb{N}$, $s > \frac{n}{2}+k$, $s \in \mathbb{R}$, there exists $C_s > 0$ such that for any $u \in \mathscr{S}$,*

$$\|u\|_{\mathscr{C}^k} \le C_s \|u\|_s. \tag{1.2.37}$$

Thus there exists a continuous embedding

$$\mathbf{H}^s \subset \mathscr{C}^k. \tag{1.2.38}$$

Proof. From (1.2.31) and (1.2.26), for $|\alpha| < s - \frac{n}{2}$,

$$\begin{aligned} |D^\alpha u| &= (2\pi)^{-n/2} \left| \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \widehat{D^\alpha u}(\xi) d\xi \right| \le (2\pi)^{-n/2} \int_{\mathbb{R}^n} |\xi^\alpha| |\hat{u}(\xi)| d\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} (1+|\xi|)^{-(s-|\alpha|)} (1+|\xi|)^{(s-|\alpha|)} |\xi^\alpha| |\hat{u}(\xi)| d\xi \\ &\le (2\pi)^{-n/2} \left(\int_{\mathbb{R}^n} (1+|\xi|)^{-2(s-|\alpha|)} d\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{1/2}. \end{aligned} \tag{1.2.39}$$

Since $s - |\alpha| > \frac{n}{2}$, $K_{\alpha,s} := \int_{\mathbb{R}^n} (1+|\xi|)^{-2(s-|\alpha|)} d\xi$ is finite. Thus we have

$$|D^\alpha u|^2 \le (2\pi)^{-n/2} K_{\alpha,s}^{1/2} \|u\|_s^2. \tag{1.2.40}$$

Let $C_s = (2\pi)^{-n/2}\sum_{|\alpha|<s-n/2} K_{\alpha,s}^{1/2}$, we get (1.2.37).

For any $u \in \mathbf{H}^s$, there exists a series $u_j \in \mathscr{S}$, such that $u_j \to u$ under the norm $\|\cdot\|_s$. Thus for any $\varepsilon > 0$, there exists $N > 0$ such that for any $j, l > N$, $\|u_j - u_l\|_s \le \varepsilon/C_s$. By (1.2.37), we have $\|u_j - u_l\|_{\mathscr{C}^k} \le \varepsilon$. Since $\mathscr{C}^k$ is complete on a bounded domain, there exists $u' \in \mathscr{C}^k$, such that $u_j \to u'$ under the norm $\|\cdot\|_{\mathscr{C}^k}$. It is easy to see that u' does not depend on the choice of the Cauchy sequence. We identify $u' \in \mathscr{C}^k$ with $u \in \mathbf{H}^s$ to get a continuous map $i : \mathbf{H}^s \to \mathscr{C}^k$. This is also an embedding. In fact, if $i(v) = 0$, taking $u_n \in \mathscr{C}_0^\infty$ such that $\|u_n - v\|_s \to 0$, then $\|u_n\|_{\mathscr{C}^k} = \|i(u_n)\|_{\mathscr{C}^k} = \|i(u_n - v)\|_{\mathscr{C}^k} \le C\|u_n - v\|_s \to 0$. So $v = 0$.

The proof of Theorem 1.2.17 is completed. □

From (1.2.35), if $s < t$, we have

$$\|u\|_s < \|u\|_t. \tag{1.2.41}$$

Thus we have the continuous embedding

$$\mathbf{H}^t \subset \mathbf{H}^s, \quad s < t. \tag{1.2.42}$$

Moreover, by Sobolev embedding theorem, for $0 < s_1 < \cdots < s_k < \cdots$,

$$\mathscr{C}_0^\infty \subset \mathscr{S} \subset \cdots \mathbf{H}^{s_k} \subset \cdots \mathbf{H}^{s_1} \subset L^2. \tag{1.2.43}$$

Theorem 1.2.18. *For $s_2 > s > s_1$, then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that for any $u \in \mathbf{H}^{s_2}$, we have*

$$\|u\|_s^2 \le \varepsilon\|u\|_{s_2}^2 + C_\varepsilon\|u\|_{s_1}^2. \tag{1.2.44}$$

Proof. By (1.2.42), $\mathbf{H}^{s_2} \subset \mathbf{H}^s \subset \mathbf{H}^{s_1}$. Thus $u \in \mathbf{H}^s$ and $u \in \mathbf{H}^{s_1}$. From Definition 1.2.16, we only need to prove for any $\xi \in \mathbb{R}^n$, we have

$$(1 + |\xi|)^{2s} \le \varepsilon(1 + |\xi|)^{2s_2} + C_\varepsilon(1 + |\xi|)^{2s_1}. \tag{1.2.45}$$

Let $\rho = (1 + |\xi|)^2 > 0$. We need to prove $\rho^s \le \varepsilon\rho^{s_2} + C_\varepsilon\rho^{s_1}$. Set $\lambda = \varepsilon^{1/(s_2-s)}$, $C_\varepsilon = \lambda^{-(s-s_1)}$. It is equivalent to $(\lambda\rho)^{s_2-s} + (\lambda\rho)^{s-s_1} \ge 1$. This inequality holds because if $\lambda\rho \ge 1$, the first part ≥ 1, if $\lambda\rho < 1$, the second part < 1.

The proof of Theorem 1.2.18 is completed. □

From Theorems 1.2.17 and 1.2.18, we have the following corollary.

Corollary 1.2.19. *For $k \in \mathbb{N}$, $s > \frac{n}{2} + k$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that for any $u \in \mathbf{H}^s$, we have*

$$\|u\|_{\mathscr{C}^k} \le \varepsilon\|u\|_s^2 + C_\varepsilon\|u\|_0^2. \tag{1.2.46}$$

Theorem 1.2.20 (Rellich Lemma). *Let $\{u_j\}$ be a sequence of functions with* $\mathrm{supp}(u_j) \subset B_1(0)$ *and there exists constant $C > 0$, such that $\|u_j\|_t \leq C$. Then for any $s < t$, there exists a Cauchy subsequence of $\{u_j\}$ with respect to the norm $\|\cdot\|_s$, thus converges in $\mathbf{H}^s$.*

Proof. Take a smooth function $\varphi \in \mathscr{C}_0^\infty$ such that $\varphi = 1$ on B^n. Thus $\varphi u_j = u_j$. By (1.2.30), $\hat{u}_j = \hat{\varphi} * \hat{u}_j$. So for any α,

$$D_\xi^\alpha \hat{u}_j(\xi) = \int_{\mathbb{R}^n} (D_\xi^\alpha \hat{\varphi})(\xi - \eta)\hat{u}_j(\eta) d\eta. \tag{1.2.47}$$

By the Cauchy-Schwarz inequality,

$$|D_\xi^\alpha \hat{u}_j(\xi)|^2 \leq \int_{\mathbb{R}^n} (1 + |\eta|)^{-2t} |D_\xi^\alpha \hat{\varphi}|^2(\xi - \eta) d\eta \times \int_{\mathbb{R}^n} (1 + |\eta|)^{2t} |\hat{u}_j(\eta)|^2 d\eta = C_\alpha(\xi)\|u_j\|_t^2, \tag{1.2.48}$$

where $C_\alpha(\xi) = \int_{\mathbb{R}^n} (1 + |\eta|)^{-2t} |D_\xi^\alpha \hat{\varphi}|^2(\xi - \eta) d\eta$ is finite, since $\hat{\varphi} \in \mathscr{S}$, and does not depend on u.

By (1.2.48), for any α, $|D_\xi^\alpha \hat{u}_j(\xi)|$ is uniformly bounded on compact subset of $\mathbb{R}^n$. Thus $\{\hat{u}_j\}$ is uniformly equicontinuous on compact subsets. By Ascoli-Arzela Theorem, there is a subsequence of $\{\hat{u}_j\}$ which is uniformly Cauchy on compact subsets, which we also denote by $\{\hat{u}_j\}$.

Fix $r > 0$.

$$\|u_j - u_k\|_s^2 = \int_{|\xi| > r} (1 + |\xi|)^{2s} |\hat{u}_j(\xi) - \hat{u}_k(\xi)|^2 d\xi + \int_{|\xi| \leq r} (1 + |\xi|)^{2s} |\hat{u}_j(\xi) - \hat{u}_k(\xi)|^2 d\xi =: A + B. \tag{1.2.49}$$

Note that

$$A \leq (1 + r)^{2(s-t)} \int_{|\xi| > r} (1 + |\xi|)^{2t} |\hat{u}_j(\xi) - \hat{u}_j(\xi)|^2 d\xi \leq \frac{\|u_j - u_k\|_t^2}{(1 + r)^{2(t-s)}} \leq \frac{2C}{(1 + r)^{2(t-s)}}. \tag{1.2.50}$$

For any $\varepsilon > 0$, we take r large, such that $2C(1 + r)^{2(s-t)} < \varepsilon/2$. Note that

$$B \leq C' \sup_{|\xi| < r} |\hat{u}_j(\xi) - \hat{u}_k(\xi)|^2. \tag{1.2.51}$$

Since $\{\hat{u}_j\}$ is uniformly Cauchy, there exists $N > 0$ such that for $j, k > N$, $|\hat{u}_j(\xi) - \hat{u}_k(\xi)|^2 \leq \varepsilon/(2C')$. Thus for $j, k > N$, $\|u_j - u_k\|_s^2 < \varepsilon$.

The proof of Theorem 1.2.20 is completed. □

By Theorems 1.2.17, 1.2.20, we have the following corollary.

Corollary 1.2.21. *Let* $\{u_j\}$ *be a sequence of functions with* $\mathrm{supp}(u_j) \subset B_1(0)$ *and there exists constant* $C > 0$*, such that* $\|u_j\|_s \le C$*. If* $s > \frac{n}{2} + k$*, then there is a subsequence which converges to a function* $u \in \mathscr{C}_0^k$ *in the uniform* $\mathscr{C}^k$*-norm.*

The following theorem says that $\mathbf{H}^{-s} = (\mathbf{H}^s)^*$.

Theorem 1.2.22. *For* $u, v \in \mathscr{S}$*, the bilinear function*

$$(u, v) := \int_{\mathbb{R}^n} \hat{u}(\xi) \cdot \hat{v}(\xi) d\xi \tag{1.2.52}$$

has a continuous extension to $\mathbf{H}^s \times \mathbf{H}^{-s}$ *for any* $s \in \mathbb{R}$*. Moreover, it identifies* $\mathbf{H}^{-s}$ *with the dual of* $\mathbf{H}^s$*, that is,*

$$\|v\|_{-s} = \sup_{u \in \mathbf{H}^s} \frac{(u, v)}{\|u\|_s}. \tag{1.2.53}$$

Proof. For $u, v \in \mathscr{S}$, by the Schwarz inequality,

$$|(u, v)| = \left| \int_{\mathbb{R}^n} (1 + |\xi|)^s \hat{u}(\xi) \cdot (1 + |\xi|)^{-s} \hat{v}(\xi) d\xi \right| \le \|u\|_s \|v\|_{-s}. \tag{1.2.54}$$

From the argument below (1.2.40), we see that the bilinear function $(\cdot, \cdot)$ has a continuous extension to $\mathbf{H}^s \times \mathbf{H}^{-s}$ for any $s \in \mathbb{R}$.

We choose u such that $\hat{u}(\xi) = \overline{\hat{v}(\xi)}(1 + |\xi|)^{-2s}$. Then

$$\|u\|_s = \|v\|_{-s}, \quad (u, v) = \int_{\mathbb{R}^n} |\hat{v}(\xi)|^2 (1 + |\xi|)^{-2s} d\xi = \|v\|_{-s}^2. \tag{1.2.55}$$

Thus

$$\sup_{u \in \mathbf{H}^s} \frac{(u, v)}{\|u\|_s} \ge \|v\|_{-s}. \tag{1.2.56}$$

Then (1.2.53) follows from (1.2.54) and (1.2.56).

The proof of Theorem 1.2.22 is completed. □

Corollary 1.2.23. *Let* $T : \mathscr{S} \to \mathscr{S}$ *and* $T^* : \mathscr{S} \to \mathscr{S}$ *be linear maps such that* $(Tu, v) = (u, T^* v)$ *for any* $u, v \in \mathscr{S}$*. For* $s \in \mathbb{R}$*, if there exists* $C > 0$ *such that for any* $u \in \mathscr{S}$*,*

$$\|Tu\|_s \le C\|u\|_s, \tag{1.2.57}$$

we have

$$\|T^* u\|_{-s} \le C\|u\|_{-s}. \tag{1.2.58}$$

Proof. For $u, v \in \mathscr{S}$, by Theorem 1.2.22, we have

$$\|T^*v\|_{-s} = \sup_{u\in\mathbf{H}^s} \frac{(u, T^*v)}{\|u\|_s} = \sup_{u\in\mathbf{H}^s} \frac{(Tu, v)}{\|u\|_s} \leq \sup_{u\in\mathbf{H}^s} \frac{\|Tu\|_s\|v\|_{-s}}{\|u\|_s} \leq C\|v\|_{-s}. \quad (1.2.59)$$

The proof of Corollary 1.2.23 is completed. □

Proposition 1.2.24. *Let A be a smooth matrix valued function on $\mathbb{R}^n$ such that $|D^\alpha A|$ is bounded for any α. Then the map $T : \mathscr{S} \to \mathscr{S}$ defined by $Tu = Au$ extends to a bounded linear map $T : \mathbf{H}^s \to \mathbf{H}^s$ for $s \in \mathbb{Z}$.*

Proof. For $s \geq 0$, there exists $C > 0$, such that for any $u \in \mathscr{S}$,

$$\|Tu\|_s = \sum_{|\alpha|\leq s} \int_{\mathbb{R}^n} |D^\alpha(Au)|^2 dx \leq C \sum_{|\alpha|\leq s} \int_{\mathbb{R}^n} |D^\alpha u|^2 dx = C\|u\|_s. \quad (1.2.60)$$

By (1.2.28), we see that $T^*u = A^T u$. For $s < 0$, as in (1.2.60), we have $\|T^*u\|_{-s} \leq C\|u\|_{-s}$. Since $(T^*)^* = T$, by Corollary 1.2.23, we have $\|Tu\|_s \leq C\|u\|_s$.

The proof of Proposition 1.2.24 is completed. □

Now we want to establish Proposition 1.2.24 for $s \in \mathbb{R}$. We prove a lemma first.

Lemma 1.2.25 (Peetre's Inequality)**.** *For any $\xi, \eta \in \mathbb{R}^n$ and $s \in \mathbb{R}$, we have*

$$\left(\frac{1 + |\xi|}{1 + |\eta|}\right)^s \leq (1 + |\xi - \eta|)^{|s|}. \quad (1.2.61)$$

Proof. For $s \geq 0$, (1.2.61) follows from

$$(1 + |\xi|) \leq 1 + |\xi - \eta| + |\eta| \leq (1 + |\eta|)(1 + |\xi - \eta|). \quad (1.2.62)$$

For $s < 0$, we use the same argument reversing ξ and η and replace s by $-s$. □

For a smooth matrix valued function $A(x) = [a_{ij}(x)]$ on $\mathbb{R}^n$, we say $A \in \mathscr{S}$ if for any i, j, $a_{ij}(x) \in \mathscr{S}$. Then we could define $\widehat{A}(\xi) = [\widehat{a_{ij}}(\xi)]$. In this case, for $u \in \mathscr{S}$, letting $A * u = \int_{\mathbb{R}^n} A(x - y)u(y)dy$, by Proposition 1.2.14, we have

$$\widehat{Au} = \widehat{A} * \hat{u}, \quad \widehat{A * u} = \widehat{A}\,\hat{u}. \quad (1.2.63)$$

Theorem 1.2.26. *Let $A \in \mathscr{S}$ be a smooth matrix valued function. Then the map $T : \mathscr{S} \to \mathscr{S}$ defined by $Tu = Au$ extends to a bounded linear map $T : \mathbf{H}^s \to \mathbf{H}^s$ for $s \in \mathbb{R}$, i.e., there exists $C > 0$ such that for any $u \in \mathscr{S}$, we have*

$$\|Au\|_s \leq C\|u\|_s. \tag{1.2.64}$$

Proof. For any $u \in \mathscr{S}$, by (1.2.35) and (1.2.63), we have

$$\begin{aligned}
\|Au\|_s^2 &= \int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\widehat{Au}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^n} (1+|\xi|)^{2s} \left| \int_{\mathbb{R}^n} \widehat{A}(\xi-\eta)\hat{u}(\eta)d\eta \right|^2 d\xi \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{(1+|\xi|)^{2s}}{(1+|\eta|)^{2s}} |\widehat{A}(\xi-\eta)|^2 d\xi \right) |(1+|\eta|)^{2s} \hat{u}(\eta)|^2 d\eta.
\end{aligned} \tag{1.2.65}$$

From (1.2.22) and Lemma 1.2.25, for $k > |s| + \frac{n}{2}$, $k \in \mathbb{N}$, there exists $C_k > 0$, such that

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{(1+|\xi|)^{2s}}{(1+|\eta|)^{2s}} |\widehat{A}(\xi-\eta)|^2 d\xi &\leq C_k \int_{\mathbb{R}^n} (1+|\xi-\eta|)^{2|s|-2k} d\xi \\
&= C_k \int_{\mathbb{R}^n} (1+|\xi|)^{2|s|-2k} d\xi < +\infty.
\end{aligned} \tag{1.2.66}$$

Let $C = C_k \int_{\mathbb{R}^n} (1+|\xi|)^{2|s|-2k} d\xi$. By (1.2.65),

$$\|Au\|_s \leq C\|u\|_s. \tag{1.2.67}$$

The proof of Theorem 1.2.26 is completed. □

In order to define the Sobolev space on manifolds, we introduce another equivalent Sobolev norm for $s \in \mathbb{R}$, which is due to Hörmander.

Proposition 1.2.27. *For $0 < s < 1$, the s-th Sobolev norm is equivalent to the following norm for any $u \in \mathscr{S}$:*

$$\|u\|'_s := \left(\|u\|_{L^2}^2 + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}}. \tag{1.2.68}$$

Proof. By Newton-Leibniz's formula,

$$u(x) - u(y) = (x-y) \cdot \int_0^1 \nabla u(tx + (1-t)y)dt. \tag{1.2.69}$$

By (1.2.69), since $s < 1$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy \\
&\qquad\le C\sum_{|\alpha|=1}\cdot\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left(\int_0^1|(\nabla u)(tx+(1-t)y)|dt\right)^2}{|x-y|^{n+2s-2}}dxdy \\
&\qquad= C\sum_{|\alpha|=1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left(\int_0^1|(\nabla u)(tx+y)|dt\right)^2}{|x|^{n+2s-2}}dxdy \\
&\le C\sum_{|\alpha|=1}\int_{\mathbb{R}^n}\frac{1}{|x|^{n+2s-2}}\int_0^1\int_{\mathbb{R}^n}|(\nabla u)(tx+y)|^2dydtdx < +\infty. \qquad (1.2.70)
\end{aligned}$$

From (1.2.24), letting $u_x(y) := u(x+y)$, we have

$$\widehat{u_x}(\xi) = e^{i\langle x,\xi\rangle}\hat{u}(\xi). \qquad (1.2.71)$$

Thus from Plancherel's formula (1.2.28) and (1.2.71), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy = \int_{\mathbb{R}^n}\frac{1}{|x|^{n+2s}}\int_{\mathbb{R}^n}|u(x+y)-u(y)|^2dydx \\
&= \int_{\mathbb{R}^n}\frac{1}{|x|^{n+2s}}\int_{\mathbb{R}^n}|\widehat{u_x}(\xi)-\hat{u}(\xi)|^2dydx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|e^{i\langle x,\xi\rangle}-1|^2}{|x|^{n+2s}}|\hat{u}(\xi)|^2dxd\xi.
\end{aligned} \qquad (1.2.72)$$

By replacing ξ to $T\xi$, where T is an orthogonal rotation, we see that $\int_{\mathbb{R}^n}|e^{i\langle x,\xi\rangle}-1|^2|x|^{-n-2s}dx$ depends only on $|\xi|$. By replacing ξ to $a\xi$, $a\in\mathbb{R}$, we see that it is homogeneous of degree $2s$. Thus by (1.2.93),

$$\int_{\mathbb{R}^n}|e^{i\langle x,\xi\rangle}-1|^2|x|^{-n-2s}dx = C_s|\xi|^{2s}, \quad C_s > 0. \qquad (1.2.73)$$

Therefore, we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy = C_s\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{u}(\xi)|^2d\xi. \qquad (1.2.74)$$

Since there exist $c, C > 0$ such that $c(1+|\xi|^{2s}) \le (1+|\xi|)^{2s} \le C(1+|\xi|^{2s})$, we see the two norms are equivalent.

The proof of Proposition 1.2.27 is completed. □

Corollary 1.2.28. *For $s > 0$, $s = m + \sigma$ with $m \in \mathbb{N}$, $0 \le \sigma < 1$, the s-th Sobolev norm is equivalent to the following norm for any $u \in \mathscr{S}$:*

$$\|u\|'_s := \left(\sum_{|\alpha| \le m} \|D^\alpha u\|^2_{L^2} + \sum_{|\alpha| \le m} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}} dx dy \right)^{\frac{1}{2}}. \tag{1.2.75}$$

Proof. By (1.2.74), we have

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}} dx dy &= C_\sigma \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\widehat{D^\alpha u}(\xi)|^2 d\xi \\ &= C_\sigma \int_{\mathbb{R}^n} |\xi|^{2(\alpha+\sigma)} |\widehat{u}(\xi)|^2 d\xi. \end{aligned} \tag{1.2.76}$$

Then Corollary 1.2.28 follows directly. □

For $s < 0$, as in (1.2.53), we define

$$\|u\|'_s = \sup_{v \in \mathbf{H}^{-s}} \frac{(u, v)}{\|v\|'_{-s}}. \tag{1.2.77}$$

Proposition 1.2.29. *Let Ω and Ω' be bounded open subsets of $\mathbb{R}^n$. Let $\Phi : \Omega \to \Omega'$ be a diffeomorphism. Let $K \subset \Omega$ be a compact subset and $K' = \Phi(K)$. Then for $s \in \mathbb{R}$, there exists $c, C > 0$ such that for any $u \in \mathscr{C}^\infty_0(K')$,*

$$c\|u\|_s \le \|u \circ \Phi\|_s \le C\|u\|_s. \tag{1.2.78}$$

Proof. Since Φ is a diffeomorphism, we only need to prove

$$\|u \circ \Phi\|_s \le C\|u\|_s \tag{1.2.79}$$

because the other inequality follows from considering Φ^{-1}.

We denote by $U = u \circ \Phi$. Let $|D\Phi^{-1}|$ be the Jacobian determinant of Φ^{-1}. Set

$$B_1 = \sup_{K_2} |D\Phi^{-1}|, \quad B_2 = \sup_{K_1} \frac{|\Phi(x) - \Phi(y)|}{|x-y|}. \tag{1.2.80}$$

Set $x' = \Phi(x)$, $y' = \Phi(y)$. Then for $s = 0$, (1.2.79) follows from

$$\int_{\mathbb{R}^n} |U(x)|^2 dx = \int_{\mathbb{R}^n} |u(x')|^2 |D\Phi^{-1}| dx' \le B_1 \int_{\mathbb{R}^n} |u(x)|^2 dx. \tag{1.2.81}$$

For $0 < s < 1$, we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|U(x)-U(y)|^2}{|x-y|^{n+2s}}dxdy = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x')-u(y')|^2}{|x-y|^{n+2s}}|D\Phi^{-1}(x')||D\Phi^{-1}(y')|dx'dy'$$
$$\leq B_1^2 B_2^{n+2s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x')-u(y')|^2}{|x'-y'|^{n+2s}}dx'dy'. \quad (1.2.82)$$

Then by Proposition 1.2.27, we have $\|u\circ\Phi\|_s \leq C\|u\|_s$.

Now we proceed by induction. Let $\chi' \in \mathscr{C}_0^\infty(\Omega')$ such that $\chi' \equiv 1$ on K'. Assume that (1.2.79) holds for any $0 \leq s < k$, $k \in \mathbb{N}$. For $k \leq s < k+1$, $1 \leq j \leq n$, by Theorem 1.2.26, we have

$$\|D^j U(x)\|_{s-1} \leq \|D^j u(x') D^j\Phi(x)\|_{s-1} = \|D^j u(x')\chi'(x') D^j\Phi(x)\|_{s-1}$$
$$\leq C\|D^j u(x')\|_{s-1}. \quad (1.2.83)$$

Since $1+|\xi|^2 \leq (1+|\xi|)^2 \leq 2(1+|\xi|^2)$, we have

$$\|u\|_{s-1}^2 + \sum_{j=1}^n \|D^j u\|_{s-1}^2 \leq \|u\|_s^2 \leq 2\|u\|_{s-1}^2 + 2\sum_{j=1}^n \|D^j u\|_{s-1}^2. \quad (1.2.84)$$

From (1.2.83), (1.2.84) and the assumption for the induction, we have

$$\|U\|_s^2 \leq 2\|U\|_{s-1}^2 + 2\sum_{j=1}^n \|D^j U\|_{s-1}^2$$
$$\leq C\|u\|_{s-1}^2 + C\sum_{j=1}^n \|D^j u\|_{s-1}^2 \leq C\|u\|_s^2. \quad (1.2.85)$$

Therefore, (1.2.79) holds for any $s \geq 0$.

For $s < 0$, we use the duality of $\mathbf{H}^s$ and $\mathbf{H}^{-s}$. Let $\chi \in \mathscr{C}_0^\infty(\Omega')$ such that $\chi \equiv 1$ on K. Then by (1.2.56) and Theorem 1.2.26, for any $v \in \mathscr{S}$, we have

$$|(v,U)| = |(\chi_1 v, U)| = |(\chi v\circ\Phi^{-1}, u|D\Phi|)| \leq \|\chi v\circ\Phi^{-1}\|_{-s}\||D\Phi|u\|_s$$
$$\leq \|\chi v\circ\Phi^{-1}\|_{-s}\|\chi'|D\Phi|u\|_s \leq C\|v\|_{-s}\|u\|_s. \quad (1.2.86)$$

Therefore,

$$\|u\circ\Phi\|_s = \sup_{v\in\mathscr{S}} \frac{|(v,U)|}{\|v\|_{-s}} \leq C\|u\|_s. \quad (1.2.87)$$

The proof of Proposition 1.2.29 is completed.

□

Now we start to define the Sobolev space on the manifold.

Definition 1.2.30. Let M be a Riemannian manifold and E be a vector bundle over M with Hermitian metric. Let $K \subset M$ be a compact subset. Let $\{U_\alpha, \phi_\alpha\}$ be a locally finite atlas of M such that E is trivial on U_i and K is covered by finite charts. Let $\{h_\alpha\}$ be a partition of unity. Then for $s \in \mathbb{R}$, the Sobolev s-norm can be defined on $u \in \mathscr{C}_0^\infty(K, E)$ by

$$\|u\|_s^2 := \sum_\alpha \|(h_\alpha u) \circ \phi_\alpha^{-1}\|_s^2. \tag{1.2.88}$$

The completion of $\mathscr{C}_0^\infty(M, E)$ in this norm is the Sobolev $\mathbf{H}_0^s(K, E)$. If M is compact, we set $\mathbf{H}^s(M, E) := \mathbf{H}_0^s(M, E)$. We often denote it by $\mathbf{H}^s(E)$ if there is no confusion.

This norm is of course highly non-intrinsic. However, Theorem 1.2.26 shows that $\|\cdot\|_s$ is independent of the choice of local chart and partition of unity up to equivalence. Proposition 1.2.29 shows that $\|\cdot\|_s$ is independent of the choice of coordinate transformation up to equivalence.

Proposition 1.2.31. *The equivalence class of the norm $\|\cdot\|_s$ is independent of the atlas and the partition of unity.*

Proof. Let $\{V_\lambda, \psi_\lambda\}$ be a locally finite atlas of M such that E is trivial on V_λ and K is covered by finite charts.. Let $\{g_\lambda\}$ be a partition of unity with respect to this new atlas. Note that if $U_\alpha \cap V_\lambda \neq \emptyset$, we see that $\operatorname{supp}(h_\alpha \cdot g_\lambda) \subset U_\alpha \cap V_\lambda$. By Theorem 1.2.26 and Proposition 1.2.29, for $u \in \mathscr{C}^\infty(K, E)$, we have

$$\begin{aligned}
&\sum_\lambda \|(g_\lambda \cdot u) \circ \psi_\lambda^{-1}\|_s^2 = \sum_\lambda \left\| \left(\sum_\alpha h_\alpha \cdot g_\lambda \cdot u \right) \circ \psi_\lambda^{-1} \right\|_s^2 \\
&\le C \sum_{\lambda,\alpha} \left\|(h_\alpha \cdot g_\lambda \cdot u) \circ \psi_\lambda^{-1}\right\|_s^2 = C \sum_{\lambda,\alpha} \left\|(h_\alpha \cdot g_\lambda \cdot u) \circ \phi_\alpha^{-1} \circ (\phi_\alpha \circ \psi_\lambda^{-1})\right\|_s^2 \\
&\qquad \le C \sum_{\lambda,\alpha} \left\|(g_\lambda \cdot h_\alpha \cdot u) \circ \phi_\alpha^{-1}\right\|_s^2 \le C \sum_\alpha \left\|(h_\alpha \cdot u) \circ \phi_\alpha^{-1}\right\|_s^2.
\end{aligned} \tag{1.2.89}$$

The proof of Proposition 1.2.31 is completed. □

For the setting in Definition 1.2.30, we define the $\mathscr{C}^k$-norm by

$$\|u\|_{\mathscr{C}^k}^2 := \sum_\alpha \|(h_\alpha u) \circ \phi_\alpha^{-1}\|_{\mathscr{C}^k}^2. \tag{1.2.90}$$

As in Proposition 1.2.31, the equivalence class of the norm $\|\cdot\|_{\mathscr{C}^k}$ is independent of the atlas and the partition of unity.

Let ∇ be a connection on E. Given $u \in \mathscr{C}^\infty(M,E)$, we have $\nabla u \in \mathscr{C}^\infty(M, T^*M \otimes E)$. Using the tensor product connection on $T^*M \otimes E$, we have $\nabla\nabla u \in \mathscr{C}^\infty(M, T^*M \otimes T^*M \otimes E)$. This process continues, for any $s \in \mathbb{N}$, we define

$$\|u\|_k''^2 := \sum_{j=1}^{k} \int_M |\underbrace{\nabla \cdots \nabla}_{j \text{ times}} u|^2 dv_x. \tag{1.2.91}$$

Proposition 1.2.32. *The norm $\|\cdot\|_k''$ in (1.2.91) is equivalent to $\|\cdot\|_k$ in (1.2.60) for $k \in \mathbb{N}$.*

Proof. By (1.2.91), we have

$$\begin{aligned}\|u\|_k''^2 := \sum_{j=1}^{k} \sum_{\lambda} \int_{\phi_\lambda(V_\lambda)} & g_\lambda \circ \psi_\lambda^{-1} \\ & \cdot \left| \sum_{1 \le i_1, \cdots, i_j \le n} (D^{i_1} + A_{i_1}) \cdots (D^{i_j} + A_{i_j})(u \circ \psi_\lambda^{-1}) \right|^2 dv_x. \end{aligned} \tag{1.2.92}$$

Then Proposition 1.2.32 follows from (1.2.23) and (1.2.88). □

From Proposition 1.2.32, the norm in (1.2.91) is independent of the metrics on TM and E and the connection ∇ up to equivalence. Sometimes, we use (1.2.91) as the definition of the Sobolev norm.

For $k \in \mathbb{N}$, the uniform $\mathscr{C}^k$-norm of $u \in \mathscr{C}_0^k(K,E)$ is defined by

$$\|u\|_{\mathscr{C}^k}''^2 := \sup_K \sum_{|\alpha| \le k} |\underbrace{\nabla \cdots \nabla}_{j \text{ times}} u|^2. \tag{1.2.93}$$

As in Proposition 1.2.32, this norm is equivalent to that in (1.2.90). Since K is compact, this norm is complete.

Now we generalize Theorems 1.2.17, 1.2.18, 1.2.20, 1.2.22 and Proposition 1.2.24 to the global case. The proof of the following theorem is obvious, which is left as an exercise.

Theorem 1.2.33. *Let E and F be vector bundles over a manifold of dimension n.*

(1) (Rellich's theorem) For any $s, t \in \mathbb{R}$, $s < t$, the inclusion map

$$\iota : \mathbf{H}_0^t(K,E) \to \mathbf{H}_0^s(K,E) \tag{1.2.94}$$

is a compact operator, i.e., ι sends any bounded subset of $\mathbf{H}_0^t(K,E)$ to relatively compact subset of $\mathbf{H}_0^s(K,E)$, equivalently, a set with compact closure.

(2) (Sobolev embedding theorem) For each $k \in \mathbb{N}$ and each $s > \frac{n}{2} + k$, there is a continuous inclusion $\mathbf{H}_0^s(K,E) \subset \mathscr{C}_0^k(K,E)$, that is, for any $u \in \mathscr{C}_0^\infty(K,E)$,

$$\|u\|_{\mathscr{C}^k} \le C_s \|u\|_s. \tag{1.2.95}$$

Furthermore, by (1), every sequence $\{u_j \in \mathscr{C}_0^\infty(K,E)\}$ which is bounded in the $\|\cdot\|_s$ norm has a subsequence which converges in the uniform $\mathscr{C}^k$-norm.

(3) For $s_2 > s > s_1$, then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that for any $u \in \mathbf{H}_0^{s_2}(K,E)$, we have

$$\|u\|_s^2 \le \varepsilon \|u\|_{s_2}^2 + C_\varepsilon \|u\|_{s_1}^2. \tag{1.2.96}$$

In particular, for $s > \frac{n}{2} + k$, $k \ge 1$,

$$\|u\|_{\mathscr{C}^k}^2 \le \varepsilon \|u\|_s^2 + C_\varepsilon \|u\|_0^2. \tag{1.2.97}$$

(4) For any Riemannian volume element dv on compact manifold M, the bilinear map on $\mathscr{C}^\infty(M,E) \times \mathscr{C}^\infty(M,E^*)$ given by setting

$$(u, v^*) = \int_M v^*(u) dv \tag{1.2.98}$$

has a continuous extension to $\mathbf{H}^s(E) \times \mathbf{H}^{-s}(E^*)$ for any $s \in \mathbb{R}$. Moreover, it identifies $\mathbf{H}^{-s}(E^*)$ with the dual of $\mathbf{H}^s(E)$, that is,

$$\|v^*\|_{-s} = \sup_{u \in \mathbf{H}^s(E)} \frac{(u, v^*)}{\|u\|_s}. \tag{1.2.99}$$

(5) Multiplication $T_A u := Au$ by any $A \in \mathscr{C}_0^\infty(K, \mathrm{Hom}(E,F) := F \otimes E^*)$ extends to a bounded linear map $T_A : \mathbf{H}_0^s(K,E) \to \mathbf{H}_0^s(K,F)$ for all $s \in \mathbb{R}$.

1.3 Pseudodifferential operator on vector bundle

Let $P = \sum_{|\alpha|\le m} A^\alpha(x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}$ be a differential operator on $\mathscr{S}$ such that $|D^\beta A^\alpha(x)|$ is bounded for all α, β. Thus from (1.2.22), for $u \in \mathscr{S}$, $Pu \in \mathscr{S}$. Then by (1.2.26) and (1.2.31), we have

$$\begin{aligned} Pu(x) = \sum_{|\alpha|\le m} i^{|\alpha|} A^\alpha(x) D_x^\alpha u(x) &= (2\pi)^{-n/2} \sum_{|\alpha|\le m} i^{|\alpha|} A^\alpha(x) \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \widehat{D^\alpha u}(\xi) d\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi) d\xi, \quad (1.3.1) \end{aligned}$$

where

$$p(x,\xi) = \sum_{|\alpha|\le m} i^{|\alpha|} A^\alpha(x)\xi^\alpha. \tag{1.3.2}$$

The matrix-valued function $p(x,\xi)$ is called the (total) symbol of P. We denote by

$$\mathrm{sym}(P) = p(x,\xi). \tag{1.3.3}$$

Let $Q : \mathscr{S} \to \mathscr{S}$ be another differential operator with symbol $q(\xi)$, i.e., A^α's are constants. Then since $\widehat{A^\alpha u}(\xi) = A^\alpha \hat{u}(\xi)$, we have $\widehat{Qu}(\xi) = q(\xi)\hat{u}(\xi)$.

$$\begin{aligned} PQu(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\widehat{Qu}(\xi) d\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi) q(\xi)\hat{u}(\xi) d\xi. \quad (1.3.4) \end{aligned}$$

Thus

$$\mathrm{sym}(P\circ Q) = \mathrm{sym}(P)\cdot \mathrm{sym}(Q). \tag{1.3.5}$$

In the theory of PDE, the main problem is to solve the equation

$$Pu = f. \tag{1.3.6}$$

From (1.3.4), naively, if $p(\xi)$ is invertible and independent of x, by (1.2.24), letting $q(\xi) := p(\xi)^{-1}$ and

$$Qf(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} q(\xi)\hat{f}(\xi) d\xi, \tag{1.3.7}$$

we have

$$PQf(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{f}(\xi) d\xi = f(x). \tag{1.3.8}$$

Thus

$$u = Qf \tag{1.3.9}$$

is a solution of (1.3.6).

There are two problems for this process. Firstly, $p(\xi)$ is not invertible at $\xi = 0$. For this problem, we will assume that $p(\xi)$ is invertible outside 0, and use a cut-off function to construct the $q(\xi)$ to handle it. It is the main content in the next section. Secondly, we need to construct a home for Q living in and study the case that p, q depend on x. This is the purpose of this section.

Definition 1.3.1. Fix $m \in \mathbb{R}$. A smooth matrix-valued function $p(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be a symbol of order m if for each α, α', there exists $C_{\alpha,\alpha'} > 0$ such that

$$|D_x^\alpha D_\xi^{\alpha'} p(x, \xi)| \leq C_{\alpha,\alpha'} (1 + |\xi|)^{m-|\alpha'|} \tag{1.3.10}$$

for all x, ξ. Let Sym^m be the space of these symbols.

Now we construct the operator from such $p(x, \xi)$ as in (1.3.7).

Proposition 1.3.2. *For each $p \in \mathrm{Sym}^m$, the formula*

$$Pu(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x, \xi) \hat{u}(\xi) d\xi \tag{1.3.11}$$

defines a linear operator $P : \mathscr{S} \to \mathscr{S}$. If p has compact x-support, this operator has a continuous extension $P : \mathbf{H}^{s+m} \to \mathbf{H}^s$ for any $s \in \mathbb{R}$.

Proof. For $u \in \mathscr{S}$, $\hat{u} \in \mathscr{S}$. For $N \in \mathbb{N}$, from (1.2.22) and (1.3.10), for any $k \in \mathbb{N}$,

$$\begin{aligned}
|x|^{2N} |D_x^\alpha Pu(x)| &= (2\pi)^{-n/2} \left| \sum_{\beta+\gamma=\alpha} \int_{\mathbb{R}^n} \frac{\alpha!}{\beta!\gamma!} |x|^{2N} D_x^\beta e^{i\langle x,\xi\rangle} D_x^\gamma p(x, \xi) \hat{u}(\xi) d\xi \right| \\
&= (2\pi)^{-n/2} \left| \sum_{\beta+\gamma=\alpha} \int_{\mathbb{R}^n} \frac{\alpha!}{\beta!\gamma!} \left(\Delta_\xi^N e^{i\langle x,\xi\rangle}\right) \left(\xi^\beta D_x^\gamma p(x, \xi)\right) \hat{u}(\xi) d\xi \right| \\
&= (2\pi)^{-n/2} \left| \sum_{\beta+\gamma=\alpha} \int_{\mathbb{R}^n} \frac{\alpha!}{\beta!\gamma!} e^{i\langle x,\xi\rangle} \Delta_\xi^N (\xi^\beta D_x^\gamma p(x, \xi) \hat{u}(\xi)) d\xi \right| \\
&\leq C_k \sum_{\beta+\gamma=\alpha} \int_{\mathbb{R}^n} (1 + |\xi|)^{m+|\beta|} (1 + |\xi|)^{-k} d\xi. \qquad (1.3.12)
\end{aligned}$$

Take k large enough, the right hand side of (1.3.12) is finite. Thus $Pu \in \mathscr{S}$.

Now we prove the second part.

We firstly try to use the definition:

$$\begin{aligned}\|Pu\|_s &= \int_{\mathbb{R}^n} (1+|\xi|)^{2s}|\widehat{Pu}(\xi)|^2 d\xi \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}(1+|\xi|)^{2s}\left|\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}Pu(x)dx\right|^2 d\xi \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}(1+|\xi|)^{2s}\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle x,\xi-\eta\rangle}p(x,\eta)\hat{u}(\eta)d\eta dx\right|^2 d\xi \quad (1.3.13)\end{aligned}$$

But in this way, $\int_{\mathbb{R}^n}((1+|\xi|)^{2s})d\xi$ may be not finite. It is not easy for us to use $e^{-i\langle x,\xi\rangle}$ to control it.

Now we use another equivalent definition (1.2.53):

$$\|Pu\|_s = \sup_v \frac{(Pu,v)}{\|v\|_{-s}}. \quad (1.3.14)$$

From (1.2.52) and (1.3.13),

$$\begin{aligned}(Pu,v) &= \int_{\mathbb{R}^n}\widehat{Pu}(\xi)\cdot\hat{v}(\xi)d\xi \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle x,\xi-\eta\rangle}p(x,\eta)\hat{u}(\eta)d\eta dx\right)\cdot\hat{v}(\xi)d\xi \quad (1.3.15)\end{aligned}$$

Set

$$\Psi(\xi,\eta) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi-\eta\rangle}p(x,\eta)dx. \quad (1.3.16)$$

Then

$$|(Pu,v)| \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\Psi(\xi,\eta)||\hat{u}(\eta)||\hat{v}(\xi)|d\xi d\eta. \quad (1.3.17)$$

We show an estimate of $\Psi(\xi,\eta)$ as follows. For any α,

$$\begin{aligned}\zeta^\alpha\int_{\mathbb{R}^n} e^{-i\langle x,\zeta\rangle}p(x,\eta)dx &= (-1)^{|\alpha|}\int_{\mathbb{R}^n} D_x^\alpha e^{-i\langle x,\zeta\rangle}p(x,\eta)dx \\ &= \int_{\mathbb{R}^n} e^{-i\langle x,\zeta\rangle}D_x^\alpha p(x,\eta)dx. \quad (1.3.18)\end{aligned}$$

Since p has compact x-support, by (1.3.10) and (1.3.18), for any $t \in \mathbb{N}$, there exists $C_t > 0$, such that

$$|\Psi(\xi,\eta)| \le C_t(1+|\eta|)^m(1+|\xi-\eta|)^{-t}. \tag{1.3.19}$$

Note that

$$(1+|\eta|)(1+|\xi-\eta|) \ge 1+|\eta|+|\xi-\eta| \ge 1+|\xi|. \tag{1.3.20}$$

Thus

$$\frac{1+|\xi|}{1+|\eta|} \le (1+|\xi-\eta|). \tag{1.3.21}$$

Let

$$\Psi'(\xi,\eta) = \Psi(\xi,\eta)\cdot(1+|\eta|)^{-s-m}\cdot(1+|\xi|)^s. \tag{1.3.22}$$

Then from (1.3.19)-(1.3.22),

$$|\Psi'(\xi,\eta)| \le C_t\frac{(1+|\xi|)^s}{(1+|\eta|)^s}(1+|\xi-\eta|)^{-t} \le C_t(1+|\xi-\eta|)^{|s|-t}. \tag{1.3.23}$$

Thus taking t large enough, $\int_{\mathbb{R}^n}|\Psi'(\xi,\eta)|d\xi$ and $\int_{\mathbb{R}^n}|\Psi'(\xi,\eta)|d\eta$ are all finite.

From (1.3.17) and (1.3.22), we have

$$\begin{aligned}|(Pu,v)| \le& \left(\int_{R^n}\left(\int_{\mathbb{R}^n}|\Psi'(\xi,\eta)|d\xi\right)(1+|\eta|)^{s+m}\hat{u}(\eta)d\eta\right)^{1/2}\\ &\times\left(\int_{R^n}\left(\int_{\mathbb{R}^n}|\Psi'(\xi,\eta)|d\eta\right)(1+|\xi|)^{-s}\hat{v}(\xi)d\xi\right)^{1/2}\\ &\le C\|u\|_{s+m}\|v\|_{-s}.\end{aligned} \tag{1.3.24}$$

Therefore, we get this proposition from (1.3.14). □

Remark that the constant C in (1.3.24) depends on s.

Definition 1.3.3. The operator P defined in (1.3.11) is called a **pseudodifferential operator of order** m on $\mathbb{R}^n$. In particular, a differential operator is a pseudodifferential operator. The space of the pseudodifferential operators of order m is denoted by $\Psi\mathrm{DO}_m$. A linear map $f : \mathscr{S} \to \mathscr{S}$ is called an **(infinitely) smoothing operator** if it could extend to $f : \mathbf{H}^s \to \mathbf{H}^{s+m}$ for any s and m. Two pseudodifferential operators P and P' are called equivalent if $P - P'$ is an (infinitely) smoothing operator.

Our next aim is to define the composition of $P, P' \in \Psi\mathrm{DO}$. This is one of the key point in solving (1.3.6).

Since the pseudodifferential operator is defined by the symbol, we study the "symbol calculus".

Definition 1.3.4. Let P be a pseudodifferential operator with symbol p. Then p is said to have a formal development

$$p \sim \sum_{j=1}^{\infty} p_j, \quad p_j \in \mathrm{Sym}^{m_j}, \tag{1.3.25}$$

if for each $m \in \mathbb{Z}$, there exists K such that $p - \sum_{j=1}^{k} p_j \in \mathrm{Sym}^{-m}$ for any $k \geq K$.

The following proposition says that the set of symbols is complete under the addition in some sense.

Proposition 1.3.5. *Any formal series $\sum_{j=1}^{\infty} p_j$, $p_j \in \mathrm{Sym}^{m_j}$, $m_j \to -\infty$, is the formal development of a pseudodifferential operator. This operator is unique up to equivalence.*

Proof. We can assume that $m_{j+1} < m_j$ for all j. Fix a smooth function $\varphi : \mathbb{R}^n \to [0,1]$, such that $\varphi(\xi) = 0$ for $|\xi| \leq 1$ and $\varphi(x) = 1$ for $|\xi| \geq 2$. For any sequence $\{r_j\}_{j=1}^{\infty}$ such that $\lim_{t\to\infty} r_j = +\infty$, the symbol

$$p(x,\xi) = \sum_{j=1}^{\infty} \varphi(\xi/r_j) p_j(x,\xi) \tag{1.3.26}$$

is well defined since the sum is finite for each (x,ξ). We plan to choose r_j such that $p(x,\xi)$ is the symbol of a pseudodifferential operator.

From (1.3.10), we have

$$\begin{aligned} |D_x^{\alpha} D_{\xi}^{\alpha'} \big(\varphi(\xi/r_j) p_j(x,\xi)\big)| &\leq C_{\alpha'} \sum_{\beta+\gamma=\alpha'} \left| D_{\xi}^{\beta} \varphi\big(\xi/r_j\big) \right| \cdot \left| D_x^{\alpha} D_{\xi}^{\gamma} p_j(x,\xi) \right| \\ &\leq C_{\alpha'} \sum_{\beta+\gamma=\alpha'} C_{j,\alpha,\gamma} \left| D_{\xi}^{\beta} \varphi\big(\xi/r_j\big) \right| \cdot (1+|\xi|)^{m_j - |\gamma|}. \end{aligned} \tag{1.3.27}$$

Let $\tilde{\varphi}_j(\xi) = \varphi(\xi/r_j)$. By induction, we could prove that

$$\left| D_{\xi}^{\beta} \varphi\big(\xi/r_j\big) \right| \leq \frac{C_{\beta}}{r_j^{|\beta|}} \|\tilde{\varphi}_j\|_{\mathscr{C}^{|\beta|}} \leq \frac{C_{\beta}}{r_j^{|\beta|}} \|\tilde{\varphi}_j\|_{\mathscr{C}^{|\alpha'|}}. \tag{1.3.28}$$

Thus

$$|D_x^\alpha D_\xi^{\alpha'}\big(\varphi(\xi/r_j)p_j(x,\xi)\big)| \leq C_{\alpha'}\|\tilde{\varphi}_j\|_{\mathscr{C}^{|\alpha'|}} \cdot \sum_{\beta+\gamma=\alpha'} \frac{C_{j,\alpha,\gamma}C_\beta}{r_j^{|\beta|}} \cdot (1+|\xi|)^{m_j-|\gamma|}. \tag{1.3.29}$$

Note that if $|\xi| < r_j$, $D_x^\alpha D_\xi^{\alpha'}\big(\varphi(\xi/r_j)p_j(x,\xi)\big) = 0$. Thus we could assume $|\xi| \geq r_j$ in the right hand side of (1.3.29). Set $m_0 = m_1$. Then

$$\begin{aligned} |D_x^\alpha D_\xi^{\alpha'}\big(\varphi(\xi/r_j)p_j(x,\xi)\big)| &\leq C_{\alpha'}\|\tilde{\varphi}_j\|_{\mathscr{C}^{|\alpha'|}} \cdot \sum_{\beta+\gamma=\alpha'} \frac{C_{j,\alpha,\gamma}C_\beta}{r_j^{|\beta|}} \cdot \frac{(1+|\xi|)^{m_{j-1}-|\alpha'|}}{(1+|\xi|)^{m_{j-1}-m_j-|\beta|}} \\ &\leq C_{\alpha'}\|\tilde{\varphi}_j\|_{\mathscr{C}^{|\alpha'|}} \cdot \frac{\sum_{\beta+\gamma=\alpha'} C_{j,\alpha,\gamma}C_\beta}{r_j^{m_{j-1}-m_j}} \cdot (1+|\xi|)^{m_{j-1}-|\alpha'|}. \end{aligned} \tag{1.3.30}$$

Let $C_{j,\alpha,\alpha'} := \sum_{\beta+\gamma=\alpha'} C_{j,\alpha,\gamma}C_\beta$. For $j > 1$, set $r_j > (2^j\|\tilde{\varphi}_j\|_{\mathscr{C}^j} \max_{|\alpha|,|\alpha'|\leq j}\{C_{j,\alpha,\alpha'}\})^{1/(m_{j-1}-m_j)}$ such that $\lim_{j\to+\infty} r_j = +\infty$. Then if $|\alpha|, |\alpha'| \leq j$, there exists $C_{\alpha,\alpha'} > 0$ such that

$$|D_x^\alpha D_\xi^{\alpha'}\big(\varphi(\xi/r_j)p_j(x,\xi)\big)| \leq \frac{C_{\alpha,\alpha'}}{2^j}(1+|\xi|)^{m_{j-1}-|\alpha'|}. \tag{1.3.31}$$

Let $k = \max\{|\alpha|, |\alpha'|\}$. Then there exists $C_{\alpha,\alpha'} > 0$ such that

$$\begin{aligned} |D_x^\alpha D_\xi^{\alpha'} p(x,\xi)\big)| &\leq \left(\sum_{j=1}^k C_{\alpha'}\|\tilde{\varphi}_j\|_{\mathscr{C}^{|\alpha'|}} \frac{\sum_{\beta+\gamma=\alpha'} C_{j,\alpha,\gamma}C_\beta}{r_j^{m_{j-1}-m_j}}\right)(1+|\xi|)^{m_1-|\alpha'|} \\ &+ \sum_{j=k+1}^\infty \frac{C_{\alpha,\alpha'}}{2^j}(1+|\xi|)^{m_1-|\alpha'|} \leq C_{\alpha,\alpha'}(1+|\xi|)^{m_1-|\alpha'|}. \end{aligned} \tag{1.3.32}$$

Therefore, $p \in \mathrm{Sym}^{m_1}$.

Following the same process, we could obtain that $p - \sum_{j=1}^k \tilde{\varphi}_j p_j \in \mathrm{Sym}^{m_k}$. Since $m_j \to -\infty$, for any $m \in \mathbb{Z}$, there exists $K > 0$ such that for any $k > K$, $p - \sum_{j=1}^k \tilde{\varphi}_j p_j \in \mathrm{Sym}^m$. Since $(1-\tilde{\varphi}_j)p_j \in \mathrm{Sym}^{-\infty}$, we have $p - \sum_{j=1}^k p_j \in \mathrm{Sym}^m$. Thus $\sum_{j=1}^\infty p_j$ is the formal development of a pseudodifferential operator.

If P' is another pseudodifferential operator with symbol p' has the same formal development, then for any $m \in \mathbb{Z}$, $p - p' \in \mathrm{Sym}^{-m}$. Thus $p - p' \in \mathrm{Sym}^{-\infty}$.

The proof of Proposition 1.3.5 is completed. □

From (1.3.1), we have

$$(Pu)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} p(x,\xi)u(y)dyd\xi. \tag{1.3.33}$$

The following lemma is technical. But in Shubin's famous book, he strongly urges the reader to carefully look at this proof. Therefore we present it here.

Lemma 1.3.6. *Let $a(x,y,\xi)$ be a smooth matrix-valued function on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ with compact x- and y-support. Fix $m \in \mathbb{R}$ and assume that for each α, β, γ, there is a constant $C_{\alpha,\beta,\gamma} > 0$ such that*

$$|D_x^\alpha D_y^\beta D_\xi^\gamma a| \le C_{\alpha,\beta,\gamma}(1+|\xi|)^{m-|\gamma|}. \tag{1.3.34}$$

Then the operator $K : \mathscr{S} \to \mathscr{S}$ given by

$$(Ku)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a(x,y,\xi)u(y)dyd\xi \tag{1.3.35}$$

is a pseudodifferential operator whose symbol k has asymptotic development

$$k(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha D_y^\alpha a)(x,x,\xi). \tag{1.3.36}$$

Proof. From (1.2.24), (1.2.29), (1.2.30) and (1.3.35), we have

$$\begin{aligned}
(Ku)(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \left(\int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} a(x,y,\xi)u(y)dy \right) d\xi \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \widehat{a_y u}(\xi) d\xi = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} (\hat{a}_y * \hat{u})(\xi)d\xi \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \int_{\mathbb{R}^n} \hat{a}_y(x,\xi-\eta,\xi)\hat{u}(\eta)d\eta d\xi \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\eta\rangle} \left(\int_{\mathbb{R}^n} e^{i\langle x,\xi-\eta\rangle} \hat{a}_y(x,\xi-\eta,\xi)d\xi \right) \hat{u}(\eta)d\eta.
\end{aligned} \tag{1.3.37}$$

We need to check the interchange of integrations in the last equation is allowed. From (1.3.16) and (1.3.19), since a is with compact x- and y-support, for any $l_1 \in \mathbb{N}$, there exists $C_{l_1} > 0$ such that

$$\begin{aligned}
|\hat{a}_y(x,\xi-\eta,\xi)| &= (2\pi)^{-n/2} \left| \int_{\mathbb{R}^n} e^{i\langle \eta-\xi,s\rangle} a(x,s,\xi)ds \right| \\
&\le C_{l_1}(1+|\xi|)^m(1+|\xi-\eta|)^{-l_1}.
\end{aligned} \tag{1.3.38}$$

Since $\hat{u} \in \mathscr{S}$, for any $l_2 \in \mathbb{N}$, there exists $C_{l_2} > 0$ such that

$$|\hat{u}(\eta)| \leq C_{l_2}(1+|\eta|)^{-l_2}. \tag{1.3.39}$$

Thus from (1.3.21), for $l_1, l_2 \in \mathbb{N}$, such that $m+n/2 < l_1 < l_2 - n/2$, we have

$$|\hat{a}_y(x, \xi-\eta, \xi)||\hat{u}(\eta)| \leq C_{l_1} C_{l_2}(1+|\xi|)^{m-l_1}(1+|\eta|)^{l_1-l_2}. \tag{1.3.40}$$

Thus the interchange of integrations is allowed.

Let

$$k(x,\eta) = \int_{\mathbb{R}^n} e^{i\langle x, \xi-\eta\rangle} \hat{a}_y(x, \xi-\eta, \xi) d\xi = \int_{\mathbb{R}^n} e^{i\langle x, \zeta\rangle} \hat{a}_y(x, \zeta, \zeta+\eta) d\zeta. \tag{1.3.41}$$

Then from (1.3.11), if $k(x,\eta)$ satisfies (1.3.10), K is a pseudodifferential operator with symbol $k(x,\eta)$.

For $l \in \mathbb{N}$, we have the Taylor expansion in the third variable,

$$\hat{a}_y(x,\zeta,\zeta+\eta) = \sum_{|\alpha|\leq l} \frac{i^{|\alpha|}}{\alpha!} (D_\eta^\alpha \hat{a}_y)(x,\zeta,\eta)\zeta^\alpha + R_l(x,\zeta,\zeta+\eta). \tag{1.3.42}$$

***Remark of Taylor expansion**: For function $F \in \mathscr{S}$, the Taylor expansion is*

$$F(x) = \sum_{|\alpha|\leq l} \frac{1}{\alpha!} F^{(\alpha)}(0) x^\alpha + R_l(x), \tag{1.3.43}$$

where

$$R_l(x) = \sum_{|\mu|=l+1} \frac{l+1}{\mu!} x^\mu \cdot \int_0^1 (1-t)^l F^{(\mu)}(tx) dt. \tag{1.3.44}$$

We explain (1.3.44) here in some details. By Taylor extension and the integration remainder in one variable, for $\varphi \in \mathscr{C}^\infty(\mathbb{R})$, we have

$$\varphi(t) = \varphi(0) + \varphi'(0)t + \cdots + \frac{1}{k!}\varphi^{(k)}(0)t^k + r_k(t),$$

where

$$r_k(t) = \frac{1}{k!}\int_0^t (t-s)^k \varphi^{(k+1)}(s) ds.$$

Set $\varphi(t) = F(tx)$. Then

$$\varphi'(t) = \sum_{|J|=1} F^{(J)}(tx)x^J, \quad \varphi''(t) = \sum_{|J|=1}\sum_{|K|=1} F^{(J+K)}(tx)x^{J+K} = \sum_{|J|=2} F^{(J)}(tx)x^J.$$

By induction, we have

$$\varphi^{(k)}(t) = \sum_{|J|=k} F^{(J)}(tx)x^J.$$

So we have

$$F(x) = \sum_{|J|\le l} \frac{1}{|J|!} F^{(J)}(0)x^J + \frac{1}{k!}\int_0^1 (1-s)^k \sum_{|J|=k+1} F^{(J)}(sx)x^J ds.$$

Note that

$$\sum_{|J|=k} F^{(J)}x^J = \sum_{|\alpha|=k} \nu(\alpha)F^{(\alpha)}x^\alpha,$$

where

$$\nu(\alpha) = \binom{k}{\alpha_1}\cdot\binom{k-\alpha_1}{\alpha_2}\cdots\binom{k-\alpha_1-\cdots-\alpha_{n-1}}{\alpha_n} = \frac{k!}{\alpha!}.$$

Thus we get (1.3.43) and (1.3.44).

By (1.3.44),

$$R_l(x,\zeta,\zeta+\eta) = \sum_{|\mu|=l+1} \frac{(l+1)i^{l+1}}{\mu!}\int_0^1 (1-t)^l (D_\eta^\mu \hat{a}_y)(x,\zeta,t\zeta+\eta)\zeta^\mu dt. \tag{1.3.45}$$

Since

$$\int_{\mathbb{R}^n} e^{i\langle x,\zeta\rangle}(D_\eta^\alpha \hat{a}_y)(x,\zeta,\eta)\zeta^\alpha d\zeta = \int_{\mathbb{R}^n} e^{i\langle x,\zeta\rangle}\zeta^\alpha(\widehat{D_\eta^\alpha a_y})(x,\zeta,\eta)d\zeta$$
$$= \int_{\mathbb{R}^n} e^{i\langle x,\zeta\rangle}(\widehat{D_y^\alpha D_\eta^\alpha a_y})(x,\zeta,\eta)d\zeta = (D_y^\alpha D_\eta^\alpha a_y)(x,x,\eta), \tag{1.3.46}$$

from (1.3.41), (1.3.42), (1.3.45) and (1.3.46), we have

$$k(x,\eta) = \sum_{|\alpha|\le l} \frac{i^{|\alpha|}}{\alpha!}(D_y^\alpha D_\eta^\alpha a_y)(x,x,\eta) + r_l(x,\eta), \tag{1.3.47}$$

where

$$r_l(x,\eta) = \int_{\mathbb{R}^n} e^{i\langle x,\zeta\rangle} R_l(x,\zeta,\zeta+\eta)d\zeta. \tag{1.3.48}$$

By Proposition 1.3.5, we only need to prove that for any $l \in \mathbb{N}$, $r_l(x,\eta) \in \mathrm{Sym}^{m-(l+1)}$.

From (1.3.45) and (1.3.48),

$$|D_x^\alpha D_\eta^\beta r_l(x,\eta)| \le \int_{\mathbb{R}^n} |D_x^\alpha D_\eta^\beta R_l(x,\zeta,\zeta+\eta)|d\zeta$$
$$\le \sum_{|\mu|=l+1} \frac{(l+1)}{\mu!}\int_{\mathbb{R}^n}\int_0^1 (1-t)^l |(D_x^\alpha D_\eta^{\mu+\beta}\hat{a}_y)(x,\zeta,t\zeta+\eta)\zeta^\mu| d\zeta dt. \tag{1.3.49}$$

From (1.3.34),

$$
\begin{aligned}
|\zeta^\gamma(D_x^\alpha D_\eta^\beta \hat{a}_y)(x,\zeta,t\zeta+\eta)| &= (2\pi)^{-n/2}\left|\int_{\mathbb{R}^n} e^{-i\langle y,\zeta\rangle}\zeta^\gamma(D_x^\alpha D_\eta^\beta a)(x,y,t\zeta+\eta)dy\right| \\
&= (2\pi)^{-n/2}\left|\int_{\mathbb{R}^n} D_y^\gamma e^{-i\langle y,\zeta\rangle}(D_x^\alpha D_\eta^\beta a)(x,y,t\zeta+\eta)dy\right| \\
&= (2\pi)^{-n/2}\left|\int_{\mathbb{R}^n} e^{-i\langle y,\zeta\rangle}(D_x^\alpha D_y^\gamma D_\eta^\beta a)(x,y,t\zeta+\eta)dy\right| \\
&\le C_{\alpha\beta\gamma}\,\mathrm{vol}(y-\mathrm{supp}(a))(1+|t\zeta+\eta|)^{m-|\beta|} \\
&\le C_{\alpha\beta\gamma}(1+|t\zeta+\eta|)^{m-|\beta|}. \qquad (1.3.50)
\end{aligned}
$$

From (1.3.49) and (1.3.50), taking $l+1>m$, $\gamma > n+2(l+1)+|\beta|+m$,

$$
\begin{aligned}
&|D_x^\alpha D_\eta^\beta r_l(x,\eta)| \le C_{\alpha\beta\gamma l}\int_{\mathbb{R}^n}\int_0^1 (1-t)^l(1+|t\zeta+\eta|)^{m-(l+1)-|\beta|}(1+|\zeta|)^{-\gamma}|\zeta|^{l+1}d\zeta dt \\
&\le C_{\alpha\beta\gamma l}\int_{\mathbb{R}^n}\int_0^1 (1-t)^l(1+|\eta|)^{m-(l+1)-|\beta|}(1+t|\zeta|)^{l+1+|\beta|-m}(1+|\zeta|)^{l+1-\gamma}d\zeta dt \\
&\le C_{\alpha\beta\gamma l}\int_{\mathbb{R}^n}(1+|\eta|)^{m-(l+1)-|\beta|}(1+|\zeta|)^{2(l+1)+|\beta|-m-\gamma}d\zeta \\
&\le C_{\alpha\beta\gamma l}(1+|\eta|)^{m-(l+1)-|\beta|}. \qquad (1.3.51)
\end{aligned}
$$

The proof of Lemma 1.3.6 is completed. □

Remark 1.3.7. If the function $a(x,y,\xi)$ in Lemma 1.3.6 vanishes for all (x,y) in a neighborhood of the diagonal, then the corresponding operator K given by (1.3.35) is infinitely smoothing.

For a differential operator P, from the definition, we see easily that P is local, that is, for $u \in \mathscr{C}_0^\infty$,

$$\mathrm{supp}(Pu) \subset \mathrm{supp}(u). \qquad (1.3.52)$$

Remark that a surprising theorem by Peetre says that if a linear operator satisfies (1.3.52), it is a differential operator. So we cannot expect the pseudodifferential operator is local. But we can prove that it is ε-local up to a smoothing operator.

For $A \subset \mathbb{R}^n$ and $\varepsilon > 0$, we set

$$A_\varepsilon := \{x \in \mathbb{R}^n : \mathrm{dist}(x,A) \le \varepsilon\}. \qquad (1.3.53)$$

An operator $P : \mathscr{S} \to \mathscr{S}$ is called ε-local if for any $u \in \mathscr{C}_0^\infty$,

$$\mathrm{supp}(Pu) \subset \mathrm{supp}(u)_\varepsilon. \qquad (1.3.54)$$

Proposition 1.3.8. *For $P \in \Psi\mathrm{DO}_m$ with symbol p, which has compact x-support, for any $\varepsilon > 0$, there exists $P_\varepsilon \in \Psi\mathrm{DO}_m$ such that it is equivalent to P and ε-local.*

Proof. Choose a smooth real-valued function ψ on $\mathbb{R}^n \times \mathbb{R}^n$ such that $\psi \equiv 1$ in a neighborhood of the diagonal and $\psi(x,y) = 0$ if $|x-y| \geq \varepsilon$. Then

$$(P_\varepsilon u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \psi(x,y) p(x,\xi) u(y) dy d\xi \tag{1.3.55}$$

is ε-local. By (1.3.33) and Lemma 1.3.6, P_ε is a pseudodifferential operator with symbol $p_\varepsilon \sim p$. Thus P_ε is equivalent to P.

The proof of Proposition 1.3.8 is completed. □

Proposition 1.3.9. *Let $\chi_1, \chi_2 \in \mathscr{C}_0^\infty(\mathbb{R}^n, \mathbb{R})$. For $P \in \Psi\mathrm{DO}_m$,*

$$P^{\chi_1,\chi_2}(u) := \chi_1 P(\chi_2 u) \in \Psi\mathrm{DO}_m\,. \tag{1.3.56}$$

Proof. From (1.3.56),

$$(P^{\chi_1,\chi_2} u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \chi_1(x) p(x,\xi) \chi_2(y) u(y) dy d\xi. \tag{1.3.57}$$

Set $a(x,y,\xi) = \chi_1(x) p(x,\xi) \chi_2(y)$ in Lemma 1.3.6, we obtain Proposition 1.3.9. □

Proposition 1.3.10. *Let P be a pseudodifferential operator and $u \in \mathbf{H}^s$ for some $s \in \mathbb{R}$. Then for any open subset $U \subset \mathbb{R}^n$, if $u|_U \in \mathscr{C}^\infty$, then $Pu|_U \in \mathscr{C}^\infty$.*

Proof. For $x \in U$, choose $\chi_1, \chi_2 \in \mathscr{C}_0^\infty(\mathbb{R}^n, \mathbb{R})$ such that $x \in \mathrm{supp}(\chi_1) \subset \mathrm{supp}(\chi_2)$, $\chi_1 \equiv 1$ near x and $\chi_2 \equiv 1$ on a neighborhood of $\mathrm{supp}(\chi_1)$. By Proposition 1.3.2, we have $P(\chi_2 u) \in \mathscr{C}^\infty$. Thus $\chi_1 P(\chi_2 u) \in \mathscr{C}^\infty$. Since

$$\chi_1 P((1-\chi_2)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \chi_1(x) p(x,\xi)(1-\chi_2)(y) u(y) dy d\xi, \tag{1.3.58}$$

and $\chi_1(x)(1-\chi_2)(y)$ vanishes in a neighborhood of the diagonal, by Remark 1.3.7, we have $\chi_1 P((1-\chi_2)u) \in \mathscr{C}^\infty$. Therefore, $\chi_1 P u \in \mathscr{C}^\infty$, which means that Pu is smooth near x. □

Note that the pseudodifferential operator is not local. Sometimes, in order to define it on manifolds, we need some conditions to guarantee that we can do analysis on a local chart.

Definition 1.3.11. For $P \in \Psi\mathrm{DO}_m$, if there exists a compact subset $K \subset \mathbb{R}^n$ such that for any $u \in \mathscr{C}_0^\infty$, $\mathrm{supp}(Pu) \subset K$ and $Pu = 0$ whenever $\mathrm{supp}(u) \cap K = \emptyset$, we say P has support in K. The set of such operators is denoted by $\Psi\mathrm{DO}_{K,m}$.

If P is an m-th order differential operator with the supports of all coefficients in K, then $P \in \Psi\mathrm{DO}_{K,m}$.

Let $K' \subset K$ be a compact subset such that there exists $\varepsilon > 0$ such that $\mathrm{dist}(K', \partial K) > \varepsilon$. Let $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ such that $\psi(x,y) = 0$ if $x \notin K'$ or $|x-y| \geq \varepsilon$. Then for any $p \in \mathrm{Sym}^m$, the pseudodifferential operator associated with $\psi(x,y)p(x,\xi)$ in (1.3.35) is an element of $\Psi\mathrm{DO}_{K,m}$.

Definition 1.3.12. For $P \in \Psi\mathrm{DO}_{K,m}$, its formal adjoint P^* is defined by

$$(Pu, v)_{L^2} = (u, P^*v)_{L^2}, \tag{1.3.59}$$

for any $u, v \in \mathscr{C}_0^\infty(K)$.

Proposition 1.3.13. *For $P \in \Psi\mathrm{DO}_{K,m}$ with symbol p, its formal adjoint $P^* \in \Psi\mathrm{DO}_{K,m}$ has symbol p^* with formal development*

$$p^* \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \bar{p}^T, \tag{1.3.60}$$

where $(\cdot)^T$ denotes the transposed matrix. In particular, the formal adjoint is unique up to smoothing operators.

Proof. For $u, v \in \mathscr{C}_0^\infty(K)$,

$$\begin{aligned}(Pu, v)_{L^2} &= \int_{\mathbb{R}^n} \left\langle (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} p(x,\xi)u(y)dyd\xi, v(x) \right\rangle dx \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \langle p(x,\xi)u(y), v(x)\rangle dyd\xi dx \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left\langle u(y), e^{-i\langle x-y,\xi\rangle} \overline{p(x,\xi)}^T v(x) \right\rangle dyd\xi dx. \end{aligned} \tag{1.3.61}$$

Formally, we could write $P^*v = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle x-y,\xi\rangle}\overline{p(x,\xi)}^T v(x) d\xi dx$. But usually, $\overline{p(x,\xi)}^T$ does not satisfy (1.3.34)the condition of Lemma 1.3.6. We use the following trick to overcome this obstruction. Fix $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^n, \mathbb{R})$ such that $\phi \equiv 1$ on K. Then $\phi u = u$. From (1.3.61), we have

$$\begin{aligned}(Pu, v)_{L^2} &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left\langle \phi(y)u(y), e^{-i\langle x-y,\xi\rangle} \overline{p(x,\xi)}^T v(x) \right\rangle dyd\xi dx \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left\langle u(y), e^{-i\langle x-y,\xi\rangle} \phi(y)\overline{p(x,\xi)}^T v(x) \right\rangle dyd\xi dx. \end{aligned} \tag{1.3.62}$$

In this case, from (1.3.10), we have

$$|D_x^\alpha D_y^\beta D_\xi^\gamma \phi(y)\overline{p(x,\xi)}^T| \leq C_{\alpha\beta\gamma}(1+|\xi|)^{m-|\gamma|}. \tag{1.3.63}$$

Thus from Lemma 1.3.6,

$$(P^*v)(y) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle y-x,\xi\rangle}\phi(y)\overline{p(x,\xi)}^T v(x)d\xi dx, \tag{1.3.64}$$

satisfying $(Pu,v)_{L^2} = (u,P^*v)_{L^2}$, is a pesudodifferential operator with symbol

$$p^*(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha \phi(x)\overline{p(y,\xi)}^T\bigg|_{x=y} = \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \overline{p(x,\xi)}^T. \tag{1.3.65}$$

The last equality holds because $p(x,\xi)$ has x-support in K.

If P_1^* is another formal adjoint of P, then the symbol of it has the same formal development as p^*. Thus $P_1^* - P^*$ is a smoothing operator.

The proof of Proposition 1.3.13 is completed. □

From Proposition 1.3.13, we could obtain another formula of $\widehat{Pu}(\xi)$ as follows. For $u,v \in \mathscr{S}$,

$$\begin{aligned}(\widehat{Pu},\hat{v})_{L^2} = (u,P^*v)_{L^2} &= \int_{\mathbb{R}^n}\left\langle u(x),(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}p^*(x,\xi)\hat{v}(\xi)d\xi\right\rangle dx \\ &= \int_{\mathbb{R}^n}\left\langle (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\overline{p^*(x,\xi)}^T u(x)dx, \hat{v}(\xi)\right\rangle d\xi. \end{aligned}\tag{1.3.66}$$

Therefore, we have

$$\widehat{Pu}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\overline{p^*(x,\xi)}^T u(x)dx. \tag{1.3.67}$$

Proposition 1.3.14. *For $P \in \Psi\mathrm{DO}_{K,l}$ and $Q \in \Psi\mathrm{DO}_{K,m}$ with symbols p and q respectively, the composition $P\circ Q \in \Psi\mathrm{DO}_{K,l+m}$ has symbol*

$$\mathrm{Sym}(P\circ Q) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha p)(D_x^\alpha q). \tag{1.3.68}$$

Proof. By (1.3.11) and (1.3.67), we have

$$\begin{aligned}(PQu)(x) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}p(x,\xi)\widehat{Qu}(\xi)d\xi \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}p(x,\xi)\overline{q^*(y,\xi)}^T u(y)dyd\xi. \end{aligned}\tag{1.3.69}$$

Since $P \in \Psi\mathrm{DO}_{K,l}$ and $Q \in \Psi\mathrm{DO}_{K,m}$, there exists $C_{\alpha\beta\gamma} > 0$, such that

$$|D_x^\alpha D_y^\beta D_\xi^\gamma p(x,\xi)\overline{q^*(y,\xi)}^T| \le C_{\alpha,\beta,\gamma}(1+|\xi|)^{m+l-|\gamma|}. \tag{1.3.70}$$

Thus by Lemma 1.3.6, $P \circ Q \in \Psi\mathrm{DO}_{K,l+m}$.

Since $(P^*)^* = P$, by (1.3.36) and (1.3.60), we have

$$\begin{aligned}
\mathrm{Sym}(PQ) &\sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha D_y^\alpha p(x,\xi)\overline{q^*(y,\xi)}^T)|_{x=y} \\
&= \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}(D_\xi^\beta p(x,\xi))(D_\xi^\gamma D_x^\alpha \overline{q^*(x,\xi)}^T) \\
&= \sum_\alpha \sum_{\beta+\gamma=\alpha}\frac{i^{|\beta|+|\gamma|}}{\beta!\gamma!}(D_\xi^\beta p(x,\xi))(D_\xi^\gamma D_x^\beta D_x^\gamma \overline{q^*(x,\xi)}^T) \\
&= \sum_\beta \frac{i^{|\beta|}}{\beta!}(D_\xi^\beta p(x,\xi))D_x^\beta\left(\sum_\gamma \frac{i^{|\gamma|}}{\gamma!}D_\xi^\gamma D_x^\gamma \overline{q^*(x,\xi)}^T\right) \\
&\sim \sum_\beta \frac{i^{|\beta|}}{\beta!}(D_\xi^\beta p(x,\xi))(D_x^\beta q(x,\xi)).
\end{aligned} \tag{1.3.71}$$

The proof of Proposition 1.3.14 is completed. □

Proposition 1.3.15. *Let $\phi : U \to V$ be a diffeomorphism between two open sets of $\mathbb{R}^n$. Then for each compact subset $K \subset U$, ϕ induces a map $\phi_* : \Psi\mathrm{DO}_{K,m} \to \Psi\mathrm{DO}_{\phi(K),m}$ by*

$$(\phi_* P)(u) := P(u\circ\phi)\circ\phi^{-1}. \tag{1.3.72}$$

Proof. Let $\psi = \phi^{-1}$. For $x \in \phi(K)$, write $x' = \psi(x)$. Then

$$\begin{aligned}
x' - y' = \psi(x) - \psi(y) &= \int_0^1 \frac{d}{dt}\psi(tx+(1-t)y)dt \\
&= \int_0^1 \nabla\psi(tx+(1-t)y)dt\cdot(x-y).
\end{aligned} \tag{1.3.73}$$

Set $\Psi(x,y) := \int_0^1 \nabla\psi(tx+(1-t)y)dt$. Then it is a smooth matrix-valued function. Since $\Psi(x,x) = (\partial\psi^i/\partial x^j)_x$ and ψ is a diffeomorphism, the matrix $\Psi(x,y)$ is invertible in a neighborhood U of the diagonal.

Let $J(x) := |\det(\partial\psi^i/\partial x^j)_x|$ denote the Jacobian of ψ. Let $\chi \in \mathscr{C}_0^\infty(U)$ such that $\chi \equiv 1$ on a smaller neighborhood of the diagonal. Then for $P \in$

$\Psi\mathrm{DO}_{K,m}$ and $u \in \mathscr{C}_0^\infty(\phi(K), \mathbb{C}^p)$,

$$\begin{aligned}[(\phi_* P)u](x) &= [P(u \circ \phi)](x') \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x'-y',\xi\rangle}(\phi^*\chi + \phi^*(1-\chi))(x',y')p(x',\xi)u(\phi(y'))dy'd\xi \\ &\quad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\Psi(x,y)^T\xi\rangle}\chi(x,y)p(\psi(x),\xi)u(y)J(y)dyd\xi \\ &\quad + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x'-y',\xi\rangle}\phi^*(1-\chi)(x',y')p(x',\xi)u(\phi(y'))dy'd\xi \\ &\qquad =: (P_1 u)(x) + (P_2 u)(x). \end{aligned} \tag{1.3.74}$$

By Remark 1.3.7, since P has compact support, P_2 is a smoothing operator. Let $\zeta = \Psi(x,y)^T\xi$. Then

$$\begin{aligned} P_1 u(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} & e^{i\langle x-y,\zeta\rangle}\chi(x,y)p(\psi(x),(\Psi(x,y)^T)^{-1}\zeta)J(y) \\ & \cdot |\det(\Psi(x,y)^T)^{-1}|u(y)dyd\zeta. \end{aligned} \tag{1.3.75}$$

Write

$$a(x,y,\zeta) := \chi(x,y)p(\psi(x),(\Psi(x,y)^T)^{-1}\zeta)J(y)|\det(\Psi(x,y)^T)^{-1}|. \tag{1.3.76}$$

Then by Lemma 1.3.6, we see that $P_1 \in \Psi\mathrm{DO}_m$. It is easy to verify that P_1, P_2 have compact supports in $\phi(K)$.

The proof of Proposition 1.3.15 is completed. □

Now we define the pseudodifferential operator on manifolds.

Let M be a smooth manifold. Let E and F be complex vector bundles over M.

Definition 1.3.16. A continuous[1] linear map $P : \mathscr{C}_0^\infty(M,E) \to \mathscr{C}^\infty(M,F)$ is called a pseudodifferential operator of order m if for each coordinate system $\{U_i,\phi_i\}$, for any $\varphi,\psi \in \mathscr{C}_0^\infty(U_i)$, $(\phi_i^{-1})^*(\varphi P\psi)$ is a pseudodifferential operator of order m. The linear space of all such operators is denoted by $\Psi\mathrm{DO}_m(E,F)$. The element in $\Psi\mathrm{DO}_{-\infty}(E,F)$ is called a smoothing operator. Two pseudodifferential operators are called equivalent if they differ by a smoothing operator.

[1] $\{\varphi_k\} \subset \mathscr{C}_0^\infty$ tends to 0 if there exists a compact subset $K \subset \mathbb{R}^n$ such that $\mathrm{supp}(\varphi_k) \subset K$ for any k and for any α, $D^\alpha\varphi_k \to 0$ uniformly on $x \in K$. With this topology, $\mathscr{C}_0^\infty$ is usually denoted by $\mathscr{D}$. $\{\varphi_k\} \subset \mathscr{C}^\infty$ tends to 0 if for any compact subset $K \subset \mathbb{R}^n$, for any $\varepsilon > 0$ and α, there exists $N > 0$ such that if $k \geq N$, $\sup_K |D^\alpha\varphi_k| < \varepsilon$. With this topology, $\mathscr{C}^\infty$ is usually denoted by $\mathscr{E}$. We assume that $P : \mathscr{D} \to \mathscr{E}$ is continuous.

By this definition, a differential operator is a pseudodifferential operator.

Proposition 1.3.17. *Let* $P \in \Psi\mathrm{DO}_m(E,F)$. *For any open set* $U \subset M$, $u|_U \in \mathscr{C}^\infty$ *implies* $Pu|_U \in \mathscr{C}^\infty$.

Proof. Take $x \in U$. Choose a local coordinate system $\{U_i, \phi_i\}$ such that $x \in U_{i_0}$ and $x \notin \cup_{i\neq i_0}\overline{U_i}$. Let $\{h_i\}$ be a partition of unity with respect to $\{U_i\}$. Let $W = U \cap (\cup_{i\neq i_0}\overline{U_i})^c$. If u is smooth near x, then so is $h_{i_0}u$. Therefore, $Pu(y) = h_{i_0}(y)Ph_{i_0}(h_{i_0}u)(y)$ is smooth near x.

The proof of Proposition 1.3.17 is completed. □

Proposition 1.3.18. *(1) If for any* $s, m \in \mathbb{R}$, *any compact set* $K \subset M$, P *extends to a bounded linear map* $P : \mathbf{H}^s_0(K,E) \to \mathbf{H}^{s-m}(K,F)$, *then* P *is a smoothing operator.*

(2) Let $\varphi, \psi \in \mathscr{C}^\infty_0(M)$. *If* $\mathrm{supp}(\varphi)\cap\mathrm{supp}(\psi) = \emptyset$, *then* $\varphi P\psi$ *is a smoothing operator for any* $P \in \Psi\mathrm{DO}(E,F)$.

Proof. (1) For any $u \in \mathbf{H}^s_0(K,E)$, $Pu \in \mathscr{C}^\infty(K,F)$. Thus for any $\varphi, \psi \in \mathscr{C}^\infty_0(U_i)$, $\varphi P\psi u \in \mathscr{C}^\infty(U_i,F)$. Thus $\varphi P\psi$ is a smoothing operator. By Definition 1.3.16, P is a smoothing operator.

(2) For any $u \in \mathbf{H}^s_0(K,E)$, for any $x \in \mathrm{supp}(\varphi)$, $\psi u = 0$ near x. By Proposition 1.3.17, $\varphi P\psi u$ is smooth near x. If $x \notin \mathrm{supp}(\varphi)$, $\varphi P\psi u(x) = 0$. Thus $\varphi P\psi u$ is smooth on M. So $\varphi P\psi$ is a smoothing operator. □

Given a Riemannian volume element dv on M, we can define a formal adjoint $P^* : \mathscr{C}^\infty_0(M,F^*) \to \mathscr{C}^\infty(M,E^*)$ of operator $P : \mathscr{C}^\infty_0(M,E) \to \mathscr{C}^\infty(M,F)$ by

$$\int_M \langle Pu, v\rangle dv = \int_M \langle u, P^*v\rangle dv \tag{1.3.77}$$

for $u \in \mathscr{C}^\infty_0(M,E)$ and $v \in \mathscr{C}^\infty_0(M,F^*)$.

Theorem 1.3.19. *Let* E, F *and* G *be vector bundles over a smooth manifold* M *and let* $P \in \Psi\mathrm{DO}_m(E,F)$ *and* $Q \in \Psi\mathrm{DO}_l(F,G)$ *with the compact support* K. *The following statements hold:*

(1) The operator P *extends to a bounded linear map* $P : \mathbf{H}^s_0(K,E) \to \mathbf{H}^{s-m}(K,F)$ *for any* $s \in \mathbb{R}$.

(2) $Q \circ P \in \Psi\mathrm{DO}_{m+l}(E,G)$.

(3) $P^* \in \Psi\mathrm{DO}_m(F^*,E^*)$ *for any* dv.

(4) A diffeomorphism $\phi : M \to M$ *induces a linear map* $\phi_* : \Psi\mathrm{DO}_m(\phi^*E, \phi^*F) \to \Psi\mathrm{DO}_m(E,F)$ *by* $\phi^*[(\phi_*P)u] = P(\phi^*u)$.

Proof. (1) Let $\{U_i, \phi_i\}$ be a local coordinate system. We only consider the part which covers K. We could assume that $U_i = \phi_i^{-1}(B_0(4))$, $V_i = \phi_i^{-1}(B_0(1))$ and $K \subset \cup V_i$. Let $\{h_i\}$ be a partition of unity with respect to $\{V_i\}$. For $u \in \mathscr{C}_0^\infty(K, E)$,

$$\|Pu\|_s = \sum_j \|h_j Pu\|_s \leq \sum_{i,j,k} \|h_j P h_k h_i u\|_s. \tag{1.3.78}$$

If $\mathrm{supp}(h_j) \cap \mathrm{supp}(h_i) \neq \emptyset$, $\mathrm{supp}(h_j) \subset U_i$. So $\|h_j P h_k h_i u\|_s \leq \|h_i u\|_{s-m}$. If $\mathrm{supp}(h_j) \cap \mathrm{supp}(h_i) = \emptyset$, $h_j P h_k h_i$ is a smoothing operator. Thus

$$\|Pu\|_s \leq \sum_{i,j,k} \|h_j P h_k h_i u\|_s \leq C \sum_i \|h_i u\|_{s-m} = C\|u\|_{s-m}. \tag{1.3.79}$$

(2) follows from $\varphi Q P \psi = \sum_i \varphi Q h_i^{1/2} h_i^{1/2} P \psi$.
(3) follows from $\varphi P^* \psi = (\psi P \varphi)^*$.
(4) follows from $\phi^*[(\varphi \phi_* P \psi) u] = ((\phi^{-1})^* \varphi) P ((\phi^{-1})^* \psi)(\phi^* u)$.
The proof of Theorem 1.3.19 is completed. □

In the rest of this section, we study the symbol of pseudodifferential operator on the manifold. We want to glue the symbols on each chart together.

We start from the differential operators. Let $P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, F)$ be a differential operator such that on a coordinate chart U_i,

$$P|_{U_i} = \sum_{|\alpha| \leq m} A_\alpha^{(i)} \frac{\partial}{\partial x^{\alpha,(i)}}. \tag{1.3.80}$$

What is the relation between $A_\alpha^{(i)}$ and $A_\beta^{(j)}$?

From (1.1.5), for example,

$$\begin{aligned} \frac{\partial}{\partial x_1^{(i)}} &= \frac{\partial x_{k_1}^{(j)}}{\partial x_1^{(i)}} \cdot \frac{\partial}{\partial x_{k_1}^{(j)}}, \\ \frac{\partial^2}{\partial x_2^{(i)} \partial x_1^{(i)}} &= \frac{\partial}{\partial x_2^{(i)}} \left(\frac{\partial x_{k_1}^{(j)}}{\partial x_1^{(i)}} \right) \cdot \frac{\partial}{\partial x_{k_1}^{(j)}} + \frac{\partial x_{k_1}^{(j)}}{\partial x_1^{(i)}} \frac{\partial x_{k_2}^{(j)}}{\partial x_2^{(i)}} \frac{\partial^2}{\partial x_{k_2}^{(j)} \partial x_{k_1}^{(j)}}, \cdots \end{aligned} \tag{1.3.81}$$

If the order is higher than one, terms are more and more crazy. We have to do the coordinate transformation modulo the $(m-1)$-th order terms. That is,

$$\frac{\partial^m}{\partial x_{l_1}^{(i)} \cdots \partial x_{l_m}^{(i)}} = \frac{\partial x_{k_1}^{(j)}}{\partial x_{l_1}^{(i)}} \cdots \frac{\partial x_{k_m}^{(j)}}{\partial x_{l_m}^{(i)}} \frac{\partial^q}{\partial x_{k_1}^{(j)} \cdots \partial x_{k_m}^{(j)}} + (m-1)\text{-order terms.} \tag{1.3.82}$$

Thus the only transformations of the top order are meaningful,

$$A^{(j)}_{k_1\cdots k_m} = A^{(i)}_{l_1\cdots l_m} \frac{\partial x^{(j)}_{k_1}}{\partial x^{(i)}_{l_1}} \cdots \frac{\partial x^{(j)}_{k_m}}{\partial x^{(i)}_{l_m}}. \tag{1.3.83}$$

As in (1.3.2), we write

$$\sigma_\xi(P)|_{U_j} = \sum_{|\alpha|=m} i^{|\alpha|} A^{(j)}_\alpha(x) \xi^{\alpha,(j)}. \tag{1.3.84}$$

If $\sigma_\xi(P)$ is globally defined, the necessary condition is

$$\xi^{l,(i)} = \frac{\partial x^{(j)}_k}{\partial x^{(i)}_l} \cdot \xi^{k,(j)} \tag{1.3.85}$$

From (1.1.14), we have

$$\xi^{l,(i)} dx^{(i)}_l = \xi^{k,(j)} dx^{(j)}_k. \tag{1.3.86}$$

Thus if $E = F = \mathbb{C}$, we can regard $\sigma_\xi(P)$ as a function on $\xi \in T^*M$. In fact, let $\{U_j \times \mathbb{C}^p\}$ be an atlas of T^*M with diffeomorphisms

$$\Phi_j : T^*M|_{U_j} \to U_j \times \mathbb{C}^p, \quad (x, \xi) \mapsto (\phi_j(x), \xi^{(j)}). \tag{1.3.87}$$

Then the transition function is $\text{diag}\{\phi_{ij}, (D(\phi_{ij})^{-1})^T\}$. By (1.1.2), $\sigma_\xi(P)$ is a function on $\xi \in T^*M$. For smooth map $f : M \to N$, if $\pi' : E \to N$ is a vector bundle over N, we can define the pull-back bundle by

$$f^*(E) = \{(m, v) \in M \times E : f(m) = \pi'(v)\}. \tag{1.3.88}$$

Easy to see that f^*E is a vector bundle over M.

Let $\pi : T^*M \to M$ be the natural projection. In general, $\sigma_\xi(P) \in \mathscr{C}^\infty(T^*M, \text{Hom}(\pi^*E, \pi^*F))$, i.e., $\sigma_\xi(P)$ is a bundle map[2]

$$\sigma_\xi(P) : \pi^*E \to \pi^*F, \tag{1.3.89}$$

which is called the **principal symbol** of P. From (1.3.75), it is obvious that for differential operators $P, P' : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, F)$, $Q : \mathscr{C}^\infty(M, F) \to \mathscr{C}^\infty(M, G)$ and $t_1, t_2 \in \mathbb{C}$, we have

$$\begin{aligned} \sigma_\xi(t_1 P + t_2 P') &= t_1 \sigma_\xi(P) + t_2 \sigma_\xi(P'), \\ \sigma_\xi(Q \circ P) &= \sigma_\xi(Q) \circ \sigma_\xi(P). \end{aligned} \tag{1.3.90}$$

[2]A bundle map is a map between two bundles over the same manifold which restricts to one point on the base manifold is a linear transform.

Now we extend the definition of the principal symbol to the pseudodifferential operator on manifold.

Let $P = \sum P_i \in \Psi\mathrm{DO}_m(E, F)$ and P_i is defined on U_i from the map $\phi_i : U_i \to \mathbb{R}^n$. From (1.3.36) and (1.3.76), after a diffeomorphism ϕ_{ij}, since $\chi(x, y) \equiv 1$ near the diagonal, we have

$$\mathrm{Sym}(\phi_{ij}^* P)(x^{(j)}, \xi^{(j)}) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha J(y) |\det\Theta| p(\phi_{ij}(x^{(j)}), \Theta\xi^{(j)})|_{y=x}, \tag{1.3.91}$$

where $\Theta = [(\partial x^{(i)}/\partial x^{(j)})^T]^{-1}$ and $J(y) = |\det(\partial x^{(i)}/\partial x^{(j)})_y|$. Then Since $|\det\Theta| = J(x)^{-1}$, we have

$$\mathrm{Sym}(\phi_{ij}^* P)(x^{(j)}, \xi^{(j)}) \sim p(x^{(i)}, \Theta\xi^{(j)}) \mod \mathrm{Sym}^{m-1}. \tag{1.3.92}$$

Therefore, as in (1.3.89), we could define the principle symbol $\sigma_\xi(P) \in \mathscr{C}^\infty(T^*M, \mathrm{Hom}(\pi^*E, \pi^*F))$.

Definition 1.3.20. We say $p \in \mathscr{C}^\infty(T^*M, \mathrm{Hom}(\pi^*E, \pi^*F))$ is a symbol of order m if p defines an element of Sym^m in each local coordinate chart. From Proposition 1.3.15, the definition is independent of the choice of the atlas. The vector space of all such symbols of order m is denoted by $\mathrm{Sym}^m(E, F)$.

If $P \in \Psi\mathrm{DO}_m(E, F)$, then $\sigma_\xi(P) \in \mathrm{Sym}^m(E, F)/\mathrm{Sym}^{m-1}(E, F)$.

1.4 Elliptic operator

1.4.1 Parametrix

Definition 1.4.1. Let P be a differential operator. We say P is elliptic if for any non-zero cotangent vector $\xi \in T^*M$, the principal symbol $\sigma_\xi(P) : E \to F$ is invertible.

Definition 1.4.2. Let $P \in \Psi\mathrm{DO}_m$ with symbol p. We say P is elliptic if there exists a constant $c > 0$ such that for all $|\xi| \geq c$, the matrix inverse of $p(x, \xi)$ exists and satisfies

$$|p(x,\xi)^{-1}| \leq C(1 + |\xi|)^{-m} \tag{1.4.1}$$

for some constant $C > 0$.

Definition 1.4.3. Let $P \in \Psi\mathrm{DO}_m(E, F)$. We say P is elliptic if its principal symbol $\sigma_\xi(P) \in \mathrm{Sym}^m(E, F)/\mathrm{Sym}^{m-1}(E, F)$ has a representative p and for some Riemannian metric, there exists a constant $c > 0$, such that for all $|\xi| \geq c$, the matrix inverse of $p(x, \xi)$ exists and satisfies

$$|p(\xi)^{-1}| \leq C(1 + |\xi|)^{-m} \tag{1.4.2}$$

for some constant $C > 0$.

Remark that the symbol $p(x, \xi)$ in conditions (1.4.1) and (1.4.2) could be replaced by the principal symbol $\sigma_\xi(P)$.

It is easy to see that if P is a differential operator, Definition 1.4.3 is equivalent to Definition 1.4.1.

From Theorem 1.3.13, if $P \in \Psi\mathrm{DO}_m(E, F)$, $P^* \in \Psi\mathrm{DO}_m(F^*, E^*)$. Since $\sigma_\xi(P^*) = \overline{\sigma_\xi(P)}^T$, P is elliptic if and only if P^* is elliptic.

Lemma 1.4.4. *Let $P \in \Psi\mathrm{DO}_m$ be a elliptic operator. Then there exists an operator $Q \in \Psi\mathrm{DO}_{-m}$, unique up to equivalence, such that*

$$PQ = \mathrm{Id} - S', \quad QP = \mathrm{Id} - S, \tag{1.4.3}$$

where $S, S' \in \Psi\mathrm{DO}_{-\infty}$.

Proof. We only need to prove that for any compact subset $K \subset \mathbb{R}^n$, on $\mathscr{C}_0^\infty(M, E)$, (1.4.3) holds. Since the composition of a smoothing operator and a pseudodifferential operator is a smoothing operator, by Proposition 1.3.8, we may assume that P and Q have compact support.

Let c be the constant in Definition 1.4.2. Let $\chi : \mathbb{R}^+ \to [0,1]$ be a smooth cut-off function such that $\chi(t) = 0$ for $t \le c$ and $\chi(t) = 1$ for $t \ge 2c$. For $x \in K$, set

$$q_0(x,\xi) = \chi(|\xi|)p(x,\xi)^{-1}. \tag{1.4.4}$$

We claim that

$$q_0 \in \operatorname{Sym}^{-m}. \tag{1.4.5}$$

In fact, for $\alpha = \beta = 0$, from (1.4.1), on $|\xi| \ge c$, we have

$$|D_x^\alpha D_\xi^\beta p(x,\xi)^{-1}| \le C(1+|\xi|)^{-m-|\beta|}. \tag{1.4.6}$$

We write $\gamma = (\alpha, \beta) \in \mathbb{N}^{2n}$ and $D^\gamma = D_x^\alpha D_\xi^\beta$ On $|\xi| \ge c$, we have

$$0 = D^\gamma(p \cdot p^{-1}) = \sum_{\gamma'+\gamma''=\gamma} \frac{\gamma!}{\gamma'!\gamma''!} D^{\gamma'}p \cdot D^{\gamma''}(p^{-1}). \tag{1.4.7}$$

Thus we have

$$D^\gamma p^{-1} = -p^{-1} \cdot \sum_{\gamma'+\gamma''=\gamma, \gamma' \neq 0} \frac{\gamma!}{\gamma'!\gamma''!} D^{\gamma'}p \cdot D^{\gamma''}(p^{-1}). \tag{1.4.8}$$

Now we take induction on $|\gamma|$. If $|\gamma| = 0$, (1.4.6) holds. We assume that (1.4.6) holds for $|\gamma| \le k-1$. Then by (1.4.8), we have

$$\begin{aligned} |D^\gamma(p^{-1})| \le C \sum_{|\beta'|+|\beta''|=|\beta|} (1+|\xi|)^{-m}(1+|\xi|)^{m-|\beta'|}(1+|\xi|)^{-m-|\beta''|} \\ \le C(1+|\xi|)^{-m-|\beta|}. \end{aligned} \tag{1.4.9}$$

Therefore, on $|\xi| \ge c$, (1.4.6) holds for any α, β. Thus the claim (1.4.5) follows from

$$|D^\gamma q(x,\xi)| \le \sum_{\gamma'+\gamma''=\gamma} \frac{\gamma!}{\gamma'!\gamma''!} |D^{\gamma'}\chi||D^{\gamma''}(p^{-1})| \le C(1+|\xi|)^{m-\beta}. \tag{1.4.10}$$

By induction, we set

$$q_k = -\sum_{j=0}^{k-1} \left\{ \sum_{|\alpha|+j=k} \frac{i^{|\alpha|}}{\alpha!} (D_\xi^\alpha q_j)(D_x^\alpha p) \right\} \cdot q_0. \tag{1.4.11}$$

From (1.4.5), we have

$$q_k \in \mathrm{Sym}^{m-k}. \tag{1.4.12}$$

Let $Q_k \in \Psi\mathrm{DO}_{m-k}$ be the pseudodifferential operator with symbol p_k. Then we have

$$\begin{aligned}
\mathrm{Sym}(Q_0P) &\sim 1 + \sum_{|\alpha|=1} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q_0)(D_x^\alpha p) + \sum_{|\alpha|=2} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q_0)(D_x^\alpha p) + \cdots \\
\mathrm{Sym}(Q_1P) &\sim \quad -\sum_{|\alpha|=1} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q_0)(D_x^\alpha p) q_0 p + \sum_{|\alpha|=1} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q_1)(D_x^\alpha p) + \cdots \\
\mathrm{Sym}(Q_2P) &\sim \qquad -\sum_{j=0}^{1} \left\{ \sum_{|\alpha|+j=2} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q_j)(D_x^\alpha p) \right\} \cdot q_0 p_0 + \cdots \\
&\cdots\ \cdots
\end{aligned} \tag{1.4.13}$$

From (1.4.13), letting

$$q \sim \sum q_k, \tag{1.4.14}$$

and $Q \in \Psi\mathrm{DO}_{-m}$ be the pseudodifferential operator with symbol q, we have $\mathrm{Sym}(QP) = 1$. Thus $QP - \mathrm{Id}$ is a smoothing operator. Remark that all constants in this proof are independent of the compact subset.

From the same argument, we could construct $Q' \in \Psi\mathrm{DO}_{-m}$ such that $PQ' - \mathrm{Id}$ is a smoothing operator. Thus

$$Q \sim Q(PQ') \sim (QP)Q' \sim Q'. \tag{1.4.15}$$

The proof of Lemma 1.4.4 is completed. □

Theorem 1.4.5. *Assume that $P \in \Psi\mathrm{DO}_m(E,F)$ is elliptic. Then there exists $Q \in \Psi\mathrm{DO}_{-m}(F,E)$, up to equivalence, such that*

$$PQ = \mathrm{Id} - S', \quad QP = \mathrm{Id} - S, \tag{1.4.16}$$

where S, S' are smoothing operators. The operator Q is called a ***parametrix*** *of P.*

Proof. We take a coordinate system $\{U_i, \phi_i\}$ such that E, F are trivial on U_i. Let $\{\psi_i\}$ be a partition of unity with respect to $\{U_i\}$. By Definition 1.4.3, for any compact subset K of U_i, $P : \mathscr{C}_0^\infty(K,E) \to \mathscr{C}^\infty(K,F)$ is elliptic. By

Lemma 1.4.4, there exists $Q_i \in \Psi\mathrm{DO}_{-m} : \mathscr{C}_0^\infty(U_i, E) \to \mathscr{C}^\infty(U_i, F)$, such that $PQ_i = \mathrm{Id} - S_i$, where $S_i : \mathscr{C}_0^\infty(U_i, E) \to \mathscr{C}^\infty(U_i, F)$ is a smoothing operator. As in the proof of Lemma 1.4.4, we may assume that $Q_i\psi_i$ has compact support K_i such that $\mathrm{supp}(\psi_i) \subset K_i \subset U_i$. Let $\varphi, \varphi_i \in \mathscr{C}_0^\infty(U_i)$ such that $\varphi \equiv 1$ on K_i and $\varphi_i \equiv 1$ on $\mathrm{supp}(\varphi)$. Then

$$\varphi_i P \varphi Q_i \psi_i = \varphi_i P Q_i \psi_i = \psi_i - \varphi_i S_i \psi_i. \tag{1.4.17}$$

Note that $\varphi_i S_i \psi_i \in \Psi\mathrm{DO}_{-\infty}(E, F)$. By Proposition 1.3.18 (2), $(1-\varphi_i)PQ_i\psi_i = (1-\varphi_i)P\varphi Q_i \psi_i \in \Psi\mathrm{DO}_{-\infty}(E,F)$. Thus

$$\begin{aligned} Q' &:= \sum Q_i \psi_i \in \Psi\mathrm{DO}_{-m}(E,F), \\ S' &:= \sum \varphi_i S_i \psi_i + \sum (1-\varphi_i) P Q_i \psi_i \in \Psi\mathrm{DO}_{-\infty}(E,F). \end{aligned} \tag{1.4.18}$$

Then

$$\begin{aligned} PQ' = P(\sum Q_i\psi_i) &= \sum PQ_i\psi_i = \sum \varphi_i PQ_i \psi_i + \sum (1-\varphi_i)PQ_i\psi_i \\ &= \sum \psi_i - \sum \varphi_i S_i \psi_i + \sum (1-\varphi_i) PQ_i \psi_i = \mathrm{Id} - S'. \end{aligned} \tag{1.4.19}$$

From (1.4.19), For $P^* \in \Psi\mathrm{DO}_m(F^*, E^*)$, there exists $Q^* \in \Psi\mathrm{DO}_{-m}(E^*, F^*)$ such that $P^*Q^* = \mathrm{Id} - S^*$, where $S \in \Psi\mathrm{DO}_{-\infty}(E^*, E^*)$. Taking the adjoint, we have $QP = \mathrm{Id} - S$. As in (1.4.15), we have $Q \sim Q'$.

The proof of Theorem 1.4.5 is completed. □

Let $L^2_{loc}(M, E)$ be the space of locally L^2-integrable sections of E on X (L^2-integrable on any bounded subset of X).

Theorem 1.4.6 (Elliptic regularity)**.** *Let $P \in \Psi\mathrm{DO}_m(E, F)$ be a elliptic operator.*

(1)Let $u \in L^2_{loc}(M,E)$ with compact support K. If $Pu \in \mathbf{H}_0^s(K, F)$, then $u \in \mathbf{H}_0^{s+m}(K, E)$.

(2) For any open subset $U \subset M$, if $Pu \in \mathscr{C}^\infty(U, F)$, then $u \in \mathscr{C}^\infty(U, E)$.

(3) If $Pu = \lambda u$ for some $\lambda \in \mathbb{C}$ and $m > 0$, then u is smooth.

Proof. (1) From Theorem 1.4.5, there exists $Q \in \Psi\mathrm{DO}_{-m}(F, E)$, such that $\mathrm{Id} = QP + S$, where S is a smoothing operator. So

$$\|u\|_{s+m} \leq \|QPu\|_{s+m} + \|Su\|_{s+m} \leq C\|Pu\|_s + C\|u\|_0 \leq \infty. \tag{1.4.20}$$

Thus $u \in \mathbf{H}_0^{s+m}(K, E)$.

(2) For $U \subset M$, if $Pu \in \mathscr{C}^\infty(U, F)$, by Proposition 1.3.17, $QPu \in \mathscr{C}^\infty(U, E)$ and $Su \in \mathscr{C}^\infty(U, E)$. Thus by $\mathrm{Id} = QP + S$, $u \in \mathscr{C}^\infty(U, E)$.

(3) If $m > 0$, $P - \lambda\,\mathrm{Id} \in \Psi\mathrm{DO}_m(E, F)$. By (2), since $(P - \lambda\,\mathrm{Id})u = 0$ is smooth, u is smooth.

The proof of Theorem 1.4.6 is completed. □

Theorem 1.4.7 (Fundamental elliptic estimate)**.** *Let $P \in \Psi\mathrm{DO}_m(E,F)$ be a elliptic operator. For any $s \in \mathbb{R}$, there exists $C > 0$ such that for any $u \in \mathbf{H}_0^{s+m}(K,E)$, we have*

$$\|u\|_{s+m} \le C(\|Pu\|_s + \|u\|_s). \tag{1.4.21}$$

Thus if M is compact, the norms $\|\cdot\|_{s+m}$ and $\|P\cdot\|_s + \|\cdot\|_s$ are equivalent.

Proof. From Theorem 1.4.5, there exists $Q \in \Psi\mathrm{DO}_{-m}(F,E)$, such that $\mathrm{Id} = QP + S$, where S is a smoothing operator. So

$$\|u\|_{s+m} \le \|QPu\|_{s+m} + \|Su\|_{s+m} \le C\|Pu\|_s + C\|u\|_s. \tag{1.4.22}$$

The proof of Theorem 1.4.7 is completed. □

Obviously, Theorems 1.4.6 and 1.4.7 hold for $M = \mathbb{R}^n$. We state it here.

Theorem 1.4.8. *Let $P \in \Psi\mathrm{DO}_m$ be a elliptic operator.*

(1)Let $u \in L^2$. If $Pu \in \mathbf{H}^s$, then $u \in \mathbf{H}^{s+m}$.

(2) For any open subset $U \subset \mathbb{R}^n$, if $Pu \in \mathscr{C}^\infty(U,\mathbb{C}^p)$, then $u \in \mathscr{C}^\infty(U,\mathbb{C}^p)$.

(3) If $Pu = \lambda u$ for some $\lambda \in \mathbb{C}$ and $m > 0$, then u is smooth.

(4) For any $s \in \mathbb{R}$, compact set $K \subset \mathbb{R}^n$, there exists $C, C' > 0$ such that for any $u \in \mathbf{H}^{s+m}$, $\mathrm{supp}(u) \subset K$, we have

$$\|u\|_{s+m} \le C(\|Pu\|_s + \|u\|_s) \le C'\|u\|_{s+m}. \tag{1.4.23}$$

Corollary 1.4.9 (Inner gradient estimate)**.** *Let P be an elliptic differential operator of order $m > 0$ defined on an open subset Ω of $\mathbb{R}^n$. Then for any compact subset $K \subset \Omega$ and $k \in \mathbb{N}$, there exists $C_{K,k} > 0$ such that for any solution u of the equation $Pu = 0$, we have*

$$\|u\|_{K,\mathscr{C}^k} \le C_{K,k}\|u\|_{\Omega,\mathscr{C}^0}, \quad \|u\|_{K,\mathscr{C}^k} \le C_{K,k}\|u\|_{\Omega,L^2}. \tag{1.4.24}$$

Proof. Choose $\varphi \in \mathscr{C}_0^\infty(\Omega)$ such that $\varphi \equiv 1$ on K. Then

$$P(\varphi u) = \varphi Pu + P'u = P'u, \tag{1.4.25}$$

where P' is a differential operator of order $m-1$. By Theorem 1.4.7,

$$\begin{aligned}\|u\|_{K,s} \le \|\varphi u\|_{\Omega,s} &\le C(\|P(\varphi u)\|_{\Omega,s-m} + \|\varphi u\|_{\Omega,s-m}) \\ &= C(\|P'u\|_{\Omega,s-m} + \|\varphi u\|_{\Omega,s-m}) \le C'\|u\|_{\Omega,s-1}.\end{aligned} \tag{1.4.26}$$

Take a sequence $K \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \subset\subset \Omega_N = \Omega$. Using (1.4.26) repeatedly, by Sobolev embedding theorem, we get the inner gradient estimate (1.4.24).

The proof of Corollary 1.4.9 is completed. □

1.4.2 Laplacian

In this subsection, we introduce the most important elliptic operator: Laplacian.

For vector fields X, Y on manifold M, we define the vector field $[X, Y]$ by

$$[X, Y]f = X(Yf) - Y(Xf), \tag{1.4.27}$$

for any $f \in \mathscr{C}^\infty(M)$. Assume that M is a Riemannian manifold with Riemannian metric g^{TM}. Then there is a canonical connection $\nabla^{TM} : \mathscr{C}^\infty(M, TM) \to \mathscr{C}^\infty(T^*M \otimes TM)$, called the **Levi-Civita connection** defined by

$$\begin{aligned}2(\nabla_X Y, Z) = ([X, Y], Z) - ([Y, Z], X) + ([Z, X], Y)\\ + X(Y, Z) + Y(Z, X) - Z(X, Y),\end{aligned} \tag{1.4.28}$$

where X, Y, Z are vector fields and $(\cdot, \cdot) = g^{TM}(\cdot, \cdot)$. It is the unique connection which preserves the Riemannian metric:

$$d(X, Y) = (\nabla X, Y) + (X, \nabla Y) \tag{1.4.29}$$

for vector fields X, Y, and which is torsion free:

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{1.4.30}$$

On $\mathbb{R}^n$, the Laplace operator is defined by

$$\Delta = -\frac{\partial^2}{\partial x_i^2}. \tag{1.4.31}$$

Let E be a vector bundle over a Riemannian manifold M, with connection ∇^E. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of (T_xM, g^{TM}), i.e., locally e_i is a vector field and at $x \in M$, $(e_i, e_j) = \delta_{ij}$. Then naively we want to define the Laplace operator on vector bundles by

$$\Delta^E = -\nabla^E_{e_i}\nabla^E_{e_i}. \tag{1.4.32}$$

As usual, we need to check that the operator is independent of the basis chosen. Let $\{e'_i\}_{i=1}^n$ be an orthonormal basis of (T_xM, g^{TM}) such that $e'_i = h_{ij}e_j$, where (h_{ij}) is an orthogonal matrix. Then using this basis,

$$\begin{aligned}-\nabla^E_{e'_i}\nabla^E_{e'_i} &= -h_{ik}\nabla^E_{e_k}(h_{ij}\nabla^E_{e_j}) = -h_{ik}h_{ij}\nabla^E_{e_k}\nabla^E_{e_j} - h_{ik}\nabla^E_{e_k(h_{ij})e_j}\\ &= -\delta_{jk}\nabla^E_{e_k}\nabla^E_{e_j} - h_{ik}\nabla^E_{\nabla_{e_k}(h_{ij}e_j) - h_{ij}\nabla_{e_k}e_j}\\ &= -\nabla^E_{e_i}\nabla^E_{e_i} - \nabla^E_{\nabla_{e'_i}e'_i} + \nabla^E_{\nabla_{e_i}e_i},\end{aligned} \tag{1.4.33}$$

where ∇ is any connection on TM. Thus

$$-\nabla^E_{e_i}\nabla^E_{e_i} + \nabla^E_{\nabla_{e_i}e_i} \tag{1.4.34}$$

does not depend on the choice of the basis. For the convenience and the uniqueness, we take $\nabla = \nabla^{TM}$, the Levi-Civita connection.

Definition 1.4.10. The Laplacian Δ^E on $\mathscr{C}^\infty(M,E)$ is the second order differential operator

$$\Delta^E = -\nabla^E_{e_i}\nabla^E_{e_i} + \nabla^E_{\nabla^{TM}_{e_i}e_i}. \tag{1.4.35}$$

Locally, the second order term of the Laplacian is just (1.4.31). So the principal symbol

$$\sigma_\xi(\Delta^E) = |\xi|^2 \cdot \mathrm{Id}_E\,. \tag{1.4.36}$$

By Definition 1.4.1, Δ^E is a elliptic operator.

Proposition 1.4.11. *For $u_1, u_2 \in \mathscr{C}^\infty_0(M,E)$, if ∇^E is a Hermitian connection, then we have*

$$\int_M (\Delta^E u_1, u_2)dv = \int_M (\nabla^E_{e_i}u_1, \nabla^E_{e_i}u_2)dv = \int_M (u_1, \Delta^E u_2)dv. \tag{1.4.37}$$

Proof. Take $\alpha \in \mathscr{C}^\infty(M, T^*M)$ such that

$$\alpha(X) = (\nabla^E_X u_1, u_2), \tag{1.4.38}$$

for vector field X. Then

$$e_i(\alpha(e_i)) = (\nabla^E_{e_i}u_1, \nabla^E_{e_i}u_2) - (\Delta^E u_1, u_2) + \alpha(\nabla^{TM}_{e_i}e_i). \tag{1.4.39}$$

Note that $\alpha(e_i)e_i$ is a vector field which is independent of the choice of the basis. Then

$$\beta = i_{\alpha(e_i)e_i}dv = \sum_i (-1)^i \alpha(e_i) e^1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e^n. \tag{1.4.40}$$

is a global defined differential form. Since $e^i \wedge \nabla^{T^*M}_{e_i}$ satisfies three conditions in Proposition 1.1.11 by replacing d to $e^i \wedge \nabla^{T^*M}_{e_i}$, we have an important formula

$$d = e^i \wedge \nabla^{T^*M}_{e_i}. \tag{1.4.41}$$

Thus

$$\begin{aligned} d\beta &= e_i(\alpha(e_i))e^1\wedge\cdots\wedge e^n+\sum_{i,k}(-1)^i\alpha(e_i)e^k\wedge\nabla^{T^*M}_{e_k}(e^1\wedge\cdots\wedge\hat{e}_i\wedge\cdots\wedge e^n)\\ &= e_i(\alpha(e_i))e^1\wedge\cdots\wedge e^n+\sum_i\alpha(e_i)\nabla^{T^*M}_{e_i}(e^1\wedge\cdots\wedge e^n)\\ &\quad -\sum_{i,k}\alpha(e_i)(\nabla^{T^*M}_{e_k}e^k,e_i)e^1\wedge\cdots\wedge e^n. \end{aligned} \tag{1.4.42}$$

Note that

$$\langle\nabla^{T^*M}_{e_i}e^k,e^k\rangle=\frac{1}{2}(\langle\nabla^{T^*M}_{e_i}e^k,e^k\rangle+\langle e^k,\nabla^{T^*M}_{e_i}e^k\rangle)=e_i(\langle e^k,e^k\rangle)=e_i(1)=0. \tag{1.4.43}$$

Thus

$$\nabla^{T^*M}_{e_i}(e^1\wedge\cdots\wedge e^n)=\sum_k\langle\nabla^{T^*M}_{e_i}e^k,e^k\rangle e^1\wedge\cdots\wedge e^n=0. \tag{1.4.44}$$

Since

$$(\nabla^{T^*M}_{e_k}e^k,e_i)=-\langle e_k,\nabla^{TM}_{e_k}e_i\rangle=\langle\nabla^{TM}_{e_k}e^k,e_i\rangle, \tag{1.4.45}$$

from (1.4.42)-(1.4.44), we have

$$d\beta=(e_i(\alpha(e_i))-\alpha(\nabla^{TM}_{e_i}e_i))e^1\wedge\cdots\wedge e^n. \tag{1.4.46}$$

From (1.4.39) and (1.4.46), we have

$$(\nabla^E_{e_i}u_1,\nabla^E_{e_i}u_2)dv-(\Delta^E u_1,u_2)dv=d\beta. \tag{1.4.47}$$

Therefore, our proposition follows from (1.4.47) and the Stokes formula.

The proof of Proposition 1.4.11 is completed. □

Definition 1.4.12. The **generalized Laplacian** (Schrödinger operator) H associated with ∇^E is of the form

$$H=\Delta^E+Q, \tag{1.4.48}$$

where Q is a section of $\mathrm{End}(E)=E^*\otimes E$ on M with lower bound, i.e., there exists $C>0$, such that for any $u\in\mathscr{C}_0^\infty(M,E)$,

$$(Qu,u)_{L^2}\geq -C\|u\|^2. \tag{1.4.49}$$

We say H is symmetric if ∇^E is a Hermitian connection and $Q^* = Q$. From Proposition 1.4.11, for symmetric H, we have

$$\int_M (Hu_1, u_2)dv = \int_M (u_1, Hu_2)dv. \tag{1.4.50}$$

Theorem 1.4.13 (Gårding's inequality)**.** *Let $K \subset M$ be a compact subset. For symmetric H, there exists $C > 0$ such that for any $u \in \mathscr{C}_0^\infty(K, E)$,*

$$\|u\|_1^2 \leq C\big((Hu, u)_{L^2} + \|u\|_0^2\big). \tag{1.4.51}$$

Proof. From Proposition 1.4.11, (1.4.48)and (1.4.49), we have

$$\begin{aligned}(Hu, u)_{L^2} = (\Delta u, u)_{L^2} + (Qu, u)_{L^2} = \|u\|_1^2 + (Qu, u)_{L^2} - \|u\|_0^2 \\ \geq \|u\|_1^2 - (C+1)\|u\|_0^2.\end{aligned} \tag{1.4.52}$$

The proof of Theorem 1.4.13 is completed. □

1.4.3 Fredholm operator

Let $T : H_1 \to H_2$ be a bounded linear map between Hilbert spaces. The kernel of T is

$$\mathrm{Ker}(T) := \{v \in H_1 : Tv = 0\}. \tag{1.4.53}$$

The range of T is

$$\mathrm{Im}(T) := \{Tv \in H_2 : v \in H_1\}. \tag{1.4.54}$$

The cokernel of T is the quotient space

$$\mathrm{Coker}(T) := H_2/\overline{\mathrm{Im}(T)}. \tag{1.4.55}$$

Definition 1.4.14. We say a bounded linear map $T : H_1 \to H_2$ is a **Fredholm operator** if its kernel and cokernel are finite dimensional and its range is closed. The index of the Fredholm operator is defined by

$$\mathrm{ind}(T) := \dim \mathrm{Ker}\, T - \dim \mathrm{Coker}\, T. \tag{1.4.56}$$

Lemma 1.4.15. *Let $P : H_1 \to H_2$ and $Q : H_2 \to H_1$ be bounded linear maps such that $QP = 1 - S_1$ and $PQ = 1 - S_2$, where S_1 and S_2 are compact operators. Then P and Q are Fredholm operators.*

Proof. If $\mathrm{Ker}(P)$ is infinite dimensional, we can choose an orthonormal basis $v_1, v_2, \cdots$ of $\mathrm{Ker}\, P$. Since $S_1|_{\mathrm{Ker}\, P} = \mathrm{Id}$, we can not find a convergent subsequence of $\{S_1(v_i) = v_i\}$. It is a contradiction with the fact that S_1 is a compact operator. Thus $\ker P$ is finite dimensional.

Let V be the orthonormal complement of $\overline{\mathrm{Im} P}$ in H_2. Then $V \simeq \mathrm{Coker}(P)$. Let P^* be the adjoint of P. If $v \in \mathrm{Ker}\, P^*$, then for any $u \in H_1$,

$$0 = \langle P^* v, u\rangle = \langle v, Pu\rangle. \tag{1.4.57}$$

Thus $v \in V$. If $v \in V$, by (1.4.57) $v \in \mathrm{Ker}\, P^*$. So we have

$$\mathrm{Ker}\, P^* \simeq \mathrm{Coker}\, P. \tag{1.4.58}$$

Since S_2 is compact, S_2^* is compact[3]. Since $Q^* P^* = \mathrm{Id} - S_2^*$, we have $\dim(\mathrm{Coker}\, P) = \mathrm{Ker}(P^*) < +\infty$.

At last, we prove that $\mathrm{Im} P$ is closed. Let $v_k = Pu_k$, $k \in \mathbb{N}$ be a sequence such that $v_k \to v$ in H_2. We need to prove that $v = Pu$ for some $u \in H_1$. We may assume that $u_k \in (\mathrm{Ker}\, P)^{\perp}$.

We claim that $\{u_k\}$ is bounded. Otherwise, by passing to a subsequence, we can assume that $\|u_k\| \to \infty$. So $P(u_k/\|u_k\|) = v_k/\|u_k\| \to 0$. Since S_1 is compact, by passing to a subsequence, $\lim_{k\to\infty}(u_k/\|u_k\|) = \lim_{k\to\infty} S_1(u_k/\|u_k\|) = w$. Note that $\|w\| = 1$. However, by continuity, $Pw = 0$. Since $P|_{(\mathrm{Ker}\, P)^{\perp}}$ is injective, $w = 0$. It is a contradiction. Therefore, $\{u_k\}$ is bounded.

Since $\{u_k\}$ is bounded, by passing to a subsequence, $S_1 u_k \to u_\infty$. Since $Qv_k \to Qv$ and $Qv_k = QPu_k = u_k - S_1 u_k$, we have $PQv = \lim_{k\to+\infty}(Pu_k - PS_1 u_k) = v - Pu_\infty$. Thus $v = P(u_\infty + Qv) \in \mathrm{Im} P$. Therefore $\mathrm{Im} P$ is closed and P is Fredholm.

By symmetry, Q is also Fredholm.

The proof of Lemma 1.4.15 is completed. □

Theorem 1.4.16. *Assume that M is compact. Let $P \in \Psi\mathrm{DO}_m(E, F)$ be an elliptic operator. Then for each $s \in \mathbb{R}$, $P_s : \mathbf{H}^s(E) \to \mathbf{H}^{s-m}(F)$ is Fredholm and* $\mathrm{ind}(P_s)$ *is independent of $s \in \mathbb{R}$.*

Proof. By Rellich Theorem, the smoothing operator is compact. Thus by Lemma 1.4.15, P_s is Fredholm.

By Theorem 1.4.6 (3), $\mathrm{Ker}\, P$ consists of smooth sections. Thus its dimension is independent of s. The same is for $\dim \mathrm{Coker}\, P = \dim \mathrm{Ker}\, P^*$.

The proof of Theorem 1.4.16 is completed. □

[3]"Functional Analysis" by Zhang gongqing, Theorem 4.1.5.

Since $\mathrm{ind}(P_s)$ is independent of s, we denote it by $\mathrm{ind}(P)$.

Add some properties of Fredholm operators, especially $\mathrm{ind}(T + R) = \mathrm{ind}(T)$ for R compact. We could use it to prove that if $\mathrm{ind}(P) \neq 0$ for elliptic operator P, then $\sigma(P) = \mathbb{C}$.

Remark that the famous Atiyah-Singer index theorem explains $\mathrm{ind}(P)$ as a topological formula.

Chapter 2

Spectral Theory

In the first chapter, we show that the operators we studied are bounded between two Sobolev spaces. However, if we want to study the eigenvalues of an operator, we must assume that it is a linear operator on one Hilbert space. Unfortunately, if the operator has a positive order, it is not bounded on a Hilbert space. But for elliptic operator, we could prove that it is a closed unbounded operator on the canonical L^2-space. In this chapter, we will study the spectral theory, roughly speaking, the eigenvalues, of the closed unbounded operator.

2.1 Compact operator

spectral theory, Hilbert-Schimidt, Trace class, kernel function, Schatten norm...
"Classes of linear operator I" "PDE I" Taylor, APP A.

2.2 Symmetry and Self-adjoint

2.2.1 Closed operator

As we discussed in Section 1.2.1, for a closed unbounded operator $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$, from the closed graph theorem: Theorem 1.2.11, T will only be defined on a linear subspace of $\mathcal{H}$. This subspace, which we denote by $D(T)$, is called the **domain** of the operator T (for the definition of the domain, we don't need T is closed).

Warning: To study an unbounded operator on a Hilbert space, we must first fix the domain and then see how it acts on that space.

Definition 2.2.1. The graph of the linear operator T is the set of pairs $\{\langle u, Tu\rangle \in \mathcal{H} \times \mathcal{H} : u \in D(T)\}$. The graph of T, denoted by $\Gamma(T)$, is a Hilbert space with inner product

$$(\langle u_1, v_1\rangle, \langle u_2, v_2\rangle) = (u_1, u_2) + (v_1, v_2). \tag{2.2.1}$$

The corresponding norm is denoted by $\|\cdot\|_\Gamma$. From (2.2.1),

$$\|\langle u, Tu\rangle\|_\Gamma^2 = \|u\|^2 + \|Tu\|^2. \tag{2.2.2}$$

Recall that in Definition 1.2.9, we say T is closed if for $u_k \in D(T)$, $u_k \to u$, $Tu_k \to v$, we have $u \in D(T)$ and $v = Tu$. The following proposition follows directly from Definition 1.2.9. In fact, in many literatures, this is the definition of the closed operator.

Proposition 2.2.2. *The linear operator T is closed if and only if $\Gamma(T)$ is a closed subspace of $\mathcal{H} \times \mathcal{H}$.*

Proof. Assume T is closed. If $\langle u_k, Tu_k\rangle$ converges in $\mathcal{H} \times \mathcal{H}$ with respect to the norm (2.2.1), since $\|\langle u_k, Tu_k\rangle\|_\Gamma^2 = \|u_k\|^2 + \|Tu_k\|^2$, we see that u_k and Tu_k converge. Let $u_k \to u$, $Tu_k \to v$, since T is closed, $u \in D(T)$ and $Tu = v$. Thus $\langle u_k, Tu_k\rangle \to \langle u, Tu\rangle \in \Gamma(T)$.

Assume that $\Gamma(T)$ is closed. If for $u_k \in D(T)$, $u_k \to u$, $Tu_k \to v$, since $\Gamma(T)$ is closed, the limit of $\langle u_k, Tu_k\rangle$ exists in $\Gamma(T)$, which is $\langle u, v\rangle$. Thus T is closed.

The proof of Proposition 2.2.2 is closed. □

Definition 2.2.3. Let T_1 and T_2 be linear operators on $\mathcal{H}$. We say T_2 is an extension of T_1, which we write $T_1 \subset T_2$, if $D(T_1) \subset D(T_2)$, i.e., $D(T_1) \subset D(T_2)$ and for any $u \in D(T_1)$, $T_2 u = T_1 u$.

Definition 2.2.4. An operator T is closable if it has a closed extension. The smallest closed extension, which exists obviously, is called the closure of T, denoted by $\overline{T}$.

In the followings, we will prove that the elliptic pesudodifferential operators on $\mathbb{R}^n$ or compact manifolds are closable. This is our main purpose to study closed operators here.

Proposition 2.2.5. *If M is compact, the elliptic pesudodifferential operator $P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, E)$ of order $m > 0$ is closable. We will also denote the closure by P for the simplicity. In this case, $D(P) = \mathbf{H}^m(M, E)$.*

Proof. From the elliptic estimate (1.4.21), the norms $\|P\cdot\|_0 + \|\cdot\|_0$ and $\|\cdot\|_m$ are equivalent. From Definition 2.2.1, the norm $\|P\cdot\|_0 + \|\cdot\|$ is equivalent to the norm of the graph $\Gamma(P)$. Thus the closure of $\Gamma(P)$ is $\{(u, Pu) : u \in \mathbf{H}^m(M, E)\}$.

The proof of Proposition 2.2.5 is completed. □

In the same way, we have the corresponding result for $M = \mathbb{R}^n$.

Proposition 2.2.6. *The elliptic pesudodifferential operator $P : \mathscr{C}^\infty \to \mathscr{C}^\infty$ of order $m > 0$ is closable. We will also denote the closure by P for the simplicity. In this case, $D(P) = \mathbf{H}^m$.*

Remark 2.2.7. (1) If M is a general noncompact manifold, the case is complex. We need additional conditions to obtain the elliptic estimates. We will not discuss it in this note.

(2) If $m = 0$, by Proposition 1.3.2, P is bounded. If $m < 0$, by Rellich's theorem and Proposition 1.3.2, P is compact operator. They are easier to handle than closed operator.

(3) From Propositions 2.2.5 and 2.2.6, we see that in general, even if T is closed, $D(T)$ may be not a closed space.

Proposition 2.2.8. *If T is closable, then $\Gamma(\overline{T}) = \overline{\Gamma(T)}$. Thus we could obtain the closure of T be taking the closure of $\Gamma(T)$.*

Proof. Suppose that S is a closed extension of T. Then $\overline{\Gamma(T)} \subset \overline{\Gamma(S)} = \Gamma(S)$. Thus if $\langle 0, v\rangle \in \overline{\Gamma(T)}$, $v = 0$. Let $A = \{u : \exists v, s.t. \langle u, v\rangle \in \overline{\Gamma(T)}\}$. Thus for $u \in A$, there exists unique $v \in \mathcal{H}$ such that $\langle u, v\rangle \in \overline{\Gamma(T)}$. Define R by $Ru = v$. Then $\Gamma(R) = \overline{\Gamma(T)}$. So R is a closed extension of T. Since $R \subset S$ for any closed extension S, we have $\overline{T} = R$.

The proof of Proposition 2.2.8 is completed. □

Remark that for general linear operator T, the closure of $\Gamma(T)$ may not be a graph of an operator.

Proposition 2.2.9. *A linear operator T is closable if and only if for $u_k \in D(T)$, $u_k \to 0$, $Tu_k \to v$, we have $v = 0$.*

Proof. If $u_k \in D(T)$, $u_k \to 0$, $Tu_k \to v$, then $\langle 0, v\rangle \in \overline{\Gamma(T)}$. Since T is closable, by Proposition 2.2.8, $\langle 0, v\rangle \in \Gamma(\overline{T})$. Thus $v = 0$.

For the other direction, if $\langle u, v_1\rangle, \langle u, v_2\rangle \in \overline{\Gamma(T)}$, then there exist $u_k \to u$, $u'_k \to u$ such that $Tu_k \to v_1$ and $Tu'_k \to v_2$. Note that $u_k - u'_k \in D(T)$. Thus $u_k - u'_k \to 0$ and $T(u_k - u'_k) \to v_1 - v_2$. So $v_1 = v_2$. Thus we could define a operator R such that for any $\langle u, v\rangle \in \overline{\Gamma(T)}$, $v = Ru$. Since $\Gamma(R) = \overline{\Gamma(T)}$

is closed, by Proposition 2.2.2, R is a closed operator, which is a closed extension of T.

The proof of Proposition 2.2.9 is completed. □

Now we summarize some properties of the closed operator.

Proposition 2.2.10. *(1) If T is a $1-1$ closed operator, then T^{-1} is closed.*

(2) If T is closed, then Ker(T) *is a closed space.*

(3) If T is closed, $D(T) = H$, then T is bounded.

Proof. (1) Rotate $\Gamma(T)$ and use Proposition 2.2.2.

(2) If $u_k \in \mathrm{Ker}(T)$, $u_k \to u$, then $Tu_k \equiv 0$. Since T is closed, $Tu = 0$.

(3) It is the closed graph theorem (Theorem 1.2.11).

The proof of Proposition 2.2.10 is completed. □

2.2.2 Symmetric and self-adjoint

Now we study the adjoint of a linear operator. In order to get a well-defined adjoint operator, we need the condition that the domain is dense in $\mathcal{H}$.

Definition 2.2.11. We say T is densely defined if $D(T)$ is dense in $\mathcal{H}$.

From the closed graph theorem, for unbounded closed operator, $D(T) \neq \mathcal{H}$. We can always assume that $D(T)$ is dense in $\mathcal{H}$. If not, we consider the Hilbert space $\overline{D(T)}$.

Remark that since the space of the smooth functions is dense in the L^2-space, all elliptic operators and pseudodifferential operators are densely defined.

Definition 2.2.12. Let T be a densely defined linear operator on $\mathcal{H}$. Let $D(T^*)$ be the set of $v \in \mathcal{H}$ for which there exists $w \in \mathcal{H}$ such that for any $u \in D(T)$,

$$(Tu, v) = (u, w). \tag{2.2.3}$$

For each such $u \in D(T^*)$, we define $T^*v = w$. The operator T^* is called the adjoint of T.

Obviously, T^* is linear. Since T is densely defined, T^* is well-defined: if for any $u \in D(T)$, $(u, w) = (u, w')$, then $w = w'$. In general $D(T^*)$ may not be dense.

Proposition 2.2.13. *Let T, S be a densely defined linear operators on $\mathcal{H}$.*

(1) The element $v \in D(T^*)$ if and only if there exists a constant $C_v > 0$ such that for any $u \in D(T)$, $(Tu, v) \leq C_v\|u\|$.

(2) The adjoint T^* is closed.

(3) If $S \subset T$, then $T^* \subset S^*$.

(4) T is closable if and only if $D(T^*)$ is dense, in which case $\overline{T} = T^{**}$.

(5) If T is closable, then $(\overline{T})^* = T^*$.

(6) T is closed if and only if T^* is densely defined and $T = T^{**}$.

Proof. (1) If $u \in D(T^*)$, by (2.2.3), $(Tu, v) \leq \|w\|\|u\|$. If for any $u \in D(T)$, $(Tu, v) \leq C_v\|u\|$, then by Riesz representation theorem[1], since (Tu, v) is bounded linear on u, there exists $w \in \mathcal{H}$ such that $(Tu, v) = (u, w)$.

(2) We define a operator V on $\mathcal{H}\times\mathcal{H}$ by $V\langle u, v\rangle = \langle -v, u\rangle$. Then $V(\Gamma(T))$ is a linear space. We claim that

$$\Gamma(T^*) = (V(\Gamma(T)))^{\perp}, \tag{2.2.4}$$

where $\cdot^{\perp}$ is the complement with respect to the inner product (2.2.1). In fact, if $u \in D(T^*)$, for any $\langle w, Tw\rangle \in \Gamma(T)$,

$$\begin{aligned}(\langle u, T^*u\rangle, V\langle w, Tw\rangle) &= (\langle u, T^*u\rangle, \langle -Tw, w\rangle)\\ &= -(u, Tw) + (T^*u, w) = 0.\end{aligned} \tag{2.2.5}$$

Thus $\Gamma(T^*) \subset (V(\Gamma(T)))^{\perp}$. On the other hand, if $\langle u, v\rangle \in (V(\Gamma(T)))^{\perp}$, then for any $\langle w, Tw\rangle \in \Gamma(T)$,

$$0 = (\langle u, v\rangle, V\langle w, Tw\rangle) = (\langle u, v\rangle, \langle -Tw, w\rangle) = -(u, Tw) + (v, w). \tag{2.2.6}$$

Thus $\langle u, v\rangle \in \Gamma(T^*)$. Then (2.2.4) holds.

Note that $(V(\Gamma(T)))^{\perp}$ is a closed subspace. So is $\Gamma(T^*)$. From Proposition 2.2.2, T^* is closed.

(3) Since $\Gamma(S) \subset \Gamma(T)$, $V(\Gamma(S)) \subset V(\Gamma(T))$. Thus (3) follows from (2.2.4).

(4) We claim that for any subspace A in $\mathcal{H} \times \mathcal{H}$,

$$V(A^{\perp}) = V(A)^{\perp}. \tag{2.2.7}$$

This follows from

$$\begin{aligned}(\langle u, v\rangle, V\langle w, t\rangle) &= (\langle u, v\rangle, \langle -t, w\rangle) = -(u, t) + (v, w)\\ &= (\langle v, -u\rangle, \langle w, t\rangle) = (V\langle u, v\rangle, \langle w, t\rangle),\end{aligned} \tag{2.2.8}$$

[1] Theorem 2.2.1 in ”Functional analysis” by Zhang

for any $(w,t) \in A$.

If $D(T^*)$ is dense, by (2), T^{**} is well-defined and closed. By (2.2.4) and (2.2.7),

$$\Gamma(T^{**}) = \left(V\left(\Gamma(T^*)\right)\right)^\perp = \left(V\left(V\left(\Gamma(T)\right)^\perp\right)\right)^\perp = \left(V^2\left(\Gamma(T)^\perp\right)\right)^\perp$$
$$= \left(\Gamma(T)^\perp\right)^\perp = \overline{\Gamma(T)}. \quad (2.2.9)$$

From Proposition 2.2.8, T is closable and $\overline{T} = T^{**}$.

If $D(T^*)$ is not dense, take $v \in D(T^*)^\perp$, $v \neq 0$. Then $\langle v, 0\rangle \in \Gamma(T^*)^\perp$. So $V(\Gamma(T^*))^\perp$ is not a graph of an operator. Since by (2.2.9), $\overline{\Gamma(T)} = V(\Gamma(T^*))^\perp$, we see that T is not closable.

(5) If T is closable, from (2) and (4),

$$T^* = \overline{T^*} = T^{***} = \overline{T}^*. \quad (2.2.10)$$

(6) follows directly from (4).

The proof of Proposition 2.2.13 is completed. □

Definition 2.2.14. Let T be a densely defined linear operator on $\mathcal{H}$. If $T \subset T^*$, we say T is **symmetric**. If $T = T^*$, we say T is **self-adjoint**. If T is closable and $\overline{T}$ is self-adjoint, we say T is **essentially self-adjoint**.

If T is symmetric, $D(T) \subset D(T^*)$. Thus T^* is densely defined. By Proposition 2.2.13 (2) and (4), we have

Proposition 2.2.15. *(1) Symmetric operator is closable.*

(2) Self-adjoint operator is closed.

The following proposition is obvious.

Proposition 2.2.16. *A densely defined linear operator T is symmetric if and only if for any $u, v \in D(T)$,*

$$(Tu, v) = (u, Tv). \quad (2.2.11)$$

Proposition 2.2.17. *A densely defined operator T is essentially self-adjoint if and only if $T \subset T^{**} = T^*$.*

Proof. Since T is closable. By Proposition 2.2.13 (4), $\overline{T} = T^{**}$.

If T is essentially self-adjoint, $\overline{T}$ is self-adjoint. By Proposition 2.2.13 (5), $T^{**} = \overline{T} = (\overline{T})^* = T^*$.

If $T \subset T^{**} = T^*$, $(\overline{T})^* = T^* = T^{**} = \overline{T}$. Thus $\overline{T}$ is self-adjoint.

The proof of Proposition 2.2.17 is completed. □

Proposition 2.2.18. *Let T be a densely defined symmetric operator. The following statements are equivalent.*
(1) T is self-adjoint.
(2) $D(T) = D(T^*)$.
(3) $T = T^{**} = T^*$.

Proof. Obvious. □

Remark 2.2.19. We compare the properties of closed, essentially self-adjoint and self-adjoint for a densely defined symmetric operator. Let T be a densely defined symmetric operator. Then

- T is closed $\Longleftrightarrow T = T^{**} \subset T^*$;
- T is essentially self-adjoint $\Longleftrightarrow T \subset T^{**} = T^*$;
- T is self-adjoint $\Longleftrightarrow T = T^{**} = T^*$.

For symmetric operator T on $\mathcal{H}$, for $\mu > 0$, we have

$$\|(T \pm i\mu \,\mathrm{Id})x\|^2 = ((T \pm i\mu \,\mathrm{Id})x, (T \pm i\mu \,\mathrm{Id})x) = \mu^2\|x\|^2 + \|Tx\|^2. \quad (2.2.12)$$

So for any $\mu > 0$, we have

$$\mathrm{Ker}(T \pm i\mu \,\mathrm{Id}) = 0. \quad (2.2.13)$$

Lemma 2.2.20. *If T is a closed symmetric operator, then* $\mathrm{Im}(T \pm i\mu \,\mathrm{Id})$ *is closed.*

Proof. If $(T + i\mu \,\mathrm{Id})x_k \to y$, since $\|(T + i\mu \,\mathrm{Id})(x_k - x_j)\|^2 = \mu^2\|x_k - x_j\|^2 + \|T(x_k - x_j)\|^2$, $x_k \to x$, $Tx_k \to w$. Since T is closed, $w = Tx$. Thus $y = w + i\mu x = (T + i\mu \,\mathrm{Id})x$. The proof for $T - i\mu \,\mathrm{Id}$ is the same.
The proof of Lemma 2.2.20 is completed. □

Lemma 2.2.21. *Let T be a densely defined symmetric operator. Then for any $\mu > 0$,*

$$\mathrm{Ker}(T^* \pm i\mu \,\mathrm{Id}) = \mathrm{Im}(T \mp i\mu \,\mathrm{Id})^{\perp}. \quad (2.2.14)$$

Proof. If $v \in \mathrm{Ker}(T^* + i\mu \,\mathrm{Id})$, $v \in D(T^*)$. For any $u \in D(T)$,

$$((T - i\mu \,\mathrm{Id})u, v) = (u, (T^* + i\mu \,\mathrm{Id})v) = 0. \quad (2.2.15)$$

Thus $v \in \mathrm{Im}(T - i\mu \,\mathrm{Id})^{\perp}$.
If $v \in \mathrm{Im}(T - i\mu \,\mathrm{Id})^{\perp}$, also as in (2.2.15), $v \in \mathrm{Ker}(T^* + i\mu \,\mathrm{Id})$.
The proof for $T^* - i\mu \,\mathrm{Id}$ is the same.
The proof of Lemma 2.2.21 is completed. □

Proposition 2.2.22. *Let T be a densely defined symmetric operator. Then the following statements are equivalent:*

(1) T is self-adjoint;

(2) T is closed and there exists $\mu > 0$ such that $\mathrm{Ker}(T^* \pm i\mu\,\mathrm{Id}) = 0$;

(3) there exists $\mu > 0$ such that $\mathrm{Im}(T \mp i\mu\,\mathrm{Id}) = H$.

Proof. For (1) $\Longrightarrow$ (2), we use Proposition 2.2.15 and (2.2.13).

For (2) $\Longrightarrow$ (3), we use Lemmas 2.2.20 and 2.2.21.

For (3) $\Longrightarrow$ (1), by Lemma 2.2.21, $\mathrm{Ker}(T^* \pm i\mu\,\mathrm{Id}) = 0$. For any $v \in D(T^*)$, there exists $u \in D(T)$ such that $(T^* \mp i\mu\,\mathrm{Id})v = (T \mp i\mu\,\mathrm{Id})u$. Since $T \subset T^*$, we have $(T^* \mp i\mu\,\mathrm{Id})(v - u) = 0$. Thus $v = u \in D(T)$. So $D(T) = D(T^*)$. By Proposition 2.2.18, T is self-adjoint.

The proof of Proposition 2.2.22 is completed. □

Proposition 2.2.23. *Let T be a densely defined symmetric operator. Then the following statements are equivalent:*

(1) T is essentially self-adjoint;

(2) there exists $\mu > 0$ such that $\mathrm{Ker}(T^* \pm i\mu\,\mathrm{Id}) = 0$;

(3) there exists $\mu > 0$ such that $\overline{\mathrm{Im}(T \mp i\mu\,\mathrm{Id})} = H$.

Proof. For (1) $\Longrightarrow$ (2), by Proposition 2.2.17, T^* is self-adjoint. From (2.2.13), we get (2).

(2)$\Leftrightarrow$(3) is obvious.

For (3) $\Longrightarrow$ (1), for any $v \in D(T^*)$, there exists $\{u_n\} \subset \mathcal{H}$ such that

$$\lim_{n\to\infty}(T \mp i\mu\,\mathrm{Id})u_n = (T^* \mp i\mu\,\mathrm{Id})v. \tag{2.2.16}$$

It implies that $\lim_{n\to\infty}(T^* \mp i\mu\,\mathrm{Id})(u_n - v) = 0$. Thus $\lim_{n\to\infty}\|(T^* \mp i\mu\,\mathrm{Id})(u_n - v)\|^2 = \lim_{n\to\infty}\mu^2\|u_n - v\|^2 + \lim_{n\to\infty}\|T^*(u_n - v)\|^2 = 0$. So $u_n \to v$, $Tu_n \to T^*v$. Thus $(v, T^*v) \in \overline{\Gamma(T)} = \Gamma(\overline{T})$. So $v \in D(\overline{T}) = D(T^{**})$, which means that $T^* \subset T^{**}$.

On the other hand, from Proposition 2.2.13, $T^{**} \subset T^*$. So $T^* = T^{**}$. From Proposition 2.2.17, T is essentially self-adjoint.

The proof of Proposition 2.2.23 is completed. □

Theorem 2.2.24 (von Neumann)**.** *Let T be a densely defined closed symmetric operator. We have*

$$D(T^*) = D(T) \oplus \mathrm{Ker}(T^* - i\,\mathrm{Id}) \oplus \mathrm{Ker}(T^* + i\,\mathrm{Id}). \tag{2.2.17}$$

Moreover, for $u = u_0 + u_+ + u_- \in D(T^*)$ with decomposition as in (2.2.17), we have $T^*u = Tu_0 + iu_+ - iu_-$.

Proof. Denote by $D_\pm := \mathrm{Ker}(T^* \mp i\,\mathrm{Id})$.

We first claim that $D(T)$, D_+ and D_- are linear independent. In fact, if $u_0 + u_+ + u_- = 0$, where $u_0 \in D(T)$, $u_\pm \in D_\pm$, since $(T^* - i\,\mathrm{Id})u_- = -2iu_-$, we have $(T - i\,\mathrm{Id})u_0 = 2iu_-$. But from Lemma 2.2.21, we have $u_- \in \mathrm{Ker}(T^* + i\,\mathrm{Id}) = \mathrm{Im}(T - i\,\mathrm{Id})^\perp$. Thus $u_- = 0$. Similarly, we get $u_+ = 0$. Therefore, $D(T)$, D_+ and D_- are linear independent.

Obviously, $D(T) \oplus D_+ \oplus D_- \subset D(T^*)$. We claim that $D(T^*) \subset D(T) \oplus D_+ \oplus D_-$. In fact, from Lemmas 2.2.20 and 2.2.21, $\mathcal{H} = \mathrm{Im}(T - i\,\mathrm{Id}) \oplus D_-$. For any $u \in D(T^*)$, $v = (T^* - i\,\mathrm{Id})u$ has decomposition $v = v_1 + v_2$, where $v_1 \in \mathrm{Im}(T - i\,\mathrm{Id})$, $v_2 \in D_-$. Take u_0 such that $(T - i\,\mathrm{Id})u_0 = v_1$. Let $u_- = -(2i)^{-1}v_2$. Since $T^*u_- = -iu_-$, $(T^* - i\,\mathrm{Id})u_- = -2iu_- = v_2$. Then $(T^* - i\,\mathrm{Id})(u_0 + u_-) = v$. So $(T^* - i\,\mathrm{Id})(u - u_0 - u_-) = 0$. Let $u_+ = u - u_0 - u_- \in D_+$. We obtain the claim.

The proof of Theorem 2.2.24 is completed. □

2.2.3 Friedrichs extension

Proposition 2.2.25. *Essentially self-adjoint operator has unique self-adjoint extension.*

Proof. Let T be a essentially self-adjoint operator, then $\bar{T} = T^{**} = T^*$ is a self-adjoint extension. If T_0 be another self-adjoint extension, then $T^*\bar{T} \subset T_0$. Since $T_0 = T_0^* \subset T^{**} = T^*$, we have $\bar{T} = T_0$.

The proof of Proposition 2.2.25 is completed. □

In general, a symmetric operator may have many self-adjoint extensions. In this subsection, we introduce the Friedrich extension.

Definition 2.2.26. Let V be a dense linear subset of $\mathcal{H}$. Let $a : V \times V \to \mathbb{C}$ be a sesquilinear form, i.e., for any $b, c \in \mathbb{C}$, $u, v \in V$,

$$a(bu, cv) = b\bar{c} \cdot a(u, v), \tag{2.2.18}$$

such that a is positive definite, i.e., there exists $\alpha > 0$, such that for any $v \in V$,

$$a(v, v) \geq \alpha \|v\|^2. \tag{2.2.19}$$

Lemma 2.2.27. *If a is positive definite, a is symmetric, i.e., for any $u, v \in V$,*

$$a(u, v) = \overline{a(v, u)}. \tag{2.2.20}$$

Proof. For any $\lambda \in \mathbb{C}$,

$$0 \le a(\lambda u + v, \lambda u + v) = |\lambda|^2 a(u,u) + \lambda a(u,v) + \bar{\lambda} a(v,u) + a(v,v). \quad (2.2.21)$$

Thus $\lambda a(u,v) + \bar{\lambda} a(v,u) \in \mathbb{R}$. It is equivalent to

$$\lambda a(u,v) - \overline{\bar{\lambda} a(v,u)} = \lambda(a(u,v) - \overline{a(v,u)}) \in \mathbb{R}. \quad (2.2.22)$$

Since $\lambda \in \mathbb{C}$ is arbitrary taken, we have $a(u,v) = \overline{a(v,u)}$.

The proof of Lemma 2.2.27 is completed. □

Lemma 2.2.28 (Schwarz inequality)**.** *If a is positive definite, for any $u, v \in V$,*

$$|a(u,v)|^2 \le a(u,u)a(v,v). \quad (2.2.23)$$

Proof. From (2.2.21), for any $\lambda \in \mathbb{C}$,

$$|\lambda|^2 a(u,u) + 2\mathrm{Re}(\lambda a(u,v)) + a(v,v) \ge 0. \quad (2.2.24)$$

If $a(u,v) = re^{i\theta}$, we take $\lambda = te^{-i\theta}$. Then (2.2.24) is

$$a(u,u)t^2 + 2rt + a(v,v) \ge 0 \quad (2.2.25)$$

for any $t \in \mathbb{R}$. Thus

$$r^2 \le a(u,u)a(v,v). \quad (2.2.26)$$

The proof of Lemma 2.2.28 is completed. □

From Lemmas 2.2.27 and 2.2.28, we see that $a(\cdot,\cdot)$ is an inner product on V. It induces a norm on V:

$$\|v\|_a^2 = a(v,v)^{1/2} \quad (2.2.27)$$

for any $v \in V$. From (2.2.19), this norm is stronger than the normal norm of the Hilbert space.

Definition 2.2.29. Let a be a positive definite sesquilinear form. We denote by $D(a) := V$ the domain of a. If $D(a)$ is complete with respect to the norm $\|\cdot\|_a$, we say a is closed. In this case, $D(a)$ is a Hilbert space with respect to the norm $\|\cdot\|_a$.

For a symmetric operator T, since for any $u \in D(T)$, $(Tu,u) = (u,Tu) = \overline{(Tu,u)}$, we have $(Tu,u) \in \mathbb{R}$. We say a self-adjoint operator T is positive definite if there exists $\alpha > 0$ such that for any $u \in D(T)$, $(Tu,u) \ge \alpha\|u\|^2$.

Proposition 2.2.30. *Let a be a closed positive definite sesquilinear form with domain V. Then there exists unique positive definite self-adjoint operator T, such that $D(T) \subset V$ and for any $u \in D(T)$, $v \in V$,*

$$(v, Tu) = a(v, u). \tag{2.2.28}$$

Proof. Set

$$D(T) := \{u \in V : \exists C_u > 0,\ s.t.\ \forall v \in V, |a(v, u)| \leq C_u \|v\|\}. \tag{2.2.29}$$

By Riesz representation theorem, there exists unique $u^* \in \mathcal{H}$ such that

$$a(v, u) = (v, u^*). \tag{2.2.30}$$

We define $Tu = u^*$ for any $u \in D(T)$.

Obviously T is linear. We claim that T is densely defined. Since $\|\cdot\|_a$ is stronger than $\|\cdot\|$, for this claim, we only need to prove that $D(T)$ is dense in V with respect to $\|\cdot\|_a$, which is equivalent to prove that if $v_0 \in V$ and for any $u \in D(T)$, $a(v_0, u) = 0$, then $v_0 = 0$. In fact, for any $w \in \mathcal{H}$,

$$|(v, w)| \leq \|v\|\|w\| \leq \frac{1}{\sqrt{\alpha}}\|w\|\|v\|_a. \tag{2.2.31}$$

From Riesz representation theorem with respect to the inner product a, there exists $u_0 \in V$,

$$(v, w) = a(v, u_0). \tag{2.2.32}$$

Thus

$$\mathrm{Im}(T) = \mathcal{H}. \tag{2.2.33}$$

So if $0 = a(v_0, u) = (v_0, Tu)$ for any $u \in D(T)$, $v_0 = 0$. We obtain the claim.

For any $u, v \in D(T)$,

$$(v, Tu) = a(v, u) = \overline{a(u, v)} = \overline{(u, Tv)} = (Tv, u). \tag{2.2.34}$$

So T is symmetric.

Now we prove that T is self-adjoint. We only need to prove that $D(T^*) \subset D(T)$. If $u \in D(T^*)$, then there exists $u^* \in \mathcal{H}$ such that for any $v \in D(T)$,

$$(u^*, v) = (u, Tv). \tag{2.2.35}$$

From (2.2.33), there exists $w \in D(T)$ such that $u^* = Tw$. Thus $(w, Tv) = (Tw, v) = (u^*, v) = (u, Tv)$ for any $v \in D(T)$. From (2.2.33) again, we have $u = w \in D(T)$.

Since a is positive definite, T is positive definite.

At last, we prove the uniqueness. If T' be another self-adjoint operator satisfying all conditions. For any $u \in D(T')$, $T'u \in \mathcal{H} = \mathrm{Im}(T)$. Thus there exists $w \in D(T)$ such that $T'u = Tw$. Thus for any $v \in V$,

$$a(v,u) = (v, T'u) = (v, Tw) = a(v, w). \tag{2.2.36}$$

Since a is positive definite, by taking $v = u - w$, we get $u = w \in D(T)$. So $T' \subset T$. In the same way, $T \subset T'$. Thus $T = T'$.

The proof of Proposition 2.2.30 is completed. □

Definition 2.2.31. Let T be a symmetric operator. If there exists $c \in \mathbb{R}$ such that for any $u \in D(T)$,

$$(u, Tu) \geq c(u,u), \tag{2.2.37}$$

we say T is bounded from below. We also write $T \geq c$.

Theorem 2.2.32 (Friedrichs extension)**.** *Let T be a symmetric operator. If $T \geq -M$, then we can construct a self-adjoint extension $\hat{T}$ of T, called the* ***Friedrich extension****, such that $\hat{T} \geq -M$.*

Proof. We assume that $T \geq 1$ first. In this case,

$$(u, Tu) \geq \|u\|^2 \tag{2.2.38}$$

for any $u \in D(T)$. Let

$$a(v,u) = (v, Tu) \tag{2.2.39}$$

for any $u, v \in D(T)$. Let V be the closure of $D(T)$ with respect to $\|\cdot\|_a$. Then a could be extended on $V \times V$, denoted by $\hat{a}(\cdot,\cdot)$. Then $\hat{a}$ is a positive definite sesquilinear form.

We claim that $V \subset \mathcal{H}$. In this case, $\hat{a}$ is closed and we can use Proposition 2.2.30. In fact, let $i : D(T) \to \mathcal{H}$ be the embedding. For any $v \in V$, there exists $\{u_n\} \subset D(T)$ such that $\lim_{n\to\infty} u_n = v$ with respect to $\|\cdot\|_a$. Thus $\{u_n\}$ is a Cauchy sequence with respect to $\|\cdot\|_a$. From (2.2.38), $\{u_n\}$ is also a Cauchy sequence with respect to $\|\cdot\|$. Thus there exists $v^* \in \mathcal{H}$ such that $\lim_{n\to\infty} u_n = v$ with respect to $\|\cdot\|$. So we can embed V into $\mathcal{H}$ by identifying v with v^*. This process is similar as we did in the proof of the Sobolev embedding theorem.

Now $\hat{a}$ is a closed positive definite sesquilinear form. From Proposition 2.2.30, there exists a unique positive definite self-adjoint operator $\hat{T}$ such that

$$(v, \hat{T}u) = \hat{a}(v,u) \tag{2.2.40}$$

for any $u \in D(\hat{T})$ and $v \in V$. Easy to see that $\hat{T} \geq 1$.

For $u \in D(T)$, from (2.2.39), for any $v \in D(T)$,

$$|a(v,u)| = |(v,Tu)| \leq \|Tu\|\|v\|. \tag{2.2.41}$$

This inequality also holds for any $v \in V$. In fact, if $u_n \to v$ with respect to $\|\cdot\|_a$, $u_n \in D(T)$, then from (2.2.38), $u_n \to v$ with respect to $\|\cdot\|$. We could take limit in (2.2.41). By (2.2.29), (2.2.41) holds for any $v \in V$ implies that $u \in D(\hat{T})$. Since

$$(v,Tu) = a(v,u) = (v,\hat{T}u) \tag{2.2.42}$$

for any $v \in D(T)$, we have $Tu = \hat{T}u$. So $T \subset \hat{T}$, $\hat{T}$ is the self-adjoint extension of T.

In general case, if $T \geq -M$, let $T' = T + (M+1)\,\mathrm{Id}$. Then $T' \geq 1$. So T' has a self-adjoint extension $\hat{T}'$ and $\hat{T}' \geq 1$. Then $\hat{T} := \hat{T}' - (M+1)\,\mathrm{Id}$ is a self-adjoint extension of T and $\hat{T} \geq -M$.

The proof of Theorem 2.2.32 is completed. □

From Definition 1.4.12, the generalized Laplacian H is symmetric and bounded from below. From Theorem 2.2.32, it has Friedrich extension. (This is a reason why we need Q is bounded from below in the definition of the generalized Laplacian.) In fact, it is its unique self-adjoint extension.

Corollary 2.2.33. *The symmetric generalized Laplacian H on a compact manifold in Definition 1.4.12 is essentially self-adjoint. We also denote the unique self-adjoint extension by H.*

Proof. We only need to prove that the Friedrich extension $\hat{H} = \overline{H}$. Assume that $H \geq -M$. Then the norm $\|\cdot\|_a$ we considered in the proof of Theorem 2.2.32 is $(\cdot, H\cdot) + (1+M)\|\cdot\|$, which is equivalent to $\|\cdot\|_1$ from the Garding inequality, Theorem 1.4.13. Thus the dense subspace V in the proof of Theorem 2.2.32 is $\mathbf{H}^1$. From (2.2.29), if $u \in D(\hat{H})$, there exists $C_u > 0$, such that for any $v \in \mathbf{H}^1$, $|(v, \hat{H}u)| \leq C_u\|v\|$, which means that

$$\|\hat{H}u\| = \sup_{v \in \mathscr{C}^\infty} \frac{(v, \hat{H}u)}{\|v\|} \leq C_u < \infty. \tag{2.2.43}$$

So $\hat{H}u \in \mathbf{H}^0(E) = L^2(M,E)$. From elliptic estimate, $\|u\|_2 \leq C(\|\hat{H}u\| + \|u\|) < +\infty$. Thus $u \in \mathbf{H}^2(E)$. Therefore $D(\hat{H}) \subset \mathbf{H}^2(E)$.

From Proposition 2.2.5, $D(\overline{H}) = \mathbf{H}^2(E)$. Since $\overline{H}$ is the smallest closed extension of H, we have $\mathbf{H}^2(E) = D(\overline{H}) \subset D(\hat{H})$. So $D(\overline{H}) = D(\hat{H}) = \mathbf{H}^2(E)$. Since $\hat{H}$ and $\overline{H}$ are completed with respect to the equivalent norms, we have $\hat{H} = \overline{H}$.

The proof of Corollary 2.2.33 is completed. □

2.2.4 Perturbation

Definition 2.2.34. Let A and B be densely defined operators on $\mathcal{H}$. If $D(A) \subset D(B)$ and there exist $a, b > 0$ such that for any $u \in D(A)$,

$$\|Bu\| \leq a\|Au\| + b\|u\|, \tag{2.2.44}$$

we say B is A-bounded with bound a.

Theorem 2.2.35. *Let A and B be densely defined operators on $\mathcal{H}$ and B is A-bounded with bound $a < 1$. Then $A + B$ on $D(A)$ is closable if and only if A is closable. In this case,*

$$D(\overline{A+B}) = D(\overline{A}). \tag{2.2.45}$$

Proof. From (2.2.44), for any $u \in D(A)$,

$$\begin{aligned}(1-a)\|Au\| - b\|u\| \leq \|Au\| - \|Bu\| \leq \|(A+B)u\| \\ \leq \|Au\| + \|Bu\| \leq (1+a)\|Au\| + b\|u\|.\end{aligned} \tag{2.2.46}$$

So if $a < 1$, for $\{u_n\} \subset D(A)$ converges, $\{Au_n\}$ is a Cauchy sequence if and only if $\{(A+B)u_n\}$ is a Cauchy sequence. If $u_n \to 0$, $Au_n \to 0$ if and only if $(A+B)u_n \to 0$. From Proposition 2.2.9, $A+B$ on $D(A)$ is closable if and only if A is closable.

If A is closable, $A+B$ is closable. If $u \in D(\overline{A})$, there exists $\{u_n\} \subset D(A)$, $u_n \to u$ and Au_n convergences. From (2.2.46), $(A+B)u_n$ also convergences. So $u \in D(\overline{A+B})$. In the same way, $D(\overline{A+B}) \subset D(\overline{A})$. So $D(\overline{A+B}) = D(\overline{A})$.

The proof of Theorem 2.2.35 is completed. □

Corollary 2.2.36. *If B is A-bounded with bound $a < 1$. Then $A + B$ is closed if and only if A is closed.*

Proof. Obvious. □

Theorem 2.2.37 (Kato-Rellich Theorem)**.** *If A is self-adjoint, B is symmetric and B is A-bounded with bound $a < 1$, then $A + B$ is self-adjoint.*

Proof. From Proposition 2.2.22, we only need to prove that there exists $\mu_0 > 0$ such that $\mathrm{Im}(A + B \pm i\mu_0 \,\mathrm{Id}) = \mathcal{H}$.

For any $\mu > 0$, from (2.2.13), $\mathrm{Ker}(A \pm i\mu\,\mathrm{Id}) = 0$. Since A is self-adjoint, by Proposition 2.2.22, $\mathrm{Im}(A \pm i\mu\,\mathrm{Id}) = \mathcal{H}$. Then $(A \pm i\mu\,\mathrm{Id})^{-1}$ is well defined on $\mathcal{H}$. For any $u \in \mathcal{H}$, taking $x = (A \pm i\mu\,\mathrm{Id})^{-1}u$ in (2.2.12), we get

$$\|A(A \pm i\mu\,\mathrm{Id})^{-1}\| \leq 1, \quad \|(A \pm i\mu\,\mathrm{Id})^{-1}\| \leq \frac{1}{\mu}. \tag{2.2.47}$$

Note that

$$A + B \pm i\mu\,\mathrm{Id} = (B(A \pm i\mu\,\mathrm{Id})^{-1} - \mathrm{Id})(A \pm i\mu\,\mathrm{Id}). \tag{2.2.48}$$

From Definition 2.2.34 and (2.2.47),

$$\|B(A \pm i\mu)\,\mathrm{Id})^{-1}\| \leq a\,\|A(A \pm i\mu\,\mathrm{Id})^{-1}\| + b\,\|(A \pm i\mu\,\mathrm{Id})^{-1}\| \leq a + \frac{b}{\mu}. \tag{2.2.49}$$

Since $a < 1$, we could take $\mu_0 > 0$ large enough such that $a+b\mu_0^{-1} < 1$. In this case, $(B(A\pm i\mu\,\mathrm{Id})^{-1} - \mathrm{Id})$ is invertible. From (2.2.48), since $\mathrm{Im}(A\pm i\mu_0\,\mathrm{Id}) = \mathcal{H}$, we have $\mathrm{Im}(A + B \pm i\mu_0\,\mathrm{Id}) = \mathcal{H}$.

The proof of Theorem 2.2.37 is completed. □

2.2.5 Hodge decomposition

Proposition 2.2.38. *Let $P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, E)$ be an essentially self-adjoint elliptic differential operator over a compact Riemannian manifold of order $m > 0$. Then there is an L^2-orthogonal direct sum decomposition*

$$\mathscr{C}^\infty(M, E) = \mathrm{Ker}\,P \oplus \mathrm{Im}P. \tag{2.2.50}$$

Proof. With respect to the L^2-decomposition $L^2(M, E) = \mathrm{Ker}\,P \oplus (\mathrm{Ker}\,P)^\perp$, for any $u \in \mathscr{C}^\infty(M, E)$, we can write $u = u_0 + u_1$ and $Pu_0 = 0$. From the elliptic regularity Theorem 1.4.6, u_0 is smooth. So is u_1.

Since P is essentially self-adjoint,

$$(\mathrm{Ker}\,P)^\perp = (\mathrm{Ker}\,\overline{P})^\perp = \mathrm{Im}\overline{P}^* = \mathrm{Im}\overline{P} = \overline{P}(\mathbf{H}^m(E)). \tag{2.2.51}$$

Thus there exists $v \in \mathbf{H}^m(E)$ such that $\overline{P}(v) = u_1$. From the elliptic regularity Theorem 1.4.6 again, v is smooth. So $\mathscr{C}^\infty(M, E) \subset \mathrm{Ker}\,P \oplus P(\mathscr{C}^\infty(M, E))$. The other direction is obvious.

The proof of Proposition 2.2.38 is completed. □

For more details of Hodge decomposition, please see my another note on Kähler Geometry.

Lemma 2.2.39. *For any $s \in \mathbb{R}$, there exists $C_s > 0$, such that for any $u \in \mathrm{Im}P$,*

$$\|Pu\|_s \geq C_s\|u\|_s. \tag{2.2.52}$$

Proof. We assume $s \geq 0$ first. If (2.2.52) does not hold, there exists $\{u_j\} \subset \mathrm{Im}P$ such that $\|u_j\|_s = 1$ and $\|Pu_j\|_s \to 0$. We may assume that $\|Pu_j\|_s \leq 1$ uniformly. From the elliptic estimates, $\|u_j\|_{s+m} \leq c(\|Pu_j\|_s + \|u_j\|_s) \leq 2c$. From Rellich theorem, there exists a subsequence of $\{u_j\}$, which we also denote by $\{u_j\}$, converges to $u \in (\mathrm{Ker}\, P)^\perp$ with respect to $\|\cdot\|_s$. Since P is closable, $Pu = 0$. So $u = 0$ and $\lim u_j = 0$ with respect to $\|\cdot\|$. Since $\|u_j\|_s$ converges, it must converge to $u = 0$, which is a contradiction with $\|u_j\|_s = 1$.

If $s < 0$, for any $u \in \mathrm{Im}P$, from (1.2.56) and (2.2.50),

$$\begin{aligned}\|u\|_s = \sup_{v\in \mathrm{Im}P} \frac{(u,v)}{\|v\|_{-s}} &= \sup_{w\in \mathrm{Im}P} \frac{(u,Pw)}{\|Pw\|_{-s}} = \sup_{w\in \mathrm{Im}P} \frac{(Pu,w)}{\|Pw\|_{-s}} \\ &\leq \sup_{w\in \mathrm{Im}P} \frac{\|Pu\|_s\|w\|_{-s}}{C_s\|w\|_{-s}} = C_s^{-1}\|Pu\|_s. \end{aligned} \tag{2.2.53}$$

The proof of Lemma 2.2.39 is completed. □

Let $P : \mathscr{C}^\infty(M,E) \to \mathscr{C}^\infty(M,E)$ be an essentially self-adjoint elliptic differential operator over a compact Riemannian manifold of order $m > 0$. From (2.2.50), $P : \mathrm{Im}P \to \mathrm{Im}P$ is an isomorphism. Then it has an inverse P^{-1}. Let $S : \mathscr{C}^\infty(M,E) \to \mathrm{Ker}\, P$ be the orthogonal decomposition with respect to the decomposition (2.2.50). Set

$$G := P^{-1} \circ (\mathrm{Id} - S) : \mathscr{C}^\infty(M,E) \to \mathscr{C}^\infty(M,E), \tag{2.2.54}$$

called **Green operator**. It is easy to see that

$$PG = GP = \mathrm{Id} - S. \tag{2.2.55}$$

Proposition 2.2.40. *For any $s \in \mathbb{R}$, there exists $C > 0$ such that for any $u \in \mathscr{C}^\infty(M,E)$,*

$$\|Gu\|_{s+m} \leq C\|u\|_s. \tag{2.2.56}$$

Then it extends a continuous map $G : \mathbf{H}^s(E) \to \mathbf{H}^{s+m}(E)$.

Proof. From Proposition 2.2.38, for any $u \in \mathscr{C}^\infty(M,E)$, there exists $u_0 \in \mathrm{Ker}\, P$, $u_1 \in \mathrm{Im}P$ such that $u = u_0 + u_1$. There exists $v_1 \in \mathrm{Im}P$ such that $Pv_1 = u_1$. From the elliptic estimate, Lemma 2.2.39 and (2.2.55),

$$\begin{aligned}\|Gu\|_{s+m} &= \|GPv_1\|_{s+m} = \|v_1\|_{s+m} \leq c(\|Pv_1\|_s + \|v_1\|_s) \\ &\leq c(1 + C_s^{-1})\|Pv_1\|_s = c(1+C_s^{-1})\|u_1\|_s \leq c(1+C_s^{-1})\|u\|_s. \end{aligned} \tag{2.2.57}$$

The proof of Proposition 2.2.40 is completed. □

2.2.6 Spectrum

Definition 2.2.41. Let T be a closed operator. Define

$$\rho(T) := \{\lambda \in \mathbb{C} : \operatorname{Ker}(\lambda \cdot \operatorname{Id} - T) = 0,\ \overline{\operatorname{Im}(\lambda \cdot \operatorname{Id} - T)} = \mathcal{H},\\ (\lambda \cdot \operatorname{Id} - T)^{-1} \text{ is bounded}\}. \tag{2.2.58}$$

It is called the **resolvent set** of T. The inverse $(\lambda \cdot \operatorname{Id} - T)^{-1}$ is called the resolvent of T at λ. The set

$$\sigma(T) := \mathbb{C} \backslash \rho(T) \tag{2.2.59}$$

is called the **spectrum** of T. If T is only closable, we define

$$\sigma(T) := \sigma(\overline{T}). \tag{2.2.60}$$

Definition 2.2.42. Let T be a closed operator. Define

$$\begin{aligned} \sigma_p(T) &:= \{\lambda \in \mathbb{C} : \operatorname{Ker}(\lambda \cdot \operatorname{Id} - T) \neq 0\}; \\ \sigma_r(T) &:= \{\lambda \in \mathbb{C} : \operatorname{Ker}(\lambda \cdot \operatorname{Id} - T) = 0,\ \overline{\operatorname{Im}(\lambda \cdot \operatorname{Id} - T)} \neq \mathcal{H}\}; \\ \sigma_c(T) &:= \{\lambda \in \mathbb{C} : \operatorname{Ker}(\lambda \cdot \operatorname{Id} - T) = 0,\ \overline{\operatorname{Im}(\lambda \cdot \operatorname{Id} - T)} = \mathcal{H}, \\ &\qquad (\lambda \cdot \operatorname{Id} - T)^{-1} \text{ is unbounded}\}, \end{aligned} \tag{2.2.61}$$

which are called **point spectrum, residual spectrum** and **continuous spectrum** respectively. Obviously,

$$\sigma(T) = \sigma_p(T) \sqcup \sigma_r(T) \sqcup \sigma_c(T). \tag{2.2.62}$$

Proposition 2.2.43. *If T is self-adjoint, then $\sigma(T) \subset \mathbb{R}$.*

Proof. It is easy to see that if we replace $\pm i\mu$ by $\lambda = v \pm i\mu \in \mathbb{C}$, Proposition 2.2.22 also holds. From (2.2.47), if $\operatorname{Im}\lambda \neq 0$, $\lambda \in \rho(T)$.

The proof of Proposition 2.2.43 is completed. □

Proposition 2.2.44. *If T is self-adjoint, then $\sigma_r(T) = \emptyset$.*

Proof. From Proposition 2.2.43, if $\lambda \in \sigma_r(T)$, $\lambda \in \mathbb{R}$. So $\operatorname{Ker}(\lambda \cdot \operatorname{Id} - T) = \operatorname{Im}(\lambda \cdot \operatorname{Id} - T)^{\perp}$.

The proof of Proposition 2.2.44 is completed. □

Definition 2.2.45. Let T be a self-adjoint operator. Define

$$\begin{aligned} \sigma_{ess}(T) &:= \sigma_c(T) \cup \{\lambda \in \sigma_p(T) : \dim \operatorname{Ker}(\lambda \cdot \operatorname{Id} - T) = +\infty\}; \\ \sigma_d(T) &:= \{\lambda \in \sigma_p(T) : 0 < \dim \operatorname{Ker}(\lambda \cdot \operatorname{Id} - T) < +\infty\}, \end{aligned} \tag{2.2.63}$$

which are called **essential spectrum** and **discrete spectrum** respectively. The element of $\sigma_d(T)$ is called an eigenvalue of T. Obviously,

$$\sigma(T) = \sigma_{ess}(T) \sqcup \sigma_d(T). \tag{2.2.64}$$

Theorem 2.2.46 (Spectral theorem for compact operators). [2] *Let $\mathcal{H}$ be a infinite dimensional Hilbert space. Let B be a compact self-adjoint operator on $\mathcal{H}$. Then there exists a complete orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ of $\mathcal{H}$ and a real sequence $\{\lambda_1, \cdots, \lambda_n, \cdots\}$ (at most countable) such that $B\varphi_j = \lambda_j \varphi_j$ and $\lim_{j\to+\infty} \lambda_j = 0$ (if the real sequence is not discrete).*

Theorem 2.2.47. *Let $P : \mathscr{C}^{\infty}(M, E) \to \mathscr{C}^{\infty}(M, E)$ be an essentially self-adjoint elliptic differential operator over a compact Riemannian manifold of order $m > 0$. Then*

(1) for $\lambda \in \sigma_d(P)$, the λ-eigenspace $E_\lambda = \operatorname{Ker}(\lambda \operatorname{Id} - P)$ is finite dimensional and consist of smooth sections;

(2) there is a direct sum decomposition

$$L^2(M, E) = \bigoplus_{\lambda \in \sigma_d(P)} E_\lambda. \tag{2.2.65}$$

(3) the spectrum

$$\sigma(P) = \sigma_d(P) \tag{2.2.66}$$

and it is discrete.

Proof. (1). Since $\lambda \operatorname{Id} - P$ is elliptic, (1) follows from Theorem 1.4.6 (3).

(2). Let E_s be the closure of ImP with respect to $\|\cdot\|_s$. Then $L^2(M, E) = \operatorname{Ker} P \oplus E_0$ and from the Rellich theorem and Proposition 2.2.46, the Green operator

$$G : E_0 \to E_m \hookrightarrow E_0 \tag{2.2.67}$$

is a compact self-adjoint operator. From Theorem 2.2.46, since $\ker G = 0$, there exists a complete orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ of E_0 and a real sequence $\{\lambda_1^{-1}, \cdots, \lambda_n^{-1}, \cdots\}$ such that $G\varphi_j = \lambda_j^{-1}\varphi_j$ and $\lim_{j\to+\infty} \lambda_j^{-1} = 0$. So we get (2).

(3) From Proposition 2.2.38, $\mathscr{C}^{\infty}(M, E) = \operatorname{Ker}(P - \lambda \operatorname{Id}) \oplus \operatorname{Im}(P - \lambda \operatorname{Id})$. If $\operatorname{Ker}(P - \lambda \operatorname{Id}) = 0$, $\mathscr{C}^{\infty}(M, E) = \operatorname{Im}(P - \lambda \operatorname{Id})$. From Lemma 2.2.39, there exists $C > 0$, such that for any $u \in \mathscr{C}^{\infty}(M, E)$, $\|(P - \lambda \operatorname{Id})u\| \geq C\|u\|$. So for any $u \in \mathscr{C}^{\infty}(M, E)$, $\|(P - \lambda \operatorname{Id})^{-1}u\| \leq C^{-1}\|u\|$. So $\sigma_c(P) = \emptyset$.

The proof of Theorem 2.2.47 is completed.

□

2.2.7 Hodge decomposition on non-compact manifold

[MM] Chapter 3.

[2] Theorem 4.4.7 in "Functional analysis I" by Zhang Gongqing

2.3 Functional Calculus

2.3.1 Functional calculus for bounded operators

In this subsection, we suppose that $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on Hilbert space $\mathcal{H}$. Note that all results in this subsection hold for Banach spaces. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators. In expressions like $\lambda \operatorname{Id} - A$, we usually write $\lambda - A$, omitting the symbol Id for the simplicity.

Definition 2.3.1. The **resolvent set** $\rho(A)$ is defined by

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is invertible}\}. \tag{2.3.1}$$

If $\lambda \in \rho(A)$, $(\lambda - A)^{-1}$ is called the **resolvent** of A. The spectrum is

$$\sigma(A) = \mathbb{C} \backslash \rho(A). \tag{2.3.2}$$

Proposition 2.3.2. *The spectrum $\sigma(A)$ is a bounded closed subset of $\mathbb{C}$.*

Proof. If $r > \|A\|$, then $r - A = r(1 - A/r)$ and $\|A/r\| < 1$. Since $\sum_{k=1}^{\infty} \|A/r\|^k < +\infty$, then $1 + \sum_{k=1}^{\infty} (A/r)^k \in \mathcal{H}$ and $r^{-1}(1 + \sum_{k=1}^{\infty} (A/r)^k)$ is the inverse of $r - A$. So $\sigma(A) \subset B_0(\|A\|)$ is bounded.

Let U be the set of invertible elements in $\mathcal{B}(\mathcal{H})$. Then U is open. Since the map $f : \mathbb{C} \to \mathcal{B}(\mathcal{H})$, $\lambda \mapsto \lambda - A$, is continuous and $\rho(A) = f^{-1}(U)$, we have $\rho(A)$ is open. So $\sigma(A)$ is closed.

The proof of Proposition 2.3.2 is completed. □

In the proof of Proposition 2.3.2, we consider a continuous map $f : \mathbb{C} \to \mathcal{B}(\mathcal{H})$. The norm of the bounded linear operator makes $\mathcal{B}(\mathcal{H})$ a Banach space, which is a topological space with topology induced by the norm. We need to say more here.

Differentiation

For $\lambda_0 \in \mathbb{C}$, we could define the derivative of f at λ_0 by

$$f'(\lambda_0) := \lim_{z \to 0} z^{-1}(f(\lambda_0 + z) - f(\lambda_0)) \tag{2.3.3}$$

if the limit exists with respect to the operator norm. Let Ω be an open subset of $\mathbb{C}$. The function f is called **differentiable** if for each $\lambda_0 \in \Omega$, the derivative of f at λ_0 exists. It is called differentiable on Ω if it is differentiable at each $\lambda_0 \in \Omega$.

Integral

Let $\Gamma = \{\Gamma(t) : t \in [0,1]\}$ be a rectifiable curve in Ω. Then $\int_\Gamma f(\lambda)d\lambda$ is defined as the limit in $\mathcal{B}(\mathcal{H})$ of sums of the form

$$\sum_j (\Gamma(t_j) - \Gamma(t_{j-1}))f(\Gamma(t_j)) \in \mathcal{B}(\mathcal{H}) \tag{2.3.4}$$

as we did in the course of Mathematical Analysis, where $\{t_0, \cdots, t_n\}$ is a partition of $[0,1]$. In the same way, we could show that if f is continuous, the limit exists.

An open subset $\Delta \subset \mathbb{C}$ is called a **Cauchy domain** if it is a disjoint union of a finite number of open connected sets $\Delta_1, \cdots, \Delta_r$, such that $\overline{\Delta_i} \cap \overline{\Delta_j} = \emptyset$ if $i \neq j$ and for each j the boundary of Δ_j consists of a finite number of non-intersecting closed rectifiable Jordan curves which are oriented in a way that Δ_j belongs to the inner domains of the curves. The oriented boundary of a bounded Cauchy domain in $\mathbb{C}$ is called a **Cauchy contour**. Usually we integrate a continuous function on a Cauchy contour. In fact, for any compact subset $K \subset \mathbb{C}$ and its any open neighborhood Ω, there exists a Cauchy domain Δ such that $K \subset \Delta \subset\subset \Omega$[3].

Analyticity

The function f is said to be **analytic** at $\lambda_0 \in \Omega$ if in some neighborhood U of λ_0 in Ω,

$$f(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0) f_n, \quad \lambda \in U. \tag{2.3.5}$$

Here $f_0, f_1 \cdots \in \mathcal{B}(\mathcal{H})$ and the series (2.3.5) converges with respect to the operator norm. We say f is analytic on Ω if it is analytic at each $\lambda_0 \in \Omega$.

Theorem 2.3.3 (Cauchy Integral Formula)**.** *Assume that $f : \Omega \to \mathcal{B}(\mathcal{H})$ is analytic on Ω. Let Γ be a Cauchy contour such that Γ and its inner domain Δ are in Ω. Then for any $\lambda_0 \in \Delta$,*

$$f(\lambda_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\lambda)}{\lambda - \lambda_0} d\lambda. \tag{2.3.6}$$

In particular,

$$\frac{1}{2\pi i} \int_\Gamma f(\lambda) d\lambda = 0. \tag{2.3.7}$$

[3]cf. GTM 011, Proposition VIII.1.1 ”Functions of one complex variables” by Conway.

Proof. Take an arbitrary continuous linear functional F on $\mathcal{B}(\mathcal{H})$. Then $F \circ f$ is an analytic function in the usual sense. From the usual Cauchy integral formula, we have

$$F[f(\lambda_0)] = \frac{1}{2\pi i} \int_\Gamma \frac{F[f(\lambda)]}{\lambda - \lambda_0} d\lambda. \tag{2.3.8}$$

On the other hand, from the definition of the integral (2.3.4), we have

$$F\left[\frac{1}{2\pi i} \int_\Gamma \frac{f(\lambda)}{\lambda - \lambda_0} d\lambda\right] = \frac{1}{2\pi i} \int_\Gamma \frac{F[f(\lambda)]}{\lambda - \lambda_0} d\lambda. \tag{2.3.9}$$

Since F is an arbitrary continuous linear functional on $\mathcal{B}(\mathcal{H})$, from the Hahn-Banach theorem[4], (2.3.8) and (2.3.9), we obtain (2.3.6).

If we replace $f(\lambda)$ in (2.3.9) by $f(\lambda)(\lambda - \lambda_0)$, we get (2.3.7).

The proof of Theorem 2.3.3 is completed. □

Theorem 2.3.4. *The function f is analytic on Ω if and only if f is differentiable on Ω.*

Proof. We only need to prove that differentiable implies analytic, which is a classical result in complex analysis in the usual sense.

Assume that f is differentiable on Ω. For any $\lambda_0 \in \Omega$, we could choose an oriented circle $\Gamma \subset \Omega$ with center at λ_0 and radius r such that its inner domain is also in Ω. Let F be an arbitrary continuous linear functional on $\mathcal{B}(\mathcal{H})$. Then the function $F \circ f$ is differentiable on Ω, and hence analytic on Ω. From the Cauchy integral formula,

$$F[f(\mu)] = \frac{1}{2\pi i} \int_\Gamma \frac{F[f(\lambda)]}{\lambda - \mu} d\lambda = F\left[\frac{1}{2\pi i} \int_\Gamma \frac{f(\lambda)}{\lambda - \mu} d\lambda\right], \quad |\mu - \lambda_0| < r. \tag{2.3.10}$$

So the Hahn-Banach theorem implies that

$$f(\mu) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\lambda)}{\lambda - \mu} d\lambda, \quad |\mu - \lambda_0| < r. \tag{2.3.11}$$

Since

$$\frac{1}{\lambda - \mu} = \frac{1}{(\lambda - \lambda_0)\left(1 - \frac{\mu - \lambda_0}{\lambda - \lambda_0}\right)} = \sum_{n=0}^{\infty} \frac{(\mu - \lambda_0)^n}{(\lambda - \lambda_0)^{n+1}}, \tag{2.3.12}$$

[4] Corollary 2.4.5 in "Functional Analysis I"

we have

$$f(\mu) = \sum_{n=0}^{\infty} (\mu - \lambda_0)^n \left(\frac{1}{2\pi i} \int_\Gamma \frac{f(\lambda)}{(\lambda - \lambda_0)^{n+1}} d\lambda \right). \tag{2.3.13}$$

So f is analytic on Ω.

The proof of Theorem 2.3.4 is completed. □

Lemma 2.3.5. *For $\lambda, \mu \in \rho(A)$, we have the resolvent equation*

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}. \tag{2.3.14}$$

Proof. Multiply $(\lambda - A)$ on the left and $(\mu - A)$ on the right. □

Corollary 2.3.6. *The resolvent $(\lambda - A)^{-1}$ is analytic on $\lambda \in \rho(A)$.*

Proof. By Theorem 2.3.4 and Lemma 2.3.5, obvious. □

Definition 2.3.7. Let Ω be an open neighborhood of $\sigma(A)$. If $f : \Omega \to \mathbb{C}$ is analytic, for a Cauchy domain Δ with boundary Γ such that $\sigma(A) \subset \Delta \subset\subset \Omega$, we define $f(A) \in \mathcal{B}(\mathcal{H})$ by

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - A)^{-1} d\lambda. \tag{2.3.15}$$

From the Cauchy integration formula, $f(A)$ is independent of the choice of the Cauchy domain satisfying $\sigma(A) \subset \Delta \subset\subset \Omega$.

Let $\mathrm{Hol}(A)$ be the set of complex-valued functions which are analytic in a neighborhood of $\sigma(A)$. From the definition, it is easy to see that For any $f \in \mathrm{Hol}(A)$, $\alpha \in \mathbb{C}$,

$$(\alpha f)(A) = \alpha f(A). \tag{2.3.16}$$

Theorem 2.3.8 (Riesz Functional Calculus)**.** *(1) For any $f, g \in \mathrm{Hol}(A)$,*

$$(f + g)(A) = f(A) + g(A), \quad (f \cdot g)(A) = f(A)g(A). \tag{2.3.17}$$

(2) If $f \equiv 1$, then $f(A) = \mathrm{Id}$, i.e.,

$$\frac{1}{2\pi i} \int_\Gamma (\lambda - A)^{-1} d\lambda = \mathrm{Id}\,. \tag{2.3.18}$$

(3) If $f(z) = z^k$ for any $z \in \mathbb{C}$, $f(A) = A^k$, i.e.,

$$\frac{1}{2\pi i} \int_\Gamma \lambda^k (\lambda - A)^{-1} d\lambda = A^k. \tag{2.3.19}$$

(4) If $f, f_1, f_2, \cdots$ are analytic on Ω, and $f_n \to f$ uniformly on compact subsets of Ω, then $\|f_n(A) - f(A)\| \to 0$ as $n \to +\infty$.

(5) If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ has radius of convergence $R > \|A\|$, then $f \in \mathrm{Hol}(A)$ and

$$f(A) = \sum_{k=0}^{\infty} a_k A^k. \tag{2.3.20}$$

Proof. The proof of the first equation of (2.3.17) is obvious. Let Ω be the open neighborhood of $\sigma(A)$ such that f and g are all analytic on Ω. Let Γ_1 and Γ_2 be Cauchy contours such that $\Gamma_1 = \partial\Delta_1$, $\Gamma_2 = \partial\Delta_2$ and $\sigma(A) \subset \Delta_1 \subset\subset \Delta_2 \subset\subset \Omega$. Then from Lemma 2.3.5,

$$\begin{aligned} f(A)g(A) &= \left(\frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)(\lambda - A)^{-1} d\lambda\right)\left(\frac{1}{2\pi i}\int_{\Gamma_2} g(\mu)(\mu - A)^{-1} d\mu\right) \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} f(\lambda)g(\mu)(\lambda - A)^{-1}(\mu - A)^{-1} d\mu d\lambda \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} f(\lambda)g(\mu)\frac{(\lambda - A)^{-1}}{\mu - \lambda} d\mu d\lambda \\ &\quad - \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} f(\lambda)g(\mu)\frac{(\mu - A)^{-1}}{\mu - \lambda} d\mu d\lambda =: A - B. \end{aligned} \tag{2.3.21}$$

So

$$\begin{aligned} A &= \frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)(\lambda - A)^{-1}\left(\frac{1}{2\pi i}\int_{\Gamma_2}\frac{g(\mu)d\mu}{\mu - \lambda}\right)d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_1}(f \cdot g)(\lambda)(\lambda - A)^{-1} d\lambda = (f \cdot g)(A) \end{aligned} \tag{2.3.22}$$

and

$$B = \frac{1}{2\pi i}\int_{\Gamma_2} g(\mu)(\mu - A)^{-1}\left(\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\lambda)d\lambda}{\mu - \lambda}\right)d\mu = 0. \tag{2.3.23}$$

(2) and (3). Let $f(z) = z^k$, $k \geq 0$. Let $\Gamma(t) = Re^{2\pi i t}$, $0 \leq t \leq 1$, $R > \|A\|$. Then

$$\begin{aligned} f(A) &= \frac{1}{2\pi i}\int_{\Gamma}\lambda^k(\lambda - A)^{-1} d\lambda = \frac{1}{2\pi i}\int_{\Gamma}\lambda^{k-1}(1 - A/\lambda)^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma}\lambda^{k-1}\sum_{n=0}^{\infty}\frac{A^n}{\lambda^n} d\lambda. \end{aligned} \tag{2.3.24}$$

Since the infinite series converges uniformly on Γ,

$$f(A) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\lambda^{n-k+1}} d\lambda \right) A^n. \tag{2.3.25}$$

If $n \neq k$, $\int_{\Gamma} \frac{1}{\lambda^{n-k+1}} d\lambda = 0$. If $n = k$, $\int_{\Gamma} \frac{1}{\lambda^{n-k+1}} d\lambda = 2\pi i$. So $f(A) = A^k$.

(4). Let Δ be a Cauchy domain such that $\sigma(A) \subset \Delta \subset\subset \Omega$. Let $\Gamma = \partial\Delta = \cup_k \Gamma_k$, where Γ_k's are closed rectifiable Jordan curves. For k fixed,

$$\begin{aligned} &\left\| \int_{\Gamma_k} f_n(\lambda)(\lambda - A)^{-1} d\lambda - \int_{\Gamma_k} f(\lambda)(\lambda - A)^{-1} d\lambda \right\| \\ &\qquad = \left\| \int_0^1 (f_n(\Gamma_k(t)) - f(\Gamma_k(t)))(\Gamma_k(t) - A)^{-1} d\Gamma_k(t) \right\| \end{aligned} \tag{2.3.26}$$

Since $\|(\Gamma_k(t) - A)^{-1}\|$ is continuous on $t \in [0, 1]$ and bounded for any t, there exists $C > 0$ such that for any $t \in [0, 1]$, $\|(\Gamma_k(t) - A)^{-1}\| \leq C$. Let $|\Gamma_k|$ be the length of Γ_k. Then from (2.3.26), we have

$$\begin{aligned} &\left\| \int_{\Gamma_k} f_n(\lambda)(\lambda - A)^{-1} d\lambda - \int_{\Gamma_k} f(\lambda)(\lambda - A)^{-1} d\lambda \right\| \\ &\qquad \leq C|\Gamma_k| \max_{\lambda \in \Gamma_k} |f_n(\lambda) - f(\lambda)|. \end{aligned} \tag{2.3.27}$$

From the conditions, we have $\|f_n(A) - f(A)\| \to 0$ as $n \to +\infty$.

(5). Let $f_n(z) = \sum_{k=1}^{n} a_k z^k$. Then from (1)-(3), $f_n(A) = \sum_{k=1}^{n} a_k A^k$. Since $f(z) = \sum_{k=0}^{\infty} a_k z^k$ has radius of convergence $R > \|A\|$, $f \in \mathrm{Hol}(A)$ and $f_n(z) \to f(z)$ uniformly on compact subsets of $\{z : |z| < \mathbb{R}\}$. From (4), $f_n(A) \to f(A)$.

The proof of Theorem 2.3.8 is completed. □

Corollary 2.3.9. *Let $\sigma \subset \sigma(A)$ be a closed subset such that $\tau := \sigma(A) \backslash \sigma$ is also closed. Let Γ be a Cauchy contour such that σ is in the inner domain of Γ and τ is out of Γ. Then the operator*

$$P_\sigma := \frac{1}{2\pi i} \int_{\Gamma} (\lambda - A)^{-1} d\lambda \tag{2.3.28}$$

is a projection, i.e.,

$$P_\sigma^2 = P_\sigma. \tag{2.3.29}$$

Furthermore, we have

$$P_\sigma + P_\tau = \mathrm{Id}, \quad P_\sigma P_\tau = 0. \tag{2.3.30}$$

Proof. We could take a Cauchy domain $\Delta = \Delta_1 \cup \Delta_2$ such that $\sigma \subset \Delta_1$, $\tau \subset \Delta_2$ and $\overline{\Delta_1} \cap \overline{\Delta_2} = \emptyset$. Let $\Gamma_i = \partial\Delta_i$ for $i = 1, 2$. Take $f \equiv 1$ on Δ_1 and $\equiv 0$ on Δ_2. Then $f \in \mathrm{Hol}(A)$. So

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma_1\cup\Gamma_2} f(\lambda)(\lambda - A)^{-1}d\lambda = \frac{1}{2\pi i}\int_{\Gamma_1}(\lambda - A)^{-1}d\lambda = P_\sigma. \quad (2.3.31)$$

Then from the second equality of (2.3.17), $P_\sigma^2 = P_\sigma$.

Take $g \equiv 1$ on Δ_2 and $\equiv 0$ on Δ_1. Then $g \in \mathrm{Hol}(A)$ and $g(A) = P_\tau$. Then from the first equality of (2.3.17), $P_\sigma + P_\tau = \mathrm{Id}$. In this case, $P_\sigma P_\tau = P_\sigma(1 - P_\sigma) = P_\sigma - P_\sigma = 0$.

The proof of Corollary 2.3.9 is completed. □

Theorem 2.3.10 (Spectral Mapping theorem)**.** *If* $f \in \mathrm{Hol}(A)$, *then*

$$\sigma(f(A)) = f(\sigma(A)). \quad (2.3.32)$$

Proof. If $\lambda_0 \in \sigma(A)$, let

$$g(\lambda) = \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} \in \mathrm{Hol}(A). \quad (2.3.33)$$

If $f(\lambda_0) \notin \sigma(f(A))$, then $f(A) - f(\lambda_0)$ is invertible. From (2.2.31), $(\lambda_0 - A)^{-1} = g(A)(f(\lambda_0) - f(A))^{-1}$ is bounded, which is a contradiction of $\lambda_0 \in \sigma(A)$. So $f(\sigma(A)) \subset \sigma(f(A))$.

For the other direction, if $\mu \notin f(\sigma(A))$, then $g(\lambda) = (f(\lambda) - \mu)^{-1} \in \mathrm{Hol}(A)$. So $g(A)(f(A) - \mu) = \mathrm{Id}$. So $\mu \notin \sigma(f(A))$.

The proof of Theorem 2.3.10 is completed. □

Corollary 2.3.11. *Let* $\sigma \subset \sigma(A)$ *be a closed subset such that* $\tau := \sigma(A)\backslash\sigma$ *is also closed. Let* $M = \mathrm{Im}P_\sigma$ *and* $L = \mathrm{Ker}\, P_\sigma$. *Then* $\mathcal{B}(\mathcal{H}) = M \oplus L$, *the spaces* M *and* L *are* A*-invariant and*

$$\sigma(A|_M) = \sigma, \quad \sigma(A|_L) = \tau. \quad (2.3.34)$$

Proof. From (2.3.30), $L = \mathrm{Im}P_\tau$ and $\mathcal{B}(\mathcal{H}) = M \oplus L$. The spaces M and L are A-invariant. Take $f = z$ on Δ_1 and $\equiv 0$ on Δ_2. Then $f \in \mathrm{Hol}(A)$. So $A|_M = AP_\sigma = f(A)$. So (2.3.34) follows from Theorem 2.3.10.

The proof of Corollary 2.3.11 is completed. □

2.3.2 Functional Calculus for unbounded operators

In this subsection, we assume that A is an unbounded operator on $\mathcal{H}$ with domain $D(A)$. Remark that we don't need A is self-adjoint in this subsection.

Proposition 2.3.12. *The resolvent set $\rho(A)$ is open. If $\rho(A) \neq \emptyset$, then the resolvent $(\lambda - A)^{-1}$ is analytic on $\lambda \in \rho(A)$. Moreover, if $\lambda_0 \in \rho(A)$ and $|\lambda - \lambda_0| \leq \|(\lambda - A)^{-1}\|$, then $\lambda \in \rho(A)$ and*

$$(\lambda - A)^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n (\lambda_0 - A)^{-(n+1)}. \tag{2.3.35}$$

Here this series converges in the operator norm. For $\lambda, \mu \in \rho(A)$, we have the resolvent equation

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}. \tag{2.3.36}$$

Proof. The proof is the same as the bounded case. □

If A is unbounded, the spectrum $\sigma(A)$ is closed but unbounded. We need to compactify $\sigma(A)$ as follows.

Let $\mathbb{C}_\infty$ be the Riemann sphere, $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, endowed with the usual topology. The set $\mathbb{C}_\infty$ is a compact topological space and for any $\lambda \in \mathbb{C}$, the Möbius transformation

$$\eta_\lambda(z) := (\lambda - z)^{-1} \tag{2.3.37}$$

is a homeomorphism on $\mathbb{C}_\infty$.

Proposition 2.3.13. *Let A be an unbounded linear operator with non-empty resolvent set $\rho(A)$. Then for $\lambda \in \rho(A)$,*

$$\eta_\lambda(\sigma(A) \cup \{\infty\}) = \sigma((\lambda - A)^{-1}). \tag{2.3.38}$$

Proof. Note that $\eta_\lambda(\infty) = 0 \in \sigma((\lambda - A)^{-1})$, because $((\lambda - A)^{-1})^{-1} = \lambda - A$ is unbounded. For $z \neq \lambda$, $z \neq \infty$,

$$z - A = (\lambda - z)\left((\lambda - z)^{-1} - (\lambda - A)^{-1}\right)(\lambda - A). \tag{2.3.39}$$

So $z \in \sigma(A)$ if and only if $\eta_\lambda(z) = (\lambda - z)^{-1} \in \sigma((\lambda - A)^{-1})$.

The proof of Proposition 2.3.13 is completed. □

Since η_λ is a homeomorphism and $\sigma((\lambda - A)^{-1})$ is compact, we have $\sigma(A) \cup \{\infty\}$ is compact in $\mathbb{C}_\infty$.

Definition 2.3.14. Let Ω be an open neighborhood of $\sigma(A)$ in $\mathbb{C}$. Let Δ be a Cauchy domain such that $\sigma \subset \Delta \subset\subset \Omega$. Let $\Gamma = \partial\Delta$ be the Cauchy contour. In this case, some connected components are not closed. Let f be an analytic function on Ω. We assume that on any open connected component Γ_i of Γ,

$$|f(\Gamma_i(t))| \in \mathscr{S}(\mathbb{R}) \tag{2.3.40}$$

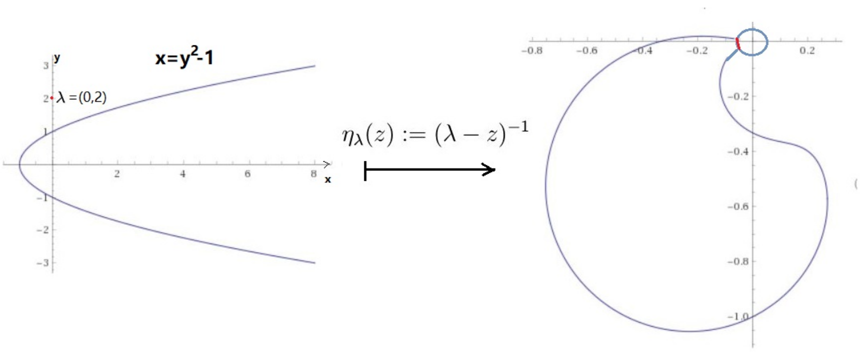
is a rapidly decreasing function (cf. (1.2.22)). We define

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - A)^{-1}d\lambda, \tag{2.3.41}$$

Proposition 2.3.15. *The definition (2.3.41) is well-defined and does not depend on the Cauchy contour satisfying (2.3.40). Moreover, for $\lambda \notin \overline{\Delta}$, we have*

$$f(A) = \frac{1}{2\pi i}\int_{\eta_\lambda(\Gamma)} f\circ\eta_\lambda^{-1}(\mu)(\mu - (\lambda - A)^{-1})^{-1}d\mu. \tag{2.3.42}$$

Roughly speaking, we have $f(A) = (f\circ\eta_\lambda^{-1})((\lambda - A)^{-1})$.



Proof. We only prove (2.3.42). From (2.3.42), we could see that (2.3.41) is well-defined and does not depend on the Cauchy contour satisfying (2.3.40).

Let $B = (\lambda - A)^{-1}$, $z \in \rho(A)$, $z \neq \lambda$ and $\mu = (\lambda - z)^{-1}$. From (2.3.39),

$$(z - A)^{-1} = \mu B(\mu - B)^{-1} = \mu(\mu(\mu - B)^{-1} - \mathrm{Id}). \tag{2.3.43}$$

By taking $z = \eta_\lambda^{-1}(\mu) = \lambda - \mu^{-1}$, from (2.3.43), we have

$$\frac{1}{2\pi i}\int_\Gamma f(z)(z - A)^{-1}dz$$
$$= \frac{1}{2\pi i}\int_{\eta_\lambda(\Gamma)} f(\eta_\lambda^{-1}(\mu))((\mu - B)^{-1} - \mu^{-1})d\mu. \tag{2.3.44}$$

Let $\Gamma_\varepsilon = \partial(\eta_\lambda(\Delta)\backslash B_0(\varepsilon))$ for $\varepsilon > 0$ small enough. Then from (2.3.40), we have

$$\frac{1}{2\pi i}\int_{\eta_\lambda(\Gamma)} f(\eta_\lambda^{-1}(\mu))\mu^{-1}d\mu = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} f(\eta_\lambda^{-1}(\mu))\mu^{-1}d\mu. \tag{2.3.45}$$

Since $f \circ \eta_\lambda^{-1}$ is holomorphic on $\eta_\lambda(\Omega)$, by Cauchy integral formula, we have

$$\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} f(\eta_\lambda^{-1}(\mu))\mu^{-1}d\mu = 0 \tag{2.3.46}$$

for any $\varepsilon > 0$. So

$$\frac{1}{2\pi i}\int_{\Gamma} f(z)(z-A)^{-1}dz = \frac{1}{2\pi i}\int_{\eta_\lambda(\Gamma)} f(\eta_\lambda^{-1}(\mu))(\mu - B)^{-1}d\mu. \tag{2.3.47}$$

The proof of Proposition 2.3.15 is completed. □

Remark 2.3.16. Remark that the assumption (2.3.40) is very strong. In fact, from the proof of Proposition 2.3.15, we only need the condition that the integral in (2.3.41) is well-defined.

From Theorems 2.3.8, 2.3.10 and Proposition 2.3.15, we obtain the following result.

Theorem 2.3.17. *For f, g be the analytic functions in Definition 2.3.14, we have*

$$(f+g)(A) = f(A) + g(A), \tag{2.3.48}$$

$$(\alpha f)(A) = \alpha f(A), \quad \forall \alpha \in \mathbb{C}, \tag{2.3.49}$$

$$(fg)(A) = f(A)g(A). \tag{2.3.50}$$

$$\sigma(f(A)) = f(\sigma(A)). \tag{2.3.51}$$

From Corollaries 2.3.9 and 2.3.11, we obtain the following result.

Theorem 2.3.18. *Let A be a closed operator with spectrum $\sigma(A) = \sigma \cup \tau$, where σ is contained in a bounded Cauchy domain Δ such that $\overline{\Delta} \cap \tau = \emptyset$. Let $\Gamma = \partial\Delta$. Then*

$$P_\sigma := \frac{1}{2\pi i}\int_\Gamma (\lambda - A)^{-1}d\lambda \tag{2.3.52}$$

is a projection. Let $M = \mathrm{Im}P_\sigma$ and $L = \mathrm{Ker}\, P_\sigma$. The spaces M and L are A-invariant and

$$\sigma(A|_M) = \sigma, \quad \sigma(A|_L) = \tau. \tag{2.3.53}$$

Furthermore, $M \subset D(A)$ and $A|_M$ is bounded.

2.3.3 Spectral decomposition for non-self-adjoint elliptic operator

Now we extend the spectral decomposition theorem (Theorem 2.2.47) to the non-self-adjoint case using the functional calculus.

Theorem 2.3.19. *Let $P : \mathscr{C}^\infty(M,F) \to \mathscr{C}^\infty(M,F)$ be an elliptic differential operator over a compact Riemannian manifold of order $m > 0$. Then for the spectrum $\sigma(P)$, there are two possibilities:*

(a) $\sigma(P) = \mathbb{C}$;

(b) $\sigma(P)$ is a discrete (maybe empty) subset of $\mathbb{C}$.

If (b) holds and $\lambda_0 \in \sigma(P)$, then there is a decomposition $L^2(M,F) = E_{\lambda_0} \oplus E'_{\lambda_0}$ such that the following conditions are satisfied:

(1) $E_{\lambda_0} \subset \mathscr{C}^\infty(M,F)$, $\dim E_{\lambda_0} < +\infty$, and E_{λ_0} is an invariant subspace of P such that there exists $N \in \mathbb{N}_+$ with $(P-\lambda_0)^N E_{\lambda_0} = 0$, i.e., the operator $P|_{E_{\lambda_0}}$ has only the eigenvalue λ_0 and is equal to the direct sum of Jordan cells of degree $\leq N$;

(2) E'_{λ_0} is a closed subspace of $L^2(M,F)$, invariant with respect to $\overline{P}$, i.e., $\overline{P}(D(\mathbf{H}^m(F) \cap E'_{\lambda_0})) \subset E'_{\lambda_0}$, and if we denote by $P'_{\lambda_0} := \overline{P}|_{E'_{\lambda_0}}$ as an unbounded operator in E'_{λ_0} with domain $\mathbf{H}^s(F) \cap E'_{\lambda_0}$, then $P'_{\lambda_0} - \lambda_0$ has a bounded inverse, i.e., $\lambda_0 \notin \sigma(P)|_{E'_{\lambda_0}}$.

Proof. Let $\lambda_0 \in \mathbb{C}\backslash\sigma(P)$, with loss of generality, assume that $\lambda_0 = 0$. So we have a bounded inverse $\overline{P}^{-1}$. Since $\overline{P}$ has positive order, by Rellich theorem, $\overline{P}^{-1}$ is a compact operator. Since

$$\overline{P} - \lambda = (\lambda^{-1} - \overline{P}^{-1})\lambda^{-1}\overline{P}, \tag{2.3.54}$$

$\lambda \in \sigma(P)$ if and only if $\lambda \neq 0$ and $\lambda^{-1} \in \sigma(\overline{P}^{-1})$. From the spectral theory of the compact operator, we see that $\sigma(P)$ is discrete.

Let $\sigma(P) \neq \mathbb{C}$, $\lambda_0 \in \sigma(A)$. Without loss of generality, we assume $\lambda_0 = 0$ again. Let Γ be the contour around 0 and not containing any other point of $\sigma(A)$. From Theorem 2.3.18, $P_0 = \frac{1}{2\pi i}\int_\Gamma (\lambda - A)^{-1} d\lambda$ is a projection, $E_{\lambda_0} = P_0(L^2(M,F))$, $E'_{\lambda_0} = (1-P_0)(L^2(M,F))$ and E_{λ_0}, E'_{λ_0} are $\overline{P}$-invariant. From (2.3.53), $\lambda_0 \notin \sigma(P)|_{E'_{\lambda_0}}$. Since P is elliptic, $E_{\lambda_0} \subset \mathscr{C}^\infty(M,F)$ and is finite dimensional. From (2.3.53), $\sigma(P|_{E_{\lambda_0}}) = \lambda_0$. So $P|_{E_{\lambda_0}}$ is a linear transform on finite dimensional linear space E_{λ_0} with single eigenvalue λ_0. From the Jordan decomposition theorem in linear algebra, there exists $N \in \mathbb{N}_+$ with $(P - \lambda_0)^N E_{\lambda_0} = 0$.

The proof of Theorem 2.3.19 is completed. □

Remark that if $\operatorname{ind}(P) \neq 0$, $\sigma(P) = \mathbb{C}$. Because $\operatorname{ind}(P-\lambda) = \operatorname{ind}(P)$.

2.3.4 Complex powers of an elliptic operator

In this subsection, we introduce an important example of functional calculus of unbounded operator: complex powers of an elliptic operator, P^z, $z \in \mathbb{C}$, $\mathrm{Re}z < 0$.

Let $P : \mathscr{C}^\infty(M, F) \to \mathscr{C}^\infty(M, F)$ be an elliptic differential operator over a compact Riemannian manifold of order $m > 0$. We assume that $0 \notin \sigma(P)$. From Theorem 2.3.19, $\sigma(P)$ is a discrete set. We assume that there exists $\varepsilon > 0$ small enough and angle

$$\Lambda = \{\lambda \in \mathbb{C} : \pi - 2\varepsilon \leq \arg \lambda \leq \pi + 2\varepsilon\} \tag{2.3.55}$$

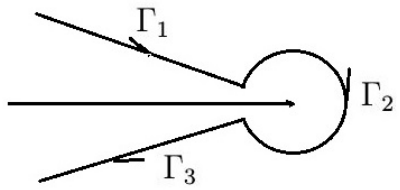
such that

$$\sigma(P) \cap \Lambda = \emptyset. \tag{2.3.56}$$

For $\lambda \in \mathbb{C}$, $z \in \mathbb{C}$, we have

$$\lambda^z = e^{z \ln \lambda} = e^{z \ln |\lambda| + iz \arg \lambda} = |\lambda|^z e^{iz \arg \lambda}. \tag{2.3.57}$$

Take $\rho > 0$ small enough that $B_0(2\rho) \cap \sigma(P) = \emptyset$. Consider the contour



where

$$\begin{aligned} \Gamma_1 : &\quad \lambda = re^{i(\pi-\varepsilon)}, \quad +\infty > r > \rho, \\ \Gamma_2 : &\quad \lambda = \rho e^{i\varphi}, \quad \pi - \varepsilon > \varphi > -\pi + \varepsilon, \\ \Gamma_3 : &\quad \lambda = re^{i(-\pi+\varepsilon)}, \quad \rho < r < +\infty. \end{aligned} \tag{2.3.58}$$

As in Definition 2.3.14, we define

$$P^z = \frac{1}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3} \lambda^z (\lambda - P)^{-1} d\lambda. \tag{2.3.59}$$

Then

$$\begin{aligned} P^z = \frac{1}{2\pi i} \int_{+\infty}^{\rho} & r^{\mathrm{Re}z} e^{-\mathrm{Im}z(\pi-\varepsilon)} e^{i(\mathrm{Im}z \ln r + \mathrm{Re}z(\pi-\varepsilon))} (re^{i(\pi-\varepsilon)} - P)^{-1} dr \\ &+ \frac{1}{2\pi i} \int_{\pi-\varepsilon}^{-\pi+\varepsilon} \rho e^{-\varphi \mathrm{Im}z} e^{i\varphi \mathrm{Re}z} (\rho e^{i\varphi} - P)^{-1} i\rho e^{i\varphi} d\varphi \\ + \frac{1}{2\pi i} \int_{\rho}^{+\infty} & r^{\mathrm{Re}z} e^{-\mathrm{Im}z(-\pi+\varepsilon)} e^{i(\mathrm{Im}z \ln r + \mathrm{Re}z(-\pi+\varepsilon))} (re^{i(-\pi+\varepsilon)} - P)^{-1} dr \end{aligned} \tag{2.3.60}$$

Lemma 2.3.20. *For $\lambda \in \Lambda$,*

$$\|(P-\lambda)^{-1}\| \leq \frac{C}{|\lambda|}. \tag{2.3.61}$$

Proof. Add it in the future. □

From Lemma 2.3.20, Remark 2.3.16, (2.3.57) and (2.3.60), we see that P^z in (2.3.59) is well-defined for $\mathrm{Re} z < 0$ and bounded.

From the functional calculus Theorem 2.3.17, we have the following result.

Theorem 2.3.21. *(1) If $\mathrm{Re} z < 0$, $\mathrm{Re} w < 0$, then $P^z P^w = P^{z+w}$.*

(2) If $k \in \mathbb{Z}$, $k > 0$, then $P^{-k} = (P^{-1})^k$.

(3) If $\mathrm{Re} z < 0$, P^z is holomophic on $\mathrm{Re} z < 0$.

2.4 Spectral Theorem

Chapter 3

Heat Kernels

In this chapter, we assume that the manifold M is compact and the generalized Laplacian H is not necessarily symmetric.

Note that if M is non-compact and H is symmetric, we can study the heat kernel following the lines of Spectral theorem and the Schwartz kernel theorem. We will not discuss it here.

3.1 heat kernels

3.1.1 What is kernel?

Let H be a generalized Laplacian on a vector bundle E over a compact Riemannian oriented manifold M.

Let E and F be Hermitian vector bundles over M. Let p_1 and p_2 be the projections from $M \times M$ onto the first and the second factor M respectively. We denote by

$$E \boxtimes F := p_1^* E \otimes p_2^* F \tag{3.1.1}$$

over $M \times M$.

Definition 3.1.1. A continuous section $P(x, y)$ on $F \boxtimes E^*$) is called a kernel. Using $P(x, y)$, we could define an operator $P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, F)$ by

$$(Pu)(x) = \int_{y \in M} \langle P(x, y), u(y) \rangle_E dy. \tag{3.1.2}$$

The kernel $P(x, y)$ is also called the kernel of P, which is also denoted by $\langle x | P | y \rangle$.

Proposition 3.1.2. *If P has a kernel $P(x,y)$, then the adjoint operator P^* has a kernel $P^*(x,y) = P(y,x)^* \in \mathscr{C}^\infty(M \times M, E^* \boxtimes F)$*[1].

Proof. For $u \in L^2(M,E)$, $v \in L^2(M,F^*)$, we have

$$\begin{aligned}(Pu,v)_{L^2} &= \int_{x\in M} \left\langle \int_{y\in M} \langle P(x,y), u(y)\rangle_E dy, v(x) \right\rangle_F dx \\ &= \int_{y\in M} \left\langle u(y), \int_{x\in M} \langle P(x,y)^*, v(x)\rangle_F dx \right\rangle_E dy \\ &= \int_{x\in M} \left\langle u(x), \int_{y\in M} \langle P(y,x)^*, v(y)\rangle_F dy \right\rangle_E dx = (u, P^*v)_{L^2}. \end{aligned} \tag{3.1.3}$$

So for any $v \in L^2(M,F^*)$,

$$P^*v = \int_{y\in M} \langle P(y,x)^*, v(y)\rangle_F dy. \tag{3.1.4}$$

The proof of Proposition 3.1.2 is completed. □

Proposition 3.1.3. *If P has a smooth kernel, then P is a smoothing operator.*

Proof. For any $m \in \mathbb{R}$, $\alpha \in \mathbb{N}^n$, $u \in \mathscr{C}^\infty(M,E)$, from Theorem 1.2.33 (3), we have

$$|D_x^\alpha Pu(x)| = \left| \int_{y\in M} \langle D_x^\alpha P(x,y), u(y)\rangle_E dy \right| \leq \|D_x^\alpha P(x,y)\|_{y,m} \|u\|_{-m}. \tag{3.1.5}$$

So for any $s, m \in \mathbb{N}$, $u \in \mathscr{C}^\infty(M,E)$,,

$$\|Pu\|_s \leq \sum_{|\alpha|\leq s} \Big\| \|D_x^\alpha P(x,y)\|_{y,m} \Big\|_x \|u\|_{-m}. \tag{3.1.6}$$

Since $P(x,y)$ is smooth on x and y, from (1.2.91), $\Big\| \|D_x^\alpha P(x,y)\|_{y,m} \Big\|_x$ is uniformly bounded. So P is a smoothing operator.

The proof of Proposition 3.1.3 is completed. □

3.1.2 Schwarz kernel theorem

"Topological vector spaces, distributions and kernels " by Treves

[1] From (3.1.1), $F \boxtimes E^* = \mathrm{Hom}(p_1^*E, p_2^*F)$. For $A \in F \boxtimes E^* = \mathrm{Hom}(p_1^*E, p_2^*F)$, we take $A^* \in \mathrm{Hom}(p_2^*F, p_1^*E) = E^* \boxtimes F$ as the transpose of the matrix.

3.1.3 Heat kernel for symmetric generalized Laplacian

Now we study the solution of the heat equation on manifold with initial condition:

$$\begin{cases} \left(\frac{\partial}{\partial t} + H\right) u(t,x) = 0, \ t > 0, \\ \lim_{t\to 0} u(t,x) = u(x) \in L^2(M,E). \end{cases} \tag{3.1.7}$$

If H is symmetric, from Theorem 2.2.47, there exists a complete orthonormal basis $\{\varphi_i\} \subset L^2(M,E)$ such that $H\varphi_i = \lambda_i\varphi_i$, where $\{\lambda_i\} = \sigma(H)$. In this case, if $u = \varphi_i$, then

$$u(t,x) = e^{-t\lambda_i}u(x) \tag{3.1.8}$$

is the unique solution of (3.1.7). In general, for $u = \sum_i a_i\varphi_i \in L^2(M,E)$, the unique solution of (3.1.7) is

$$u(t,x) = \sum_i a_i e^{-t\lambda_i}\varphi_i(x). \tag{3.1.9}$$

In spired of (3.1.8), for $t > 0$, we define the **heat operator** $e^{-tH} : L^2(M,E) \to L^2(M,E)$ by

$$u(t,x) = e^{-tH}u(x). \tag{3.1.10}$$

For any $s,t > 0$, from (3.1.9) and (3.1.10), we have

$$e^{-tH}e^{-sH} = e^{-(s+t)H}, \tag{3.1.11}$$

which means that the heat operators form a semi-group.

Let

$$e^{-tH}(x,y) = \sum_i e^{-t\lambda_i}\varphi_i(x) \boxtimes \varphi_i(y)^*. \tag{3.1.12}$$

Then from (3.1.9)-(3.1.12), formally,

$$\begin{aligned} e^{-tH}u(x) &= \sum_i a_i e^{-t\lambda_i}\varphi_i(x) = \sum_i a_i e^{-t\lambda_i}\varphi_i(x)\int_{y\in M}\left\langle \varphi_i(y)^*, \sum_j a_j\varphi_j(y)\right\rangle dy \\ &= \int_{y\in M}\left\langle \sum_i e^{-t\lambda_i}\varphi_i(x)\boxtimes\varphi_i(y)^*, \sum_j a_j\varphi_j(y)\right\rangle dy \\ &= \int_{y\in M}\langle e^{-tH}(x,y), u(y)\rangle dy. \end{aligned} \tag{3.1.13}$$

So if (3.1.12) is uniformly convergent in $\mathscr{C}^r$-norm for any $r \in \mathbb{N}$, i.e., $e^{-tH}(x,y)$ is smooth on x and y, $e^{-tH}(x,y)$ is the smooth kernel of the heat operator e^{-tH}, called the **heat kernel**. In the followings, we will prove that (3.1.12) is uniformly convergent in $\mathscr{C}^r$-norm for any $r \in \mathbb{N}$.

Lemma 3.1.4. *Let P be a self-adjoint elliptic differential operator of order $m > 0$ on Hermitian vector bundle E over compact Riemannian manifold M. If we order the eigenvalues $|\lambda_1| \le |\lambda_2| \le \cdots$, then*

(1) for any $l \in \mathbb{N}$, there exists $C_l > 0$ such that for $k \in \mathbb{N}$, $s > (2l+n)/2m$, $s \in \mathbb{N}$, we have

$$\|\varphi_k\|_{\mathscr{C}^l} \le C_l(1 + |\lambda_k|^s), \tag{3.1.14}$$

where φ_k is the eigenfunction with respect to λ_k such that $\|\varphi_k\|_{L^2} = 1$;

(2) there exist $C > 0$ and $\varepsilon > 0$ such that for any $k \in \mathbb{N}$,

$$|\lambda_k| \ge Ck^{\varepsilon}. \tag{3.1.15}$$

Proof. (1) From the Sobolev embedding theorem and the elliptic estimate, there exist $C'_l, C' > 0$ such that

$$\|\varphi_k\|_{\mathscr{C}^l} \le C'_l\|\varphi_k\|_{ms} \le C'_lC'(\|\varphi_k\|_0 + \|P^s\varphi_k\|_0) = C'_lC'(1 + |\lambda_k|^s). \tag{3.1.16}$$

(2) If we replace P by P^s, we replace λ_i by λ_i^s. So we only need to prove (2) for $m > n/2$. Set

$$F(k) := \mathrm{span}_{i\le k}\{\varphi_i\} \subset \mathscr{C}^\infty(M,E). \tag{3.1.17}$$

From Sobolev embedding theorem and the elliptic estimate, for $\varphi \in F(k)$, for any $x \in M$, we have

$$|\varphi(x)| \le C\|\varphi\|_m \le C(\|P\varphi\|_0 + \|\varphi\|_0) \le C(1 + |\lambda_k|)\|\varphi\|_0. \tag{3.1.18}$$

Let $\varphi = \sum_j c_j\varphi_j$. Then

$$\left|\sum_{1\le j\le k} c_j\varphi_j(x)\right| = |\varphi(x)| \le C(1 + |\lambda_k|)\left(\sum_j |c_j|^2\right)^{1/2}. \tag{3.1.19}$$

Choose a local orthonormal frame of E and decompose $\varphi_j(x)$ into components $\varphi_{\nu,j}$ for $1 \le \nu \le p$, where $\dim E = p$. Then

$$|\varphi(x)|^2 = \sum_{1\le\nu\le p}\left|\sum_{1\le j\le k} c_j\varphi_{\nu,j}(x)\right|^2. \tag{3.1.20}$$

We fix ν and take $c_j = \varphi^*_{\nu,j}(x)$. Then from (3.1.19) and (3.1.20), we have

$$\sum_{1\le j\le k} |\varphi_{\nu,j}(x)|^2 \le C(1+|\lambda_k|)\left(\sum_j |\varphi_{\nu,j}(x)|^2\right)^{1/2}. \tag{3.1.21}$$

So

$$\sum_{1\le j\le k} |\varphi_{\nu,j}(x)|^2 \le C^2(1+|\lambda_k|)^2. \tag{3.1.22}$$

Taking sum over ν, we have

$$\sum_{1\le j\le k} |\varphi_j(x)|^2 \le pC^2(1+|\lambda_k|)^2. \tag{3.1.23}$$

Integral (3.1.23) over M, we have

$$k \le pC^2 \operatorname{vol}(M)(1+|\lambda_k|)^2. \tag{3.1.24}$$

So we obtain (2).

The proof of Lemma 3.1.4 is completed. □

Proposition 3.1.5. *The heat kernel exists if H is symmetric. More precisely, (3.1.12) is uniformly convergent in $\mathscr{C}^r$-norm for any $r \in \mathbb{N}$, i.e., $e^{-tH}(x,y)$ is smooth on x and y.*

Proof. Since H is bounded from below, there are only finitely negative eigenvalues. Since we consider the convergence when $i \to \infty$, we may assume that $H > 0$. From Lemma 3.1.4, for $s > (l+l'+n)/m$, $s \in \mathbb{N}$,

$$\|e^{-t\lambda_k}\varphi_k(x) \boxtimes \varphi_k(y)^*\|_{\mathscr{C}^l_x,\mathscr{C}^{l'}_y} \le C_l C_{l'} e^{-t\lambda_k}(1+\lambda_k)^s. \tag{3.1.25}$$

From (3.1.15), for any $s' \le s$, $s \in \mathbb{N}$, we have

$$e^{-t\lambda_k}\lambda_k^{s'} \le C_k t^{-s'} e^{-t\lambda_k/2} \le C_l t^{-s'} e^{-tCk^\varepsilon}. \tag{3.1.26}$$

From (3.1.25) and (3.1.26), we have

$$\sum_k \|e^{-t\lambda_k}\varphi_k(x) \boxtimes \varphi_k(y)^*\|_{\mathscr{C}^l_x,\mathscr{C}^{l'}_y} \le Ct^{-s'} \sum_k e^{-tCk^\varepsilon} < +\infty. \tag{3.1.27}$$

Remark that the convergence of $\sum_k e^{-Ck^\varepsilon}$ is equivalent to that of $\int_0^{+\infty} e^{-Cx^\varepsilon}dx = C^{-\varepsilon^{-1}}(\varepsilon^{-1}-1)\int_0^{+\infty} y^{\varepsilon^{-1}-1}e^{-y}dy$, which is a Gamma function.

The proof of Proposition 3.1.5 is completed. □

Note that from (3.1.7) and (3.1.13), for any $u(x) \in L^2(M, E)$,

$$\left(\frac{\partial}{\partial t} + H\right)(e^{-tH}u)(x) = \int_{y \in M} \left(\frac{\partial}{\partial t} + H_x\right) e^{-tH}(x, y)u(y)dy = 0. \quad (3.1.28)$$

So we have

$$\left(\frac{\partial}{\partial t} + H_x\right) e^{-tH}(x, y) = 0. \quad (3.1.29)$$

3.1.4 Heat kernel on Euclidean space

Let $M = \mathbb{R}$, $E = \mathbb{C}$ and $H = \Delta = -\frac{d^2}{dx^2}$. From the knowledge of the PDE course for undergraduates, the heat kernel is

$$e^{-t\Delta}(x, y) = \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{(x-y)^2}{4t}}. \quad (3.1.30)$$

Thus for $u \in L^2(\mathbb{R})$,

$$\left(e^{-t\Delta}u\right)(x) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} u(y)dy. \quad (3.1.31)$$

Proposition 3.1.6. *For even $l \in \mathbb{N}$, if $\|u\|_{\mathscr{C}^{l+1}} \leq +\infty$, then there exists $C > 0$ such that*

$$\left| e^{-t\Delta}u - \sum_{k=0}^{l/2} \frac{(-t)^k}{k!} \Delta^k u \right| \leq Ct^{l/2+1}. \quad (3.1.32)$$

This is another explanation why we write heat operator as e^{-tH}.

Proof. From (3.1.31),

$$\left(e^{-t\Delta}u\right)(x) = \frac{1}{(4\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{v^2}{4}} u(x + \sqrt{t}v)dv. \quad (3.1.33)$$

By Taylor expansion,

$$u(x + \sqrt{t}v) = \sum_{k=0}^{l} \frac{(\sqrt{t}v)^k}{k!} u^{(k)}(x) + \frac{(\sqrt{t}v)^{l+1}}{l!} \int_0^1 (1 - s)^l u^{(l+1)}(x + s\sqrt{t}v)ds. \quad (3.1.34)$$

Since $\|u\|_{\mathscr{C}^{l+1}} \leq +\infty$,

$$\left| u(x+\sqrt{t}v) - \sum_{k=0}^{l} \frac{t^{k/2}v^k}{k!} u^{(k)}(x) \right| \leq \frac{t^{(l+1)/2}v^{l+1}}{l!} \|u\|_{l+1}. \tag{3.1.35}$$

From (3.1.33) and (3.1.35), we have

$$\left| \left(e^{-t\Delta}u\right)(x) - \sum_{k=0}^{l} \frac{1}{(4\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{v^2}{4}} v^k dv \cdot \frac{t^{k/2}}{k!} u^{(k)}(x) \right|$$
$$\leq \frac{1}{(4\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{v^2}{4}} v^{l+1} dv \cdot \frac{t^{(l+1)/2}}{l!} \|u\|_{l+1}. \tag{3.1.36}$$

Let

$$A(t) = \sum_{k=0}^{+\infty} \frac{1}{(4\pi)^{\frac{1}{2}}} \frac{t^k}{k!} \int_{\mathbb{R}} e^{-\frac{v^2}{4}} v^k dv = \frac{1}{(4\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{v^2}{4}+tv} dv. \tag{3.1.37}$$

Let $u = v - 2t$. We have

$$A(t) = \frac{1}{(4\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{u^2}{4}+t^2} du = e^{t^2}. \tag{3.1.38}$$

So we have

$$\frac{1}{(4\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{v^2}{4}} v^k dv = \begin{cases} \frac{k!}{(k/2)!}, & \text{if } k \text{ even;} \\ 0, & \text{if } k \text{ odd.} \end{cases} \tag{3.1.39}$$

From (3.1.36) and (3.1.39), we obtain (3.1.32).

The proof of Proposition 3.1.6 is completed. □

3.1.5 Non-symmetric heat kernel

In this subsection, the generalized Laplacian may be non-symmetric.

Compare with the symmetric case, we define the heat kernel by summarizing the properties that the kernel of an operator e^{-tH} must have.

Definition 3.1.7. A heat kernel for H is a continuous section $e^{-tH}(x,y)$ of the bundle $E \boxtimes E^*$ over $\mathbb{R}_+ \times M \times M$, satisfying the following properties:

(1) $e^{-tH}(x,y)$ is $\mathscr{C}^1$ with respect to t and $\mathscr{C}^2$ with respect to x, i.e., $\frac{\partial}{\partial t} e^{-tH}(x,y)$ is continuous and $\frac{\partial^2}{\partial x_i \partial x_j} e^{-tH}(x,y)$ are continuous for any coordinate system of x;

(2) $(\frac{\partial}{\partial t} + H_x) e^{-tH}(x,y) = 0$;

(3) Let e^{-tH} be the operator defined as in (3.1.2), called the **heat operator**. Then for any $s \in \mathscr{C}^\infty(M,E)$, $\lim_{t\to 0} e^{-tH}s = s$ with respect to the $\mathscr{C}^0$-norm.

We need to prove that the heat kernel in Definition 3.1.7 exists and is unique. We first assume the existence and study the uniqueness. In the next section, we will prove that for any generalized Laplacian, the heat kernel always exists and smooth on t, x, y.

Lemma 3.1.8. *Assume that H^* has a heat kernel. If $s(t, x)$ is a map from $\mathbb{R}_+$ to the space of sections of E which is $\mathscr{C}^1$ in t and $\mathscr{C}^2$ in x (in the meaning of Definition 3.1.7 (1)), such that $\lim_{t\to 0} s(t, x) = 0$ and which satisfies the heat equation $(\frac{\partial}{\partial t} + H_x)s(t, x) = 0$, then $s(t, x) = 0$.*

Proof. For any $u \in \mathscr{C}^\infty(M, E^*)$, $0 < \theta < t$, let

$$f(\theta) = \int_{M\times M} \langle s(\theta, x), e^{-(t-\theta)H^*}(x, y)u(y)\rangle dxdy. \tag{3.1.40}$$

From the heat equation in Definition 3.1.7, we have

$$\begin{aligned} \frac{\partial}{\partial\theta} f(\theta) = &\int_{M\times M} \left\langle \frac{\partial}{\partial\theta} s(\theta, x), e^{-(t-\theta)H^*}(x, y)u(y)\right\rangle dxdy \\ &+ \int_{M\times M} \left\langle s(\theta, x), \frac{\partial}{\partial\theta} e^{-(t-\theta)H^*}(x, y)u(y)\right\rangle dxdy \\ &= \int_{M\times M} \left\langle -H_x s(\theta, x), e^{-(t-\theta)H^*}(x, y)u(y)\right\rangle dxdy \\ &+ \int_{M\times M} \left\langle s(\theta, x), H_x^* e^{-(t-\theta)H^*}(x, y)u(y)\right\rangle dxdy = 0. \end{aligned} \tag{3.1.41}$$

Since $\lim_{\theta\to 0} s(\theta, x) = 0$, $\lim_{\theta\to 0} f(\theta) = 0$. So

$$0 = f(t) = \int_M \langle s(\theta, x), u(x)\rangle dx \tag{3.1.42}$$

for any $u \in \mathscr{C}^\infty(M, E^*)$. Thus for any $t > 0$, $s(t, x) = 0$.

The proof of Lemma 3.1.8 is completed. □

Proposition 3.1.9. *(1) If H^* has a heat kernel, then H has at most one heat kernel.*

(2) If H and H^* have heat kernels, then

$$e^{-tH^*}(x, y) = e^{-tH}(y, x)^*. \tag{3.1.43}$$

(3) If H and H^* have heat kernels, then $\{e^{-tH}\}_{t>0}$ form a semi-group.

Proof. For any $u \in \mathscr{C}^\infty(M, E^*)$, $s \in \mathscr{C}^\infty(M, E)$, $0 < \theta < t$, let

$$f(\theta) = \int_M \langle e^{-\theta H} s(x), e^{-(t-\theta)H^*} u(x) \rangle dx. \tag{3.1.44}$$

As in (3.1.41), $f'(\theta) = 0$. So

$$(e^{-tH} s, u)_{L^2} = f(t) = \lim_{t \to 0} f(0) = (s, e^{-tH^*} u)_{L^2}. \tag{3.1.45}$$

So

$$\exp(-tH^*) = \exp(-tH)^*. \tag{3.1.46}$$

From Proposition 3.1.2, we get (1) and (2).

For (3), set

$$s_t = e^{-tH} e^{-\theta H} s - e^{-(t+\theta)H} s. \tag{3.1.47}$$

Then $(\partial_t + H) s_t = 0$. Since $\lim_{t \to 0} s_t = 0$, by Lemma 3.1.8, $s_t = 0$ for any $t > 0$.

The proof of Proposition 3.1.9 is completed. □

Proposition 3.1.10. *Assume that H and H^* have heat kernels. Let Δ be a connected Cauchy domain with $\Gamma = \partial \Delta$ such that $\lim_{t \to \pm\infty} \mathrm{Re}(\Gamma(t)) = +\infty$. Assume that $\int_\Gamma e^{-t\lambda} \|(\lambda - H)^{-1}\| d\lambda < +\infty$. Then the heat operator*

$$e^{-tH} = \frac{1}{2\pi i} \int_\Gamma e^{-t\lambda} (\lambda - H)^{-1} d\lambda. \tag{3.1.48}$$

So our notation e^{-tH} is compatible with that in functional calculus.

Proof. Note that the operators on both sides of (3.1.48) are bounded for $t > 0$. Let

$$f(t) = \frac{1}{2\pi i} \int_\Gamma e^{-t\lambda} (\lambda - H)^{-1} d\lambda. \tag{3.1.49}$$

Then we have

$$H f(t) = f(t) H. \tag{3.1.50}$$

By Cauchy integral formula and $\lim_{t \to \pm\infty} \mathrm{Re}(\Gamma(t)) = +\infty$, $\int_\Gamma e^{-t\lambda} d\lambda = 0$. Since $\lambda(\lambda - H)^{-1} = (\lambda - H)^{-1} H + \mathrm{Id}$, for any $s \in \mathscr{C}^\infty(M, E)$,

$$\begin{aligned} \frac{\partial}{\partial t} f(t) s &= \frac{1}{2\pi i} \int_\Gamma \frac{\partial}{\partial t} e^{-t\lambda} (\lambda - H)^{-1} s d\lambda = -\frac{1}{2\pi i} \int_\Gamma \lambda e^{-t\lambda} (\lambda - H)^{-1} s d\lambda \\ &= -\frac{1}{2\pi i} \int_\Gamma e^{-t\lambda} (\lambda - H)^{-1} H s d\lambda = -f(t) H s = -H f(t) s. \end{aligned} \tag{3.1.51}$$

So for any $s \in \mathscr{C}^\infty(M,E)$, $f(t)s$ satisfies the heat equation.

For $\theta, t > 0$, since $f(\theta)e^{-tH}s$ and $e^{-tH}f(\theta)s$ satisfy the heat equation om t, and $\lim_{t\to 0} f(\theta)e^{-tH}s = \lim_{t\to 0} e^{-tH}f(\theta)s = f(\theta)s$, by Lemma 3.1.8, we have

$$f(\theta)e^{-tH} = e^{-tH}f(\theta). \tag{3.1.52}$$

From functional calculus Theorem 2.3.17, for $\theta, t > 0$, we have

$$f(\theta + t) = f(\theta)f(t) = f(t)f(\theta). \tag{3.1.53}$$

From (3.1.53), $\lim_{t\to 0} f(t)f(\theta)s = f(\theta)s$. So from Lemma 3.1.8 again, we have

$$f(t)f(\theta) = e^{-tH}f(\theta). \tag{3.1.54}$$

By (3.1.52)-(3.1.54), for $t > 0$ fixed and any $\theta > 0$,

$$f(\theta)(e^{-tH} - f(t)) = 0. \tag{3.1.55}$$

Taking $\theta \to 0$, we obtain (3.1.48).

The proof of Proposition 3.1.10 is completed. $\square$

3.2 Construction of the heat kernels

3.2.1 Toy model

In this section, we will construct the heat kernel for generalized Laplacian H on $V := \mathscr{C}^\infty(M, E)$.

We study the toy model first: let V be a finite dimensional vector space and H be a linear endomorphism; we construct $P_t = e^{-tH}$.

Definition 3.2.1. The k-simplex

$$\Delta_k := \{(t_1, \cdots, t_k) : 0 \le t_1 \le \cdots \le t_k \le 1\} \subset \mathbb{R}^k. \tag{3.2.1}$$

We often parametrize Δ_k by the coordinates

$$\sigma_0 = t_1, \sigma_i = t_{i+1} - t_i,\ 1 \le i \le k-1,\ \sigma_k = 1 - t_k, \tag{3.2.2}$$

such that $\sigma_1 + \cdots + \sigma_k = 1$ and $0 \le \sigma_i \le 1$. For $t > 0$, the rescaled simplex

$$t\Delta_k := \{(t_1, \cdots, t_k) : 0 \le t_1 \le \cdots \le t_k \le t\} \subset \mathbb{R}^k. \tag{3.2.3}$$

Let v_k be the volume of Δ_k. since $v_1 = 1$ and

$$v_k = \int_0^1 \mathrm{vol}(t_k \Delta_{k-1}) dt_k = \int_0^1 t_k^{k-1} v_{k-1} dt_k = \frac{v_{k-1}}{k}, \tag{3.2.4}$$

we have

$$v_k = \frac{1}{k!}. \tag{3.2.5}$$

Let $K_t : \mathbb{R}_+ \to \mathrm{End}(V)$ be an approximate solution of the heat equation for small t in the sense that for some small $\alpha > 0$, there exists $C > 0$ such that

$$R_t = \frac{dK_t}{dt} + HK_t \le Ct^\alpha, \tag{3.2.6}$$

and $K_0 = 1$. The function K_t is also called a parametrix for the heat equation. The function R_t is called the remainder.

Let

$$Q_t^1 = \int_0^t K_{t-t_1} R_{t_1} dt_1. \tag{3.2.7}$$

Then

$$\frac{dQ_t^1}{dt} = R_t + \int_0^t R_{t-t_1} R_t dt_1 - HQ_t^1. \tag{3.2.8}$$

So from (3.2.6)-(3.2.8),

$$\left(\frac{d}{dt} + H\right)(K_t - Q_t^1) = -\int_0^t R_{t-t_1} R_t dt_1 = O(t^{2\alpha+1}). \tag{3.2.9}$$

Following this way, we could make the error term smaller and smaller:

Theorem 3.2.2. *Let $Q_t : \mathbb{R}_+ \to \mathrm{End}(V)$ be defined by*

$$Q_t^k := \int_{t\Delta_k} K_{t-t_k} R_{t_k - t_{k-1}} \cdots R_{t_2 - t_1} R_{t_1} dt_1 \cdots dt_k \tag{3.2.10}$$

and $Q_t^0 = K_t$. Then

$$P_t = e^{-tH} = \sum_{k=0}^{+\infty} (-1)^k Q_t^k \tag{3.2.11}$$

and

$$P_t = K_t + O(t^{1+\alpha}). \tag{3.2.12}$$

Proof. Let

$$R^{(k)}(s) := \int_{s\Delta_{k-1}} R_{s-t_{k-1}} \cdots R_{t_2-t_1} R_{t_1} dt_1 \cdots dt_k. \tag{3.2.13}$$

Then as in (3.2.8), we have

$$\left(\frac{d}{dt} + H\right) Q_t^k = R^{(k+1)}(t) + R^{(k)}(t). \tag{3.2.14}$$

From (3.2.6) and (3.2.3),

$$R^{(k)}(t) = O(t^{k\alpha}). \tag{3.2.15}$$

Since $\mathrm{vol}(t\Delta_k) = t^k/k!$, there exists $C_0 > 0$ such that

$$|Q_t^k| \le C_0 C^k t^{k\alpha} \frac{t^k}{k!}. \tag{3.2.16}$$

So the right hand side of (3.2.11) converges. Since $Q_0^0 = 1$ and $Q_0^k = 0$ for $k > 0$, we obtain (3.2.11). From (3.2.16) again, we have $P_t = K_t + O(t^{1+\alpha})$.

The proof of Theorem 3.2.2 is completed. □

The original version of Theorem 3.2.2 is the Volterra series for the exponential of a perturbed operator. If $H = H_0 + H_1 \in \mathrm{End}(V)$, for $K_t = e^{-tH_0}$,

$$R_t = \left(\frac{d}{dt} + H\right) e^{-tH_0} = H_1 e^{-tH_0}. \tag{3.2.17}$$

So from Theorem 3.2.2, we have

$$e^{-t(H_0+H_1)} = e^{-tH_0} + \sum_{k=1}^{\infty} (-1)^k I_k, \tag{3.2.18}$$

where

$$\begin{aligned} I_k :&= \int_{t\Delta_k} e^{-(t-t_k)H_0} H_1 e^{-(t_k - t_{k-1})H_0} \cdots H_1 e^{-t_1 H_0} dt_1 \cdots dt_k \\ &= \int_{t\Delta_k} e^{-\sigma_0 H_0} H_1 e^{-\sigma_1 H_0} \cdots H_1 e^{-\sigma_k H_0} d\sigma_1 \cdots d\sigma_k. \end{aligned} \tag{3.2.19}$$

So

$$\begin{aligned} e^{-t(H_0+H_1)} &= \sum_{k=0}^{\infty} (-t)^k \int_{\Delta_k} e^{-\sigma_0 t H_0} H_1 e^{-\sigma_1 t H_0} \cdots H_1 e^{-\sigma_k t H_0} d\sigma_1 \cdots d\sigma_k \\ &= e^{-tH_0} - t \int_0^1 e^{(1-\sigma)tH_0} H_1 e^{-\sigma t H_0} d\sigma + \cdots \end{aligned} \tag{3.2.20}$$

3.2.2 Estimates of the parametrix

For $V = \mathscr{C}^\infty(M, E)$, we study the kernel instead of the operator.

We leave the proof of the following theorem to the next subsection.

Theorem 3.2.3. *For every $N \in \mathbb{Z}_+$, there exists a smooth one-parameter family of smooth kernels $k_t^N(x, y)$, such that for every $\ell \in \mathbb{N}$,*

(1) for every $T > 0$, there exists $C > 0$ such that for $0 < t < T$, $u \in \mathscr{C}^\infty(M, E)$, we have

$$\|K_t^N u\|_{\mathscr{C}^\ell} \le C \|u\|_{\mathscr{C}^\ell}, \tag{3.2.21}$$

where K_t^N is the operator associated with $k_t^N(x, y)$;

(2) for $u \in \mathscr{C}^\infty(M, E)$,

$$\lim_{t \to 0} \|K_t u - u\|_{\mathscr{C}^\ell} = 0; \tag{3.2.22}$$

(3) for any $s \in \mathbb{N}$, there exists $C(\ell, s) > 0$ such that the kernel

$$r_t^N(x, y) := (\partial_t + H_x) k_t^N(x, y) \tag{3.2.23}$$

satisfies the estimate

$$\|\partial_t^s r_t^N\|_{\mathscr{C}^\ell} \leq C(\ell, s) t^{N-n/2-\ell/2-s} \tag{3.2.24}$$

for N large enough.

In order to simplify the notation, we omit the symbols "N" and "$dt_1 \cdots dt_k$" if there is no confuse.

Let K_t and R_t be the corresponding operators with respect to k_t and r_t.

As in (3.2.10) and (3.2.13), we consider

$$Q_t^k := \int_{t\Delta_k} K_{t-t_k} R_{t_k - t_{k-1}} \cdots R_{t_2 - t_1} R_{t_1}, \tag{3.2.25}$$

which is defined by the kernel

$$q_t^k(x, y) = \int_{t\Delta_k} \int_{M^k} k_{t-t_k}(x, z_k) r_{t_k - t_{k-1}}(z_k, z_{k-1}) \cdots r_{t_1}(z_1, y). \tag{3.2.26}$$

Let

$$r_t^{k+1}(x, y) = \int_{t\Delta_k} \int_{M^k} r_{t-t_k}(x, z_k) r_{t_k - t_{k-1}}(z_k, z_{k-1}) \cdots r_{t_1}(z_1, y). \tag{3.2.27}$$

Lemma 3.2.4. *For $s \in \mathbb{N}$, if $N > (n+\ell)/2 + s$, then ,*

$$\|\partial_t^s r_t^{k+1}\|_{\mathscr{C}^\ell(M\times M)} \leq C^{k+1} t^{(k+1)(N-n/2)-\ell/2-s} \operatorname{vol}(M)^k \frac{t^k}{k!}. \tag{3.2.28}$$

Proof. If $N > (n+\ell)/2$, by (3.2.24), r_t and its derivatives up to order ℓ extend continuously to $t = 0$. Using (3.2.24) again, we obtain Lemma 3.2.4.

The proof of Lemma 3.2.4 is completed. □

Lemma 3.2.5. *Assume that $N > (n+\ell)/2$ and that $\ell \geq 1$.*

(1) There exists $\widetilde{C} > 0$ such that for every $k \geq 1$,

$$\|q_t^k\|_{\mathscr{C}^\ell(M\times M)} \leq \widetilde{C} C^k \operatorname{vol}(M)^{k-1} t^{k(N-n/2)-\ell/2} \frac{t^k}{(k-1)!}. \tag{3.2.29}$$

(2) The kernel $q_t^k(x, y)$ is $\mathscr{C}^1$ on t and

$$(\partial_t + H_x) q_t^k(x, y) = r_t^{k+1}(x, y) + r_t^k(x, y). \tag{3.2.30}$$

Proof. Let

$$b(t,s,x,y) = \int_{z\in M} k_{t-s}(x,z) r_s^k(z,y) = K_{t-s,x} r_s^k(x,y). \tag{3.2.31}$$

Then from (3.2.26) and (3.2.27),

$$q_t^k(x,y) = \int_0^t b(t,s,x,y)ds. \tag{3.2.32}$$

From Theorem 3.2.3 (1), Lemma 3.2.4 and (3.2.31), for $0 \le s \le t$,

$$\|b(t,s)\|_{\mathscr{C}^\ell(M\times M)} \le C'C^k \operatorname{vol}(M)^{k-1} t^{k(N-n/2)-\ell/2} \frac{t^{k-1}}{(k-1)!}. \tag{3.2.33}$$

So we obtain (1) from (3.2.32) and (3.2.33).

From (3.2.23) and (3.2.31), $b(t,s,x,y)$ is $\mathscr{C}^1$ on s and t and smooth on x. From (3.2.23),

$$(\partial_t + H_x) b(t,s,x,y) = \int_{z\in M} r_{t-s}(x,z) r_s^k(z,y) = r^{k+1}(x,y). \tag{3.2.34}$$

Then (3.2.30) follows from (3.2.32) and (3.2.34).

The proof of Lemma 3.2.5 is completed. □

Theorem 3.2.6. *Assume that the kernel $k_t^N(x,y)$ satisfies the conditions of Theorem 3.2.3 with $N > n/2+1$.*

(1) For any ℓ such that $N > (n+\ell+1)/2$,

$$p_t(x,y) = \sum_{k=0}^{\infty} (-1)^k q_t^k(x,y) \tag{3.2.35}$$

converges in the $\mathscr{C}^{\ell+1}(M\times M)$-norm and defines a $\mathscr{C}^1$-map from $\mathbb{R}_+$ to $\mathscr{C}^\ell(M\times M, E\boxtimes E^*)$ such that

$$(\partial_t + H_x) p_t(x,y) = 0. \tag{3.2.36}$$

(2) When $t \to 0$,

$$\|\partial_t^s (p_t - k_t^N)\|_{\mathscr{C}^\ell(M\times M)} = O(t^{(N-n/2)-s-\ell/2+1}). \tag{3.2.37}$$

(3) The kernel p_t is a heat kernel for the operator H.

Proof. From (3.2.29),

$$\sum_{k=0}^{\infty} \|q_t^k\|_{\mathscr{C}^{\ell+1}(M\times M)} \leq \|k_t\|_{\mathscr{C}^{\ell+1}(M\times M)} + \widetilde{C}Ct^{N-n/2-\ell/2+1/2}e^{C\operatorname{vol}(M)t^{N-n/2+1}} < +\infty. \quad (3.2.38)$$

So (3.2.25) converges in $\mathscr{C}^{\ell+1}(M\times M)$-norm. From (3.2.28), (3.2.29) and (3.2.30),

$$\begin{aligned}\|\partial_t q_t^k\|_{\mathscr{C}^{\ell}(M\times M)} &\leq \|r_t^{k+1}\|_{\mathscr{C}^{\ell}(M\times M)} + \|r_t^k\|_{\mathscr{C}^{\ell}(M\times M)} + \|q_t^k\|_{\mathscr{C}^{\ell+2}(M\times M)} \\ &\leq C^{k+1}t^{(k+1)(N-n/2)-\ell/2}\operatorname{vol}(M)^k\frac{t^k}{k!} + C^k t^{k(N-n/2)-\ell/2}\operatorname{vol}(M)^{k-1}\frac{t^{k-1}}{(k-1)!} \\ &\quad + \widetilde{C}C^k\operatorname{vol}(M)^{k-1}t^{k(N-n/2)-\ell/2-1}\frac{t^k}{(k-1)!} \\ &\leq Ct^{N-n/2-\ell/2}(C\operatorname{vol}(M)t^{N-n/2+1}/k+1+\widetilde{C}) \\ &\quad \cdot C^{k-1}t^{(k-1)(N-n/2)}\operatorname{vol}(M)^{k-1}\frac{t^{k-1}}{(k-1)!}. \end{aligned} \quad (3.2.39)$$

So there exist $C_0, C_1 > 0$ such that

$$\sum_{k=0}^{\infty} \|\partial_t q_t^k\|_{\mathscr{C}^{\ell}(M\times M)} \leq \|\partial_t k_t\|_{\mathscr{C}^{\ell}(M\times M)} + (C_0t^{N-n/2+1}+C_1)t^{N-n/2-\ell/2}e^{C\operatorname{vol}(M)t^{N-n/2+1}} < +\infty. \quad (3.2.40)$$

Thus p_t is $\mathscr{C}^1$ on t from $\mathbb{R}_+$ to $\mathscr{C}^\ell(M\times M)$. As in Theorem 3.2.2, we have (3.2.26).

As in (3.2.39), we have

$$\|\partial_t^s q_t^k\|_{\mathscr{C}^{\ell}(M\times M)} \leq \|\partial_t^{s-1} r_t^{k+1}\|_{\mathscr{C}^{\ell}(M\times M)} + \|\partial_t^{s-1} r_t^k\|_{\mathscr{C}^{\ell}(M\times M)} + \|\partial_t^{s-1} q_t^k\|_{\mathscr{C}^{\ell+2}(M\times M)} = O(t^{k(N-n/2)-\ell/2-s+1}). \quad (3.2.41)$$

From (3.2.41), we get (3.2.37).

For (3), we only need to check the initial condition. Since k_t^N satisfies the initial condition, from (3.2.37), we get (3).

The proof of Theorem 3.2.6 is completed. □

3.2.3 Formal solution

In this subsection, we will prove Theorem 3.2.3. We will start from some basic results in Riemannian Geometry.

Let g^{TM} be the metric on M. Usually we denote it by g for simplicity. we consider a smooth path $x_t : [0,1] \to M$ and define its length as

$$L(x_t) = \int_0^1 |\dot{x}_t| dt. \tag{3.2.42}$$

The Riemannian distance between $x_0, x_1 \in M$ is the infimum of $L(x_t)$ over all smooth paths connecting them, denoted by $d(x,y)$. Let ∇ be the Levi-Civita connection. A smooth path is a geodesic if for any $t \in [0,1]$,

$$\nabla_{\dot{x}_t} \dot{x}_t = 0. \tag{3.2.43}$$

Given $\mathbf{x} = \dot{x}_0 \in T_{x_0}M$ small enough, the solution of (3.2.43) is unique. We write $x_1 = \exp_{x_0} \mathbf{x}$. Since the derivative $\exp_{x_0,*}$ is an isomorphism, by the inverse function theorem, $\exp_{x_0}$ defines a diffeomorphism from a small ball around zero to a neighborhood of x_0 in M. Let inj_{x_0} be the radius of the largest ball such that $\exp_{x_0}$ is a diffeomorphism. Let $\mathrm{inj} = \inf_{x \in M} \mathrm{inj}_{\mathrm{x}}$. Since M is compact, $\mathrm{inj} > 0$. Choose an orthonormal frame of $T_{x_0}M$, on which the coordinate functions are $\mathbf{x}_i$ and the partial derivatives are $\frac{\partial}{\partial \mathbf{x}_i}$. On $B(0,\varepsilon) \subset T_{x_0}M$, we define the metric by $\exp_{x_0}^*(g)$. If we consider $B(0,\varepsilon)$ as a chart of M at x_0, we have $\frac{\partial}{\partial \mathbf{x}_i} = \frac{\partial}{\partial x_i}$. Usually, we simply denoted it by ∂_i. With respect to this coordinates, we have

$$(\partial_i, \partial_j) = g_{ij}. \tag{3.2.44}$$

Let

$$\mathcal{R} = \sum_i \mathbf{x}_i \partial_i \in T_{\mathbf{x}}(T_{x_0}M). \tag{3.2.45}$$

Then $\exp_{x_0,*} \mathcal{R} \in T_{\exp_{x_0} \mathbf{x}} M$, which we also denote by $\mathcal{R}$.

In order to distinguish the points on $T_{x_0}M$ and those on M, we write $x = \exp_{x_0} \mathbf{x}$. Let x_t be the geodesic connecting x_0 and x, and let $Y(t) \in T_{x_t}M$ be a vector field along x_t. if for any $t \in [0,1]$,

$$\nabla_{\dot{x}_t} Y(t) = 0, \tag{3.2.46}$$

we say $Y(1)$ **is the parallel transport of** $Y(0)$ **along** x_t. Since $\mathbf{x} \in B(0, \mathrm{inj})$, the solution of (3.2.46) is unique associated with initial condition. So $Y(1)$ is uniquely determined by $Y(0)$. We write

$$Y(1) = \tau(x, x_0) Y(0). \tag{3.2.47}$$

Let

$$e_i(x) := \tau(x, x_0) \partial_i. \tag{3.2.48}$$

Lemma 3.2.7. *(1)* $\{e_i(x)\}$ *is an orthonormal frame of* T_xM.
(2) $e_i(x) = \partial_i + O(|\mathbf{x}|)$.
(3) $\nabla_{\mathcal{R}} e_i = 0$.

Proof. (1) Let $Y_0(t), Y_1(t) \in T_{x_t}M$ be vector fields along x_t satisfying (3.2.46). Since ∇^{TM} preserves the metric,

$$\dot{x}_t(Y_0(t), Y_1(t)) = (\nabla_{\dot{x}_t} Y_0(t), Y_1(t)) + (Y_0(t), \nabla_{\dot{x}_t} Y_1(t)) = 0. \tag{3.2.49}$$

So $(Y_0(t), Y_1(t))$ is a constant along x_t. Let $Y_0(0) = \partial_i$ and $Y_1(0) = \partial_j$, we get (1).

(2) Let $e_i(x) = f_{ij}(x)\partial_j$. Note that $f_{ij}(0) = \delta_{ij}$. So $f_{ij}(x) = \delta_{ij} + O(|\mathbf{x}|)$. We get (2).

(3) Let $x_t = \exp_{x_0} t\mathbf{x}$. Then $\mathcal{R} = |\mathcal{R}|\dot{x}_t$. Since $\nabla_{\dot{x}_1} e_i = 0$, we get (3).

The proof of Lemma 3.2.7 is completed. □

Lemma 3.2.8. *(1)* $\nabla_{\mathcal{R}}\mathcal{R} = \mathcal{R}$.
(2) $\mathcal{R} = \sum_i \mathbf{x}_i e_i$, *and thus* $(\mathcal{R}, \mathcal{R}) = |\mathbf{x}|^2$.
(3) $(\mathcal{R}, \partial_i) = \mathbf{x}_i$, *and thus* $\mathbf{x}_i = g_{ij}\mathbf{x}_j$.
(4) $d(x_0, x) = |\mathbf{x}|$. *(Note that* $|\mathbf{x}|$ *only depends on* $g(x_0)$ *but* $d(x_0, x)$ *depends on* $g(x_t)$ *for* $t \in [0, 1]$.*)*

Proof. (1) The curve $\mathbf{x}_t = t\mathbf{x}$ is a geodesic. Note that $\mathcal{R}(\mathbf{x}_t) = t\dot{\mathbf{x}}_t$. We also simply denote by $\nabla = \exp_{x_0}^*(\nabla)$. So by (3.2.43),

$$\nabla_{\mathcal{R}}\mathcal{R} = t\nabla_{\dot{\mathbf{x}}_t}(t\dot{\mathbf{x}}_t) = t\dot{\mathbf{x}}_t = \mathcal{R}. \tag{3.2.50}$$

For the second equality, we consider function $f(x_t) = t$ and then $\nabla_{\dot{\mathbf{x}}_t} t = \nabla_{\dot{\mathbf{x}}_t} f = \frac{\partial}{\partial t} f(x_t) = 1$.

(2) From Lemma 3.2.7 (3) and (1),

$$\mathcal{R}(\mathcal{R}, e_i) = (\nabla_{\mathcal{R}}\mathcal{R}, e_i) + (\mathcal{R}, \nabla_{\mathcal{R}} e_i) = (\mathcal{R}, e_i). \tag{3.2.51}$$

From Lemma 3.2.7 (2),

$$(\mathcal{R}, e_i) = \sum_j \mathbf{x}_j(\partial_j, e_i) = \mathbf{x}_i + O(|\mathbf{x}|^2). \tag{3.2.52}$$

Since $\mathcal{R}(\mathbf{x}_{i_1} \cdots \mathbf{x}_{i_k}) = k\mathbf{x}_{i_1} \cdots \mathbf{x}_{i_k}$, from (3.2.51), there is no $O(|\mathbf{x}|^2)$ term in (3.2.52). So $(\mathcal{R}, e_i) = \mathbf{x}_i$. Since $\{e_i(x)\}$ is an orthonormal frame by Lemma 3.2.7 (1), we have $\mathcal{R} = \sum_i \mathbf{x}_i e_i$ and $(\mathcal{R}, \mathcal{R}) = |\mathbf{x}|^2$.

(3) Note that

$$[\mathcal{R}, \partial_i] = -\partial_i(\mathbf{x}_j)\partial_j = -\partial_i. \tag{3.2.53}$$

Since ∇ is torsion free,

$$(\mathcal{R}, \nabla_{\mathcal{R}}\partial_i) = (\mathcal{R}, \nabla_{\partial_i}\mathcal{R}) + (\mathcal{R}, [\mathcal{R}, \partial_i]) = \frac{1}{2}\partial_i|\mathcal{R}|^2 - (\mathcal{R}, \partial_i). \tag{3.2.54}$$

So from (2) and (3.2.54),

$$\mathcal{R}(\mathcal{R}, \partial_i) = (\nabla_{\mathcal{R}}\mathcal{R}, \partial_i) + (\mathcal{R}, \nabla_{\mathcal{R}}\partial_i) = \frac{1}{2}\partial_i|\mathcal{R}|^2 = \mathbf{x}_i. \tag{3.2.55}$$

Since $(\mathcal{R}, \partial_i) = \sum_j \mathbf{x}_j(\partial_i, \partial_j) = \mathbf{x}_i + O(|\mathbf{x}|^2)$, by $\mathcal{R}(\mathbf{x}_{i_1} \cdots \mathbf{x}_{i_k}) = k\mathbf{x}_{i_1} \cdots \mathbf{x}_{i_k}$ again, from (3.2.55), we have $(\mathcal{R}, \partial_i) = \mathbf{x}_i$, and thus $\mathbf{x}_i = g_{ij}\mathbf{x}_j$.

(4) We use the fact that locally the geodesic is the shortest path. Let $\mathbf{x}_t = t\mathbf{x}$. From (2),

$$\begin{aligned} d(x_0, x) = \int_0^1 |\dot{\mathbf{x}}_t| dt = \int_0^1 t^{-1}|R(\dot{\mathbf{x}}_t)| dt \\ = \int_0^1 t^{-1}|\mathbf{x}_t| dt = \int_0^1 |\mathbf{x}| dt = |\mathbf{x}|. \end{aligned} \tag{3.2.56}$$

The proof of Lemma 3.2.8 is completed. □

Let

$$j(\mathbf{x}) = \det{}^{1/2}(g_{ij}(\mathbf{x})). \tag{3.2.57}$$

Then the pull back of the volume form on $T_{x_0}M$

$$dx = j(\mathbf{x})d\mathbf{x}. \tag{3.2.58}$$

In other words, we have

$$j(\mathbf{x}) = |\det(d_{\mathbf{x}} \exp_{x_0})|. \tag{3.2.59}$$

Take $\varepsilon <$ inj. Let $V_y = \mathrm{Im}(\exp_y|_{B(0,\varepsilon)})$. For $x \in V_y$, we define a neighborhood of the diagonal of $M \times M$ by

$$U_\varepsilon = \{(x, y) \in M \times M : x \in V_y\}. \tag{3.2.60}$$

If $(x, y) \in U_\varepsilon$, $d(x, y) < \varepsilon$.

As in (3.1.30), let

$$q_t(x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{d(x,y)^2}{4t}} \in \mathscr{C}^\infty(\mathbb{R}_+ \times U_\varepsilon), \tag{3.2.61}$$

which is modelled on the Euclidean heat kernel. To construct an approximate solution to the heat equation for H, we plan to find a formal solution to the heat equation as a series of the form

$$k_t(x,y) = q_t(x,y)\sum_{t=0}^{\infty} t^i \Psi_i(x,y,H), \tag{3.2.62}$$

where the coefficients $(x,y) \mapsto \Psi_i(x,y,H)$ are smooth sections of the bundle $E \boxtimes E^*$ over U_ε.

We would like to have $(\partial_t + H)k_t(x,y) = 0$.

Proposition 3.2.9. *For any time dependent section s_t of E over U_ε, we have*

$$(\partial_t + H)(q_t \cdot s_t) = q_t \cdot \big(\partial_t + H + t^{-1}\nabla^E_{\mathcal{R}} + (2t)^{-1}\mathcal{R}(\log j)\big)s_t. \tag{3.2.63}$$

Proof. From (1.4.35),

$$\begin{aligned}\Delta^E(q_t s_t) &= -\nabla^E_{e_i}\nabla^E_{e_i}(q_t s_t) + \nabla^E_{\nabla_{e_i}e_i}(q_t s_t)\\ &= -\nabla^E_{e_i}(e_i(q_t)s_t - q_t\nabla^E_{e_i}s_t) + (\nabla_{e_i}e_i)(q_t)s_t + q_t\nabla^E_{\nabla_{e_i}e_i}s_t\\ &= (\Delta^E q_t)s_t - 2e_i(q_t)\nabla^E_{e_i}s_t + q_t\Delta^E s_t.\end{aligned} \tag{3.2.64}$$

From (3.2.64),

$$(\partial_t + H)(q_t s_t) = ((\partial_t + \Delta)q_t)s_t - 2(dq_t, \nabla^E s_t) + q_t(\partial_t + H)s_t. \tag{3.2.65}$$

Write $x = \exp_y \mathbf{x}$. Then from Lemma 3.2.8 (4),

$$q_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}}. \tag{3.2.66}$$

Then

$$\partial_t q_t = \left(-\frac{n}{2t} + \frac{|\mathbf{x}|^2}{4t^2}\right) q_t. \tag{3.2.67}$$

From (3.2.66),

$$\begin{aligned}\Delta q_t &= -\frac{1}{(4\pi t)^{n/2}} e_i\left(-\frac{1}{4t}e^{-\frac{|\mathbf{x}|^2}{4t}} e_i(|\mathbf{x}|^2)\right) - \frac{1}{(4\pi t)^{n/2}}\frac{1}{4t}e^{-\frac{|\mathbf{x}|^2}{4t}}(\nabla_{e_i}e_i)(|\mathbf{x}|^2)\\ &= -\frac{1}{4t}q_t \cdot \left(\Delta(|\mathbf{x}|^2) + \frac{1}{4t}\big(e_i(|\mathbf{x}|^2)\big)^2\right).\end{aligned} \tag{3.2.68}$$

We calculate $\Delta(|\mathbf{x}|^2)$ first. Let $g^{ij} = (g^{-1})_{ij}$. Then $g^{ij}g_{jk} = \delta_{ik}$. From Lemma 3.2.8 (3),

$$g^{ij}\mathbf{x}_j = g^{ij}g_{jk}\mathbf{x}_k = \mathbf{x}_i. \tag{3.2.69}$$

For any $\phi \in \mathscr{C}_0^\infty(T_yM)$, from (1.4.37), (3.2.58) and (3.2.69),

$$\begin{aligned}\int_{T_yM} \phi\Delta(|\mathbf{x}|^2)dx &= \int_{T_yM} (d\phi, d(|\mathbf{x}|^2))dx = 2\int_{T_yM} (\partial_i\phi)g^{ij}\mathbf{x}_j j(\mathbf{x})d\mathbf{x} \\ &= 2\int_{T_yM} (\partial_i\phi)\mathbf{x}_i j(\mathbf{x})d\mathbf{x} = -2\int_{T_yM} \phi\partial_i(\mathbf{x}_i j(\mathbf{x}))d\mathbf{x} \\ &= -2\int_{T_yM} \phi(n + \mathcal{R}(\log j))dx. \end{aligned}\tag{3.2.70}$$

So

$$\Delta(|\mathbf{x}|^2) = -2(n + \mathcal{R}(\log j)). \tag{3.2.71}$$

On the other hand, from (3.2.69),

$$\big(e_i(|\mathbf{x}|^2)\big)^2 = g^{ij}\partial_i(|\mathbf{x}|^2)\partial_j(|\mathbf{x}|^2) = 4g^{ij}\mathbf{x}_i\mathbf{x}_j = 4\mathbf{x}_j\mathbf{x}_j = 4|\mathbf{x}|^2. \tag{3.2.72}$$

By (3.2.68), (3.2.71) and (3.2.72), we have

$$\Delta q_t = \left(\frac{1}{2t}(n + \mathcal{R}(\log j)) - \frac{|\mathbf{x}|^2}{4t^2}\right) q_t. \tag{3.2.73}$$

From (3.2.66),

$$\partial_i(q_t) = -\frac{\partial_i(|\mathbf{x}|^2)}{4t} q_t = -\frac{\mathbf{x}_i}{2t} q_t. \tag{3.2.74}$$

So from (3.2.70),

$$(dq_t, \nabla^E s_t) = g^{ij}\partial_i(q_t)\nabla^E_{\partial_j} s_t = -\frac{1}{2t} q_t \nabla^E_{\mathcal{R}} s_t. \tag{3.2.75}$$

Therefore, (3.2.63) is obtained from (3.2.67), (3.2.73) and (3.2.75).

The proof of Proposition 3.2.9 is completed. □

Let

$$B = j^{1/2} \circ H \circ j^{-1/2}. \tag{3.2.76}$$

Let

$$\Phi_t = j^{1/2} s_t. \tag{3.2.77}$$

Proposition 3.2.10. *The following identity holds,*

$$\big(\partial_t + H + t^{-1}\nabla^E_{\mathcal{R}} + (2t)^{-1}\mathcal{R}(\log j)\big)s_t = j^{-1/2}\cdot\big(\partial_t + B + t^{-1}\nabla^E_{\mathcal{R}}\big)\Phi_t. \tag{3.2.78}$$

Proof. From (3.2.76),

$$H \circ j^{-1/2} = j^{-1/2}B. \tag{3.2.79}$$

From (3.2.77),

$$j^{-1/2}\nabla^E_{\mathcal{R}}\Phi_t = j^{-1/2}\nabla_{\mathcal{R}} j^{1/2}s_t = \frac{1}{2}j^{-1}\mathcal{R}(j)s_t + \nabla^E_{\mathcal{R}}s_t. \tag{3.2.80}$$

So (3.2.78) follows from (3.2.79) and (3.2.80).

The proof of Proposition 3.2.10 is completed. □

Definition 3.2.11. Let $\Phi_t(x, y)$ be a formal power series in t whose coefficients are smooth sections of $E \boxtimes E^*$ on U_ε. We say $q_t(x, y)j^{-1/2}(\mathbf{x})\Phi_t(x, y)$ is a **formal solution** of the heat equation around y if $x \mapsto \Phi_t(x, y)$, considered as a section of the bundle $E \boxtimes E^*$ over V_y, satisfies the equation

$$\Big(\partial_t + B + t^{-1}\nabla^E_{\mathcal{R}}\Big)\Phi_t(\cdot, y) = 0. \tag{3.2.81}$$

Let $x_t = \exp_y t\mathbf{x}$ be the geodesic connecting y and x, and let $Y(t) \in E_{x_t}$. As in (3.2.46), if for any $t \in [0, 1]$,

$$\nabla^E_{\dot{x}_t}Y(t) = 0, \tag{3.2.82}$$

we say $Y(1)$ **is the parallel transport of** $Y(0)$ **along** x_t **with respect to** ∇^E. As before, $Y(1)$ is uniquely determined by $Y(0)$. We write

$$Y(1) = \tau^E(x, y)Y(0). \tag{3.2.83}$$

In this case, $\tau^E(x, y) : E_y \to E_x$ is a linear isomorphism.

Theorem 3.2.12. *There exists a unique formal solution $k_t(x, y)$ of the heat equation*

$$(\partial_t + H_x)k_t(x, y) = 0 \tag{3.2.84}$$

of the form

$$k_t(x, y) = q_t(x, y)j^{-1/2}(\mathbf{x})\sum_{i=0}^{\infty} t^i\Phi_i(x, y), \tag{3.2.85}$$

such that $\Phi_0(y,y) = \mathrm{Id}_E$. *Furthermore, we have the following recursive formula for* Φ_i*:*

$$\tau^E(x,y)^{-1}\Phi_i(x,y) = -\int_0^1 s^{i-1}\tau^E(x_s,y)^{-1}(B_x\cdot\Phi_{i-1})(x_s,y)ds. \tag{3.2.86}$$

In particular, $\Phi_0(x,y) = \tau^E(x,y)$.

Proof. From Definition 3.2.11, k_t in (3.2.85) is a formal solution if and only if

$$\Big(\partial_t + B + t^{-1}\nabla^E_{\mathcal{R}}\Big)\sum_{i=0}^{\infty} t^i\Phi_i(x,y) = 0. \tag{3.2.87}$$

Note that the equation (3.2.87) is equivalent to the system of equations:

$$\begin{aligned}&\nabla^E_{\mathcal{R}}\Phi_0 = 0,\\ &(\nabla^E_{\mathcal{R}} + i)\Phi_i = -B_x\Phi_{i-1}, \quad i>0.\end{aligned} \tag{3.2.88}$$

The parallel transport $\tau^E(x,y)$ along x_s satisfies the equation $\nabla^E_{\mathcal{R}}\tau^E = 0$ and $\tau^E(y,y) = \mathrm{Id}_E$. So from the uniqueness of the differential equation with initial condition, we have $\Phi_0(x,y) = \tau^E(x,y)$.

Let $\{Y_{y,j}\}_j$ be a basis of E_y. Since $\tau^E(x_s,y) : E_y \to E_{x_s}$ is a linear isomorphism, $\{Y_{x_s,j} = \tau^E(x_s,y)Y_{y,j} \in E_{x_s}\}$ is a basis of E_{x_s}. We could write

$$\Phi_i(x_s,y) = X_j(s)Y_{x_s,j}, \quad -B_x\Phi_{i-1}(x_s,y) = Z_j(s)Y_{x_s,j}. \tag{3.2.89}$$

Since $\mathcal{R}(x_s) = s\dot{x}_s$ and $\nabla^E_{\mathcal{R}}Y_{x_s,j} = 0$, we have

$$\nabla^E_{\mathcal{R}}\Phi_i(x_s,y) = \mathcal{R}(X_j(s))Y_{x_s,j} = sX_j'(s)Y_{x_s,j}. \tag{3.2.90}$$

So the second equation in (3.2.88) is equivalent to the differential equation

$$sX_j'(s) + iX_j(s) = Z_j(s). \tag{3.2.91}$$

The solution of $X_j(s)$ in (3.2.91) is

$$X_j(s) = s^{-i}\int_0^s v^{i-1}Z_j(v)dv + Cs^{-i}. \tag{3.2.92}$$

Since $X_j(s)$ is not singular for $s\to 0$, we get $C=0$ in (3.2.92). Observe that $\tau^E(x,y)^{-1}\Phi_i(x,y) = X_j(1)Y_{y,j}$ and $\tau^E(x_s,y)^{-1}(B_x\cdot\Phi_{i-1})(x_s,y) = Z_j(s)Y_{y,j}$. We obtain (3.2.86).

The proof of Theorem 3.2.12 is completed. □

Let $\psi : \mathbb{R}_+ \to [0,1]$ be a smooth cut-off function such that

$$\psi(s) = \begin{cases} 1, & \text{if } s < \varepsilon^2/4, \\ 0, & \text{if } s > \varepsilon^2. \end{cases} \tag{3.2.93}$$

We write $j(x,y) = j(\mathbf{x})$. Then we construct $k_t^N(x,y)$ in Theorem 3.2.3 by

$$k_t^N(x,y) = \psi(d(x,y)^2) q_t(x,y) j^{-1/2}(x,y) \sum_{i=0}^{N} t^i \Phi_i(x,y). \tag{3.2.94}$$

The following theorem is stronger than Theorem 3.2.3.

Theorem 3.2.13. *Let ℓ be an even positive integer.*

(1) For any $T > 0$, the kernels k_t^N, $0 < t < T$ define a uniformly bounded family of operators K_t^N on $\mathscr{C}^\ell(M,E)$, and

$$\lim_{t\to 0} \|K_t^N s - s\|_{\mathscr{C}^\ell} = 0. \tag{3.2.95}$$

(2) There exist differential operators D_k of order less than or equal to $2k$ such that D_0 is the identity and such that for any $s \in \mathscr{C}^{\ell+1}(M,E)$,

$$\left\| K_t^N s - \sum_{k=0}^{\ell/2-j} t^k D_k s \right\|_{\mathscr{C}^{2j}} = O\left(t^{(\ell+1)/2-j}\right). \tag{3.2.96}$$

(3) The kernel $r_t^N(x,y) = (\partial_t + H_x) k_t^N(x,y)$ satisfies the estimates

$$\|\partial_t^k r_t^N\|_{\mathscr{C}^\ell} < C t^{(N-n/2)-k-\ell/2}, \tag{3.2.97}$$

where the constant $C > 0$ only depends on ℓ and k.

Proof. We write $y = \exp_x \mathbf{y}$, with $\mathbf{y} \in T_xM$. Let

$$\Psi_i(x,\mathbf{y}) := \psi(\|\mathbf{y}\|^2) j^{1/2}(\mathbf{y}) \Phi_i(x, \exp_x \mathbf{y}) \tau^E(x,y)^{-1} \in \operatorname{End}(E_x). \tag{3.2.98}$$

Let $s \in \mathscr{C}^\infty(M,E)$. For $y \in B(x,\varepsilon)$, we write $s(x,\mathbf{y}) = \tau^E(x,y)s(y) \in E_x$. Then from (3.2.58), for $\mathbf{y} = \sqrt{t}v$,

$$\begin{aligned}(K_t^N s)(x) &= (4\pi t)^{-n/2} \int_M e^{-d(x,y)^2/4t} \sum_{i=0}^{N} t^i \psi(d(x,y)^2) j^{-1/2}(x,y) \Phi_i(x,y) s(y) dy \\ &= (4\pi t)^{-n/2} \int_{T_xM} e^{-|\mathbf{y}|^2/4t} \sum_{i=0}^{N} t^i \Psi_i(x,\mathbf{y}) s(x,\mathbf{y}) d\mathbf{y} \\ &= (4\pi)^{-n/2} \int_{T_xM} e^{-|v|^2/4} \sum_{i=0}^{N} t^i \Psi_i(x, t^{1/2}v) s(x, t^{1/2}v) dv. \end{aligned} \tag{3.2.99}$$

From (3.2.98), we see that $\Psi_0(x,0) = \mathrm{Id}_{E_x}$.

Let $\{Y_j\}$ be a basis of E_y and $s(y) = s_j Y_j$. Then $s(x,\mathbf{y}) = s_j \tau^E(x,y)Y_j$. Let $C = \max_{y\in B(x,\varepsilon)} \sum_{|\alpha|\le\ell} |D_x^\alpha(\tau^E(x,y))|$. Then $\|s(x,\mathbf{y})\|_{\mathscr{C}_x^\ell} \le C|s(y)| \le C\|s\|_{\mathscr{C}^0} \le C\|s\|_{\mathscr{C}^\ell}$. So from (3.2.99), there exists $C' > 0$ such that for $0 \le t \le T$,

$$\|K_t^N s\|_{\mathscr{C}^\ell} \le C'\|\Psi\|_{\mathscr{C}_x^\ell} \left(\sum_{i=0}^N T^i\right)\left(\int_{\mathbb{R}^n} e^{-|v|^2/4} dv\right)\cdot \|s\|_{\mathscr{C}^\ell}. \tag{3.2.100}$$

So K_t is uniformly bounded on $\mathscr{C}^\ell(M,E)$. In this case, for $|\alpha|\le\ell$,

$$\lim_{t\to 0} D^\alpha(K_t^N s - s) = D^\alpha \lim_{t\to 0}(K_t^N s - s). \tag{3.2.101}$$

From (3.2.99),

$$\lim_{t\to 0} K_t^N s = (4\pi)^{-n/2}\int_{T_xM} e^{-|v|^2/4}\Psi_0(x,0)s(x,0)dv = s(x). \tag{3.2.102}$$

Therefore, we get (3.2.95).

For (2), set $\sigma = t^{1/2}$ and

$$f(\sigma,v) = \sum_{i=0}^N \sigma^{2i}\Psi_i(x,\sigma v)s(x,\sigma v). \tag{3.2.103}$$

Taylor expansion at $\sigma = 0$, from (1.3.43), we have

$$\begin{aligned} f(\sigma,v) = \sum_{|\alpha|\le l}\sum_{\beta+\gamma=\alpha}\sum_{i=0}^N \sigma^{2i}\frac{\alpha!}{\beta!\gamma!}\Psi_i^{(\beta)}(x,0)s^{(\gamma)}(x,0)(\sigma v)^\alpha \\ + \sum_{|\mu|=l+1}\sum_{\beta+\gamma=\mu}\sum_{i=0}^N \sigma^{2i}\frac{l+1}{\mu!}(\sigma v)^\mu\cdot\frac{\alpha!}{\beta!\gamma!} \\ \cdot\int_0^1 (1-s)^l\Psi_i^{(\beta)}(x,sv)s^{(\gamma)}(x,sv)ds. \end{aligned} \tag{3.2.104}$$

If $|\alpha|\le\ell$ is odd,

$$\int_{T_xM} e^{-|v|^2/4}v^\alpha dv = 0. \tag{3.2.105}$$

So we only need to consider the even case. In this case, let

$$D_k := (4\pi)^{-n/2}\sum_{2i+|\alpha|=2k}\sum_{|\alpha|\le l}\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}\Psi_i^{(\beta)}(x,0)\int_{T_xM} e^{-|v|^2/4}v^\alpha dv\cdot D_x^\gamma. \tag{3.2.106}$$

In particular, $D_0 = \mathrm{Id}$. Since $s^{(\gamma)}(x,0) = D_x^\gamma s(x)$, from (3.2.99), (3.2.104) and (3.2.106), we have

$$\left\| K_t^N s - \sum_{k=0}^{\ell/2} t^k D_k s \right\|_{\mathscr{C}^0} = O\left(t^{(\ell+1)/2}\right). \tag{3.2.107}$$

In the same way, we have

$$\left\| K_t^N s - \sum_{k=0}^{\ell/2-j} t^k D_k s \right\|_{\mathscr{C}^{2j}} = O\left(t^{(\ell+1)/2-j}\right). \tag{3.2.108}$$

Note that in this case, $s \in \mathscr{C}^{\ell+1}(M,E)$, so our estimate is for $O\left(t^{(\ell+1)/2-j}\right)$.

From Propositions 3.2.9 and 3.2.10,

$$r_t^N(x,y) = q_t(x,y) j^{-1/2}(x,y) \Big(\partial_t + B_x + t^{-1}\nabla_{\mathcal{R}}^E\Big) \left(\psi(d(x,y)^2) \sum_{i=0}^{N} t^i \Phi_i(x,y)\right). \tag{3.2.109}$$

If $d(x,y) \le \varepsilon/2$, $\partial_x^\alpha \psi(d(x,y)^2) \equiv 0$. If $d(x,y) > \varepsilon/2$, for any k, we have

$$\partial_x^\alpha \psi(d(x,y)^2) e^{-d(x,y)^2/4t} < \partial_x^\alpha \psi(d(x,y)^2) e^{-\varepsilon^2/16t} = O(t^k). \tag{3.2.110}$$

From (3.2.87), the terms on the right hand side of (3.2.109), which do not involve a derivative of $\psi(d(x,y)^2)$ cancel, except for one remaining term $t^N q_t(x,y)(B_x\Phi_N)(x,y)$, which may be bounded by $t^{N-n/2}$. So we have

$$\|r_t^N\|_{\mathscr{C}^0} < Ct^{(N-n/2)}. \tag{3.2.111}$$

The estimate of $\|\partial_t^k r_t^N\|_{\mathscr{C}^\ell}$ is similar, once we observe that

$$\partial_t e^{-x^2/t} = t^{-1}(x^2/t)e^{-x^2/t} = O(t^{-1}), \tag{3.2.112}$$

and

$$\partial_x e^{-x^2/t} = t^{-1/2}(-2x/t^{1/2})e^{-x^2/t} = O(t^{-1/2}). \tag{3.2.113}$$

The proof of Theorem 3.2.13 is completed.

□

Now we summarize the properties of heat kernels.

Theorem 3.2.14. *Let $p_t(x,y)$ be the heat kernel of H. Then there exist $\Phi_i \in \mathscr{C}^\infty(M \times M, E \boxtimes E^*)$ such that for every $N > n/2$, the kernel $k_t^N(x,y)$ defined by*

$$\frac{1}{(4\pi t)^{n/2}} e^{-\frac{d(x,y)^2}{4t}} \psi(d(x,y)^2) j^{-1/2}(x,y) \sum_{i=1}^{N} t^i \Phi_i(x,y) \tag{3.2.114}$$

is asymptotic to $p_t(x,y)$:

$$\|\partial_t^k (p_t(x,y) - k_t^N(x,y))\|_{\mathscr{C}^\ell} = O(t^{N-n/2-\ell/2-k}). \tag{3.2.115}$$

The leading term $\Phi_0(x,y) = \tau^E(x,y)$.

The following corollary is the generalization of Proposition 3.1.6.

Corollary 3.2.15. *Let P_t be the heat operator associated with $p_t(x,y)$. Then for $k \in \mathbb{N}$, $s \in \mathscr{C}^\infty(M,E)$,*

$$\left\| P_t s - \sum_{i=0}^{k} \frac{(-tH)^i}{i!} s \right\|_{\mathscr{C}^j} = O(t^{k+1}). \tag{3.2.116}$$

Proof. The heat equation $(\partial_t + H)P_t s = 0$ implies that in (3.2.96), $D_k = (-H)/k!$.

The proof of Corollary 3.2.15 is completed. □

3.3 Trace of the heat kernel

We assume that H is symmetric in this subsection.

Definition 3.3.1. Let $\mathcal{H}$ be a separated Hilbert space and K be a bounded linear operator on $\mathcal{H}$. Let $\{e_i\}$ be an orthonormal basis of $\mathcal{H}$. We define

$$\mathrm{Tr}[K] := (Ke_i, e_i). \tag{3.3.1}$$

We say K is trace class if $\mathrm{Tr}[K]$ is finite and independent of the choice of the orthonormal basis.

Theorem 3.3.2. *The heat operator* $\exp(-tH)$ *is trace class and for* $t > 0$*,*

$$\mathrm{Tr}[\exp(-tH)] = \int_M \mathrm{Tr}^E[\exp(-tH(x,x))]dx. \tag{3.3.2}$$

Theorem 3.3.3 (Weyl law)**.** *Let* $\lambda_1 \leq \lambda_2 \leq \cdots$ *be the eigenvalues of* H*. As* $t \to 0$*, we have*

$$\sum_{i=1}^{\infty} e^{-t\lambda_i} = (4\pi t)^{-n/2} \dim E \cdot \mathrm{vol}(M) + O(t^{-n/2+1}). \tag{3.3.3}$$

Theorem 3.3.4 (Karamata)**.** *Let* $d\mu(\lambda)$ *be a positive measure on* $\mathbb{R}_+$ *such that the integral*

$$\int_0^\infty e^{-t\lambda} d\mu(\lambda) \tag{3.3.4}$$

converges for $t > 0$*, and such that*

$$\lim_{t\to 0} t^\alpha \int_0^\infty e^{-t\lambda} d\mu(\lambda) = C \tag{3.3.5}$$

for some positive constants α *and* C*. If* f *is a continuous function on* $[0,1]$*, then*

$$\lim_{t\to 0} t^\alpha \int_0^\infty f(e^{-t\lambda}) e^{-t\lambda} d\mu(\lambda) = \frac{C}{\Gamma(\alpha)} \int_0^\infty f(e^{-t}) t^{\alpha-1} e^{-t} dt. \tag{3.3.6}$$

Using Karamata's theorem, we obtain the following restatement of Weyl's theorem.

Corollary 3.3.5. *Let* $N(\lambda)$ *be the number of eigenvalues of* H *that are less than* λ*. When* λ *is large enough,*

$$N(\lambda) \sim \frac{\dim(E)\,\mathrm{vol}(M)}{(4\pi)^{n/2}\Gamma(n/2+1)} \lambda^{n/2}. \tag{3.3.7}$$

3.4 Finite propagation

3.5 Zeta function of a Laplacian