λ-RING STRUCTURE IN DIFFERENTIAL K-THEORY

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Abstract. We prove that any differential K^0-ring for closed manifolds admits a λ-ring structure.

0. Introduction

The notion of λ-rings was first introduced by Grothendieck [9] in 1956 and later studied by Atiyah and Tall [2]; see also [3].

For a commutative ring R with identity, a λ-structure on R consists of a countable family of maps $\lambda^n : R \to R$, $n \in \mathbb{N}$, satisfying two sets of conditions (see Definition 1.2). If only a subset of these conditions is satisfied, we refer to it as a pre-λ-ring structure (Definition 1.1). The λ-ring structure is closely related to Adams operations: if R is a λ-ring, one can associate naturally Adams operations. Conversely, for a pre-λ-ring R, with compatible Adams operations, if R is torsion free, the pre-λ-ring structure of R is a λ-ring structure (Theorem 1.3).

It is well known that, although the topological K-ring is not torsion free, it carries a natural λ-ring structure [2]. The corresponding Adams operations plays a key role in the proof of Adams' celebrated theorem on the Hopf invariant [1].

In [8], Gillet and Soulé introduced arithmetic K-theory in Arakelov geometry. This theory extends Grothendieck's K-theory by adding conjugation-invariant Hermitian metrics on homomorphic vector bundles over the complex points and differential forms of type (p, p) modulo $\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}$. A pre-$\lambda$-ring structure on the arithmetic K-ring was constructed in [8], which was later shown by Roessler [14] to be a λ-ring structure.

Differential K-theory, introduced by Freed-Hopkins [6] and developed further by Hopkins-Singer [10], Simons-Sullivan [16], Bunke-Schick [5], Freed-Lott [7], etc., provides a smooth analogue of arithmetic K-theory. It extends topological K-theory by adding Hermitian metrics and connections on complex vector bundles and differential forms modulo exact forms. In [4], Bunke defined Adams operations on the differential K-ring directly in a sophisticated topological way and obtained an associated Adams-Riemann-Roch theorem. In [13], the authors explicitly constructed a pre-λ-structure on the differential K-ring explicitly, which plays important role in their work on the localization formula for η-invariants. In this paper, we prove in Theorem 2.4 that this pre-λ-structure is in fact a λ-ring structure, and we show that the corresponding Adams operations yield a constructive realization of those defined in [4].

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1. λ-RING STRUCTURE

Definition 1.1. [3, (1.1)-(1.3)] For a commutative ring R with identity, a pre-λ-ring structure is defined by a countable set of maps $\lambda^n : R \to R$ with $n \in \mathbb{N}$ such that for all $x, y \in R$,

(1) $\lambda^0(x) = 1$;
(2) $\lambda^1(x) = x$;
(3) $\lambda^n(x+y) = \sum_{j=0}^n \lambda^j(x)\lambda^{n-j}(y)$.

If R has a pre-λ-ring structure, we call it a pre-λ-ring.

Remark that in [2, §1] the pre-λ-ring here is called the λ-ring.

Let R be a pre-λ-ring. If t is an indeterminate, we define for $x \in R$,

$$\lambda_t(x) = \sum_{n\geq 0} \lambda^n(x)t^n. \tag{1.1}$$

Then the relations (1) and (3) show that λ_t is a homomorphism from the additive group of R into the multiplicative group $1 + R[[t]]^+$, of formal power series in t with constant term 1, i.e.,

$$\lambda_t(x+y) = \lambda_t(x)\lambda_t(y) \quad \text{for any } x, y \in R. \tag{1.2}$$

Let $\xi_1, \cdots, \xi_q, \zeta_1 \cdots, \zeta_r$ be indeterminates and let s_i and σ_i be the i-th elementary symmetric functions in $\xi_1, \cdots, \xi_q$ and $\zeta_1 \cdots, \zeta_r$ respectively. Let $P_n(s_1, \cdots, s_n; \sigma_1, \cdots, \sigma_n)$ be the coefficient of t^n in $\prod_{ij}(1 + \xi_i\zeta_j t)$. Let $P_{n,m}(s_1, \cdots, s_{mn})$ be the coefficient of t^n in

$$\prod_{i_1<\cdots<i_m} (1 + \xi_{i_1} \cdots \xi_{i_m} t).$$

We may assume that $r, q \geq n$ for P_n and $q \geq nm$ for $P_{n,m}$. Easy to see that P_n is a polynomial with weight n in $\{s_i\}$ and also weight n in $\{\sigma_i\}$, $P_{n,m}$ is a polynomial with weight nm in $\{s_i\}$. Both P_n and $P_{n,m}$ have integer coefficients and so may be defined in any commutative ring.

Let A be any commutative ring with identity. By [2, Lemma 1.1], $1+A[[t]]^+$ is a pre-λ-ring with new operations $(\widetilde{+}, \widetilde{\times}, \widetilde{\lambda}^n)$ (with 1 is the "zero" and $1 + t$ is the "identity") defined as follows: for $a_i, b_j \in A$,

$$\begin{aligned} \left(1 + \sum a_n t^n\right) \widetilde{+} \left(1 + \sum b_n t^n\right) &:= \left(1 + \sum a_n t^n\right) \cdot \left(1 + \sum b_n t^n\right), \\ \left(1 + \sum a_n t^n\right) \widetilde{\times} \left(1 + \sum b_n t^n\right) &:= 1 + \sum P_n(a_1, \cdots, a_n; b_1, \cdots, b_n)t^n \end{aligned} \tag{1.3}$$

and

$$\widetilde{\lambda}^m \left(1 + \sum a_n t^n\right) = 1 + \sum P_{n,m}(a_1, \cdots, a_{nm})t^n. \tag{1.4}$$

Definition 1.2. [2, Definition 1.2] We say a pre-λ-ring structure of R is a λ-ring structure if $\lambda_t : R \to 1 + R[[t]]^+$ is a homomorphism of pre-λ-rings, i.e., for any $x, y \in R$, $n \in \mathbb{N}$,

$$\lambda_t(x+y) = \lambda_t(x)\widetilde{+}\lambda_t(y),\ \lambda_t(xy) = \lambda_t(x)\widetilde{\times}\lambda_t(y),\ \lambda_t(\lambda^n(x)) = \widetilde{\lambda}^n(\lambda_t(x)), \tag{1.5}$$

which reads that for any $x, y \in R$,

$$\lambda^n(xy) = P_n(\lambda^1(x), \cdots, \lambda^n(x); \lambda^1(y), \cdots, \lambda^n(y)),\ \text{i.e., } \lambda_t(xy) = \lambda_t(x)\widetilde{\times}\lambda_t(y), \tag{1.6}$$

and

$$\lambda^m(\lambda^n(x)) = P_{m,n}(\lambda^1(x), \cdots, \lambda^{mn}(x)), \text{ i.e., } \lambda_t(\lambda^n(x)) = \widetilde{\lambda}^n(\lambda_t(x)). \tag{1.7}$$

If R has a λ-ring structure, we call it a λ-ring.

If R is a λ-ring, from (1.7) for $n = 0$, we get

$$\lambda_t(1) = 1 + t. \tag{1.8}$$

Note that in [2] the λ-ring here is called the special λ-ring. The following theorem (see [12, p.49] or [3, §V, Appendice]) is a convenient criterion to verify whether a pre-λ-ring is in fact a λ-ring.

Theorem 1.3. *Let R be a pre-λ-ring. We assume that R is torsion free, that is, for any $r \in R$, $r \neq 0$ and any $n \in \mathbb{Z}_+$, $nr = \underbrace{r + \cdots + r}_{n} \neq 0$. Let $\Psi^n : R \to R$, $n \in \mathbb{Z}_+$ be the Adams operations defined by*

$$\frac{d}{dt} \log \lambda_t(a) = \sum_{n=0}^{+\infty} (-1)^n \Psi^{n+1}(a) t^n, \quad \text{for all } a \in R. \tag{1.9}$$

Suppose that for any $n, m \in \mathbb{Z}_+$, $a, b \in R$, we have

$$\Psi^n(1) = 1, \quad \Psi^n(ab) = \Psi^n(a)\Psi^n(b), \quad \Psi^n(\Psi^m(a)) = \Psi^{nm}(a). \tag{1.10}$$

Then the pre-λ-ring structure of R is a λ-ring structure.

2. Differential K-theory

Let B be a closed manifold. Let $\underline{E} = (E, h^E, \nabla^E)$ be a triple which consists of a complex vector bundle E over B, a Hermitian metric h^E on E and a Hermitian connection ∇^E on (E, h^E). As in [13], we call such $\underline{E}$ a geometric triple on B.

For a geometric triple $\underline{E}$ on B, denote by $R^E := (\nabla^E)^2$ the curvature on E. The Chern character form (cf. e.g., [17, (1.22)])

$$\mathrm{ch}(\underline{E}) := \mathrm{Tr}\left[\exp\left(\frac{i}{2\pi} R^E\right)\right] \in \Omega^{\mathrm{even}}(B, \mathbb{R}). \tag{2.1}$$

For two geometric triples $\underline{E_0}$ and $\underline{E_1}$, which have the same underlying vector bundle E, for the obvious projection $\pi : B \times \mathbb{R} \to B$, there exists geometric triple $\underline{\pi^* E}$ on $B \times \mathbb{R}$, such that

$$\underline{\pi^* E}|_{B \times \{j\}} = \underline{E_j}, \quad j = 0, 1. \tag{2.2}$$

For $\alpha = \alpha_0 + ds \wedge \alpha_1 \in \Lambda^\bullet(T^*(B \times \mathbb{R}))$ with $\alpha_0, \alpha_1 \in \Lambda^\bullet(T^*B)$, we denote by $\{\alpha\}^{ds} := \alpha_1$. Thus the Chern-Simons form (cf. e.g., [15, Definition B.5.3])

$$\widetilde{\mathrm{ch}}(\underline{E_0}, \underline{E_1}) := \int_0^1 \{\mathrm{ch}(\underline{\pi^* E})\}^{ds} ds \in \Omega^{\mathrm{odd}}(B, \mathbb{R}) / \mathrm{Im}\, d. \tag{2.3}$$

It is independent of the choice of $\underline{\pi^* E}$ and satisfies

$$d\, \widetilde{\mathrm{ch}}(\underline{E_0}, \underline{E_1}) = \mathrm{ch}(\underline{E_1}) - \mathrm{ch}(\underline{E_0}). \tag{2.4}$$

Definition 2.1. [7, Definition 2.16] A cycle for the differential K-group of B is a pair $(\underline{E}, \phi)$ where $\underline{E} = (E, h^E, \nabla^E)$ is a geometric triple on B and ϕ is an element in $\Omega^{\mathrm{odd}}(B, \mathbb{R})/\operatorname{Im} d$. We say two cycles $(\underline{E_1}, \phi_1)$ and $(\underline{E_2}, \phi_2)$ are equivalent if there exist a geometric triple $\underline{E_3} = (E_3, h^{E_3}, \nabla^{E_3})$ and a vector bundle isomorphism over B

$$\Phi : E_1 \oplus E_3 \to E_2 \oplus E_3 \tag{2.5}$$

such that

$$\widetilde{\mathrm{ch}}\left(\underline{E_1} \oplus \underline{E_3}, \Phi^* \left(\underline{E_2} \oplus \underline{E_3}\right)\right) = \phi_2 - \phi_1. \tag{2.6}$$

We define the sum of the cycles in the natural way and denote by $\widehat{K}^0(B)$ the Grothendieck group of equivalence classes of cycles.

Denote by $[\underline{E}, \phi] \in \widehat{K}^0(B)$ the equivalence class of a cycle $(\underline{E}, \phi)$. For $[\underline{E}, \phi], [\underline{F}, \psi] \in \widehat{K}^0(B)$, set

$$[\underline{E}, \phi] \cup [\underline{F}, \psi] = [\underline{E} \otimes \underline{F}, \mathrm{ch}(\underline{E}) \wedge \psi + \phi \wedge \mathrm{ch}(\underline{F}) - d\phi \wedge \psi]. \tag{2.7}$$

This product (2.7) on $\widehat{K}^0(B)$ is well-defined, associative and commutative ([11, Proposition 4.79]). Thus $(\widehat{K}^0(B), +, \cup)$ is a commutative ring with unity $[\underline{\mathbb{C}}, 0]$, where $\underline{\mathbb{C}}$ denotes the trivial line bundle over B with trivial metric and connection.

We denote by $Z^{\mathrm{even}}(B, \mathbb{R})$ the vector space of even degree closed differential forms on B. Consider the vector space

$$\Gamma(B) := Z^{\mathrm{even}}(B, \mathbb{R}) \oplus (\Omega^{\mathrm{odd}}(B, \mathbb{R})/\operatorname{Im} d). \tag{2.8}$$

We give degree $l \geq 0$ to $Z^{2l}(B, \mathbb{R}) \oplus \left(\Omega^{2l-1}(B, \mathbb{R})/\operatorname{Im} d\right)$ with $\Omega^{-1}(\cdot) = \{0\}$. We define the multiplication on $\Gamma(B)$ by the formula

$$(\omega_1, \phi_1) * (\omega_2, \phi_2) := (\omega_1 \wedge \omega_2, \omega_1 \wedge \phi_2 + \phi_1 \wedge \omega_2 - d\phi_1 \wedge \phi_2), \tag{2.9}$$

which is commutative and associative. Then $\Gamma(B)$ is a ring with unity $(1, 0)$ which is torsion free. We define the Adams operation $\Psi_\Gamma^k : \Gamma(B) \to \Gamma(B)$ for $k \in \mathbb{N}$ by

$$\Psi_\Gamma^k(\alpha, \beta) = (k^l \alpha, k^l \beta) \quad \text{for} \quad (\alpha, \beta) \in Z^{2l}(B, \mathbb{R}) \oplus (\Omega^{2l-1}(B, \mathbb{R})/\operatorname{Im} d). \tag{2.10}$$

Then the relation

$$\lambda_t(x) = \sum_{n \geq 0} \lambda_\Gamma^n(x) t^n := \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \Psi_\Gamma^k(x) t^k}{k}\right) \tag{2.11}$$

gives a pre-λ-ring structure on $\Gamma(B)$ (cf. [13, (2.4)]). For $\alpha \in Z^{2l}(B, \mathbb{R})$, the map $\Psi_Z^k(\alpha) := k^l \alpha$ also defines Adams operations $\Psi_Z^k : Z^{\mathrm{even}}(B, \mathbb{R}) \to Z^{\mathrm{even}}(B, \mathbb{R})$ for $k \in \mathbb{Z}_+$. It induces a pre-λ-ring structure $\lambda_Z^n : Z^{\mathrm{even}}(B, \mathbb{R}) \to Z^{\mathrm{even}}(B, \mathbb{R})$ on $Z^{\mathrm{even}}(B, \mathbb{R})$ as in (2.11). From [13, Lemma 2.3], the restriction map $p : \Gamma(B) \to Z^{\mathrm{even}}(B, \mathbb{R})$ is a homomorphism of pre-λ-rings.

Lemma 2.2. *The pre-λ-ring structures on $\Gamma(B)$ and $Z^{\mathrm{even}}(B, \mathbb{R})$ are λ-ring structures. Moreover the restriction map $p : \Gamma(B) \to Z^{\mathrm{even}}(B, \mathbb{R})$ is a homomorphism of λ-rings.*

Proof. Since $\Gamma(B)$ (resp. $Z^{\mathrm{even}}(B, \mathbb{R})$) is torsion free and the Adams operations Ψ_Γ^k (resp. Ψ_Z^k) satisfy (1.9) and (1.10), from Theorem 1.3, the pre-λ-ring structure on $\Gamma(B)$ (resp.

$Z^{\mathrm{even}}(B,\mathbb{R}))$ is a λ-ring structure. Moreover, since $p\circ\Psi^k_\Gamma=\Psi^k_Z\circ p$, by (2.11) and the corresponding definition for λ^n_Z, the restriction map $p:\Gamma(B)\to Z^{\mathrm{even}}(B,\mathbb{R})$ is a homomorphism of λ-rings. The proof of Lemma 2.2 is complete. □

Let $(\underline{E},\phi)$ and $(\underline{F},\psi)$ be cycles of $\widehat{K}^0(B)$. Then by (2.7) and (2.9),

$$(\underline{E},\phi)\cup(\underline{F},\psi)=\Big(\underline{E}\otimes\underline{F},\,\big[(\mathrm{ch}(\underline{E}),\phi)*(\mathrm{ch}(\underline{F}),\psi)\big]_{\mathrm{odd}}\Big),\tag{2.12}$$

where $[\cdot]_{\mathrm{odd}}$ (resp. $[\cdot]_{\mathrm{even}}$) is the component of $\Gamma(B)$ in $\Omega^{\mathrm{odd}}(B,\mathbb{R})/\mathrm{Im}\,d$ (resp. $Z^{\mathrm{even}}(B,\mathbb{R})$).

From (2.9) and (2.11), for $\omega\in Z^{\mathrm{even}}(B,\mathbb{R})$, $\phi\in\Omega^{\mathrm{odd}}(B,\mathbb{R})/\mathrm{Im}\,d$, we have

$$\big[\lambda^k_\Gamma(\omega,0)\big]_{\mathrm{odd}}=0,\quad \big[\lambda^k_\Gamma(0,\phi)\big]_{\mathrm{even}}=0.\tag{2.13}$$

Note that from [13, Lemma 2.3, (2.11), (2.33) and (2.47)], we have

$$\big[\lambda^k_\Gamma(\mathrm{ch}(\underline{E}),\phi)\big]_{\mathrm{even}}=\mathrm{ch}(\Lambda^k(\underline{E})),\tag{2.14}$$

here $\Lambda^k(\underline{E})$ denotes the k-th exterior algebra bundle of E with corresponding metric and connection induced from $\underline{E}$. Set

$$\lambda^k(\underline{E},\phi):=\big(\Lambda^k(\underline{E}),\,\big[\lambda^k_\Gamma(\mathrm{ch}(\underline{E}),\phi)\big]_{\mathrm{odd}}\big)\,.\tag{2.15}$$

By (2.13) and (2.15),

$$\lambda^k(\underline{0},\phi)=\big(\underline{0},\,\big[\lambda^k_\Gamma(0,\phi)\big]_{\mathrm{odd}}\big)\,,\quad \lambda^k(\underline{E},0):=\big(\Lambda^k(\underline{E}),0\big)\,.\tag{2.16}$$

From the proof of [13, Theorem 2.6], the map λ^k in (2.15) induces a well-defined map $\lambda^k:\widehat{K}^0(B)\to\widehat{K}^0(B)$. Moreover by [13, Theorem 2.6], (2.15) induces a pre-λ-ring structure on $\widehat{K}^0(B)$.

From the proof of [2, Theorem 1.5 (i)], for vector bundles E and F, by the splitting principle we have

$$\Lambda^n(E\otimes F)=P_n\big(\Lambda^1(E),\cdots,\Lambda^n(E);\Lambda^1(F),\cdots,\Lambda^n(F)\big),\tag{2.17}$$

and

$$\Lambda^n(\Lambda^m(E))=P_{n,m}\big(\Lambda^1(E),\cdots,\Lambda^{nm}(E)\big).\tag{2.18}$$

as isomorphisms of complex vector bundles.

Lemma 2.3. *For geometric triples $\underline{E}$ and $\underline{F}$ on B, we have*

$$\Lambda^n(\underline{E}\otimes\underline{F})=P_n\big(\Lambda^1(\underline{E}),\cdots,\Lambda^n(\underline{E});\Lambda^1(\underline{F}),\cdots,\Lambda^n(\underline{F})\big)\tag{2.19}$$

and

$$\Lambda^n(\Lambda^m(\underline{E}))=P_{n,m}\big(\Lambda^1(\underline{E}),\cdots,\Lambda^{nm}(\underline{E})\big).\tag{2.20}$$

Proof. From (2.17) and (2.18), we only need to prove (2.19) and (2.20) locally. So we may assume that B is contractible and E, F are trivial over B. Here we only prove (2.19). The proof of (2.20) is the same.

Since E and F can be decomposed into the orthogonal sums of the complex line bundles, from the definition of P_n, the isomorphism in (2.17) is an isometry.

Let $\{s_i^E\}$ and $\{s_j^F\}$ be the orthonormal frames of E and F respectively. Then for a fixed vector field X,

$$\nabla_X^E s_i^E = \sum_k \omega_{ik}^E(X) s_k^E, \quad \nabla_X^F s_j^F = \sum_l \omega_{jl}^F(X) s_l^F, \tag{2.21}$$

where $\omega^E(X)$ and $\omega^F(X)$ are the connection matrices associated with E and F respectively. Since E and F can be decomposed into the orthogonal sums of the complex line bundles generated by s_i^E and s_j^F respectively, for any section s of $\Lambda^n(E\otimes F)$, it has the same representation by s_i^E and s_j^F as the corresponding section of $P_n\big(\Lambda^1(E), \cdots, \Lambda^n(E); \Lambda^1(F), \cdots, \Lambda^n(F)\big)$. Thus the connection matrices on two hand sides of (2.17) induced by (2.21) are the same.

The proof of Lemma 2.3 is complete. □

The main result of this paper is as follows.

Theorem 2.4. *The pre-λ-ring structure on $\widehat{K}^0(B)$ defined in (2.15) is a λ-ring structure.*

Proof. From Definition 1.2, we only need to prove that for cycles $(\underline{E}, \phi)$ and $(\underline{F}, \psi)$,

$$\lambda_t\left((\underline{E}, \phi) \cup (\underline{F}, \psi)\right) = \lambda_t(\underline{E}, \phi) \widetilde{\times} \lambda_t(\underline{F}, \psi) \tag{2.22}$$

and

$$\lambda_t(\lambda^m(\underline{E}, \phi)) = \widetilde{\lambda}^m\left(\lambda_t(\underline{E}, \phi)\right). \tag{2.23}$$

By (1.2) and (1.3),

$$\begin{aligned}\lambda_t\big((\underline{E}, \phi) \cup (\underline{F}, \psi)\big) = {} & \lambda_t\big((\underline{E}, 0) \cup (\underline{F}, 0)\big) \widetilde{+} \lambda_t\big((\underline{E}, 0) \cup (\underline{0}, \psi)\big) \\ & \widetilde{+} \lambda_t\big((\underline{0}, \phi) \cup (\underline{F}, 0)\big) \widetilde{+} \lambda_t\big((\underline{0}, \phi) \cup (\underline{0}, \psi)\big).\end{aligned} \tag{2.24}$$

Then by (1.1), (1.3), (2.9), (2.12), (2.15), (2.16) and (2.19),

$$\begin{aligned}\lambda_t\big((\underline{E}, 0) \cup (\underline{F}, 0)\big) & \overset{(2.12)}{=} \lambda_t\big((\underline{E} \otimes \underline{F}, 0)\big) \xlongequal{(1.1)(2.15)} 1 + \sum_{n \geq 1} \big(\Lambda^n(\underline{E} \otimes \underline{F}), 0\big) \cdot t^n \\ & \overset{(2.19)}{=} 1 + \sum_{n \geq 1} \Big(P_n\big(\Lambda^1(\underline{E}), \cdots, \Lambda^n(\underline{E}); \Lambda^1(\underline{F}), \cdots, \Lambda^n(\underline{F})\big), 0\Big) \cdot t^n \\ & \xlongequal{(2.9)(2.12)} 1 + \sum_{n \geq 1} P_n\Big(\big(\Lambda^1(\underline{E}), 0\big), \cdots, \big(\Lambda^n(\underline{E}), 0\big); \big(\Lambda^1(\underline{F}), 0\big), \cdots, \big(\Lambda^n(\underline{F}), 0\big)\Big) \cdot t^n \\ & \overset{(1.3)}{=} \left(1 + \sum_{n \geq 1} \big(\Lambda(\underline{E}), 0\big) t^n\right) \widetilde{\times} \left(1 + \sum_{n \geq 1} \big(\Lambda(\underline{F}), 0\big) t^n\right) \overset{(2.16)}{=} \lambda_t\big(\underline{E}, 0\big) \widetilde{\times} \lambda_t\big(\underline{F}, 0\big).\end{aligned} \tag{2.25}$$

By (2.7), (2.9) and (2.15),

$$\begin{aligned}\lambda^n\big((\underline{E}, 0) \cup (\underline{0}, \psi)\big) & \overset{(2.7)}{=} \lambda^n\big(\underline{0}, \mathrm{ch}(\underline{E})\psi\big) \overset{(2.15)}{=} \Big(\underline{0}, \big[\lambda_\Gamma^n(0, \mathrm{ch}(\underline{E})\psi)\big]_{\mathrm{odd}}\Big) \\ & \overset{(2.9)}{=} \Big(\underline{0}, \big[\lambda_\Gamma^n((\mathrm{ch}(\underline{E}), 0) * (0, \psi))\big]_{\mathrm{odd}}\Big).\end{aligned} \tag{2.26}$$

From (2.13) and (2.14), we have

$$\lambda_\Gamma^k(\mathrm{ch}(\underline{E}), 0) = (\mathrm{ch}(\Lambda^k(\underline{E})), 0). \tag{2.27}$$

From (2.13), we have

$$\lambda^k_\Gamma(0,\psi)=\big(0,[\lambda^k_\Gamma(0,\psi)]_{\mathrm{odd}}\big). \tag{2.28}$$

Recall that P_n is a polynomial with weight n in $\{\sigma_i\}$. Since $\Gamma(B)$ is a λ-ring, by (1.6), (2.9), (2.27) and (2.28), for $n\geq 1$, we get

$$\begin{aligned}
&\lambda^n_\Gamma\big((\mathrm{ch}(\underline{E}),0)*(0,\psi)\big)\overset{(1.6)}{=}P_n\big(\lambda^1_\Gamma(\mathrm{ch}(\underline{E}),0),\cdots,\lambda^n_\Gamma(\mathrm{ch}(\underline{E}),0);\lambda^1_\Gamma(0,\psi),\cdots,\lambda^n_\Gamma(0,\psi)\big)\\
&\overset{(2.27)(2.28)}{=\!=\!=\!=\!=}P_n\big((\mathrm{ch}(\Lambda^1(\underline{E})),0),\cdots,(\mathrm{ch}(\Lambda^n(\underline{E})),0);\big(0,\big[\lambda^1_\Gamma(0,\psi)\big]_{\mathrm{odd}}\big),\cdots,(0,[\lambda^n_\Gamma(0,\psi)]_{\mathrm{odd}})\big)\\
&\qquad\overset{(2.9)}{=}\big(0,\big[P_n\big((\mathrm{ch}(\Lambda^1(\underline{E})),0),\cdots,(\mathrm{ch}(\Lambda^n(\underline{E})),0);\\
&\qquad\qquad\big(0,\big[\lambda^1_\Gamma(0,\psi)\big]_{\mathrm{odd}}\big),\cdots,(0,[\lambda^n_\Gamma(0,\psi)]_{\mathrm{odd}})\big)\big]_{\mathrm{odd}}\Big).
\end{aligned}\tag{2.29}$$

By (2.12), (2.16) and (2.29), for $n\geq 1$,

$$\begin{aligned}
&P_n\big(\lambda^1(\underline{E},0),\cdots,\lambda^n(\underline{E},0);\lambda^1(\underline{0},\psi),\cdots,\lambda^n(\underline{0},\psi)\big)\\
&\overset{(2.16)}{=}P_n\,\big((\Lambda^1(\underline{E}),0),\cdots,(\Lambda^n(\underline{E}),0);\big(\underline{0},\big[\lambda^1_\Gamma(0,\psi)\big]_{\mathrm{odd}}\big),\cdots,(\underline{0},[\lambda^n_\Gamma(0,\psi)]_{\mathrm{odd}})\big)\\
&\overset{(2.12)}{=}\big(\underline{0},\big[P_n\big((\mathrm{ch}(\Lambda^1(\underline{E})),0),\cdots,(\mathrm{ch}(\Lambda^n(\underline{E})),0);\\
&\qquad\big(0,\big[\lambda^1_\Gamma(0,\psi)\big]_{\mathrm{odd}}\big),\cdots,(0,[\lambda^n_\Gamma(0,\psi)]_{\mathrm{odd}})\big)\big]_{\mathrm{odd}}\Big)\\
&\overset{(2.29)}{=}\big(\underline{0},\big[\lambda^n_\Gamma\big((\mathrm{ch}(\underline{E}),0)*(0,\psi)\big)\big]_{\mathrm{odd}}\big).
\end{aligned}\tag{2.30}$$

By (2.26) and (2.30), we get

$$\lambda^n\big((\underline{E},0)\cup(\underline{0},\psi)\big)=P_n\big(\lambda^1(\underline{E},0),\cdots,\lambda^n(\underline{E},0);\lambda^1(\underline{0},\psi),\cdots,\lambda^n(\underline{0},\psi)\big). \tag{2.31}$$

Similarly, we have

$$\lambda^n\big((\underline{0},\phi)\cup(\underline{0},\psi)\big)=P_n\big(\lambda^1(\underline{0},\phi),\cdots,\lambda^n(\underline{0},\phi);\lambda^1(\underline{0},\psi),\cdots,\lambda^n(\underline{0},\psi)\big). \tag{2.32}$$

Thus from (1.1), (1.3), (2.31) and (2.32),

$$\lambda_t\big((\underline{E},0)\cup(\underline{0},\psi)\big)=\lambda_t(\underline{E},0)\widetilde{\times}\lambda_t(\underline{0},\psi),\quad \lambda_t\big((\underline{0},\phi)\cup(\underline{0},\psi)\big)=\lambda_t(\underline{0},\phi)\widetilde{\times}\lambda_t(\underline{0},\psi). \tag{2.33}$$

To sum up, since $(\widetilde{+},\widetilde{\times})$ satisfies the distributive law (cf. [2, Lemma 1.1]), by (1.3), (2.25) and (2.33), we have

$$\begin{aligned}
&\lambda_t((\underline{E},0)\cup(\underline{F},\psi))=\lambda_t((\underline{E},0)\cup(\underline{F},0))\cdot\lambda_t((\underline{E},0)\cup(\underline{0},\psi))\\
&=\lambda_t((\underline{E},0)\cup(\underline{F},0))\widetilde{+}\lambda_t((\underline{E},0)\cup(\underline{0},\psi))=(\lambda_t(\underline{E},0)\widetilde{\times}\lambda_t(\underline{F},0))\widetilde{+}(\lambda_t(\underline{E},0)\widetilde{\times}\lambda_t(\underline{0},\psi))\\
&\qquad=\lambda_t(\underline{E},0)\widetilde{\times}\,\big(\lambda_t(\underline{F},0))\widetilde{+}\lambda_t(\underline{0},\psi))\big)=\lambda_t(\underline{E},0)\widetilde{\times}\lambda_t(\underline{F},\psi).
\end{aligned}\tag{2.34}$$

In the same way, by (1.3), (2.24), (2.33) and (2.34), we have

$$\lambda_t((\underline{E},\phi)\cup(\underline{F},\psi))=\lambda_t((\underline{E},0)\cup(\underline{F},\psi))\cdot\lambda((\underline{0},\phi)\cup(\underline{F},\psi))=\lambda_t(\underline{E},\phi)\widetilde{\times}\lambda_t(\underline{F},\psi). \tag{2.35}$$

On the other hand, by (2.12), (2.14) and (2.15), we have

$$\lambda^i\big(\underline{E},\phi\big)\cup\lambda^j\big(\underline{F},\psi\big)=\Big(\Lambda^i(\underline{E})\otimes\Lambda^j(\underline{F}),\big[\lambda^i_\Gamma(\mathrm{ch}(\underline{E}),\phi)*\lambda^j_\Gamma(\mathrm{ch}(\underline{F}),\psi)\big]_{\mathrm{odd}}\Big). \tag{2.36}$$

Note that by (2.15),

$$\lambda^i\big(\underline{E},\phi\big)+\lambda^j\big(\underline{F},\psi\big)=\Big(\Lambda^i(\underline{E})\oplus\Lambda^j(\underline{F}),\,\big[\lambda^i_\Gamma(\mathrm{ch}(\underline{E}),\phi)+\lambda^j_\Gamma(\mathrm{ch}(\underline{F}),\psi)\big]_{\mathrm{odd}}\Big). \tag{2.37}$$

Since $P_{n,m}$ is a polynomial, from (2.36) and (2.37), we have

$$\begin{aligned} &P_{n,m}\big(\lambda^1(\underline{E},\phi),\cdots,\lambda^{mn}(\underline{E},\phi)\big)\\ &\qquad=\Big(P_{n,m}\big(\Lambda^1(\underline{E}),\cdots,\Lambda^{mn}(\underline{E})\big),\,\Big[P_{n,m}\big(\lambda^1_\Gamma(\mathrm{ch}(\underline{E}),\phi),\cdots,\lambda^{mn}_\Gamma(\mathrm{ch}(\underline{E}),\phi)\big)\Big]_{\mathrm{odd}}\Big).\end{aligned} \tag{2.38}$$

Recall that $\Gamma(B)$ is a λ-ring. By (1.7) and (2.38), we see that

$$\begin{aligned} &P_{n,m}\big(\lambda^1(\underline{E},\phi),\cdots,\lambda^{mn}(\underline{E},\phi)\big)\\ &\qquad=\Big(P_{n,m}\big(\Lambda^1(\underline{E}),\cdots,\Lambda^{mn}(\underline{E})\big),\,\big[\lambda^n_\Gamma(\lambda^m_\Gamma(\mathrm{ch}(\underline{E}),\phi))\big]_{\mathrm{odd}}\Big).\end{aligned} \tag{2.39}$$

Similarly, by (1.1), (1.4), (2.14), (2.15), (2.20) and (2.39),

$$\begin{aligned} \lambda_t\big(\lambda^m(\underline{E},\phi)\big) &\overset{(2.15)}{=} \lambda_t\Big(\Lambda^m(\underline{E}),\,\big[\lambda^m_\Gamma(\mathrm{ch}(\underline{E}),\phi)\big]_{\mathrm{odd}}\Big)\\ &\xlongequal{(1.1)(2.15)} 1+\sum_{n\geq1}t^n\Big(\Lambda^n\big(\Lambda^m(\underline{E})\big),\,\Big[\lambda^n_\Gamma\Big(\,\mathrm{ch}\,\big(\Lambda^m(\underline{E})\big),\,\big[\lambda^m_\Gamma(\mathrm{ch}(\underline{E}),\phi)\big]_{\mathrm{odd}}\Big)\Big]_{\mathrm{odd}}\Big)\\ &\overset{(2.14)}{=} 1+\sum_{n\geq1}t^n\Big(\Lambda^n\big(\Lambda^m(\underline{E})\big),\,\Big[\lambda^n_\Gamma\big(\lambda^m_\Gamma\big(\,\mathrm{ch}(\underline{E}),\phi\,\big)\big)\Big]_{\mathrm{odd}}\Big)\\ &\xlongequal{(2.20)(2.39)} 1+\sum_{n\geq1}t^nP_{n,m}\big(\lambda^1(\underline{E},\phi),\cdots,\lambda^{mn}(\underline{E},\phi)\big)\overset{(1.4)}{=}\tilde{\lambda}^m\big(\lambda_t(\underline{E},\phi)\big).\end{aligned} \tag{2.40}$$

The proof of Theorem 2.4 is complete. □

3. Adams operations

If R is a λ-ring, by [3, Proposition V.7.5 i)], the Adams operations $\Psi^k:R\to R$ defined in (1.9) satisfy (1.10). From [2, §I.5 (2)], for $x\in R$,

$$\Psi^k(x)=\nu_k(\lambda^1(x),\cdots,\lambda^k(x)) \tag{3.1}$$

where $\nu_k(s_1,\cdots,s_k)=\xi_1^k+\cdots+\xi_q^k$ is the k-th Newton polynomial, s_i being the i-th elementary symmetric function in $\{\xi_j\}$. By [2, Proposition I.5.1], $\Psi^k:R\to R$ is a homomorphism of λ-rings.

For a geometric triple $\underline{E}$ on B, we denote by

$$\Psi^k(\underline{E}):=\nu_k(\Lambda^1(\underline{E}),\cdots,\Lambda^k(\underline{E})). \tag{3.2}$$

Proposition 3.1. *For $[\underline{E},\phi]\in\widehat{K}^0(B)$, $\phi=\sum_{l\geq1}\phi_l$, $\phi_l\in\Omega^{2l-1}(B,\mathbb{R})/\mathrm{Im}\,d$, $k\in\mathbb{Z}_+$, we have*

$$\Psi^k([\underline{E},\phi])=\Big[\Psi^k(\underline{E}),\sum_{l\geq1}k^l\phi_l\Big]. \tag{3.3}$$

For any $k\in\mathbb{N}$, $\Psi^k:\widehat{K}^0(B)\to\widehat{K}^0(B)$ is a homomorphism of λ-rings.

Proof. Applying (3.1) for the λ-ring $\Gamma(B)$, we get

$$\Psi^k(\mathrm{ch}(\underline{E}),\phi) = \nu_k(\lambda^1(\mathrm{ch}(\underline{E}),\phi),\cdots,\lambda^k(\mathrm{ch}(\underline{E}),\phi)). \tag{3.4}$$

Since $\widehat{K}^0(B)$ is a λ-ring, from (2.36), (2.37), (3.1), (3.2), (3.4) and ν_k is a polynomial, we have

$$\begin{aligned}\Psi^k(\underline{E},\phi) &\overset{(3.1)}{=} \nu_k(\lambda^1(\underline{E},\phi),\cdots,\lambda^k(\underline{E},\phi))\\ &\xlongequal{(2.36)(2.37)} \left(\nu_k(\Lambda^1(\underline{E}),\cdots,\Lambda^k(\underline{E})), \left[\nu_k\left(\lambda^1_\Gamma(\mathrm{ch}(\underline{E}),\phi),\cdots,\lambda^k_\Gamma(\mathrm{ch}(\underline{E}),\phi)\right)\right]_{\mathrm{odd}}\right)\\ &\xlongequal{(3.1)(3.2)(3.4)} \left(\Psi^k(\underline{E}), [\Psi^k_\Gamma(\mathrm{ch}(\underline{E}),\phi)]_{\mathrm{odd}}\right).\end{aligned} \tag{3.5}$$

Thus this proposition follows directly from (2.10). The last part is from [2, Proposition I.5.1] as explained above. □

From (3.2), if $\underline{L} = (L, h^L, \nabla^L)$ is a geometric triple on B with line bundle L, by the definition of ν_k, we have

$$\Psi^k(\underline{L}) = \underline{L^k}. \tag{3.6}$$

If $\underline{E} = \oplus_{j=1}^r \underline{L_j}$, and $\underline{L_j} = (L_j, h^{L_j}, \nabla^{L_j})$ with line bundles L_j, then by Theorem 2.4, Proposition 3.1 and (3.6),

$$\begin{aligned}[\Psi^k(\underline{E}),0] = \Psi^k([\underline{E},0]) = \Psi^k\left(\sum_{j=1}^r[\underline{L_j},0]\right) &= \sum_{j=1}^r \Psi^k([\underline{L_j},0])\\ &= \sum_{j=1}^r[\Psi^k(\underline{L_j}),0] = \sum_{j=1}^r[\underline{L_j^k},0] = [\oplus_{j=1}^r\underline{L_j^k},0].\end{aligned} \tag{3.7}$$

From Lemma 2.2 and (2.14),

$$\lambda^k_Z(\mathrm{ch}(\underline{E})) = \mathrm{ch}(\Lambda^k(\underline{E})). \tag{3.8}$$

Thus from (3.1), (3.2), (3.8) and ν_k is a polynomial, we have

$$\begin{aligned}\Psi^k_Z(\mathrm{ch}(\underline{E})) = \nu_k(\lambda^1_Z(\mathrm{ch}(\underline{E})),\cdots,\lambda^k_Z(\mathrm{ch}(\underline{E}))) &= \nu_k(\mathrm{ch}(\Lambda^1(\underline{E})),\cdots,\mathrm{ch}(\Lambda^k(\underline{E})))\\ &= \mathrm{ch}(\nu_k(\Lambda^1(\underline{E}),\cdots,\Lambda^k(\underline{E}))) = \mathrm{ch}(\Psi^k(\underline{E})).\end{aligned} \tag{3.9}$$

So the Adams operations constructed in this section satisfy the conditions in [4, (12)]. Thus the Adams operations Ψ^k in (3.3) gives a constructive realization of $\widehat{\Psi}^k$ in [4].

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