ETA FORM AND SPECTRAL SEQUENCE FOR THE COMPOSITION OF FIBRATIONS

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Abstract. In this paper, inspired by the spectral sequences occurring for signature operators with respect to the composition of fibrations, we define "spectral sequences" for family of Dirac operators and prove the equivariant family version of the adiabatic limit formula for eta invariants using the heat kernel method and the analytic localization techniques established by Bismut-Lebeau. In our formula, the remainder terms are constructed by "spectral sequences" analogous to those of Dai and Bunke-Ma.

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1. Introduction

The Bismut-Cheeger eta form serves as the family extension of the eta invariant in index theory, which originally comes from the adiabatic limit of eta invariants. This limit is initiated by E. Witten [24] for physical consideration and well studied by Bismut-Cheeger [6] and Dai [11]. In the general case of the adiabatic limit for Dirac operators in [11], a global spectral term arises from the (asymptotically) very small eigenvalues. If we consider the signature operators, this spectral term can be constructed by Leray spectral sequences.

In [10], in order to discuss the secondary index theory for flat bundles with duality, Bunke and Ma generalize the signature operators to the flat case and the adiabatic limit formula to the family case. In this case, the spectral terms are generalized to the finite dimensional eta forms constructed by spectral sequences.

In [15, 16, 17], for Dirac operators, the first author generalizes the adiabatic limit formula to the equivariant family case for a fiberwise Lie group action. In [17], the spectral terms are explained as equivariant Dai-Zhang higher spectral flows [12]. But those higher spectral flow terms cannot degenerate to the terms in [11] directly for the Dirac operator and the formula there cannot be used directly for the signature operator.

In this paper, inspired by [10], we build a framework to unify the Dirac operator case and the signature operator case and extend this construction to the equivariant family version by introducing a series of vector bundles over the base manifold which can be taken as the analogy of spectral sequences. When the base manifold is a point, our formula yields the classical adiabatic limit formula for the Dirac operator and the signature operator in [11]. Moreover, the part of our formula for "spectral sequences" can also be considered as the refinement of the remainder terms in [17].

Now we explain our result in some details.

Let $\pi_X : W \to V$ be a submersion of two closed spin manifolds with oriented closed fiber X. Let $TX := \ker(\pi_{X,*} : TW \to TV)$ be the relative tangent bundle over W with Euclidean metric g^{TX}. Then TX is also spin. Let T^HW be a horizontal subbundle of TW such that $TW = T^HW \oplus TX$. Let $S(TX)$ be the spinor bundle over W associated with the spin structure of TX and $\nabla^{S(TX)}$ be the induced connection on it. Let D_X^F be the family of Dirac operators along the fiber X twisted by a Hermitian vector bundle F with a Hermitian connection ∇^F. Assume that $\ker D_X^F$ forms a vector bundle over V. Under this assumption, the Bismut-Cheeger eta form $\tilde{\eta}(\underline{\pi_X}, \underline{S(TX) \otimes F}) \in \Omega^*(V)$ (non-equivariant version of Definition 2.2 for $\mathcal{R} = 0$) is well-defined.

Let g^{TV} be a Riemannian metric on TV with Levi-Civita connection ∇^{TV}. Let $S(TV)$ be the spinor bundle with induced connection $\nabla^{S(TV)}$. For $T>0$, let $g_T^{TW}:=\pi_X^*g^{TV}\oplus T^{-2}g^{TX}$ and $\nabla^{TW,T}$ be the Levi-civita connection of the metric g_T^{TW}. Let $\mathrm{Cl}_T(TW)$ be the Clifford algebra bundle associated with g_T^{TW}. Then $S(TW)=\pi_X^*S(TV)\widehat{\otimes}S(TX)$ is the spinor bundle with a Clifford action $c:\mathrm{Cl}_T(TW)\to\mathrm{End}(S(TW))$ (see (3.11)). In this case, the induced connection associated with $\nabla^{TW,T}$ on $S(TW)$ is

$$\begin{aligned}\nabla^{S(TW),T}&=\pi_X^*\nabla^{S(TV)}\otimes 1+1\otimes\nabla^{S(TX)}\\&\quad+\frac{1}{2T}\langle\mathcal{S}(\cdot)e_i,f_{p,X}^H\rangle c(e_i)c(f_p)+\frac{1}{4T^2}\langle\mathcal{S}(\cdot)f_{p,X}^H,f_{q,X}^H\rangle c(f_p)c(f_q),\end{aligned}\tag{1.1}$$

where $\{e_i\}$, $\{f_p\}$ are locally orthonormal frames of TX, TV, $f_{p,X}^H$ is the horizontal lift of f_p and $\mathcal{S}$ is defined in (2.5). Let $D_{W,T}^F$ be the Dirac operator associated with $\nabla^{S(TW),T}$ and ∇^F. Let $D_V^{\ker D_X}$ be the Dirac operator associated with $\nabla^{\ker D_X}$ (see (3.7) for the definition). Let $\eta(D_{W,T}^F)$ and $\eta(D_V^{\ker D_X})$ be the corresponding Atiyah-Patodi-Singer eta invariants in [1].

The famous adiabatic limit formula for Dirac operators is stated as follows.

Theorem 1.1. [6, 11] *If* $\dim W$ *is odd,*

$$\lim_{T\to+\infty}\eta\left(D_{W,T}^F\right)=2\int_V\widehat{\mathrm{A}}(TV,\nabla^{TV})\tilde{\eta}(\underline{\pi_X},\underline{S(TX)\otimes F})+\eta\big(D_V^{\ker D_X}\big)+R,\tag{1.2}$$

where R is an integer-valued remainder term and $\widehat{\mathrm{A}}(\cdot)$ is the corresponding $\widehat{\mathrm{A}}$-form (see [2, §1.5 and §4.1] *for the definition). Moreover, if $\ker D_X^F$ forms a vector bundle over V, and if $\dim\ker D_{W,T}^F$ is independent of T, then*

$$R=\sum_{\lambda\in A}\mathrm{sgn}(\lambda),\quad A:=\left\{\lambda\in\mathrm{Sp}(D_{W,T}^F):\lambda=\mathrm{O}\left(\frac{1}{T}\right)\right\},\tag{1.3}$$

where $\mathrm{Sp}(D_{W,T}^F)$ is the set of the spectrum of $D_{W,T}^F$. In particular, if D_X^F is invertible, then $R=0$.

Note that if $\dim V$ is even, then the term $\eta(D_V^{\ker D_X})$ in (1.2) vanishes.

Remark 1.2.
(1) In [6, 11], the authors use the rescaling $g_x^{TW}=g^{TX}+x^{-2}\pi_X^*g^{TV}$, $x\to 0$. Here we consider $T=x^{-1}$. Then $g_T^{TW}=x^2g_x^{TW}$. Note that when we multiply a constant on the metric, the eta invariant does not change. For the reminder term R in (1.3), in [11, Theorem 0.1], the summation runs over all eigenvalues λ_x decaying at least quadratically in x. After taking $T=x^{-1}$, these eigenvalues correspond to the eigenvalues in A, which decays at least linearly. So (1.2) is the same as the results in [6, 11].
(2) Let ${}^0\nabla^{TW}=\pi_X^*\nabla^{TV}\oplus\nabla^{TX}$. Then by [15, Proposition 4.5] (cf. also [21, (4.30)]), $\lim_{T\to+\infty}\widetilde{\widehat{\mathrm{A}}}(TW,\nabla^{TW,T},{}^0\nabla^{TW})=0$, where $\widetilde{\widehat{\mathrm{A}}}(\cdot)$ is the Chern-Simons form for the $\widehat{\mathrm{A}}$-form (cf. [22, Definition B.5.3]). The formula (1.2) can also be formulated as an equality:

$$\begin{aligned}\eta\left(D_{W,T}^F\right)&=2\int_V\widehat{\mathrm{A}}(TV,\nabla^{TV})\tilde{\eta}(\underline{\pi_X},\underline{S(TX)\otimes F})\\&\quad-2\int_W\widetilde{\widehat{\mathrm{A}}}(TW,\nabla^{TW,T},{}^0\nabla^{TW})\mathrm{ch}(F,\nabla^F)+\eta\big(D_V^{\ker D_X}\big)+R.\end{aligned}\tag{1.4}$$

We usually take $T=1$.

In fact we can drop the spin conditions and replace $S(TX)\otimes F$ and $S(TV)$ by $\mathbb{Z}_2$-graded Clifford modules $\mathcal{E}_X$ and $\mathcal{E}_V$ with Clifford connections $\nabla^{\mathcal{E}_X}$ and $\nabla^{\mathcal{E}_V}$ respectively. On Clifford module $\mathcal{E}=\pi_X^*\mathcal{E}_V\widehat{\otimes}\mathcal{E}_X$, as in (1.1),

$$\nabla^{\mathcal{E},T}={}^0\nabla^{\mathcal{E}}+\frac{1}{2T}\langle\mathcal{S}(\cdot)e_i,f_{p,X}^H\rangle c(e_i)c(f_p)+\frac{1}{4T^2}\langle\mathcal{S}(\cdot)f_{p,X}^H,f_{q,X}^H\rangle c(f_p)c(f_q)\tag{1.5}$$

is a Clifford connection associated with $\nabla^{TW,T}$. Here ${}^0\nabla^{\mathcal{E}} = \pi_X^* \nabla^{\mathcal{E}_V} \otimes 1 + 1 \otimes \nabla^{\mathcal{E}_X}$. Let $D^{\mathcal{E}}_{W,T}$ be the Dirac operator associated with the Clifford connections $\nabla^{\mathcal{E},T}$. Let $D^{\mathcal{E}_X}_X$ be the fiberwise Dirac operator. Then if $\ker D^{\mathcal{E}_X}_X$ is a vector bundle and $\dim\ker D^{\mathcal{E}}_{W,T}$ is independent of T, (1.4) can be naturally extended to the Clifford module case with reminider term (1.3).

For the assumptions about $\ker D^{\mathcal{E}_X}_X$ and $\dim\ker D^{\mathcal{E}}_{W,T}$, an important example is the case for signature operators. For $\mathcal{E}_X = \Lambda^\bullet T^*X \otimes \mathbb{C}$, $\mathcal{E}_V = \Lambda^\bullet T^*V \otimes \mathbb{C}$ and $\mathcal{E} = \Lambda^\bullet T^*W \otimes \mathbb{C}$, if $\dim X$ and $\dim V$ are both even, then the kernel of the signature operator $D^{\mathcal{E}_X}_X = d_X + d_X^*$ forms a vector bundle, and $D^{\mathcal{E}}_{W,T}$ is conjugate to $d_W + d^*_{W,T}$, whose kernel is stable. Here $d^*_{W,T}$ is the adjoint of d_W with respect to g^{TW}_T. For this case, the Clifford connection is

$$
\begin{aligned}
(1.6)\quad \nabla^{\mathcal{E},T} = {}^0\nabla^{\mathcal{E}} &+ \frac{1}{2T}\langle \mathcal{S}(\cdot)e_i, f^H_{p,X}\rangle \big(c(e_i)c(f_p) + \hat{c}(e_i)\hat{c}(f^H_{p,X})\big) \\
&+ \frac{1}{4T^2}\langle \mathcal{S}(\cdot)f^H_{p,X}, f^H_{q,X}\rangle \big(c(f_p)c(f_q) + \hat{c}(f^H_{p,X})\hat{c}(f^H_{q,X})\big),
\end{aligned}
$$

where $\hat{c}(U) = U^* \wedge + i_U$ for $U \in TW$. Observe that (1.6) is slightly different from (1.5). In [11], Dai also studies this example when $\dim X$ is even and $\dim V$ is odd. For this case, the assumptions for $\ker D^{\mathcal{E}_X}_X$ and $\ker D^{\mathcal{E}}_{W,T}$ are also satisfied. See also Section 3.4 for details. In [11, Theorem 0.3], Dai shows that the adiabatic limit formula also holds for signature case (when $\dim V$ is odd, the Clifford connection is (3.93)).

Note that if V is a point and $\dim X$ is odd,

$$
(1.7)\qquad \tilde{\eta}(\underline{\pi_X}, \underline{\mathcal{E}_X}) = \frac{1}{2}\eta(D^{\mathcal{E}_X}_X).
$$

Thus the Bismut-Cheeger eta form can be considered as the higher degree version of the eta invariant. If V is a fibration over a closed manifold S with closed fiber, then W is also a fibration over S. We generalize the eta invariants in (1.4) to the Bismut-Cheeger eta forms. In fact, we can generalize them directly to the equivariant eta forms for fiberwise compact Lie group action.

Let G be a compact Lie group. Let W, V, S be closed G-manifolds. Let $\pi_X : W \to V$, $\pi_Y : V \to S$ be equivariant submersions with closed oriented fibers X, Y. Then $\pi_Z = \pi_Y \circ \pi_X : W \to S$ is an equivariant submersion with closed oriented fiber Z. Assume that G acts on S trivially. We have the diagram of fibrations:

$$
(1.8)\qquad
\begin{array}{ccccc}
X & \longrightarrow & Z & \longrightarrow & W \\
 & & \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_X} \;\searrow{\scriptstyle \pi_Z} \\
 & & Y & \longrightarrow & V \xrightarrow{\ \pi_Y\ } S.
\end{array}
$$

Let $\underline{\pi_X} = (\pi_X, T^H_X W, g^{TX})$, $\underline{\pi_Y} = (\pi_Y, T^H_Y V, g^{TY})$ and $\underline{\pi_Z} = (\pi_Z, T^H_Z W, g^{TZ})$ be equivariant geometric data with respect to π_X, π_Y and π_Z as in (2.10). Assume that $T^H_Z W \subseteq T^H_X W$ and $g^{TZ} = \pi_X^* g^{TY} \oplus g^{TX}$. Let ∇^{TX}, ∇^{TY} and ∇^{TZ} be the corresponding connections on TX, TY and TZ as in (2.3). Set ${}^0\nabla^{TZ} := \pi_X^* \nabla^{TY} \oplus \nabla^{TX}$.

Let $\underline{\mathcal{E}_X} = (\mathcal{E}_X, h^{\mathcal{E}_X}, \nabla^{\mathcal{E}_X})$ (resp. $\underline{\mathcal{E}_Y} = (\mathcal{E}_Y, h^{\mathcal{E}_Y}, \nabla^{\mathcal{E}_Y})$) be a $\mathbb{Z}_2$-graded G-equivariant self-adjoint Clifford module of $\mathrm{Cl}(TX)$ over W (resp. Clifford module of $\mathrm{Cl}(TY)$ over V) with a G-invariant Clifford connection as in (2.10). Set $\mathcal{E} = \pi_X^* \mathcal{E}_Y \widehat{\otimes} \mathcal{E}_X$. For $T > 0$, let $g^{TZ}_T := \pi_X^* g^{TY} \oplus T^{-2} g^{TX}$ and $\nabla^{TZ,T}$ be the connection on TZ as in (2.3) associated with g^{TZ}_T. Then with respect to g^{TZ}_T, the Clifford connection

$$
(1.9)\qquad \nabla^{\mathcal{E},T} = {}^0\nabla^{\mathcal{E}} + \frac{1}{2T}\langle \mathcal{S}(\cdot)e_i, f^H_{p,X}\rangle c(e_i)c(f_p) + \frac{1}{4T^2}\langle \mathcal{S}(\cdot)f^H_{p,X}, f^H_{q,X}\rangle c(f_p)c(f_q) + \mathcal{A}_T,
$$

where $\mathcal{A}_T \in \Omega^1(W, \mathrm{End}(\mathcal{E}))$ is G-invariant and supercommutes with the Clifford action. So $(\mathcal{E}, \pi_X^* h^{\mathcal{E}_Y} \otimes h^{\mathcal{E}_X}, \nabla^{\mathcal{E}} := \nabla^{\mathcal{E},1})$ is a $\mathbb{Z}_2$-graded G-equivariant self-adjoint Clifford module of $\mathrm{Cl}(TZ)$ over W. In order to obtain reasonable results, we add assumptions for $\mathcal{A}_T$ in (3.17), which are satisfied for connections in (1.5) and (1.6).

Let $D^{\mathcal{E}_X}_X$ and $D^{\mathcal{E}}_{Z,T}$ be the families of Dirac operators along the fibers X and Z associated with $(g^{TX}, \nabla^{\mathcal{E}_X})$ and $(g^{TZ}_T, \nabla^{\mathcal{E},T})$ respectively. Assume that $\ker D^{\mathcal{E}_X}_X$ forms a vector bundle over

V. Let $\widetilde{D}_Y$ be the limit operator of the restriction of $D^{\mathcal{E}}_{Z,T}$ on $\mathcal{E}_Y \otimes \ker D^{\mathcal{E}_X}_X$ when $T \to \infty$ (see (3.22) for the definition). Then if $\mathcal{A}_T = 0$, $\widetilde{D}_Y$ is the fiberwise Dirac operator. We assume that the equivariant eta form for $\widetilde{D}_Y$, $\widetilde{\eta}'_g(\underline{\pi_Y, \mathcal{E}_Y \otimes \ker D^{\mathcal{E}_X}_X})$, is well-defined.

The purpose of this paper is to establish the following result, which we state in Theorem 3.6.

Theorem 1.3. *Under Assumptions 3.4 and 3.5, for $g \in G$, modulo exact forms on S, we have*

$$\begin{aligned}(1.10)\quad \widetilde{\eta}_g(\underline{\pi_Z, \mathcal{E}}) = \widetilde{\eta}'_g(\underline{\pi_Y, \mathcal{E}_Y \otimes \ker D^{\mathcal{E}_X}_X}) + \int_{Y^g} \widehat{\mathrm{A}}_g(TY, \nabla^{TY}) \mathrm{ch}_g(\mathcal{E}_Y/S, \nabla^{\mathcal{E}_Y}) \widetilde{\eta}_g(\underline{\pi^g_X, \mathcal{E}_X}) \\ - \int_{Z^g} \widetilde{\widehat{\mathrm{A}}}_g(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}) \mathrm{ch}_g(\mathcal{E}/S, \nabla^{\mathcal{E}}) - \int_{Z^g} \widehat{\mathrm{A}}_g(TZ, {}^0\nabla^{TZ}) \widetilde{\mathrm{ch}}_g(\mathcal{E}/S, \nabla^{\mathcal{E}}, {}^0\nabla^{\mathcal{E}}) \\ + \sum_{r=2}^{r_0} \widetilde{\eta}_g(E_r, B_r) + \widetilde{\mathrm{ch}}_g(\ker D^{\mathcal{E}}_Z, \nabla^{\infty}, \nabla^{\ker D_Z}).\end{aligned}$$

Here $\underline{\pi^g_X}$ is defined in (3.33), Y^g and Z^g are the fixed point sets of $g \in G$ on Y and Z respectively, which are assumed to be oriented, $\widehat{\mathrm{A}}_g(\cdot)$ and $\mathrm{ch}_g(\mathcal{E}/S, \nabla^{\mathcal{E}})$ are the equivariant $\widehat{\mathrm{A}}$-form and the equivariant relative Chern character form (see, e.g., [19, (1.32), (1.33)] *for the definitions) and $\widetilde{\widehat{\mathrm{A}}}_g(\cdot)$ and $\widetilde{ch}_g(\cdot)$ are the corresponding equivariant Chen-Simons forms, which are the natural equivariant extension of* [22, Definition B.5.3]. *The definitions of the last two terms above follow from (3.52) and (3.60).*

In [17, Theorem 1.6], the first author proved that if $\mathcal{A}_T = 0$ and the kernels of all fiberwise Dirac operators are vector bundles, for $g \in G$, modulo exact forms,

$$\begin{aligned}(1.11)\quad \widetilde{\eta}_g(\underline{\pi_Z, \mathcal{E}}) = \widetilde{\eta}'_g(\underline{\pi_Y, \mathcal{E}_Y \otimes \ker D^{\mathcal{E}_X}_X}) + \int_{Y^g} \widehat{\mathrm{A}}_g(TY, \nabla^{TY}) \mathrm{ch}_g(\mathcal{E}_Y/S, \nabla^{\mathcal{E}_Y}) \widetilde{\eta}_g(\underline{\pi^g_X, \mathcal{E}_X}) \\ - \int_{Z^g} \widetilde{\widehat{\mathrm{A}}}_g(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}) \mathrm{ch}_g(\mathcal{E}/S, \nabla^{\mathcal{E}}) + \widetilde{R},\end{aligned}$$

where $\widetilde{R} \in \mathrm{ch}_g(K^{\bullet}_G(S))$ is the image of the equivariant Chern character for some equivariant higher spectral flow.

In [10, Theorem 5.11], Bunke and Ma state an analogous result for generalized signature operators (without group action). The term corresponding to $\widetilde{R}$ is the sum of finite-dimensional eta forms associated to the higher pages of the Leray-Serre spectral sequence.

If S is a point, the setting in Theorem 1.1 satisfies Assumptions 3.4 and 3.5. In this case, our theorem degenerates to previous result (see Proposition 5.3).

This paper is organized as follows. In Section 2, we review the definition of the equivariant eta form and define the equivariant finite dimensional eta form. In Section 3, we state our main result in Theorem 3.6. In Section 4, we use Theorems 4.2-4.5 to prove Theorem 3.6. Section 5 is devoted to the proof of Theorem 4.5.

Notation. All manifolds in this paper are smooth and without boundary. All vector bundles in this paper are smooth. All fibrations in this paper are submersions with closed oriented fibers. We denote by d the exterior differential operator and d_S when we like to insist the base manifold S.

We use the Einstein summation convention in this paper: when an index variable appears twice in a single term and is not otherwise defined, it implies summation of that term over all the values of the index.

We use the superconnection formalism of Quillen [23]. If A is a $\mathbb{Z}_2$-graded algebra, and if $a, b \in A$, then we will note $[a, b] := ab - (-1)^{\deg(a)\deg(b)} ba$ as the supercommutator of a, b. If F, F' are two $\mathbb{Z}_2$-graded spaces, we will note $F \widehat{\otimes} F'$ as the $\mathbb{Z}_2$-graded tensor product as in [2, §1.3]. If one of F, F' is ungraded, we understand it as $\mathbb{Z}_2$-graded by taking its odd part as zero.

For the fiber bundle $\pi : W \to S$, we use the sign convention for the integration of the differential forms along the oriented fibers Z as follows: for $\alpha \in \Omega^\bullet(S)$ and $\beta \in \Omega^\bullet(W)$,

$$\int_Z (\pi^*\alpha) \wedge \beta = \alpha \wedge \int_Z \beta. \tag{1.12}$$

2. Equivariant eta form

In this section, we introduce the basic object of this paper – eta forms. In Section 2.1, we describe the geometry of a fibration and introduce the Bismut superconnection to define the equivariant Bismut-Cheeger eta forms. In Section 2.2, we introduce the equivariant version of finite dimensional eta forms for a vector bundle.

2.1. **Equivariant Bismut-Cheeger eta form.** In this subsection, we introduce the definition of the equivariant Bismut-Cheeger eta form. Since we want our main result to contain the signature operator case, the definition in [15] needs to be properly generalized.

Given a submersion of closed manifolds $\pi : W \to S$ with closed oriented fiber Z, let G be a compact Lie group which acts on W with $\pi \circ g = \pi$, $\forall g \in G$. In this case, the G-action on S is trivial. We denote by $TZ := \ker(\pi_* : TW \to TS)$ the relative tangent bundle and T^HW a horizontal subbundle of TW such that

$$TW = T^HW \oplus TZ. \tag{2.1}$$

Then T^HW and TZ are both vector bundles over W. We assume that the G-action preserves the orientation of TZ. We assume that T^HW is also G-equivariant. Then the G-action preserves the splitting (2.1). For $U \in TS$, let $U^H \in T^HW$ be its horizontal lift in T^HW such that $\pi_*U^H = U$. Let $P^{TZ} : TW \to TZ$ be the projection with respect to (2.1).

Let g^{TZ} and g^{TS} be G-invariant metrics on TZ and TS respectively. Then

$$g^{TW} := \pi^*g^{TS} \oplus g^{TZ} \tag{2.2}$$

is a G-invariant metric on TW.

Let ∇^{TW} be the Levi-Civita connection associated with (TW, g^{TW}) and

$$\nabla^{TZ} := P^{TZ}\nabla^{TW}P^{TZ}. \tag{2.3}$$

Then ∇^{TZ} is a G-invariant Euclidean connection on TZ depending only on (T^HW, g^{TZ}) (cf. [4, Theorem 1.9]). Let ∇^{TS} be the Levi-Civita connection on (TS, g^{TS}). Let

$${}^0\nabla^{TW} := \pi^*\nabla^{TS} \oplus \nabla^{TZ} \tag{2.4}$$

be a connection on TW, which is also G-invariant. We define

$$\mathcal{S} := \nabla^{TW} - {}^0\nabla^{TW}. \tag{2.5}$$

Then $\mathcal{S}$ is a 1-form on W with values in antisymmetric elements of $\mathrm{End}(TW)$. Let $\mathcal{T}$ be the torsion of ${}^0\nabla^{TW}$. Then by [4, (1.30)], for $U, V \in TS$,

$$\mathcal{T}(U^H, V^H) = -P^{TZ}[U^H, V^H] \in TZ. \tag{2.6}$$

Let $\mathrm{Cl}(TZ)$ be the Clifford algebra bundle of (TZ, g^{TZ}), whose fiber at $x \in W$ is the Clifford algebra $\mathrm{Cl}(T_xZ)$ of the Euclidean vector space (T_xZ, g^{T_xZ}). A $\mathbb{Z}_2$-graded self-adjoint Clifford module of $\mathrm{Cl}(TZ)$,

$$\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-, \tag{2.7}$$

is a $\mathbb{Z}_2$-graded complex vector bundle equipped with a Hermitian metric $h^{\mathcal{E}}$ preserving the splitting (2.7) and a fiberwise Clifford multiplication c of $\mathrm{Cl}(TZ)$ such that the action c restricted to TZ is skew-adjoint on $(\mathcal{E}, h^{\mathcal{E}})$ and anticommutes (resp. commutes) with the $\mathbb{Z}_2$-grading if the dimension of the fibres is even (resp. odd). Locally, the Clifford module $\mathcal{E}$ can be written as

$$\mathcal{E} = S(TZ)\widehat{\otimes}F, \tag{2.8}$$

where $S(TZ)$ is the (possibly nonexistent) spinor bundle and $F = F_+ \oplus F_-$ is a $\mathbb{Z}_2$-graded complex vector bundle. In this case, if $\dim Z$ is even, $S(TZ) = S_+(TZ) \oplus S_-(TZ)$ and

$$\mathcal{E}_+ = (S_+(TZ) \otimes F_+) \oplus (S_-(TZ) \otimes F_-)\,, \quad \mathcal{E}_- = (S_+(TZ) \otimes F_-) \oplus (S_-(TZ) \otimes F_+)\,;$$

if $\dim Z$ is odd,

$$\mathcal{E}_+ = S(TZ) \otimes F_+, \quad \mathcal{E}_- = S(TZ) \otimes F_-.$$

Let $\nabla^{\mathcal{E}}$ be a Clifford connection on $\mathcal{E}$ associated with ∇^{TZ}, that is, $\nabla^{\mathcal{E}}$ preserves $h^{\mathcal{E}}$ and the splitting (2.7) and for any $U \in TW$, $V \in C^\infty(W, TZ)$,

$$\left[\nabla^{\mathcal{E}}_U, c(V)\right] = c\left(\nabla^{TZ}_U V\right). \tag{2.9}$$

As in [2, Proposition 3.40] and (2.8), locally, the connection $\nabla^{\mathcal{E}}$ is uniquely determined by a connection on F. We assume that the action of G can be lifted on $\mathcal{E}$ such that it is compatible with the Clifford action and preserves the splitting (2.7). We assume that $h^{\mathcal{E}}$ and $\nabla^{\mathcal{E}}$ are G-invariant.

Notation 2.1. We denote by

$$\underline{\pi} := (\pi, T^H W, g^{TZ}), \quad \underline{\mathcal{E}} := (\mathcal{E}, h^{\mathcal{E}}, \nabla^{\mathcal{E}}) \tag{2.10}$$

the corresponding geometric data of the fibration π and the Clifford module $\mathcal{E}$ of $\mathrm{Cl}(TZ)$ introduced above.

On Clifford module $\mathcal{E}$ of $\mathrm{Cl}(TZ)$, we define a family of Dirac operators over S by

$$D^{\mathcal{E}}_Z := \sum_{i=1}^{\dim Z} c(e_i)\nabla^{\mathcal{E}}_{e_i}, \tag{2.11}$$

for $\{e_i\}_{i=1}^{\dim Z}$ a local orthonormal frame of (TZ, g^{TZ}). This definition is independent of the choice of $\{e_i\}_{i=1}^{\dim Z}$. we see that $D^{\mathcal{E}}_Z$ commutes with the G-action.

Let E_b be the space of smooth sections of $\mathcal{E}$ over Z_b, $b \in S$, equipped with the L^2-inner product

$$\langle \cdot, \cdot \rangle_{E_b} := \int_{Z_b} h^{\mathcal{E}}(\cdot, \cdot) dv_Z. \tag{2.12}$$

As in [4], we take $(E, \langle \cdot, \cdot \rangle_E)$ as an infinite dimensional vector bundle over S. Let ∇^E be the $\langle \cdot, \cdot \rangle_E$-preserving connection on E with respect to $\nabla^{\mathcal{E}}$ defined by (cf. [7, (1.16)])

$$\nabla^E_U := \nabla^{\mathcal{E}}_{U^H} - \frac{1}{2}\langle \mathcal{S}(e_i)e_i, U^H \rangle, \quad \forall\, U \in C^\infty(S, TS). \tag{2.13}$$

Let $\{f_p\}$ be a local frame of TS and $\{f^p\}$ be its dual. By (2.6), we denote by

$$c(\mathcal{T}) = \frac{1}{2}c(\mathcal{T}(f_p^H, f_q^H))f^p \wedge f^q \wedge = -\frac{1}{2}c(P^{TX}[f_p^H, f_q^H])f^p \wedge f^q \wedge. \tag{2.14}$$

Let B be the Bismut superconnection defined by (cf. [2, P.336])

$$B := D^{\mathcal{E}}_Z + \nabla^E - \frac{c(\mathcal{T})}{4} \tag{2.15}$$

on $C^\infty(S, E) \simeq C^\infty(W, \mathcal{E})$, which only depends on geometric data $(T^H W, g^{TZ}, \nabla^{\mathcal{E}})$. For $u > 0$, we denote δ_u the operator on $\Lambda^i(T^*S)\widehat{\otimes}E$ by multiplying differential forms by $u^{-i/2}$. Then for $u > 0$, we define the rescaled Bismut superconnection by

$$B_u := \sqrt{u}\delta_u \circ B \circ \delta_u^{-1} = \sqrt{u}D^{\mathcal{E}}_Z + \nabla^E - \frac{c(\mathcal{T})}{4\sqrt{u}}. \tag{2.16}$$

For a trace class element $P \in \Lambda(T^*S)\widehat{\otimes}\mathrm{End}(E)$, we denote by $\mathrm{Tr}_s^{\mathrm{odd/even}}[P]$ the part of $\mathrm{Tr}_s[P]$ which take values in odd or even forms. Set

$$\widetilde{\mathrm{Tr}}[P] := \begin{cases} \mathrm{Tr}_s[P], & \text{if } \dim Z \text{ is even;} \\ \mathrm{Tr}_s^{\mathrm{odd}}[P], & \text{if } \dim Z \text{ is odd.} \end{cases} \tag{2.17}$$

Here $\mathrm{Tr}_s[P]$ denotes the supertrace of P as in [2, §1.3].

For $\alpha \in \Omega^i(S)$, set

$$\psi_S(\alpha) = \begin{cases} \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{i}{2}} \cdot \alpha, & i \text{ is even;} \\ \frac{1}{\sqrt{\pi}} \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{i-1}{2}} \cdot \alpha, & i \text{ is odd,} \end{cases} \tag{2.18}$$

and

$$\widetilde{\psi}_S(\alpha) = \begin{cases} \frac{1}{\sqrt{\pi}}\psi_S(\alpha), & i \text{ is even;} \\ \frac{1}{2\sqrt{-1}\sqrt{\pi}}\psi_S(\alpha), & i \text{ is odd.} \end{cases} \tag{2.19}$$

For $\beta \in \Omega^\bullet(S \times \mathbb{R}_u)$, if we write $\beta = \beta_0 + du \wedge \beta_1$, with $\beta_0, \beta_1 \in \Omega^\bullet(S)$, we set

$$\{\beta\}^{du} := \beta_1. \tag{2.20}$$

Then by (2.18)-(2.20), we have

$$\{\psi_{S\times\mathbb{R}}(\beta)\}^{du} = \widetilde{\psi}_S(\beta_1). \tag{2.21}$$

For $g \in G$, let W^g be the fixed point set of g-action on W. Then $\pi|_{W^g} : W^g \to S$ is a fiber bundle with fiber Z^g. We assume that TZ^g is oriented.

Definition 2.2 (Compare with [15, Definition 2.3])**.** For G-equivariant self-adjoint $\mathcal{R} \in C^\infty(W, \mathrm{End}(\mathcal{E}))$, we set

$$B'_{u^2} = B_{u^2} + u\mathcal{R}. \tag{2.22}$$

For $g \in G$, the *equivariant Bismut-Cheeger eta form* $\widetilde{\eta}_g(\underline{\pi}, \mathcal{R}, \underline{\mathcal{E}}) \in \Omega^\bullet(S)$ with respect to $\mathcal{R}$ is defined by

$$\begin{aligned} \widetilde{\eta}_g(\underline{\pi}, \mathcal{R}, \underline{\mathcal{E}}) &:= -\int_0^{+\infty} \left\{ \psi_{S\times\mathbb{R}} \widetilde{\mathrm{Tr}} \left[g \exp\left(-\left(B'_{u^2} + du \wedge \frac{\partial}{\partial u} \right)^2 \right) \right] \right\}^{du} du \\ &= \begin{cases} \int_0^{+\infty} \widetilde{\psi}_S \mathrm{Tr}^{\mathrm{even}} \left[g \frac{\partial B'_{u^2}}{\partial u} \exp(-(B'_{u^2})^2) \right] du \in \Omega^{\mathrm{even}}(B;\mathbb{C}), & \text{if} \dim Z \text{ is odd;} \\ \int_0^{+\infty} \widetilde{\psi}_S \mathrm{Tr}_s \left[g \frac{\partial B'_{u^2}}{\partial u} \exp(-(B'_{u^2})^2) \right] du \in \Omega^{\mathrm{odd}}(B;\mathbb{C}), & \text{if} \dim Z \text{ is even,} \end{cases} \end{aligned} \tag{2.23}$$

if the integral term in (2.23) converges absolutely at 0 and $+\infty$. If $\mathcal{R} = 0$, we will simply denote the equivariant eta form by $\widetilde{\eta}_g(\underline{\pi}, \underline{\mathcal{E}})$.

In general, The absolute convergence of the integral term in (2.23) at 0 and $+\infty$ require additional conditions. As in [15, (2.77)], if $\ker(D_Z^{\mathcal{E}} + \mathcal{R})$ forms a vector bundle over S, then the integral term converges absolutely at $+\infty$. By [15, (2.72)], if $\mathcal{R} = 0$, from the equivariant version of the local family index theorem, the integral term converges absolutely at 0. There are also other cases for the absolute convergence at 0 for nontrivial $\mathcal{R}$. See the discussion for the signature operators in Section 3.4. In fact, in order to get a well-defined eta form, we can also use a cut-off function $\chi(u)$ of u and multiply it by $\mathcal{R}$ in (2.22) such that $\chi(u)$ is equal to 0 near $u = 0$ and equal to 1 near $u = +\infty$. But we will not use this trick in this paper.

2.2. **Finite dimensional eta form.** In this subsection, we introduce the definition of the equivariant finite dimensional eta form.

Let $F \to S$ be a G-equivariant $\mathbb{Z}_2$-graded vector bundle with Hermitian metric h^F, which preserves the $\mathbb{Z}_2$-grading. We assume that the G-action on S is trivial and h^F is G-invariant. Let ∇^F be a G-invariant connection preserving h^F. Take $V \in C^\infty(S, \mathrm{End}(F))$ commuting with the G-action. We assume that V either commutes or anticommutes with the $\mathbb{Z}_2$-grading. Set

$$F' := \ker V. \tag{2.24}$$

We assume that $\dim\ker V$ is locally constant. Then $F' \to S$ is a G-equivariant $\mathbb{Z}_2$-graded vector bundle. We define the equivariant geometric data on F' by the orthogonal projection $P^{\ker V} : F \to F'$ as

$$h^{F'} := P^{\ker V} \circ h^F \circ P^{\ker V}, \quad \nabla^{F'} := P^{\ker V} \circ \nabla^F \circ P^{\ker V}. \tag{2.25}$$

Then $\nabla^{F'}$ is a connection preserving $h^{F'}$.

Based on Quillen's work [23], we define the superconnection as follows.

Definition 2.3. Let σ be a quantity which commutes with $\Omega^{\mathrm{even}}(S)$ and anti-commutes with $\Omega^{\mathrm{odd}}(S)$. We define superconnection $L : \Omega^\bullet(S,F) \to \Omega^\bullet(S,F)$ by:

$$L := \begin{cases} \nabla^F + V, & \text{if } V \text{ anticommutes with the } \mathbb{Z}_2\text{-grading;} \\ \nabla^F + \sigma V, & \text{if } V \text{ commutes with the } \mathbb{Z}_2\text{-grading.} \end{cases} \tag{2.26}$$

Remark 2.4. In the sequel, in order to simpify the notations, when V commutes with the $\mathbb{Z}_2$-grading, we also usually omit σ and regard V as a quantity commutes with differential forms of even degree and anti-commutes with differential forms of odd degree.

For $u > 0$, set

$$L_u := \sqrt{u}\delta_u \circ L \circ \delta_u^{-1} = \sqrt{u}V + \nabla^F. \tag{2.27}$$

Definition 2.5. For $g \in G$, we define the *equivariant finite dimensional eta form* by

$$\begin{aligned} \widetilde{\eta}_g(F,L) &:= -\int_0^{+\infty} \left\{ \psi_{S\times\mathbb{R}} \mathrm{Tr}_s \left[g \exp\left(-\left(L_{u^2} + du \wedge \frac{\partial}{\partial u} \right)^2 \right) \right] \right\}^{du} du \\ &= \int_0^{+\infty} \widetilde{\psi}_S \mathrm{Tr}_s \left[g \frac{\partial L_{u^2}}{\partial u} \exp(-L_{u^2}^2) \right] du. \end{aligned} \tag{2.28}$$

The legitimacy of the definition follows from the equivariant version of [2, Theorem 9.7]. Moreover, by the equivariant version of [2, (9.2)],

$$d\widetilde{\eta}_g(F,L) = \mathrm{ch}_g(F', \nabla^{F'}) - \mathrm{ch}_g(F, \nabla^F). \tag{2.29}$$

3. Functoriality of eta forms

We will present our main result in this section. In Section 3.1, we investigate the geometry of a composition of fibrations, then define the Dirac operators. In Section 3.2, we construct a series of bundles for the composition of fibrations by means of Dirac operators in Section 3.1, which is an analogy of [3, (6.9)] and [21, (2.13)]. In Section 3.3 we state our main result, which can be seen as the relation of the equivariant Bismut-Cheeger and finite dimensional eta forms associated with the composition of fibrations under some assumptions. In Section 3.4, we discuss the case for signature operators, which satisfies all assumptions in our main result.

3.1. **Composition of fibrations.**

In this subsection, we introduce the geometry settings of our main result. We revisit the geometric setting in Section 1 to make the definitions clear.

Let W, V, S be closed G-manifolds. Let $\pi_X : W \to V$, $\pi_Y : V \to S$ be equivariant submersions with closed oriented fibers X, Y respectively. Then $\pi_Z := \pi_Y \circ \pi_X : W \to S$ is an equivariant submersion with closed oriented fiber, which we denote by Z. Assume that G acts on S trivially.

We denote by TX, TY, TZ the corresponding relative tangent bundles for π_X, π_Y, π_Z, and $T_X^H W$, $T_Y^H V$, $T_Z^H W$ the horizontal G-equivariant subbundles respectively. For $U \in TS$, $U' \in TV$, we shall denote by $U'^H_X \in T_X^H W$, $U_Y^H \in T_Y^H V$, $U_Z^H \in T_Z^H W$ the corresponding horizontal lifts of U', U, U such that $\pi_{X,*}(U'^H_X) = U'$, $\pi_{Y,*}(U_Y^H) = U$, $\pi_{Z,*}(U_Z^H) = U$. We assume that $T_Z^H W \subseteq T_X^H W$, such that for $U \in TS$, $(U_Y^H)_X^H = U_Z^H$. Let $T^H Z := T_X^H W \cap TZ$. We have a splitting $TZ = T^H Z \oplus TX$ such that $T^H Z \simeq \pi_X^* TY$.

Let g^{TX}, g^{TY} be two G-invariant Euclidean metrics on relative tangent bundles TX, TY respectively. We define

$$g^{TZ} := \pi_X^* g^{TY} \oplus g^{TX} \tag{3.1}$$

on TZ, which is also G-invariant. Let ∇^{TX}, ∇^{TY}, ∇^{TZ} be G-invariant connections defined in (2.3) on TX, TY, TZ respectively. Let ${}^0\nabla^{TZ}$ be the connection

$${}^0\nabla^{TZ} := \pi_X^* \nabla^{TY} \oplus \nabla^{TX}. \tag{3.2}$$

In the following, we write $\{k_\alpha\}$, $\{e_i\}$, $\{f_p\}$ the local orthonormal frames of (TS, g^{TS}), (TX, g^{TX}), (TY, g^{TY}) correspondingly, and $\{k^\alpha\}$, $\{e^i\}$, $\{f^p\}$ the dual frames.

Let $\underline{\mathcal{E}_X} = (\mathcal{E}_X, h^{\mathcal{E}_X}, \nabla^{\mathcal{E}_X})$ (resp. $\underline{\mathcal{E}_Y} = (\mathcal{E}_Y, h^{\mathcal{E}_Y}, \nabla^{\mathcal{E}_Y})$) be a $\mathbb{Z}_2$-graded G-equivariant self-adjoint Clifford module of $\mathrm{Cl}(TX)$ over W (resp. Clifford module of $\mathrm{Cl}(TY)$ over V) with a G-invariant Clifford connection. Then $\mathcal{E}_X$ is a $\mathbb{Z}_2$-graded G-equivariant complex vector bundle over W and $\mathcal{E}_Y$ is a $\mathbb{Z}_2$-graded G-equivariant complex vector bundle over V. Set

$$\mathcal{E} := \pi_X^* \mathcal{E}_Y \widehat{\otimes} \mathcal{E}_X. \tag{3.3}$$

Then $\mathcal{E}$ is a $\mathbb{Z}_2$-graded G-equivariant self-adjoint Clifford module of

$$\mathrm{Cl}(TZ) \simeq \pi_X^* \mathrm{Cl}(TY) \widehat{\otimes} \mathrm{Cl}(TX) \tag{3.4}$$

with Hermitian metric

$$h^{\mathcal{E}} := \pi_X^* h^{\mathcal{E}_Y} \otimes h^{\mathcal{E}_X}. \tag{3.5}$$

For $U \in TY$, the Clifford action $c(U)$ on $\mathcal{E}_Y$ are lifted on $\pi_X^* \mathcal{E}_Y$ as $c(U_X^H)$. Set

$${}^0\nabla^{\mathcal{E}} := \pi_X^* \nabla^{\mathcal{E}_Y} \otimes 1 + 1 \otimes \nabla^{\mathcal{E}_X}. \tag{3.6}$$

Then ${}^0\nabla^{\mathcal{E}}$ is the Clifford connection associated with ${}^0\nabla^{TZ}$, but not associated with ∇^{TZ}.

Let $D_X^{\mathcal{E}_X}$ be the family of Dirac operators with respect to $(g^{TX}, \nabla^{\mathcal{E}_X})$. For any $v \in V$, $\ker D_{X_v}^{\mathcal{E}_X}$ is a finite dimensional G-representation. We assume that $\bigsqcup_{v \in V} \ker D_{X_v}^{\mathcal{E}_X}$ forms a vector bundle $\ker D_X^{\mathcal{E}_X}$ over V.

Denote by E_X the infinite dimensional vector bundle over V associated with $\mathcal{E}_X$. We shall denote by $\langle \cdot, \cdot \rangle_{E_X}$ the L^2-inner product on E_X and ∇^{E_X} the G-invariant $\langle \cdot, \cdot \rangle_{E_X}$-preserving connection as (2.12) and (2.13). Then $\langle \cdot, \cdot \rangle_{E_X}$ induces a G-invariant Hermitian metric $h^{\ker D_X}$ on $\ker D_X^{\mathcal{E}_X}$. Let $P : E_X \to \ker D_X^{\mathcal{E}_X}$ be the orthogonal projection with respect to $\langle \cdot, \cdot \rangle_{E_X}$. We define

$$\nabla^{\ker D_X} := P \circ \nabla^{E_X} \circ P, \tag{3.7}$$

which is a G-invariant Hermitian connection on $\ker D_X^{\mathcal{E}_X}$. We define the inner product $\langle \cdot, \cdot \rangle_{\mathcal{E}_Y \otimes E_X}$, based on $h^{\mathcal{E}_Y}$ and $\langle \cdot, \cdot \rangle_{E_X}$. Set

$$\nabla^{\mathcal{E}_Y \otimes E_X} := \nabla^{\mathcal{E}_Y} \otimes 1 + 1 \otimes \nabla^{E_X} \tag{3.8}$$

on the bundle $\mathcal{E}_Y \widehat{\otimes} E_X \to V$. Set

$$h^{\mathcal{E}_Y \otimes \ker D_X} = h^{\mathcal{E}_Y} \otimes h^{\ker D_X}, \quad \nabla^{\mathcal{E}_Y \otimes \ker D_X} = \nabla^{\mathcal{E}_Y} \otimes 1 + 1 \otimes \nabla^{\ker D_X}. \tag{3.9}$$

Note that all these data constructed on $\mathcal{E}_Y \widehat{\otimes} \ker D_X^{\mathcal{E}_X}$ are G-invariant.

We define the G-invariant metrics over TZ and TW for $T \geq 1$:

$$g_T^{TZ} := \pi_X^* g^{TY} \oplus \frac{1}{T^2} g^{TX}, \quad g_T^{TW} := \pi_Z^* g^{TS} \oplus g_T^{TZ}. \tag{3.10}$$

Let $\mathrm{Cl}_T(TZ)$ be the Clifford algebra bundle associated with g_T^{TZ}. It is easy to see that the map

$$(\mathrm{Cl}_T(TZ), g_T^{TZ}) \to (\mathrm{Cl}(TZ), g^{TZ}), \quad f_p \mapsto f_p,\ Te_i \mapsto e_i \tag{3.11}$$

is an isomorphism of Clifford algebras. Then we can regard $\mathcal{E}$ as a Clifford module of $\mathrm{Cl}_T(TZ)$ through this isomorphism and denote the Clifford multiplication by c_T.

Let $\nabla^{TZ,T}$ be the G-invariant connection in (2.3) associated with $(T_Z^H W, g_T^{TZ})$. Then by [21, Theorem 5.1] (see also [15, (4.1)]),

$$\nabla^{TZ,T} = {}^0\nabla^{TZ} + P^{TX} \mathcal{S}_X P^{T^H Z} + \frac{1}{T^2} P^{T^H Z} \mathcal{S}_X P^{TZ}, \tag{3.12}$$

where $\mathcal{S}_X$ is the tensor in (2.5) associated with $\underline{\pi_X}$ and $P^{T^H Z}$ is the orthognal projection onto $T^H Z$.

Let $h_T^{\mathcal{E}_X}$ be a G-invariant Hermitian metric on $\mathcal{E}_X$ preserving the $\mathbb{Z}_2$-grading such that the Clifford action c_T restricted to TX is skew-adjoint and let $h_T^{\mathcal{E}} := \pi_X^* h^{\mathcal{E}_Y} \otimes h_T^{\mathcal{E}_X}$ as in (3.5). Let $\nabla^{\mathcal{E},T}$ be a G-invariant Clifford connection on $\mathcal{E}$ associated with $\nabla^{TZ,T}$ and $h_T^{\mathcal{E}}$. As in

[2, Proposition 3.40], [15, (4.3)] and (2.8), there exists G-equivariant $\mathcal{A}_T \in \Omega^1(W, \mathrm{End}(\mathcal{E}))$ supercommuting with $\mathrm{Cl}(TZ)$ such that

$$\nabla^{\mathcal{E},T}_{\cdot} := {}^0\nabla^{\mathcal{E}}_{\cdot} + \frac{1}{2T}\langle \mathcal{S}_X(\cdot)e_i, f^H_{p,X}\rangle c(e_i)c(f_p) + \frac{1}{4T^2}\langle \mathcal{S}_X(\cdot)f^H_{p,X}, f^H_{q,X}\rangle c(f_p)c(f_q) + \mathcal{A}_T(\cdot). \tag{3.13}$$

Here $c(e_i)$ denotes the action of $c_T(Te_i)$ on $\mathcal{E}$ through the map in (3.11). In fact, as in (2.8), the Clifford modules $\mathcal{E}_X$, $\mathcal{E}_Y$ and $\mathcal{E}$ can be locally written as

$$\begin{aligned} &\mathcal{E}_X = S(TX)\widehat{\otimes}F_X, \quad \mathcal{E}_Y = S(TY)\widehat{\otimes}F_Y, \\ &\mathcal{E} = S(TZ)\widehat{\otimes}\pi_X^*F_Y\widehat{\otimes}F_X = \pi_X^*S(TY)\widehat{\otimes}S(TX)\widehat{\otimes}\pi_X^*F_Y\widehat{\otimes}F_X, \end{aligned} \tag{3.14}$$

where $S(TX)$ and $S(TY)$ are the corresponding (possibly nonexistent) spinor bundles and F_X and F_Y are $\mathbb{Z}_2$-graded vector bundles. From the proof of [2, Proposition 3.40], a Clifford connection on $\mathcal{E}$ can be locally uniquely splitted as the sum of the canonical connection $\nabla^{S(TZ),T}$ on $S(TZ)$ with respect to $\nabla^{TZ,T}$ and a connection on $\pi_X^*F_Y\widehat{\otimes}F_X$. From [2, (3.13)] and (3.12), with respect to the metric $\nabla^{TZ,T}$,

$$\begin{aligned} \nabla^{S(TZ),T} = \pi_X^*\nabla^{S(TY)} \otimes 1 + 1 \otimes \nabla^{S(TX)} \\ + \frac{1}{2T}\langle \mathcal{S}_X(\cdot)e_i, f^H_{p,X}\rangle c(e_i)c(f_p) + \frac{1}{4T^2}\langle \mathcal{S}_X(\cdot)f^H_{p,X}, f^H_{q,X}\rangle c(f_p)c(f_q). \end{aligned} \tag{3.15}$$

Here $\nabla^{S(TY)}$ and $\nabla^{S(TX)}$ are the canonical connections on $S(TY)$ and $S(TX)$ with respect to ∇^{TY} and ∇^{TX} respectively. So there exists uniquely G-equivariant $\mathcal{A}_T$ on $\pi_X^*F_Y\widehat{\otimes}F_X$ such that (3.13) holds locally. Since other terms in (3.13) are globally defined and preserve the metric $h^{\mathcal{E}}_T$, so $\mathcal{A}_T$ is globally defined and skew-adjoint with respect to $h^{\mathcal{E}}_T$. If $T = 1$, we simply denote by

$$\nabla^{\mathcal{E}} := \nabla^{\mathcal{E},1}. \tag{3.16}$$

Let $D^{\mathcal{E}}_{Z,T}$ be the family of Dirac operators with respect to $(g^{TZ}, \nabla^{\mathcal{E},T})$. We assume that $\ker D^{\mathcal{E}}_{Z,T}$ forms a vector bundle over $S \times [1, +\infty)_T$. We simply denote by $D^{\mathcal{E}}_Z = D^{\mathcal{E}}_{Z,1}$.

In order to obtain reasonable results, we must make additional assumptions about $\mathcal{A}_T$. On the other hand, we expect the two most interesting examples, usual Dirac operator and signature operator, to hold under the assumptions. In this paper, we assume that

$$\begin{aligned} &\mathcal{A}_T(U_1) = Q_{X,1}(U_1)T^{-1} + Q_{X,2}(U_1)T^{-2}, \quad &&\text{for } U_1 \in TX; \\ &\mathcal{A}_T(U_2) = Q_Y(U_2)T^{-1}, &&\text{for } U_2 \in T^HZ; \\ &\mathcal{A}_T(U_3) = O(T^{-1}), &&\text{for } U_3 \in T^H_ZW,\ T \to +\infty, \end{aligned} \tag{3.17}$$

where $Q_{X,1}(U_1)$, $Q_{X,2}(U_1)$ and $Q_Y(U_2)$ are endomorphisms of $\mathcal{E}$ over W. Since $\mathcal{A}_T$ supercommutes with the $\mathrm{Cl}(TZ)$ action and is skew-adjoint with respect to $h^{\mathcal{E}}_T$, we see that $Q_{X,1}(U_1)$, $Q_{X,2}(U_1)$ and $Q_Y(U_2)$ supercommute with the $\mathrm{Cl}(TZ)$ action and are skew-adjoint with respect to $h^{\mathcal{E}}_1$. For the case of Dirac operator, $\mathcal{A}_T = 0$. Later in (3.94), we will see that for the signature operator, the Clifford connection satisfies the assumption (3.17).

Since $\mathcal{S}_X(e_i)e_j = \mathcal{S}_X(e_j)e_i$ and $\mathcal{S}_X(e_i)f^H_{p,X} - \mathcal{S}_X(f^H_{p,X})e_i \in TX$, from (2.11), (2.13), (3.8), (3.13) and (3.17), we have

$$\begin{aligned} D^{\mathcal{E}}_{Z,T} &= c(e_i)\nabla^{\mathcal{E},T}_{Te_i} + c(f_p)\nabla^{\mathcal{E},T}_{f^H_{p,X}} \\ &= TD^{\mathcal{E}_X}_X + \frac{1}{2}\langle \mathcal{S}_X(e_i)e_j, f^H_{p,X}\rangle c(e_i)c(e_j)c(f_p) + \frac{1}{4T}\langle \mathcal{S}_X(e_i)f^H_{p,X}, f^H_{q,X}\rangle c(e_i)c(f_p)c(f_q) \\ &\quad + Tc(e_i)\mathcal{A}_T(e_i) + c(f_p)\,{}^0\nabla^{\mathcal{E}}_{f^H_{p,X}} + \frac{1}{2T}\langle \mathcal{S}_X(f^H_{p,X})e_i, f^H_{q,X}\rangle c(f_p)c(e_i)c(f_q) + c(f_p)\mathcal{A}_T(f^H_{p,X}) \\ &= TD^{\mathcal{E}_X}_X + c(f_p)\nabla^{\mathcal{E}_Y\otimes E_X}_{f_p} + c(e_i)Q_{X,1}(e_i) \\ &\quad - \frac{1}{4T}\langle \mathcal{S}_X(e_i)f^H_{p,X}, f^H_{q,X}\rangle c(e_i)c(f_p)c(f_q) + \frac{1}{T}(c(e_i)Q_{X,2}(e_i) + c(f_p)Q_Y(f^H_{p,X})). \end{aligned} \tag{3.18}$$

Set

$$D_H := c(f_p)\nabla_{f_p}^{\mathcal{E}_Y\otimes E_X} + c(e_i)Q_{X,1}(e_i) \tag{3.19}$$

on $C^\infty(V,\mathcal{E}_Y\widehat{\otimes}E_X)\simeq C^\infty(W,\mathcal{E})$ and

$$\mathcal{C} := -\frac{1}{8}\langle \mathcal{T}_X(f_{p,X}^H, f_{q,X}^H), e_i\rangle c(e_i)c(f_p)c(f_q) + c(e_i)Q_{X,2}(e_i) + c(f_p)Q_Y(f_{p,X}^H). \tag{3.20}$$

Here $\mathcal{T}_X$ is the torsion tensor associated with $\underline{\pi_X}$ as in (2.6). Then by [4, (1.28), (1.30)], we have

$$D_{Z,T}^{\mathcal{E}} = TD_X^{\mathcal{E}_X} + D_H + \frac{\mathcal{C}}{T}. \tag{3.21}$$

By (3.21), D_H and $\mathcal{C}$ are self-adjoint with respect to $h_1^{\mathcal{E}}$.

Let $D_Y^{\mathcal{E}_Y\otimes\ker D_X}$ be the family of Dirac operators associated with $(g^{TY},\nabla^{\mathcal{E}_Y\otimes\ker D_X})$. Then by (3.19) and (3.21),

$$\widetilde{D}_Y := PD_HP = D_Y^{\mathcal{E}_Y\otimes\ker D_X} + P\,c(e_i)Q_{X,1}(e_i)\,P. \tag{3.22}$$

We assume that $\ker\widetilde{D}_Y$ forms a vector bundle over S.

As in (2.13), for $U\in C^\infty(S,TS)$, we set

$$\begin{aligned}\nabla_U^{E,T} &:= \nabla_{U_Z^H}^{\mathcal{E},T} - \frac{1}{2}\langle \mathcal{S}_Z(e_i)e_i, U_Z^H\rangle - \frac{1}{2}\langle \mathcal{S}_Z(f_{p,X}^H)f_{p,X}^H, U_Z^H\rangle,\\ {}^0\nabla_U^{E} &:= {}^0\nabla_{U_Z^H}^{\mathcal{E}} - \frac{1}{2}\langle \mathcal{S}_Z(e_i)e_i, U_Z^H\rangle - \frac{1}{2}\langle \mathcal{S}_Z(f_{p,X}^H)f_{p,X}^H, U_Z^H\rangle,\end{aligned} \tag{3.23}$$

where $\mathcal{S}_Z$ is the tensor in (2.5) associated with $\underline{\pi_Z}$. From (3.13) and (3.23), we have

$$\nabla_U^{E,T} = {}^0\nabla_U^{E} + \frac{1}{2T}\langle \mathcal{S}_X(U_Z^H)e_i, f_{p,X}^H\rangle c(e_i)c(f_p) + \mathcal{A}_T(U_Z^H). \tag{3.24}$$

Let B_T be the Bismut superconnection defined in (2.15) with respect to $(T_Z^HW,\nabla^{\mathcal{E},T})$. By [15, Proposition 5.5], we have

$$\begin{aligned}B_T = TD_X^{\mathcal{E}_X} &+ {}^0\nabla^E + D^H - \frac{c(\mathcal{T}_Y)}{4} - \frac{1}{8T}\langle \mathcal{T}_X(f_{p,X}^H, f_{q,X}^H), e_i\rangle c(e_i)c(f_p)c(f_q)\\ &+ \frac{1}{2T}\langle \mathcal{S}_X(k_{\alpha,Z}^H)e_i, f_{p,X}^H\rangle c(e_i)c(f_p)k^\alpha\wedge - \frac{1}{8T}\langle \mathcal{T}_Z(k_{\alpha,Z}^H, k_{\beta,Z}^H), e_i\rangle c(e_i)k^\alpha\wedge k^\beta\wedge\\ &+ \frac{1}{T}c(e_i)Q_{X,2}(e_i) + \frac{1}{T}c(f_p)Q_Y(f_{p,X}^H) + k^\alpha\wedge\mathcal{A}_T(k_{\alpha,Z}^H).\end{aligned} \tag{3.25}$$

Here $\mathcal{T}_Y$ and $\mathcal{T}_Z$ are the torsion tensors associated with $\underline{\pi_Y}$ and $\underline{\pi_Z}$. Let ∇^{E_Y} be the connection as in (2.13) with respect to π_Y and $\nabla^{\mathcal{E}_Y\otimes\ker D_X}$. Then by [15, (5.42)], we have

$$\nabla^{E_Y} = P\,{}^0\nabla^E P. \tag{3.26}$$

Let B_Y be the Bismut superconnection associated with $(\underline{\pi_Y},\underline{\mathcal{E}_Y})$. Set

$$B_Y' := \widetilde{D}_Y + \nabla^{E_Y} - \frac{c(\mathcal{T}_Y)}{4}. \tag{3.27}$$

Then by (3.22) and (3.27), we have

$$B_Y' = B_Y + Pc(e_i)Q_{X,1}(e_i)P. \tag{3.28}$$

From (3.22) and (3.25)-(3.27), we have

$$PB_TP = B_Y' + O\left(\frac{1}{T}\right). \tag{3.29}$$

We assume that the equivariant Bismut-Cheeger eta form with respect to $Pc(e_i)Q_{X,1}(e_i)P$:

$$\tilde{\eta}_g'(\underline{\pi_Y},\underline{\mathcal{E}_Y\otimes\ker D_X^{\mathcal{E}_X}}) := \tilde{\eta}_g(\underline{\pi_Y}, Pc(e_i)Q_{X,1}(e_i)P, \underline{\mathcal{E}_Y\otimes\ker D_X^{\mathcal{E}_X}}) \in \Omega^\bullet(S) \tag{3.30}$$

is well-defined.

3.2. **The bundles of spectral sequence.** With all these geometric data prepared, in this subsection, we will define a series of bundles over S, denoted by $\{E_r\}_{r=0,1,\cdots,\infty}$.

Compared with [3, (6.9)] and [21, (2.13)], we make the following definition.

Definition 3.1. For $r=0$, let $E_0=E$ be the infinite dimensional bundle over S with fiber $C^\infty(Z,\mathcal{E})$. For $r=1$, let E_1 be the infinite dimensional bundle over S with fiber $C^\infty(Y,\mathcal{E}_Y\widehat{\otimes}\ker D_X^{\mathcal{E}_X})$. For $r\geq 2$, we define the $C^\infty(S)$-module

$$\begin{aligned}\mathscr{E}_r := \Big\{ & s_0\in C^\infty(S,E):\ \text{There exist } s_1,\cdots,s_{r-1}\in C^\infty(S,E),\ \text{such that}\\ & D_X^{\mathcal{E}_X}s_0=0,\ D_Hs_0+D_X^{\mathcal{E}_X}s_1=0,\ \mathcal{C}s_0+D_Hs_1+D_X^{\mathcal{E}_X}s_2=0,\\ & \cdots,\mathcal{C}s_{r-3}+D_Hs_{r-2}+D_X^{\mathcal{E}_X}s_{r-1}=0\Big\}.\end{aligned}\tag{3.31}$$

We assume that $\mathscr{E}_r$ is a finite generated projective $C^\infty(S)$-module. Then it is the space of smooth sections of a vector bundle E_r over S.

By (3.22) and (3.31), for $b\in S$, $E_{2,b}=\ker\widetilde{D}_{Y_b}$, which is a finite dimensional vector space. For $r>2$, $E_{r,b}\subseteq E_{2,b}$. So for $r\geq 2$, $\dim E_{r,b}<+\infty$. From the assumption in Definition 3.1, $\{E_{r,b}\}_{b\in S}$ constitute finite dimensional complex vector bundles E_r for $r\geq 2$. Moreover, they are G-equivariant and $\mathbb{Z}_2$-graded.

In Section 3.4, we will see that when we consider signature operators, E_r is isomorphic to $\mathcal{E}_r$ in [21, (2.13)] . So for signature operator, E_r can be interpreted as terms of Leray spectral sequences [21, Proposition 2.1]. However, in our general case, there is no topological meaning for $E_r, r\geq 2$. Hence we need the assumption in Definition 3.1.

Now we construct the geometric data and the equivariant eta forms for E_r.

For $r=0$, $E_0=E$. By abusing the notation, we write h_0 for $\langle\cdot,\cdot\rangle_E$, the metric defined in (2.12) on E. We write $\nabla^0:=\nabla^E$ and $D_0:=D_X^{\mathcal{E}_X}$. Let B_0 be the Bismut superconnection associated with $(T_X^HW,g^{TX},\nabla^{\mathcal{E}_X})$ as in (2.15).

Let V^g be the fixed point set of the g-action on V for $g\in G$. Then $\pi_X|_{V^g}:W|_{V^g}\to V^g$ is a fiber bundle with fiber X. We assume that TX^g is oriented. For any $g\in G$, the equivariant Bismut-Cheeger eta form

$$\widetilde{\eta}_g(\underline{\pi_X^g},\underline{\mathcal{E}_X})\in\Omega^\bullet(V^g)\tag{3.32}$$

is well-defined. Here $\underline{\pi_X^g}$ stands for

$$\underline{\pi_X^g}:=\left(\pi_X|_{V^g},T_X^H(W|_{V^g}):=T^HW\cap T(W|_{Y^g}),g^{TX}\right).\tag{3.33}$$

For $r=1$, $E_1=\ker D_0=\ker D_X^{\mathcal{E}_X}$. Let $p_1:=P:E_0\to E_1$ be the orthogonal projection. Let h_1 be the metric on E_1 induced from h_0. Set

$$\nabla^1:=\nabla^{\mathcal{E}_Y\otimes\ker D_X}=p_1\,\nabla^0\,p_1.\tag{3.34}$$

Then it is a connection on E_1 preserving h_1.

Let $D_1:=\widetilde{D}_Y$ be the operator in (3.22). For $g\in G$, we assume that TY^g is oriented. We have assumed that

$$\widetilde{\eta}'_g(\underline{\pi_Y},\underline{\mathcal{E}_Y\otimes\ker D_X^{\mathcal{E}_X}})\in\Omega^\bullet(S)\tag{3.35}$$

defined in (3.30) is well-defined.

For $r\geq 2$, let

$$p_r:E_0\to E_r\tag{3.36}$$

be the orthogonal projection with respect to h_0. Let h_r be the metric on E_r induced from h_0 and

$$\nabla^r:=p_r\,\nabla^0\,p_r.\tag{3.37}$$

Then ∇^r preserves h_r.

Comparing with [3, (6.10)] and [21, (2.14)], for $r \geq 2$, we define D_r on E_r by

$$D_r s_0 := p_r(D_H s_{r-1} + \mathcal{C} s_{r-2}), \tag{3.38}$$

where s_{r-1}, s_{r-2} are the elements in (3.31). Then D_r commutes (resp. anticommutes) with the $\mathbb{Z}_2$-grading on E_r when $\dim Z$ is odd (resp. even).

Theorem 3.2. *The operator D_r in (3.38) is well-defined. That is, it is independent of the choice of $s_1, \cdots, s_{r-1}$ in (3.31). Moreover, D_r is self-adjoint and*

$$\ker D_r = E_{r+1}. \tag{3.39}$$

Proof. Since all operators in (3.31) are along the fiber Z, we may assume that S is a point.

For $r = 0, 1$, $D_0 = D_X^{\mathcal{E}_X}$ and $D_1 = \widetilde{D}_Y$ are well-defined. Moreover, we have $\ker D_0 = E_1$ by Definition 3.1. Since D_0 and D_1 are self-adjoint elliptic operators, by Hodge theory, we have the orthogonal decompositions,

$$E_0 = \operatorname{Im} D_0 \oplus \ker D_0 = \operatorname{Im} D_0 \oplus E_1, \quad E_1 = \operatorname{Im} D_1 \oplus \ker D_1. \tag{3.40}$$

We first prove (3.39) for $r = 1$. If $s_0 \in E_2$, by (3.31), there exists $s_1 \in E_0$, such that $D_H s_0 + D_0 s_1 = 0$. So by (3.40),

$$D_1 s_0 = p_1 D_H s_0 = P(-D_0 s_1) = 0, \tag{3.41}$$

which implies $E_2 \subseteq \ker D_1$. On the other hand, if $s_0 \in E_1$ and $D_1 s_0 = p_1 D_H s_0 = 0$, then by (3.40), we see that $D_H s_0 \in \operatorname{Im} D_0$. Thus there exists $s_1 \in E_0$, such that $D_H s_0 = D_0(-s_1)$, which implies $\ker D_1 \subseteq E_2$. Therefore, we have $\ker D_1 = E_2$.

For $r > 1$, E_r is finite-dimensional. We assume that for any $0 \leq r' \leq r-1$, $D_{r'}$ is well-defined, self-adjoint and $\ker D_{r'} = E_{r'+1}$. We will show that these assumptions imply that D_r is well-defined, self-adjoint and $\ker D_r = E_{r+1}$. Then Theorem 3.2 follows by induction.

Since $D_{r'}$ is well-defined and self-adjoint, as in (3.40), we have the orthogonal decomposition

$$E_{r'} = \operatorname{Im} D_{r'} \oplus \ker D_{r'} = \operatorname{Im} D_{r'} \oplus E_{r'+1}. \tag{3.42}$$

For $s_0 \in E_r$, the $(r-1)$-tuple $(s_1, \cdots, s_{r-1})$ in (3.31) is called a suitable $(r-1)$-tuple for $s_0 \in E_r$. Let $(s_1, \cdots, s_{r-1})$ and $(s'_1, \cdots, s'_{r-1})$ be two suitable $(r-1)$-tuples for $s_0 \in E_r$. Set $t_i := s_{i+1} - s'_{i+1}$, $0 \leq i \leq r-2$. Then by (3.31), we have

$$D_0 t_0 = 0,\ D_H t_0 + D_0 t_1 = 0,\ \cdots,\ \mathcal{C} t_{r-4} + D_H t_{r-3} + D_0 t_{r-2} = 0, \tag{3.43}$$

which means that $t_0 \in E_{r-1}$ and $(t_1, \cdots, t_{r-2})$ is a suitable $(r-2)$-tuple for $t_0 \in E_{r-1}$. Since D_{r-1} is well-defined, by (3.38) and (3.43),

$$\begin{aligned} &p_r(D_H s_{r-1} + \mathcal{C} s_{r-2}) - p_r(D_H s'_{r-1} + \mathcal{C} s'_{r-2}) \\ &\qquad = p_r(D_H t_{r-2} + \mathcal{C} t_{r-3}) = p_r p_{r-1}(D_H t_{r-2} + \mathcal{C} t_{r-3}) = p_r D_{r-1} t_0. \end{aligned} \tag{3.44}$$

By the inductive assumption, $\ker D_{r-1} = E_r$. So from (3.42), $D_{r-1} t_0 \in (\ker D_{r-1})^\perp = E_r^\perp$. Here $(\ker D_{r-1})^\perp$ and $E_r^\perp$ are the orthogonal complements in E_0 with respect to h_0. Thus $p_r D_{r-1} t_0 = 0$. From (3.44), we see that D_r is well-defined.

Now we will prove that D_r is self-adjoint. For $s_0, s'_0 \in E_r$, let $(s_1, \cdots, s_{r-1})$ and $(s'_1, \cdots, s'_{r-1})$ be the corresponding suitable $(r-1)$-tuples. Then by (3.31), since $D_0 = D_X^{\mathcal{E}_X}, D_H$ and $\mathcal{C}$ are all symmetric with respect to h_0, we have

$$
\begin{aligned}
&\langle D_r s_0, s_0' \rangle = \langle D_H s_{r-1} + \mathcal{C} s_{r-2}, s_0' \rangle = \langle \mathcal{C} s_{r-2}, s_0' \rangle + \langle s_{r-1}, D_H s_0' \rangle \\
=&\langle \mathcal{C} s_{r-2}, s_0' \rangle - \langle s_{r-1}, D_0 s_1' \rangle \\
=&\langle \mathcal{C} s_{r-2}, s_0' \rangle - \langle D_0 s_{r-1}, s_1' \rangle = \langle \mathcal{C} s_{r-2}, s_0' \rangle + \langle D_H s_{r-2}, s_1' \rangle + \langle \mathcal{C} s_{r-3}, s_1' \rangle \\
=&\langle \mathcal{C} s_{r-2}, s_0' \rangle + \langle \mathcal{C} s_{r-3}, s_1' \rangle + \langle s_{r-2}, D_H s_1' \rangle \\
=&\langle \mathcal{C} s_{r-2}, s_0' \rangle + \langle \mathcal{C} s_{r-3}, s_1' \rangle - \langle s_{r-2}, \mathcal{C} s_0' \rangle - \langle s_{r-2}, D_0 s_2' \rangle \\
=&\langle \mathcal{C} s_{r-3}, s_1' \rangle - \langle s_{r-2}, D_0 s_2' \rangle \\
=&\cdots \\
=&\langle \mathcal{C} s_0, s_{r-2}' \rangle - \langle s_1, D_0 s_{r-1}' \rangle \\
=&\langle s_0, \mathcal{C} s_{r-2}' \rangle + \langle D_H s_0, s_{r-1}' \rangle = \langle s_0, D_H s_{r-1}' + \mathcal{C} s_{r-2}' \rangle \\
=&\langle s_0, D_r s_0' \rangle.
\end{aligned} \tag{3.45}
$$

So D_r is self-adjoint. As in (3.42), we have the orthogonal decomposition $E_r = \operatorname{Im} D_r \oplus \ker D_r$.

At last, we will prove that $\ker D_r = E_{r+1}$. We will use a downward recursion process which is motivated by [3, (6.20)-(6.22)].

For $0 \le k \le r$, set

$$
\begin{aligned}
E_{r+1}^k := \{ s_0 \in E_0 :\ & \text{There exist } s_1, \cdots, s_{r-1} \in E_0, \text{ such that} \\
& D_0 s_0 = 0,\ D_H s_0 + D_0 s_1 = 0,\ \mathcal{C} s_0 + D_H s_1 + D_0 s_2 = 0, \cdots, \\
& \mathcal{C} s_{r-3} + D_H s_{r-2} + D_0 s_{r-1} = 0,\ p_k(\mathcal{C} s_{r-2} + D_H s_{r-1}) = 0 \}.
\end{aligned} \tag{3.46}
$$

For $k = r$, by (3.31) and (3.38), we see that

$$E_{r+1}^r = \ker D_r. \tag{3.47}$$

For $k = 1$, by (3.40), $p_1(\mathcal{C} s_{r-2} + D_H s_{r-1}) = 0$ means that there exists $s_r \in E_0$, such that $\mathcal{C} s_{r-2} + D_H s_{r-1} + D_0 s_r = 0$. We have

$$E_{r+1}^1 = E_{r+1}. \tag{3.48}$$

We claim that for any $1 \le k \le r$, $E_{r+1}^k = E_{r+1}^{k-1}$. Then E_{r+1}^k is independent of k. In particular, by (3.47) and (3.48), we obtain $\ker D_r = E_{r+1}$.

By (3.46), it is easy to see that $E_{r+1}^k \supseteq E_{r+1}^{k-1}$. For $s_0 \in E_{r+1}^k$, we say $(s_1, \cdots, s_{r-1})$, $s_i \in E_0$, is a k-suitable $(r-1)$-tuple for $s_0 \in E_{r+1}^k$ if they satisfy the equations in (3.46). Let $(s_1, \cdots, s_{r-1})$ be a k-suitable $(r-1)$-tuple for $s_0 \in E_{r+1}^k$. From (3.42), there exists $s_0' \in E_{k-1}$, such that

$$\mathcal{C} s_{r-2} + D_H s_{r-1} = D_{k-1} s_0'. \tag{3.49}$$

Let $(s_1', \cdots, s_{k-2}')$ be a suitable $(k-2)$-tuple for $s_0' \in E_{k-1}$. Then by (3.31) and (3.49), we have

$$
\begin{aligned}
& D_0 s_0' = 0,\ D_H s_0' + D_0 s_1' = 0,\ \mathcal{C} s_0' + D_H s_1' + D_0 s_2' = 0, \\
& \cdots, \mathcal{C} s_{k-4}' + D_H s_{k-3}' + D_0 s_{k-2}' = 0, \\
& p_{k-1}(\mathcal{C} s_{k-3}' + D_H s_{k-2}') = \mathcal{C} s_{r-2} + D_H s_{r-1} = p_{k-1}(\mathcal{C} s_{r-2} + D_H s_{r-1}).
\end{aligned} \tag{3.50}
$$

Then by (3.46) and (3.50), it is easy to see that $(s_1'', s_2'', \cdots, s_{r-1}'') = (s_1, s_2, \cdots, s_{r-1}) - (0, \cdots, 0, s_1', \cdots, s_{k-2}')$ is a $(k-1)$-suitable $(r-1)$-tuple for $s_0 \in E_{r+1}^{k-1}$. So we get $E_{r+1}^k = E_{r+1}^{k-1}$.

The proof of Theorem 3.3 is complete.

□

From the assumption in Definition 3.1, for $r \ge 2$, E_r is a G-equivariant $\mathbb{Z}_2$-graded vector bundle over S. From (3.38) and Theorem 3.2, $D_r \in C^\infty(S, \operatorname{End}(E_r))$ commutes with the group action. By Definition 2.3, we define the superconnection

$$B_r := \nabla^r + D_r, \quad r \ge 2. \tag{3.51}$$

By Definition 2.5 and Theorem 3.2, for any $g \in G$, we define

$$\widetilde{\eta}_g(E_r, B_r) \in \Omega^\bullet(S), \quad r \ge 2. \tag{3.52}$$

As $E_r \supseteq E_{r+1}$ and $\dim E_r < +\infty$ for $r \geq 2$, there exists r_0 such that for $r \geq r_0$, $E_r = E_{r_0}$. We denote by E_∞ the convergent one.

We assume that $\ker D_Z^{\mathcal{E}}$ forms a vector bundle over S. Let

$$p_\infty : E_0 \to E_\infty, \quad p : E_0 \to \ker D_Z^{\mathcal{E}} \tag{3.53}$$

be the orthogonal projections associated with h_0. We have the natural connection on $\ker D_Z^{\mathcal{E}}$:

$$\nabla^{\ker D_Z} := p\,\nabla^0\, p. \tag{3.54}$$

We assume that $\ker D_{Z,T}^{\mathcal{E}}$ is a G-equivariant vector bundle over $S \times [1, +\infty)_T$. Let

$$p^T : E_0 \to \ker D_{Z,T}^{\mathcal{E}} \tag{3.55}$$

be the orthogonal projection associated with h_0. The following lemma will be proved after Theorem 5.2.

Lemma 3.3. *There exist $C > 0$, $T_0 \geq 1$, such that for any $T \geq T_0$, $s \in E_0$,*

$$\|p^T s - p_\infty s\| \leq \frac{C}{T}\|s\|. \tag{3.56}$$

Here $\|\cdot\|$ is the L^2-norm along the fiber Z associated with $\langle\cdot,\cdot\rangle_E$. Moreover, we have

$$E_\infty \cong \ker D_Z^{\mathcal{E}}. \tag{3.57}$$

Compared with (3.54), we write

$$\nabla^{\ker D_{Z,T}} := p^T \circ \nabla^{E,T} \circ p^T. \tag{3.58}$$

From (3.17), (3.24) and (3.56), $\lim_{T\to+\infty} \nabla^{\ker D_{Z,T}}$ exists. We denote the limit by

$$\nabla^\infty = p_\infty\, {}^0\nabla^E\, p_\infty. \tag{3.59}$$

We rearrange the parameter by $s = T^{-1}$. Then $\ker D_{Z,1/s}^{\mathcal{E}}$ is a G-equivariant vector bundle over $S \times [0,1]_s$. Then we can define the equivariant Chern-Simons form $\widetilde{\mathrm{ch}}_g(\ker D_Z^{\mathcal{E}}, \nabla^\infty, \nabla^{\ker D_Z})$ as in [18, (1.29)]. Moreover (see e.g., [18, (1.30)]),

$$d\,\widetilde{\mathrm{ch}}_g(\ker D_Z^{\mathcal{E}}, \nabla^\infty, \nabla^{\ker D_Z}) = \mathrm{ch}_g(\ker D_Z^{\mathcal{E}}, \nabla^{\ker D_Z}) - \mathrm{ch}_g(E_\infty, \nabla^\infty). \tag{3.60}$$

3.3. **The main result.** In this subsection, we collect all assumptions we made in this section and state the main result.

We assume that $T_Z^H W \subseteq T_X^H W$, $g^{TZ} = g^{TX} \oplus \pi_X^* g^{TY}$. We assume that for $g \in G$, TX^g and TY^g are oriented. So TZ^g is also oriented.

Assumption 3.4. We assume that

- $\ker D_X^{\mathcal{E}_X}$ is a vector bundle over V;
- $\mathcal{A}_T$ satisfies (3.17);
- $\ker \widetilde{D}_Y$ is a vector bundle over S;
- $\mathscr{E}_r, r \geq 2$, are finitely generated projective $C^\infty(S)$-modules;
- $\ker D_{Z,T}^{\mathcal{E}}$ is a vector bundle over $S \times [1, +\infty)_T$;

Assumption 3.5. We assume that $\widetilde{\eta}'_g(\underline{\pi_Y}, \underline{\mathcal{E}_Y \otimes \ker D_X^{\mathcal{E}_X}})$ defined in (3.30) is well-defined.

Theorem 3.6. *For $g \in G$, under the assumptions above, modulo exact forms on S, we have*

$$\begin{aligned}\widetilde{\eta}_g(\underline{\pi_Z}, \underline{\mathcal{E}}) = {}& \widetilde{\eta}'_g(\underline{\pi_Y}, \underline{\mathcal{E}_Y \otimes \ker D_X^{\mathcal{E}_X}}) + \int_{Y^g} \widehat{\mathrm{A}}_g(TY, \nabla^{TY})\mathrm{ch}_g(\mathcal{E}_Y/S, \nabla^{\mathcal{E}_Y})\widetilde{\eta}_g(\underline{\pi_X^g}, \underline{\mathcal{E}_X}) \\ & - \int_{Z^g} \widetilde{\widehat{\mathrm{A}}}_g(TZ, \nabla^{TZ}, {}^0\nabla^{TZ})\mathrm{ch}_g(\mathcal{E}/S, \nabla^{\mathcal{E}}) - \int_{Z^g} \widehat{\mathrm{A}}_g(TZ, {}^0\nabla^{TZ})\widetilde{\mathrm{ch}}_g(\mathcal{E}/S, \nabla^{\mathcal{E}}, {}^0\nabla^{\mathcal{E}}) \\ & + \sum_{r=2}^{r_0} \widetilde{\eta}_g(E_r, B_r) + \widetilde{\mathrm{ch}}_g(\ker D_Z^{\mathcal{E}}, \nabla^\infty, \nabla^{\ker D_Z}).\end{aligned} \tag{3.61}$$

3.4. **Signature operator: an important example.** In this subsection, we explain that if the Clifford module is the exterior algebra bundle, for the signature operator, Assumption 3.4 is satisfied when $\dim X$ is even. If $\dim Y$ is also even, Assumption 3.5 is satisfied. In this case, Theorem 3.6 is the equivariant family version of the adiabatic limit formula for signature operators. If $\dim Y$ and $\dim Z$ are odd, Assumption 3.5 is satisfied when S is a point. In this case we regain formulas in [11, §4.1] using the language of Clifford mudules and generalize [11, Theorem 0.3] to the equivariant case.

Let $\underline{\pi_X}$, $\underline{\pi_Y}$ and $\underline{\pi_Z}$ be the fibrations with geometric data introduced in Section 3.1. We assume that $T_Z^H W \subseteq T_X^H W$, $g^{TZ} = g^{TX} \oplus \pi_X^* g^{TY}$ and for $g \in G$, TX^g and TY^g are oriented.

Assume that $\dim X = 2k$ is even. For $U \in TX$, let $U^* \in T^*X$ corresponding to U by the metric g^{TX}. Set

$$c_X(U) = U^* \wedge - i_U, \quad \widehat{c}_X(U) = U^* \wedge + i_U. \tag{3.62}$$

Then

$$\mathcal{E}_X := \Lambda^\bullet(T^*X) \otimes \mathbb{C} \tag{3.63}$$

is a $\mathbb{Z}_2$-graded self-adjoint Clifford module of $\mathrm{Cl}(TX)$ with induced metric $h^{\Lambda T^*X}$ and connection $\nabla^{\Lambda T^*X}$ associated with the Clifford multiplication c_X. As in [2, Definition 3.57], we define the chirality operator

$$\Gamma_X = (\sqrt{-1})^k e_1 \cdots e_{2k} \in \mathrm{Cl}(TX) \otimes \mathbb{C}, \tag{3.64}$$

where $\{e_i\}_{i=1}^{2k}$ is a locally oriented orthonormal frame of TX. Then Γ_X does not depend on the locally oriented orthonormal frame and $\Gamma_X^2 = 1$. For $U \in TX$, we have $\Gamma_X \cdot U = -U \cdot \Gamma_X$. The $\mathbb{Z}_2$-grading of $\mathcal{E}_X$ is defined by

$$\mathcal{E}_{X,\pm} = \{\alpha \in \mathcal{E}_X : c_X(\Gamma_X)\alpha = \pm\alpha\}. \tag{3.65}$$

Let d_X be the exterior differentiation along the fiber X and d_X^* be its formal adjoint. Then by [2, Proposition 3.58], we have

$$d_X^* = -c_X(\Gamma_X)\, d_X\, c_X(\Gamma_X). \tag{3.66}$$

In this case the Dirac operator

$$D_X^{\mathcal{E}_X} = d_X + d_X^* \tag{3.67}$$

is called the signature operator. If TX is spin with spinor bundle $S(TX)$, then we have $\mathcal{E}_X = S(TX) \otimes S(TX)$ and

$$\mathcal{E}_{X,\pm} = S_\pm(TX) \otimes S(TX). \tag{3.68}$$

The signature operator is the standard fiberwise Dirac operator with twisted bundle $S(TX)$.

If $\dim X = 2k-1$ is odd, we consider the fiber bundle $W \times \mathbb{R} \to V$ with fiber $X \times \mathbb{R}$ and with trivial geometric structure on the $\mathbb{R}$ part. Then the dimension of the fiber $X \times \mathbb{R}$ is even. In this case $\widetilde{\mathcal{E}}_X := \Lambda^\bullet(T^*(X \times \mathbb{R})) \otimes \mathbb{C}$ is a Clifford module of $\mathrm{Cl}(T(X \times \mathbb{R}))$ with Clifford action

$$\tilde{c}(U) = U^* \wedge - i_U \tag{3.69}$$

for $U \in T(X \times \mathbb{R})$ and $\mathbb{Z}_2$-grading $\widetilde{\mathcal{E}}_X = \widetilde{\mathcal{E}}_{X,+} \oplus \widetilde{\mathcal{E}}_{X,-}$ as in (3.65). Then we have a natural isomorphism

$$f_X : \widetilde{\mathcal{E}}_{X,+} \to \Lambda^\bullet T^*X \otimes \mathbb{C} \tag{3.70}$$

by sending $\alpha = \alpha_1 + dr \wedge \alpha_2 \in \widetilde{\mathcal{E}}_{X,+}$ to α_1, where $\alpha_1, \alpha_2 \in \Lambda^\bullet T^*X \otimes \mathbb{C}$.

Now we explain the isomorphism in (3.70) explicitly. In odd case, by [2, Definition 3.57], the chirality operator is defined by

$$\Gamma_X = (\sqrt{-1})^k e_1 \cdots e_{2k-1} \in \mathrm{Cl}(TX) \otimes \mathbb{C}. \tag{3.71}$$

Then Γ_X does not depend on the locally oriented orthonormal frame neither and $\Gamma_X^2 = 1$. For $U \in TX$, we have $\Gamma_X \cdot U = U \cdot \Gamma_X$. Let ∂_r be the unit vector in $T\mathbb{R}$ and dr be its dual. We orient $X \times \mathbb{R}$ by $e_1 \wedge \cdots \wedge e_{2k-1} \wedge \partial_r$. Then by (3.64) and (3.71), we have $\Gamma_{X\times\mathbb{R}} = \Gamma_X \cdot \partial_r$. Since

$$\tilde{c}(\Gamma_{X\times\mathbb{R}})\alpha = \tilde{c}(\Gamma_X)\tilde{c}(\partial_r)(\alpha_1 + dr \wedge \alpha_2) = \tilde{c}(\Gamma_X)(dr \wedge \alpha_1 - \alpha_2), \tag{3.72}$$

by (3.65), $\alpha \in \widetilde{\mathcal{E}}_{X,+}$ is equivalent to

$$\alpha = \alpha_1 - dr \wedge \tilde{c}(\Gamma_X)\alpha_1. \tag{3.73}$$

Under the isomorphism (3.70), we regard $\mathcal{E}_X := \Lambda^\bullet T^*X \otimes \mathbb{C}$ as a Clifford module of $\mathrm{Cl}(TX)$ with the Clifford multiplication (compare with [14, (I.3.10)])

$$c_X(U) := \tilde{c}(U)\tilde{c}(\partial_r), \quad U \in TX, \tag{3.74}$$

on $\widetilde{\mathcal{E}}_{X,+}$. From (3.73), for $\alpha_1 \in \Lambda^\bullet T^*X \otimes \mathbb{C}$,

$$c_X(U)\alpha_1 = \tilde{c}(U)\tilde{c}(\partial_r)(\alpha_1 - dr \wedge \tilde{c}(\Gamma_X)\alpha_1) = \tilde{c}(U)(dr \wedge \alpha_1 + \tilde{c}(\Gamma_X)\alpha_1). \tag{3.75}$$

So on $\mathcal{E}_X := \Lambda^\bullet T^*X \otimes \mathbb{C}$, we have

$$c_X(U) = \tilde{c}(U)\tilde{c}(\Gamma_X) = \tilde{c}(\Gamma_X)\tilde{c}(U), \quad U \in TX. \tag{3.76}$$

By [2, Proposition 3.58], if $\dim X$ is odd,

$$d_X^* = \tilde{c}(\Gamma_X)\, d_X\, \tilde{c}(\Gamma_X). \tag{3.77}$$

Since Γ_X is parallel with respect to the Levi-Civita connection ∇^{TX}, the connection $\nabla^{\Lambda T^*X}$ is also a Clifford connection of c_X in (3.76). So the fiberwise Dirac operator

$$D_X^{\mathcal{E}_X} = c_X(e_i)\nabla_{e_i}^{\Lambda T^*X} = \tilde{c}(\Gamma_X)(d_X + d_X^*) = \tilde{c}(\Gamma_X)d_X + d_X\tilde{c}(\Gamma_X), \tag{3.78}$$

which is called the signature operator in odd case and is the same as the signature operator A in [11, §4.1] when V is a point. If TX is spin, then by (3.68),

$$\mathcal{E}_X = S_+(T(X\times\mathbb{R}))\widehat{\otimes}S(T(X\times\mathbb{R})) \simeq S(TX)\widehat{\otimes}S(T(X\times\mathbb{R})). \tag{3.79}$$

The signature operator is the standard Dirac operator with twisted bundle $S(T(X\times\mathbb{R}))$.

For $\pi_Y : V \to S$, let $\mathcal{E}_Y := \Lambda^\bullet T^*Y \otimes \mathbb{C}$ be the Clifford module of $\mathrm{Cl}(TY)$ associated with the Clifford multiplication c_Y in the same way as c_X on $\mathcal{E}_X$. Set $\mathcal{E} := \Lambda^\bullet T^*Z \otimes \mathbb{C}$ over W. Then $\mathcal{E} = \pi_X^*\mathcal{E}_Y\widehat{\otimes}\mathcal{E}_X$ is a Clifford module of $\mathrm{Cl}(TZ) = \pi_Y^*\mathrm{Cl}(TY)\widehat{\otimes}\mathrm{Cl}(TX)$ with the graded tensor product action. For the calculation, we express the graded tensor product by ungraded one as in [6, (1.10), (1.11)] (see also [16, §2.1]). We first assume that $\dim Z$ and $\dim Y$ are odd, which is the case in [11]. By [6, (1.11)] (see also [16, (2.6), (2.7)]), from (3.4), the Clifford action of $\mathrm{Cl}(TZ)$ on $\pi_X^*\mathcal{E}_Y\widehat{\otimes}\mathcal{E}_X$ is

$$\begin{aligned} c(U) &:= c_Y(U)c_X(\Gamma_X), && \text{for } U \in TY; \\ c(U) &:= c_X(U), && \text{for } U \in TX. \end{aligned} \tag{3.80}$$

Then the Dirac operator is

$$D_Z^{\mathcal{E}} = c(f_p)\nabla_{f_{p,X}^H}^{\Lambda T^*Z} + c(e_i)\nabla_{e_i}^{\Lambda T^*Z}. \tag{3.81}$$

Let c_Z be the Clifford multiplication of $\mathrm{Cl}(TZ)$ on $\mathcal{E}$ as in (3.74). Remark that in this case $D_Z^{\mathcal{E}}$ is not the same as the signature operator

$$A_Z := c_Z(f_p)\nabla_{f_{p,X}^H}^{\Lambda T^*Z} + c_Z(e_i)\nabla_{e_i}^{\Lambda T^*Z} \tag{3.82}$$

because $c_Z(U) \neq c(U)$ for $U \in TX$. But these two Clifford representations are isomorphic and $D_Z^{\mathcal{E}}$ is the same as the signature operator A_Z up to a conjugation. In fact, from the isomorphism (3.70), originally for the $\mathrm{Cl}(TZ)$ action, c_Z acts on $\widetilde{\mathcal{E}}_{Z,+}$ and c acts on $\pi_X^*\widetilde{\mathcal{E}}_{Y,+} \otimes \mathcal{E}_X$. Note that

$$\widetilde{\mathcal{E}}_{Z,+} = \left(\pi_X^*\widetilde{\mathcal{E}}_{Y,+} \otimes \mathcal{E}_{X,+}\right) \oplus \left(\pi_X^*\widetilde{\mathcal{E}}_{Y,-} \otimes \mathcal{E}_{X,-}\right). \tag{3.83}$$

Then we obtain an isomorphism

$$f : \widetilde{\mathcal{E}}_{Z,+} \to \pi_X^*\widetilde{\mathcal{E}}_{Y,+} \otimes \mathcal{E}_X \tag{3.84}$$

by taking the $-\tilde{c}(\partial_r)$ action on the $\widetilde{\mathcal{E}}_{Y,-}$ part in (3.83). By (3.70) and (3.84), we have the isomorphism

$$\varphi : \pi_X^*\mathcal{E}_Y \otimes \mathcal{E}_X \xrightarrow{f_Y^{-1}\otimes Id} \pi_X^*\widetilde{\mathcal{E}}_{Y,+} \otimes \mathcal{E}_X \xrightarrow{f^{-1}} \widetilde{\mathcal{E}}_{Z,+} \xrightarrow{f_Z} \mathcal{E}_Z \tag{3.85}$$

and the Clifford action c on $\pi_X^*\mathcal{E}_Y \otimes \mathcal{E}_X$ is the same as c_Z on $\mathcal{E}_Z$. So we have

$$c = \varphi^{-1}\, c_Z\, \varphi. \tag{3.86}$$

Explicitly, we can calculate that

$$\varphi(\alpha\otimes\beta) = \frac{1}{2}\alpha\wedge(1+\tilde{c}(\Gamma_X))\beta + \frac{1}{2}\tilde{c}(\Gamma_Y)\alpha\wedge(1-\tilde{c}(\Gamma_X))\beta, \tag{3.87}$$

and

$$\varphi^{-1}(\alpha\wedge\beta) = \frac{1}{2}\alpha\otimes(1+\tilde{c}(\Gamma_X))\beta + \frac{1}{2}\tilde{c}(\Gamma_Y)\alpha\otimes(1-\tilde{c}(\Gamma_X))\beta. \tag{3.88}$$

From (3.86)-(3.88), since Γ_X and Γ_Y are parallel with respect to $\nabla^{\Lambda T^*Z}$, we know that φ commutes with $\nabla_U^{\Lambda T^*Z}$ for $U \in TZ$. Thus by (3.78),

$$D_Z^{\mathcal{E}} = \varphi^{-1}\, A_Z\, \varphi = \varphi^{-1}\, \tilde{c}(\Gamma_Z)(d_Z + d_Z^*)\, \varphi = \varphi^{-1}\left(\tilde{c}(\Gamma_Z)d_Z + d_Z\tilde{c}(\Gamma_Z)\right)\varphi, \tag{3.89}$$

where d_Z is the exterior differential operator along the fiber Z and d_Z^* is its dual. Following the same argument, we see that the equivariant eta form for the signature operator A_Z is the same as that for the Dirac operator $D_Z^{\mathcal{E}}$.

Set ${}^0\nabla^{\Lambda T^*Z} := \pi_X^*\nabla^{\Lambda T^*Y}\widehat{\otimes}1 + 1\widehat{\otimes}\nabla^{\Lambda T^*X}$. From (3.12),

$$\begin{aligned}\nabla_\cdot^{\Lambda T^*Z,T} = {}&{}^0\nabla_\cdot^{\Lambda T^*Z} - \langle \mathcal{S}_X(\cdot)f_{p,X}^H, e_i\rangle (f_{p,X}^H)^*\wedge i_{e_i} \\ &- \frac{1}{T^2}\langle \mathcal{S}_X(\cdot)f_{p,X}^H, f_{q,X}^H\rangle (f_{p,X}^H)^*\wedge (f_{q,X}^H)^*\wedge - \frac{1}{T^2}\langle \mathcal{S}_X(\cdot)e_i, f_{p,X}^H\rangle e^i\wedge (f_{q,X}^H)^*\wedge,\end{aligned} \tag{3.90}$$

where $(f_{p,X}^H)^*$ is the adjoint of $f_{p,X}^H$. As in (3.13) and (3.76), with respect to g_T^{TZ}, we denote by

$$\begin{aligned} c_Z(e_i) &= \left(\frac{1}{T}e^i\wedge - Ti_{e_i}\right)\tilde{c}(\Gamma_Z), \quad c_Z(f_p) = \left((f_{p,X}^H)^*\wedge - i_{f_{p,X}^H}\right)\tilde{c}(\Gamma_Z);\\ \hat{c}_Z(e_i) &= \left(\frac{1}{T}e^i\wedge + Ti_{e_i}\right)\tilde{c}(\Gamma_Z), \quad \hat{c}_Z(f_p) = \left((f_{p,X}^H)^*\wedge + i_{f_{p,X}^H}\right)\tilde{c}(\Gamma_Z).\end{aligned} \tag{3.91}$$

Since $\Gamma_Z^2 = 1$, by (3.90) and (3.91), we have

$$\begin{aligned}\nabla_\cdot^{\Lambda T^*Z,T} = {}&{}^0\nabla_\cdot^{\Lambda T^*Z} + \frac{1}{2T}\langle \mathcal{S}_X(\cdot)e_i, f_{p,X}^H\rangle (c_Z(e_i)c_Z(f_p) + \hat{c}_Z(e_i)\hat{c}_Z(f_p)) \\ &+ \frac{1}{4T^2}\langle \mathcal{S}_X(\cdot)f_{p,X}^H, f_{q,X}^H\rangle (c_Z(f_p)c_Z(f_q) + \hat{c}_Z(f_p)\hat{c}_Z(f_q)).\end{aligned} \tag{3.92}$$

By (3.86) and (3.92), since φ commutes with $\nabla_\cdot^{\Lambda T^*Z,T}$ and ${}^0\nabla_\cdot^{\Lambda T^*Z}$, we have

$$\begin{aligned}\nabla_\cdot^{\Lambda T^*Z,T} = {}&{}^0\nabla_\cdot^{\Lambda T^*Z} + \frac{1}{2T}\langle \mathcal{S}_X(\cdot)e_i, f_{p,X}^H\rangle (c(e_i)c(f_p) + \varphi^{-1}\hat{c}_Z(e_i)\hat{c}_Z(f_p)\varphi) \\ &+ \frac{1}{4T^2}\langle \mathcal{S}_X(\cdot)f_{p,X}^H, f_{q,X}^H\rangle (c(f_p)c(f_q) + \varphi^{-1}\hat{c}_Z(f_p)\hat{c}_Z(f_q)\varphi).\end{aligned} \tag{3.93}$$

Comparing with (3.13), we have

$$\begin{aligned}\mathcal{A}_T(U_1) &= \frac{1}{2T}\langle \mathcal{S}_X(U_1)e_i, f_{p,X}^H\rangle \varphi^{-1}\hat{c}_Z(e_i)\hat{c}_Z(f_p)\varphi \\ &\quad + \frac{1}{4T^2}\langle \mathcal{S}_X(U_1)f_{p,X}^H, f_{q,X}^H\rangle \varphi^{-1}\hat{c}_Z(f_p)\hat{c}_Z(f_q)\varphi, \qquad \text{for } U_1 \in TX;\\ \mathcal{A}_T(U_2) &= \frac{1}{2T}\langle \mathcal{S}_X(U_2)e_i, f_{p,X}^H\rangle \varphi^{-1}\hat{c}_Z(e_i)\hat{c}_Z(f_p)\varphi, \qquad \text{for } U_2 \in T^HZ;\\ \mathcal{A}_T(U_3) &= \frac{1}{2T}\langle \mathcal{S}_X(U_3)e_i, f_{p,X}^H\rangle \varphi^{-1}\hat{c}_Z(e_i)\hat{c}_Z(f_p)\varphi, \qquad \text{for } U_3 \in T_Z^HW.\end{aligned} \tag{3.94}$$

So the signature operator case satisfies the assumption in (3.17).

From (3.18) and (3.94), we have

$$D^{\mathcal{E}}_{Z,T} = TD^{\mathcal{E}_X}_X + c(f_p)\nabla^{\mathcal{E}_Y\otimes E_X}_{f_p} + \frac{1}{2}\langle \mathcal{S}_X(e_i)e_j, f^H_{p,X}\rangle c(e_i)\varphi^{-1}\hat{c}_Z(e_j)\hat{c}_Z(f_p)\varphi \tag{3.95}$$

$$+\frac{1}{4T}\langle \mathcal{S}_X(e_i)f^H_{p,X}, f^H_{q,X}\rangle\big(-c(e_i)c(f_p)c(f_q)+c(e_i)\varphi^{-1}\hat{c}_Z(f_p)\hat{c}_Z(f_q)\varphi+2c(f_p)\varphi^{-1}\hat{c}_Z(e_i)\hat{c}_Z(f_q)\varphi\big),$$

which is the same as [11, (4.6)], where the slightly difference comes from the different notations. By (3.22), in this case

$$\widetilde{D}_Y = D^{\mathcal{E}_Y\otimes \ker D_X}_Y + P\left(\frac{1}{2}\langle \mathcal{S}_X(e_i)e_j, f^H_{p,X}\rangle c(e_i)\varphi^{-1}\hat{c}_Z(e_j)\hat{c}_Z(f_p)\varphi\right)P, \tag{3.96}$$

which is the same as $A_B\otimes \ker A_Y$ in [11, P. 304].

Let $H^\bullet(X)$ be the graded vector bundles over V whose fiber over $v\in V$ is the cohomology $H^\bullet(X_v,\mathbb{C})$ of X_v. Let $H^\bullet(Z)$ be the graded vector bundles over S defined similarly. By (3.67), (3.89) and Hodge theory, there are isomorphisms of smooth graded G-equivariant vector bundles,

$$H^\bullet(X)\simeq \ker D^{\mathcal{E}_X}_X, \quad H^\bullet(Z)\simeq \ker D^{\mathcal{E}}_Z. \tag{3.97}$$

In particular, $\ker D^{\mathcal{E}_X}_X$ and $\ker D^{\mathcal{E}}_Z$ form G-equivariant complex vector bundles over V and S respectively. Let $d^*_{Z,T}$ be the adjoint of d_Z with respect to the metric g^{TZ}_T. Then by (3.89), we have

$$D^{\mathcal{E}}_{Z,T} = \varphi^{-1}\,\tilde{c}(\Gamma_Z)(d_Z + d^*_{Z,T})\,\varphi. \tag{3.98}$$

From (3.98) and the Hodge theory,

$$H^\bullet(Z)\simeq \ker(d_Z + d^*_{Z,T}) = \ker D^{\mathcal{E}}_{Z,T}. \tag{3.99}$$

So $\ker D^{\mathcal{E}}_{Z,T}$ is a vector bundle over $S\times(0,\infty)_T$.

Recall that E_X is the infinite dimensional bundle over V whose fiber at $v\in V$ is $C^\infty(X_v,\mathcal{E}_X|_{X_v})$. For $s\in C^\infty(V,E_X)$, and $U\in TV$, put (cf. [9, Definition 3.2])

$$\overline{\nabla}^{E_X}_U s := L_{U^H_X}s, \tag{3.100}$$

where $L_{U^H_X}$ is the Lie differentiation operator on $C^\infty(V,E_X)\simeq C^\infty(W,\mathcal{E}_X)$.

Restricted on fiber Z, by [9, Proposition 3.4], on $C^\infty(Y,E_X)$,

$$d^F_Z = d^F_X + \overline{\nabla}^{E_X} + i_{\mathcal{T}_X|_{TY}}, \tag{3.101}$$

where $i_{\mathcal{T}_X|_{TY}} = \frac{1}{2}\langle \mathcal{T}_X(f^H_{k,X}, f^H_{l,X}), e_i\rangle f^k\wedge f^l\wedge i_{e_i}$. We denote by d^H the extension of $\overline{\nabla}^{E_X}$ on $\Omega^\bullet(Y,E_X)=\Omega^\bullet(Z,F)$ by Leibniz's rule. Then on $\Omega^\bullet(Z,F)$, we have

$$d^F_Z = d^F_X + d^H + i_{\mathcal{T}_X|_{TY}}. \tag{3.102}$$

Let $d^{H,*}$ and $i^*_{\mathcal{T}_X|_{TY}}$ be the formal adjoints of d^H and $i_{\mathcal{T}_X|_{TY}}$ with respect to $\langle\cdot,\cdot\rangle_E$ in (2.12). Then by [21, (2.8), (5.28)], (3.21), (3.89), (3.101) and (3.102), we have

$$D_H = \varphi^{-1}\tilde{c}(\Gamma_Z)(d^H + d^{H,*})\varphi, \quad \mathcal{C} = \varphi^{-1}\tilde{c}(\Gamma_Z)(i_{\mathcal{T}_X|_{TY}} + i^*_{\mathcal{T}_X|_{TY}})\varphi. \tag{3.103}$$

So in this case, the vector bundles E_r in Definition 3.1 is isomorphic to E'_r in [21, (2.13)]. By [21, (2.9)], it is isomorphic to the Leray spectral sequence associated to the filtration in [21, (2.1)] on the filtered complex $(\Omega^\bullet(Z,F), d^F_Z)$, which are vector bundles over S by [21, Proposition 3.1]. Therefore, all assumptions in Assumption 3.4 are satisfied in this case.

By [9, Propositions 2.5 and 3.14],

$$\nabla^H := P\,\overline{\nabla}^{E_X}\,P \tag{3.104}$$

is a flat connection on $\ker D^{\mathcal{E}_X}_X$. In general, it is not unitary flat. Let d^H_Y be the induced exterior differentiation along the fiber Y associated with the flat connection ∇^H. Let $\nabla^{H,*}$ and $d^{H,*}_Y$ are the formal adjoints of ∇^H and d^H_Y with respect to h_1. From ([9, Proposition 3.14]), we have

$$P(d^H + d^{H,*})P = d^H_Y + d^{H,*}_Y. \tag{3.105}$$

From (3.76), (3.80) and (3.86), we have $\varphi\, D_X^{\mathcal{E}_X} = \tilde{c}(\Gamma_Z) D_X^{\mathcal{E}_X}\varphi$. So we see that

$$\varphi_Y := P\circ\varphi\circ P : \mathcal{E}_Y \otimes \ker D_X^{\mathcal{E}_X} \longrightarrow \mathcal{E}_Y\otimes \ker D_X^{\mathcal{E}_X} \tag{3.106}$$

is an isomorphism and $\varphi_Y^{-1} = P\circ\varphi^{-1}\circ P$. Since $D_X^{\mathcal{E}_X}$ commutes with $\tilde{c}(\Gamma_Z)$,

$$\tau_Y := P\,\tilde{c}(\Gamma_Z)P = P\,\tilde{c}(\Gamma_Z). \tag{3.107}$$

From (3.22), (3.78), (3.103) and (3.105)-(3.107), we have

$$\widetilde{D}_Y = \varphi_Y^{-1}\tau_Y(d_Y^H + d_Y^{H,*})\varphi_Y = \varphi_Y^{-1}(\tau_Y d_Y^H + d_Y^H\tau_Y)\varphi_Y. \tag{3.108}$$

which is the same as [11, (4.7)]. Note that $\widetilde{D}_Y^2 = \varphi_Y^{-1}(d_Y^H + d_Y^{H,*})^2\varphi_Y$. So if S is a point, from the equivariant index theorem for flat vector bundle, when $t\to 0$,

$$\mathrm{Tr}[\widetilde{D}_Y\exp(-t\widetilde{D}_Y^2)] = \mathrm{Tr}[\tau_Y(d_Y^H + d_Y^{H,*})\exp(-t(d_Y^H + d_Y^{H,*})^2)] = O(1). \tag{3.109}$$

So in this case, we generalize [11, Theorem 0.3] to the equivariant case. We believe that if S is a closed manifold, in this case, $\widetilde{\eta}_g'(\underline{\pi_Y}, \underline{\mathcal{E}_Y\otimes\ker D_X^{\mathcal{E}_X}})$ is well-defined. If it works, we can generalize [11, Theorem 0.3] to the equivariant family case.

If $\dim X$ and $\dim Y$ are all even, since we don't need to compare two different Clifford action, things are easier. In this case, $D_Z^{\mathcal{E}} = d_Z + d_Z^*$, $D_X^{\mathcal{E}_X} = d_X + d_X^*$ and

$$\widetilde{D}_Y = D_Y^{\mathcal{E}_Y\otimes\ker D_X} + P\left(\frac{1}{2}\langle \mathcal{S}_X(e_i)e_j, f_{p,X}^H\rangle c(e_i)\hat{c}(e_j)\hat{c}(f_p)\right)P = d_Y^H + d_Y^{H,*}. \tag{3.110}$$

Therefore from the equivariant family index theorem for flat vector bundles (equivariant version of [9, Theorem 3.15]), as in [15, (2.69)-(2.72)], $\widetilde{\eta}_g'(\underline{\pi_Y}, \underline{\mathcal{E}_Y\otimes\ker D_X^{\mathcal{E}_X}})$ is well-defined and Assumption 3.4 is satisfied. Thus Theorem 3.6 holds for this case.

4. The proof of Theorem 3.6

In this section, we will prove our main result Theorem 3.6. In Section 4.1, we define a second layer of adiabatic limit, then obtain a 1-form on $\mathbb{R}\times\mathbb{R}$. In Section 4.2, we follow the same strategy as in [15, §4] combined with the first author's work in [20] to prove the Theorem 3.6. There is one intermediate theorem we need to prove, which is left to the next section.

4.1. The fundamental form.

Take $\widehat{S} := \mathbb{R}_{+,T}\times\mathbb{R}_{+,u}\times S$, $\widehat{W} := \mathbb{R}_{+,T}\times\mathbb{R}_{+,u}\times W$ such that $\widehat{\pi}_Z : \widehat{W}\to\widehat{S}$ is identity on $\mathbb{R}_{+,T}\times\mathbb{R}_{+,u}$ and is π_Z on W. Let $\widehat{P}_W : \widehat{W}\to W$ be the natural projection and $\widehat{\mathcal{E}} := \widehat{P}_W^*\mathcal{E}$. Let $B_{u,T}$ be the rescaled version of B_T defined in (3.25) as in (2.16). Let $\widehat{B}$ be a superconnection defined by

$$\widehat{B}|_{(T,u)} = B_{u^2,T} + dT\wedge\frac{\partial}{\partial T} + du\wedge\frac{\partial}{\partial u}. \tag{4.1}$$

Then by the equivariant version of [2, Theorem 9.17],

$$d\,\widetilde{\mathrm{Tr}}\left[g\exp(-\widehat{B}^2)\right] = 0. \tag{4.2}$$

Definition 4.1. Let $\gamma := du\wedge\gamma^u + dT\wedge\gamma^T$ be the part of $\psi_{\widehat{S}}\widetilde{\mathrm{Tr}}[g\exp(-\widehat{B}^2)]$ of degree one with respect to the coordinate (T,u) with functions $\gamma^u,\gamma^T : \mathbb{R}_{+,T}\times\mathbb{R}_{+,u}\to\Omega^\bullet(S)$.

It follows from [15, Proposition 4.2] that there exists a smooth family $\alpha : \mathbb{R}_{+,T}\times\mathbb{R}_{+,u}\to\Omega^\bullet(S)$ such that

$$\left(du\wedge\frac{\partial}{\partial u} + dT\wedge\frac{\partial}{\partial T}\right)\gamma = dT\wedge du\wedge d_S\alpha. \tag{4.3}$$

We take $\varepsilon, A, T_0\in\mathbb{R}$ such that $0\le\varepsilon\le A<+\infty, 1\le T_0<+\infty$. Set $\Gamma = \Gamma_{\varepsilon,A,T_0}$, the contour in $\mathbb{R}_{+,T}\times\mathbb{R}_{+,u}$ with four parts Γ_1, Γ_2, Γ_3, Γ_4 and $\mathcal{U}$ the domain enclosed by Γ, as in the Figure 1.

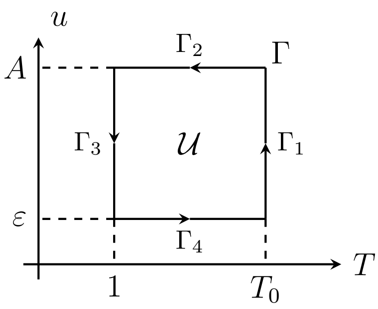

Figure 1. The contour Γ

We denote by $I_i^0 := \int_{\Gamma_i}\gamma$. Then by (4.3) and Stokes formula,

$$\sum_{i=1}^4 I_i^0 = \int_{\partial\mathcal{U}}\gamma = d_S\left(\int_{\mathcal{U}}\alpha\, dT\wedge du\right). \tag{4.4}$$

We take the limits $A\to+\infty, T_0\to+\infty$ and then $\varepsilon\to 0$ in the indicated order. Let I_i^k, $1\le i\le 4$, $1\le k\le 3$ denote the value of the part I_i^0 after taking the k-th limit if they exist.

4.2. **Intermediate results.** In this subsection, we use some intermediate results to prove our main result. With all these superconnections B_r, $r\ge 0$, we build the bridge between the form γ and equivariant eta forms $\widetilde{\eta}_g$ for E_r.

For $r=0$, we consider the fibration $\pi_X|_{V^g} := W|_{V^g}\to V^g$. Let $B_{0,t}$ be the rescaled Bismut superconnection associated with $(T_X^H(W|_{V^g}), g^{TX}, \nabla^{\mathcal{E}_X})$ as in (2.16). Set

$$\gamma_0(t) = \left\{\psi_{V^g\times\mathbb{R}}\widetilde{\mathrm{Tr}}\left[g\exp\left(-\left(B_{0,t^2}+dt\wedge\frac{\partial}{\partial t}\right)^2\right)\right]\right\}^{dt}. \tag{4.5}$$

Then by [15, (2.72), (2.77)] and [15, Definition 3.3],

$$\gamma_0(t) = \mathrm{O}(1), \text{ when } t\to 0, \quad \gamma_0(t)=\mathrm{O}(t^{-2}), \text{ when } t\to+\infty, \tag{4.6}$$

and

$$\widetilde{\eta}_g(\underline{\pi_X^g}, \underline{\mathcal{E}_X}) = -\int_0^{+\infty}\gamma_0(t)dt. \tag{4.7}$$

Note that γ_0 here is the same as γ' in [15, (2.83)], which is different from γ in [15, (2.78)] by changing the variable.

For $r=1$, we consider the fibration $\pi_Y: V\to S$. Let $B_{1,t}$ be the rescaled version of B_Y' defined in (3.27) as in (2.16). Set

$$\gamma_1(t) = \left\{\psi_{S\times\mathbb{R}}\widetilde{\mathrm{Tr}}\left[g\exp\left(-\left(B_{1,t^2}+dt\wedge\frac{\partial}{\partial t}\right)^2\right)\right]\right\}^{dt}. \tag{4.8}$$

Since $\ker\widetilde{D}_Y$ forms a vector bundle, similarly we have

$$\gamma_1(t) = \mathrm{O}(t^{-2}), \text{ when } t\to+\infty. \tag{4.9}$$

Then

$$\widetilde{\eta}_g'(\underline{\pi_Y}, \underline{\mathcal{E}_Y\otimes\ker D_X^{\mathcal{E}_X}}) = -\int_0^{+\infty}\gamma_1(t)dt. \tag{4.10}$$

For $r\ge 2$, set

$$B_{r,t} := \sqrt{t}D^r + \nabla^r \tag{4.11}$$

on E_r as in (2.27). Set

$$\gamma_r(t) = \left\{\psi_{S\times\mathbb{R}}\mathrm{Tr}_s\left[g\exp\left(-\left(B_{r,t^2}+dt\wedge\frac{\partial}{\partial t}\right)^2\right)\right]\right\}^{dt}. \tag{4.12}$$

By the equivariant version of [2, Theorem 9.7], we have

$$\gamma_r(t) = \mathrm{O}(1), \text{ when } t \to 0, \quad \gamma_r(t) = \mathrm{O}(t^{-2}), \text{ when } t \to +\infty. \tag{4.13}$$

By Definition 2.5,

$$\widetilde{\eta}_g(E_r, B_r) = -\int_0^{+\infty} \gamma_r(t)dt. \tag{4.14}$$

From Definition 4.1,

$$\gamma^u(T,u) = \left\{\psi_{S\times\mathbb{R}}\widetilde{\mathrm{Tr}}\left[g\exp\left(-\left(B_{u^2,T} + du\wedge\frac{\partial}{\partial u}\right)^2\right)\right]\right\}^{du}. \tag{4.15}$$

By [15, (2.72), (2.77)], for $T \geq 1$ fixed,

$$\gamma^u(T,u) = \mathrm{O}(1), \text{ when } u \to 0, \quad \gamma^u(T,u) = \mathrm{O}(u^{-2}), \text{ when } u \to +\infty. \tag{4.16}$$

From [15, Definition 3.3],

$$\widetilde{\eta}_g(\underline{\pi_Z}, \underline{\mathcal{E}}) = -\int_0^{+\infty} \gamma^u(1,u)du. \tag{4.17}$$

Since $\widehat{\mathrm{A}}_g(TX, \nabla^{TX})$ only depends on $g \in G$ and R^{TZ}, we can denote it by $\widehat{\mathrm{A}}_g(R^{TX})$. Let R_T^{TZ} be the curvature of $\nabla^{TZ,T}$. We define (cf. [15, (4.19)]),

$$\gamma_A(T) := -\left.\frac{\partial}{\partial s}\right|_{s=0} \widehat{\mathrm{A}}_g\left(R_T^{TZ} + s\frac{\partial\nabla^{TZ,T}}{\partial T}\right). \tag{4.18}$$

By [15, Proposition 4.5], when $T \to +\infty$, $\gamma_A(T) = \mathrm{O}(T^{-2})$, and modulo exact forms on W^g,

$$\widetilde{\widehat{\mathrm{A}}}_g(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}) = -\int_1^{+\infty} \gamma_A(T)dT. \tag{4.19}$$

Since (3.29) holds, we have the following theorem, the proof of which is the same as [15, Theorem 4.3 (i),(ii)] except for replacing D_Y by $\widetilde{D}_Y$.

Theorem 4.2. (1) *For any $u > 0$,*

$$\lim_{T\to+\infty} \gamma^u(T,u) = \gamma_1(u). \tag{4.20}$$

(2) *For fixed $0 < u_1 < u_2 < +\infty$, there exists $C > 0$ such that, for $u \in [u_1, u_2], T \geq 1$, we have*

$$|\gamma^u(T,u)| \leq C. \tag{4.21}$$

The following theorem is the analogue of [15, Theorem 4.6].

Theorem 4.3. (1) *For fixed $0 < u_1 < u_2 < +\infty$, there exist $\delta \in (0,1], C > 0$ and $T_0 \geq 1$, such that for any $u \in [u_1, u_2], T \geq T_0$, we have*

$$|\gamma^T(T,u)| \leq \frac{C}{T^{1+\delta}}. \tag{4.22}$$

(2) *For any $T \geq 0$, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\gamma^T(T\varepsilon^{-1}, \varepsilon) = \int_{Y^g} \widehat{\mathrm{A}}_g(TY, \nabla^{TY})\mathrm{ch}_g(\mathcal{E}_Y/S, \nabla^{\mathcal{E}_Y})\gamma_0(T). \tag{4.23}$$

(3) *There exists $C > 0$, such that for $\varepsilon \in (0,1]$, $\varepsilon \leq T \leq 1$,*

$$\varepsilon^{-1}\left|\gamma^T(T\varepsilon^{-1}, \varepsilon) + \int_{Z^g} \gamma_A(T\varepsilon^{-1})\mathrm{ch}_g(\mathcal{E}/S, \nabla^{\mathcal{E},T/\varepsilon})\right| \leq C. \tag{4.24}$$

(4) *There exist $\delta \in (0,1], C > 0$ such that for $\varepsilon \in (0,1]$, $T \geq 1$,*

$$\varepsilon^{-1}|\gamma^T(T\varepsilon^{-1}, \varepsilon)| \leq \frac{C}{T^{1+\delta}}. \tag{4.25}$$

Proof. We only need to point out the parts that are different from the proof of [15, Theorem 4.6]. Set

$$\mathcal{B}'_{\varepsilon,T/\varepsilon} = B^2_{\varepsilon^2,T/\varepsilon} + \varepsilon^{-1} dT \wedge \left.\frac{\partial B_{\varepsilon^2,T'}}{\partial T'}\right|_{T'=T\varepsilon^{-1}}. \tag{4.26}$$

By [15, (7.2)], we have

$$\varepsilon^{-1}\gamma^T(T/\varepsilon,\varepsilon) = \left\{\psi_S \widetilde{\mathrm{Tr}}[g\exp(-\mathcal{B}'_{\varepsilon,T/\varepsilon})]\right\}^{dT}. \tag{4.27}$$

Compare with [15, (7.31)], after a careful calculation, we have

$$\begin{aligned}
\mathcal{B}'_{\varepsilon,T/\varepsilon} = &-\Big(T\,{}^0\nabla^{\mathcal{E}}_{e_i} + \frac{\varepsilon}{2}\langle \mathcal{S}_X(e_i)e_j, f^H_{p,X}\rangle c(e_j)c(f_p) + \varepsilon Q_{X,1}(e_i) + \frac{\varepsilon^2}{T}Q_{X,2}(e_i)\\
&+\frac{\varepsilon^2}{4T}\langle \mathcal{S}_X(e_i)f^H_{p,X}, f^H_{q,X}\rangle c(f_p)c(f_q) + \frac{1}{2}\langle \mathcal{S}_3(e_i)e_j, k^H_{\alpha,Z}\rangle c(e_j)k^\alpha\wedge\\
&+\frac{\varepsilon}{2T}\langle \mathcal{S}_Z(e_i)f^H_{p,X}, k^H_{\alpha,Z}\rangle c(f_p)k^\alpha\wedge + \frac{1}{4T}\langle \mathcal{S}_Z(e_i)k^H_{\alpha,Z}, k^H_{\beta,Z}\rangle k^\alpha\wedge k^\beta\wedge\Big)^2\\
&+dT\wedge\Big(c(e_i)\,{}^0\nabla^{\mathcal{E}}_{e_i} - \frac{1}{8T^2}\big(\varepsilon^2\langle[f^H_{p,X}, f^H_{q,X}], e_i\rangle c(e_i)c(f_p)c(f_q)\\
&\quad +4\varepsilon\langle \mathcal{S}_X(k^H_{\alpha,Z})e_i, f^H_{p,X}\rangle c(e_i)c(f_p)k^\alpha\wedge + \langle[k^H_{\alpha,Z}, k^H_{\beta,Z}], e_i\rangle c(e_i)k^\alpha\wedge k^\beta\wedge\big)\\
&\quad -\frac{\varepsilon^2}{T^2}(c(e_i)Q_{X,2}(e_i) + c(f_p)Q_Y(f^H_{p,X})) + \varepsilon^{-1}k^\alpha\wedge \left.\frac{\partial \mathcal{A}_{T'}(k^H_{\alpha,Z})}{\partial T'}\right|_{T'=T\varepsilon^{-1}}\Big)\\
&-\varepsilon^2\Big({}^0\nabla^{\mathcal{E}}_{f^H_{p,X}} + \frac{\varepsilon}{2T}\langle \mathcal{S}_X(f^H_{p,X})e_i, f^H_{q,X}\rangle c(e_i)c(f_q) + \frac{1}{2T}\langle \mathcal{S}_Z(f^H_{p,X})e_i, k^H_{\alpha,Z}\rangle c(e_i)k^\alpha\wedge\\
&+\frac{1}{2\varepsilon}\langle \mathcal{S}_Y(f_p)f_q, k^H_{\alpha,Y}\rangle c(f_q)k^\alpha\wedge + \frac{1}{4\varepsilon^2}\langle \mathcal{S}_Y(f_p)k^H_{\alpha,Y}, k^H_{\beta,Y}\rangle k^\alpha\wedge k^\beta\wedge + \frac{\varepsilon}{T}Q_Y(f^H_{p,X})\Big)^2\\
&+\frac{\varepsilon^2}{4}K^Z_{T/\varepsilon} + \frac{T^2}{2}R^{\mathcal{E}/S}_{T/\varepsilon}(e_i,e_j)c(e_i)c(e_j) + T\varepsilon R^{\mathcal{E}/S}_{T/\varepsilon}(e_i, f^H_{p,1})c(e_i)c(f^H_{p,1})\\
&+\frac{\varepsilon^2}{2}R^{\mathcal{E}/S}_{T/\varepsilon}(f^H_{p,1}, f^H_{q,1})c(f^H_{p,1})c(f^H_{q,1}) + \frac{1}{2}R^{\mathcal{E}/S}_{T/\varepsilon}(k^H_{\alpha,3}, k^H_{\beta,3})k^\alpha\wedge k^\beta\wedge\\
&+\varepsilon R^{\mathcal{E}/S}_{T/\varepsilon}(f^H_{p,1}, k^H_{\alpha,3})c(f^H_{p,1})k^\alpha\wedge + T R^{\mathcal{E}/S}_{T/\varepsilon}(e_i, k^H_{\alpha,3})c(e_i)k^{\alpha,H}_3\wedge,
\end{aligned} \tag{4.28}$$

where K^Z_T is the scalar curvature for g^{TZ}_T and $R^{\mathcal{E}/S}_T$ is the twisting curvature (see [2, Proposition 3.43]) associated with $\nabla^{\mathcal{E},T}$. Note that from the assumption in (3.17), the differences between (4.28) and [15, (7.31)] are uniformly bounded for small ε. So all estimates in [15, Section 7] hold in this case.

From (3.13) and (3.17), we have

$$\lim_{\varepsilon\to 0} R^{\mathcal{E}/S}_{T/\varepsilon} = \pi^*_X R^{\mathcal{E}_Y/S} + R^{\mathcal{E}_X/S}, \tag{4.29}$$

where $R^{\mathcal{E}_Y/S}$ and $R^{\mathcal{E}_X/S}$ are twisting curvatures associated with $\nabla^{\mathcal{E}_Y}$ and $\nabla^{\mathcal{E}_X}$ respectively. Let $f_1, \cdots, f_{l'}$ be a locally frame of TY^g. Compare with [15, (7.58)], by (4.28) and (4.29),

$$
\begin{aligned}
\lim_{\varepsilon\to 0} L^3_{\varepsilon,T} = &-\Bigg(T\,{}^0\nabla^{\mathcal{E}}_{e_i} + \frac{1}{2}\sum_{1\le p\le l'}\langle \mathcal{S}_X(e_i)e_j, f^H_{p,X}\rangle c(e_j) f^p\wedge \\
&+\frac{1}{4T}\sum_{1\le p,q\le l'}\langle \mathcal{S}_X(e_i)f^H_{p,X}, f^H_{q,X}\rangle f^p\wedge f^q\wedge + \frac{1}{2}\langle \mathcal{S}_Z(e_i)e_j, k^H_{\alpha,Z}\rangle c(e_j)k^\alpha\wedge \\
&+\frac{1}{2T}\sum_{1\le p\le l'}\langle \mathcal{S}_Z(e_i)f^H_{p,X}, k^H_{\alpha,Z}\rangle f^p\wedge k^\alpha\wedge + \frac{1}{4T}\langle \mathcal{S}_Z(e_i)k^H_{\alpha,Z}, k^H_{\beta,Z}\rangle k^\alpha\wedge k^\beta\wedge\Bigg)^2 \\
&+dT\wedge\Bigg(D^X - \frac{1}{8T^2}\Bigg(\sum_{1\le p,q\le l'}\langle [f^H_{p,X}, f^H_{q,X}], e_i\rangle c(e_i) f^p\wedge f^q\wedge \\
&\quad +4\sum_{1\le p\le l'}\langle \mathcal{S}_X(k^H_{\alpha,Z})e_i, f^H_{p,X}\rangle c(e_i) f^p\wedge k^\alpha\wedge + \langle [k^H_{\alpha,Z}, k^H_{\beta,Z}], e_i\rangle c(e_i) k^\alpha\wedge k^\beta\wedge\Bigg)\Bigg) \\
&-\Bigg(\partial_p + \frac{1}{4}\sum_{1\le q,r\le l'}\langle R^{TY}(U,f_p)f_q, f_r\rangle f^q\wedge f^r\wedge \\
&+\frac{1}{4}\langle R^{TY}(U,f_p)k^H_{\alpha,Y}, k^H_{\beta,Y}\rangle k^\alpha\wedge k^\beta\wedge + \frac{1}{2}\sum_{1\le q\le l'}\langle R^{TY}(U,f_p)f_q, k^H_{\alpha,Y}\rangle f^q\wedge k^\alpha\wedge\Bigg)^2 \\
&+\frac{T^2}{4}K^X + \frac{T^2}{2}R^{\mathcal{E}_X/S}(e_i,e_j)c(e_i)c(e_j) + T\sum_{1\le p\le l'} R^{\mathcal{E}_X/S}(e_i, f^H_{p,X})c(e_i)f^p\wedge \\
&+TR^{\mathcal{E}_X/S}(e_i, k^H_{\alpha,Z})c(e_i)k^\alpha\wedge + \frac{1}{2}\sum_{1\le p,q\le l'}(\pi_X^* R^{\mathcal{E}_Y/S} + R^{\mathcal{E}_X/S})(f^H_{p,1}, f^H_{q,1})f^p\wedge f^q\wedge \\
&+\sum_{1\le p\le l'}(\pi_X^* R^{\mathcal{E}_Y/S} + R^{\mathcal{E}_X/S})(f^H_{p,X}, k^H_{\alpha,Z})f^p\wedge k^\alpha\wedge \\
&+\frac{1}{2}(\pi_X^* R^{\mathcal{E}_Y/S} + R^{\mathcal{E}_X/S})(k^H_{\alpha,Z}, k^H_{\beta,Z})k^\alpha\wedge k^\beta\wedge.
\end{aligned}
\tag{4.30}
$$

Notice that in (4.30), the terms from $\mathcal{A}_T$ do not appear here. So we obtain Theorem 4.3 (2) in the same way as in [15, Section 7].

By (3.29) and (4.28), the proofs of Theorem 4.3 (1), (3) and (4) are the same as those in [15]. □

Compared with [15, Theorem 4.4], by [15, (4.5)], (3.23), (3.58) and the equivariant version of [2, Theorem 9.19], we have

Theorem 4.4. *For $T \ge 1$,*

$$
\lim_{u\to+\infty}\gamma^T(T,u) = \left\{\psi_{S\times\mathbb{R}}\widetilde{\mathrm{Tr}}\left[g\exp\left(-\left(\nabla^{\ker D_{Z,T}} + dT\wedge\frac{\partial}{\partial T}\right)^2\right)\right]\right\}^{dT}. \tag{4.31}
$$

The last section will be devoted to prove the following theorem, which is the analogue of [15, Theorem 4.3 (iii)].

Theorem 4.5. *We have the following identity:*

$$
\lim_{T\to+\infty}\int_1^{+\infty}\gamma^u(T,u)du = \int_1^{+\infty}\gamma_1(u)du - \sum_{r=2}^{r_0}\widetilde{\eta}_g(E_r,B_r). \tag{4.32}
$$

By Theorems 4.2-4.5, we can calculate I_i^3 to prove the main result Theorem 3.6. From the same approach in [15, §4.3], we have

$$I_1^3 = -\widetilde{\eta}_g'(\underline{\pi_Y}, \underline{\mathcal{E}_Y \otimes \ker D_X^{\mathcal{E}_X}}) - \sum_{r=2}^{r_0} \widetilde{\eta}_g(E_r, B_r), \tag{4.33}$$

$$I_3^3 = \widetilde{\eta}_g(\underline{\pi_Z}, \underline{\mathcal{E}}), \tag{4.34}$$

and

$$\begin{aligned} I_4^3 = & -\int_{Y^g} \widehat{\mathrm{A}}_g(TY, \nabla^{TY}) \mathrm{ch}_g(\mathcal{E}_Y/S, \nabla^{\mathcal{E}_Y}) \widetilde{\eta}_g(\underline{\pi_X^g}, \underline{\mathcal{E}_X}) \\ & + \int_{Z^g} \widetilde{\widehat{\mathrm{A}}}_g(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}) \mathrm{ch}_g(\mathcal{E}/S, \nabla^{\mathcal{E}}) + \int_{Z^g} \widehat{\mathrm{A}}_g(TZ, {}^0\nabla^{TZ}) \widetilde{\mathrm{ch}}_g(\mathcal{E}/S, \nabla^{\mathcal{E}}, {}^0\nabla^{\mathcal{E}}). \end{aligned} \tag{4.35}$$

By Theorem 4.4 and (3.59),

$$\begin{aligned} I_2^3 &= -\lim_{A\to+\infty} \lim_{u\to+\infty} \int_1^A \gamma^T(T,u) dT \\ &= -\lim_{A\to+\infty} \int_1^A \left\{ \psi_{S\times\mathbb{R}} \widetilde{\mathrm{Tr}} \left[g \exp\left(-\left(\nabla^{\ker D_{Z,T}} + dT \wedge \frac{\partial}{\partial T} \right)^2 \right) \right] \right\}^{dT} dT \\ &= -\lim_{A\to+\infty} \widetilde{\mathrm{ch}}_g(\ker D_Z^{\mathcal{E}}, \nabla^{\ker D_{Z,A}}, \nabla^{\ker D_Z}) \\ &= -\widetilde{\mathrm{ch}}_g(\ker D_Z^{\mathcal{E}}, \nabla^{\infty}, \nabla^{\ker D_Z}). \end{aligned} \tag{4.36}$$

By (4.4) and [13, §22, Theorem 17], we have

$$\sum_{i=1}^4 I_i^3 \equiv 0 \mod d\Omega^{\bullet}(S). \tag{4.37}$$

Therefore we complete the proof of Theorem 3.6.

5. The Proof of Theorem 4.5

The purpose of this section is to prove Theorem 4.5. In Section 5.1, we analyze the resolvents of Dirac operators $D_{Z,T}^{\mathcal{E}}$ and D_r to establish the relations. In Section 5.2, we follow what Ma has done in [20] for the functoriality of holomorphic analytic torsions to give the proof.

5.1. Limits of resolvent.

For $b \in S$, we set

$$\mathbb{E}_{0,b} := C^{\infty}(Z_b, \pi_Z^* \Lambda(T^*S) \widehat{\otimes} \mathcal{E}). \tag{5.1}$$

For $\mu \in \mathbb{R}$, we will make use of geometric structures to define Sobolev space $\mathbb{E}_{0,b}^{\mu}$ of order μ.

Taking $c > 0$, for $r \geq 2$, set

$$U_r := \left\{ \lambda \in \mathbb{C} : \inf_{\mu \in \mathrm{Sp}(D_r)} |\lambda - \mu| \geq c \right\}. \tag{5.2}$$

The proof of the following theorem parallels that of [3, Theorem 6.2].

Theorem 5.1. *For $r \geq 2$, $\lambda \in U_r$, there exist linear maps*

$$\psi_{r,\lambda} : \mathbb{E}_0 \to (\mathbb{E}_0)^{r+1}, \tag{5.3}$$

such that for $s \in \mathbb{E}_0$, we write $\psi_{r,\lambda}(s) = (s_0, \cdots, s_r)$, which satisfies

$$\begin{aligned} D_X^{\mathcal{E}_X} s_0 &= 0, \\ D_H s_0 + D_X^{\mathcal{E}_X} s_1 &= 0, \\ &\vdots \\ D_X^{\mathcal{E}_X} s_{r-1} + D_H s_{r-2} + \mathcal{C} s_{r-3} &= 0, \\ -D_X^{\mathcal{E}_X} s_r - D_H s_{r-1} - \mathcal{C} s_{r-2} + \lambda s_0 &= s. \end{aligned} \tag{5.4}$$

And for any $\mu \in \mathbb{R}$, $\psi_{r,\lambda}$ can be extended to a bounded linear map from $\mathbb{E}_0^\mu$ to $(\mathbb{E}_0^\mu)^{r+1}$. Moreover, we have $s_0 \in E_r$, which can be given by

$$s_0 = (\lambda - D_r)^{-1} p_r s. \tag{5.5}$$

Let α_T, $T \in [1, +\infty]$ be a family of tensors or differential operators. We denote by $a^{(i)}$ the derivative of $\alpha_T - \alpha_\infty$ of order i. Here we only consider the spatial derivatives, excluding the T-derivative. If for any $p \in \mathbb{N}$, there exists $C > 0$ such that when $T \geq 1$, $\sup \|a^{(i)}\| \leq C/T^k, i = 0, 1, \cdots, p$, we write $\alpha_T = \alpha_\infty + \mathrm{O}\left(\frac{1}{T^k}\right)$.

We take c_1, c_2 such that

$$\bigcup_{b\in S}\bigcup_{r\geq 2} \mathrm{Sp}(D_{r,b}^{2,>0}) \subseteq (c_1, c_2), \quad (0, 2c_1) \cap \bigcup_{b\in S} \mathrm{Sp}(\widetilde{D}_{Y_b}^2) = \emptyset. \tag{5.6}$$

Set

$$U_0 := \left\{ \lambda \in \mathbb{C} : \frac{\sqrt{c_1}}{2} \leq |\lambda| \leq \sqrt{c_1}, \text{ or } \sqrt{c_2} \leq |\lambda| \leq 2\sqrt{c_2} \right\}. \tag{5.7}$$

Comparing with [3, Theorem 6.5], by Assumption 3.4, we have the following relation of resolvents of $D_{Z,T}^{\mathcal{E}}$ and D_r. The proof follows closely [3, Theorem 6.5]. We write it here for the completeness.

Theorem 5.2. *Given $r \geq 2$, for $\lambda \in U_0$, $s \in \mathbb{E}_0^0$, there exists $C > 0$, such that when $T \to +\infty$,*

$$\|(\lambda - T^{r-1} D_{Z,T}^{\mathcal{E}})^{-1} s - p_r (\lambda - D_r)^{-1} p_r s\| \leq \frac{C}{T} \|s\|. \tag{5.8}$$

Proof. Set

$$\begin{aligned} M_{r,T} : (\mathbb{E}_0)^{r+1} &\to \mathbb{E}_0, \\ (s_0, \cdots, s_r) &\mapsto s_0 + \frac{s_1}{T} + \cdots + \frac{s_r}{T^r}. \end{aligned} \tag{5.9}$$

Define

$$N_{r,T} = M_{r,T} \circ \psi_{r,\lambda}, \tag{5.10}$$

where $\psi_{r,\lambda}$ is defined in (5.3). By (5.4), we get

$$\begin{aligned} (\lambda - T^{r-1} D_{Z,T}^{\mathcal{E}}) N_{r,T} s &= (\lambda - T^r D_X^{\mathcal{E}_X} - T^{r-1} D_H - T^{r-2} \mathcal{C}) \left(s_0 + \frac{s_1}{T} + \cdots + \frac{s_r}{T^r} \right) \\ &= s + \lambda \left(\frac{s_1}{T} + \cdots + \frac{s_r}{T^r} \right) - \frac{1}{T} D_H s_r - \frac{1}{T} \mathcal{C} s_{r-1} - \frac{1}{T^2} \mathcal{C} s_r. \end{aligned} \tag{5.11}$$

Hence

$$\begin{aligned} (\lambda - T^{r-1} D_{Z,T}^{\mathcal{E}})^{-1} s &= N_{r,T} s + \\ &(\lambda - T^{r-1} D_{Z,T}^{\mathcal{E}})^{-1} \left(-\lambda \left(\frac{s_1}{T} + \cdots + \frac{s_r}{T^r} \right) + \frac{1}{T} D_H s_r + \frac{1}{T} \mathcal{C} s_{r-1} + \frac{1}{T^2} \mathcal{C} s_r \right). \end{aligned} \tag{5.12}$$

Note that the resolvent of $D_{Z,T}^{\mathcal{E}}$ satisfies the same estimate as [3, Theorem 5.28]. So for $T \gg 1$,

$$\bigcup_{b\in S} \mathrm{Sp}(D_{Z,T}^{\mathcal{E},2}|_{Z_b}) \cap [0, c_1] \subset [0, c_1/4). \tag{5.13}$$

Let $\mathbf{E}_{T,b}^{[0,a]}$, $a > 0$, be the eigenspaces of $D_{Z,T}^{\mathcal{E},2}|_{Z_b}$ associated with eigenvalues $\lambda \in [0, a]$. Then by (5.13), $\mathbf{E}_T^{[0,c_1/4]} = \bigcup_{b\in S} \mathbf{E}_{T,b}^{[0,c_1/4]}$ is a finite dimensional vector bundle over S. Let $P_T^{[0,c_1/4]}$ be the orthogonal projection onto $\mathbf{E}_T^{[0,c_1/4]}$. From the same estimate as [3, Theorem 5.28], when $T \to \infty$,

$$P_T^{[0,c_1/4]} \to p_2. \tag{5.14}$$

For $\lambda \in \mathbb{C}$, if $\mathrm{Im}(\lambda) \neq 0$, $\|(\lambda - T^{r-1} D^{\mathcal{E}}_{Z,T})^{-1}\|$ is uniformly bounded. From (5.12) and (5.14), if $\mathrm{Im}(\lambda) \neq 0$, when $T \to \infty$, in $\mathbb{E}^0_0$,

$$P_T^{[0,c_1/4]}(\lambda - T^{r-1} D^{\mathcal{E}}_{Z,T})^{-1} P_T^{[0,c_1/4]} s \to p_2 p_r (\lambda - D_r)^{-1} p_r p_2 s = p_r(\lambda - D_r)^{-1} p_r s. \tag{5.15}$$

Take $\lambda_T \in \mathrm{Sp}(D^{\mathcal{E}}_{Z,T}|_{Z_b}) \cap [-\sqrt{c_1}/2, \sqrt{c_1}/2]$. Then by [11, Theorem 1.5], λ_T is analytic in T. From (5.14), when $T \to \infty$, λ_T can be written as $\lambda_T \sim a_1 T^{-1} + a_2 T^{-2} + \cdots$. From (5.15), for $r \geq 2$, the eigenvalues of $D^{\mathcal{E}}_{Z,T}|_{Z_b}$ which are $\mathrm{O}(T^{1-r})$ can be put in one to one correspondence with the corresponding eigenvalues of $D_{r,b}$, i.e.,

$$\lambda_T \sim a_{r-1} T^{-(r-1)} + a_r T^{-r} + \cdots \in \mathrm{Sp}(D^{\mathcal{E}}_{Z,T}|_{Z_b}) \tag{5.16}$$

corresponds to $a_{r-1} \in \mathrm{Sp}(D_{r,b})$. Therefore, we can naturally obtain (5.8).

□

Theorem 5.2 is essential for our analysis. It tells us that all eigenvalues of $D^{\mathcal{E}}_{Z,T}$ which satisfy $\mathrm{O}\left(1/T^{r-1}\right)$ obey $\frac{1}{T^{r-1}}\left(\mathrm{Sp}(D_r) + \mathrm{O}(1/T)\right)$. We depict the contours δ_0, Δ_0 in Figure 2:

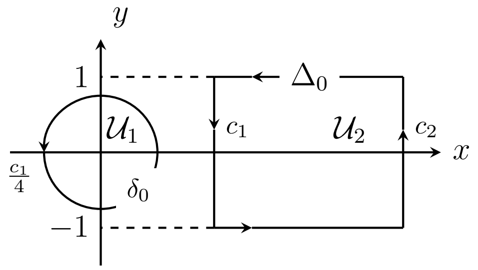

FIGURE 2. Contours δ_0, Δ_0

From the correspondence in (5.16), we know that when $T \gg 1$,

$$\mathrm{Sp}(D^{\mathcal{E},2}_{Z,T}) \cap [0, 2c_1] \subseteq \frac{\mathcal{U}_1}{T^{2(r_0-1)}} \cup \bigcup_{r=2}^{r_0} \frac{\mathcal{U}_2}{T^{2(r-1)}}. \tag{5.17}$$

In the following, we give the proof of Lemma 3.3.

Proof of Lemma 3.3. Set

$$\begin{aligned} P_{r,T} &:= \frac{1}{2\pi\sqrt{-1}} \int_{\{\lambda \in \mathbb{C}: |\lambda| = \sqrt{c_1}\}} (\lambda - T^{r-1} D^{\mathcal{E}}_{Z,T})^{-1} d\lambda \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\{\lambda \in \mathbb{C}: |\lambda| = \frac{\sqrt{c_1}}{T^{r-1}}\}} (\lambda - D^{\mathcal{E}}_{Z,T})^{-1} d\lambda. \end{aligned} \tag{5.18}$$

By Theorem 3.2 and (5.6),

$$P_r := \frac{1}{2\pi\sqrt{-1}} \int_{\{\lambda \in \mathbb{C}: |\lambda| = \sqrt{c_1}\}} p_r (\lambda - D_r)^{-1} p_r d\lambda = p_{r+1}. \tag{5.19}$$

Recall that r_0 is the index from which E_r converges. Hence $P_{r_0-1} = P_{r_0} = \cdots = p_\infty$.

By Theorem 5.2, when $T \gg 1$,

$$\|P_{r_0,T} s - P_{r_0} s\| \leq \frac{C}{T} \|s\|. \tag{5.20}$$

By (3.39), for $r \geq r_0$, $D_r = 0$. From the correspondence in (5.16), we see that for T large enough, $D^{\mathcal{E}}_{Z,T}$ has no eigenvalue in $(-\sqrt{c_1} T^{1-r_0}, \sqrt{c_1} T^{1-r_0})$ other than the eigenvalue 0. So from (5.18), we have $P_{r_0,T} = p^T$. We obtain (3.56).

By Assumption 3.4, $\dim\ker D^{\mathcal{E}}_{Z,T}$ is independent of T. According to (5.20), $\dim \mathrm{Im} P_{r_0} = \dim\ker D^{\mathcal{E}}_{Z,T}$. So $\dim\ker D^{\mathcal{E}}_{Z,\infty} = \dim E_\infty$. Under Assumption 3.4, $\ker D^{\mathcal{E}}_{Z,T} \cong \ker D^{\mathcal{E}}_Z$ are isomorphic vector bundles. Hence $\ker D^{\mathcal{E}}_Z \cong E_\infty$.

The proof of Lemma 3.3 is complete. □

In the following proposition, we show that Theorem 3.6 really extends Dai's adiabatic limit formula to the family case. Recall that by [11, Theorem 1.5], along the fiber Z, the spectrum λ of $D^{\mathcal{E}}_{Z,T}$ is analytic in T.

Proposition 5.3. If S is a point, set

$$A_r := \left\{ \lambda \in \mathrm{Sp}(D^{\mathcal{E}}_{Z,T}) : \lambda = \mathrm{O}\left(\frac{1}{T^{r-1}}\right) \right\}. \tag{5.21}$$

When $T \to +\infty$, for unity $e \in G$,

$$\widetilde{\eta}_e(E_r, B_r) = \sum_{\lambda \in A_r / A_{r+1}} \mathrm{sgn}(\lambda). \tag{5.22}$$

Proof. By Definition 2.5, when S is a point,

$$\widetilde{\eta}_e(E_r, B_r) = \int_0^{+\infty} \widetilde{\psi}_{\{\mathrm{pt}\}} \mathrm{Tr}_s \left[\frac{D_r}{2\sqrt{t}} \exp(-tD_r^2) \right] dt. \tag{5.23}$$

By Gauss integral, as

$$\frac{1}{\sqrt{\pi}} \int_0^{+\infty} \lambda \mathrm{e}^{-t\lambda^2} \frac{dt}{2\sqrt{t}} = \mathrm{sgn}(\lambda),$$

we have

$$\widetilde{\eta}_e(E_r, B_r) = \sum_{\lambda \in \mathrm{Sp}(D_r)} \mathrm{sgn}(\lambda).$$

According to Theorem 5.2, we see that $\widetilde{\eta}_e(E_r, B_r) = \sum_{\lambda \in A_r/A_{r+1}} \mathrm{sgn}(\lambda)$. Note that in (1.3), $A = A_2$. So $\sum \widetilde{\eta}_e(E_r, B_r) = R$ in (1.3). □

5.2. **Proof of Theorem 4.5.** We start from the definitions of $\gamma^u(T, u)$ and $\gamma_r(u)$. Set

$$\begin{aligned} \mathcal{B}_{u,T} &:= B^2_{u^2,T} + du \wedge \frac{\partial B_{u^2,T}}{\partial u}, \\ \mathcal{B}_1 &:= (B'_Y)^2 + du \wedge \delta^{-1}_{u^2} \frac{\partial B_{1,u^2}}{\partial u} \delta_{u^2}, \\ \mathcal{B}_r &:= B^2_r + du \wedge \delta^{-1}_{u^2} \frac{\partial B_{r,u^2}}{\partial u} \delta_{u^2}, \quad r \geq 2, \\ \mathcal{B}_{r,u,T} &:= \mathcal{B}_{T^{r-1}u,T}. \end{aligned} \tag{5.24}$$

Denote by $\mathcal{T}_{Z,T}$ the torsion of $\pi_Z^* \nabla^{TS} \oplus \nabla^{TZ,T}$. By (2.16) and (5.24),

$$\begin{aligned} \mathcal{B}_{u,T} &= u^2 \delta_{u^2} B^2_T \delta^{-1}_{u^2} + du \wedge \left(D_{Z,T} + \frac{c(\mathcal{T}_{Z,T})}{4u^2} \right) \\ &= u^2 \delta_{u^2} \left(B^2_T + u^{-2} du \wedge \left(D_{Z,T} + \frac{c(\mathcal{T}_{Z,T})}{4} \right) \right) \delta^{-1}_{u^2}. \end{aligned} \tag{5.25}$$

Note that $\mathcal{B}_{1,T} = B^2_T + du \wedge (D_{Z,T} + c(\mathcal{T}_{Z,T})/4)$. Then by Definition 4.1, (4.8), (4.12) and (5.25),

$$\begin{aligned} \gamma^u(T, u) &= \left\{ \psi_{S \times \mathbb{R}} \widetilde{\mathrm{Tr}}[g \exp(-\mathcal{B}_{u,T})] \right\}^{du} \\ &= \left\{ \psi_{S \times \mathbb{R}} \delta_{u^2} \widetilde{\mathrm{Tr}} \left[g \exp\left(-u^2 \left(B^2_T + u^{-2} du \wedge \left(D_{Z,T} + \frac{c(\mathcal{T}_{Z,T})}{4} \right) \right) \right) \right] \right\}^{du} \\ &= \left\{ u^{-2} \psi_{S \times \mathbb{R}} \delta_{u^2} \widetilde{\mathrm{Tr}}[g \exp(-u^2 \mathcal{B}_{1,T})] \right\}^{du}, \\ \gamma_r(u) &= \left\{ u^{-2} \psi_{S \times \mathbb{R}} \delta_{u^2} \widetilde{\mathrm{Tr}}[g \exp(-u^2 \mathcal{B}_r)] \right\}^{du}, \quad r \geq 1. \end{aligned} \tag{5.26}$$

The proof of the following theorem is similar to [5, Theorem 9.2] (see also [15, Lemma 5.8]).

Theorem 5.4. *For $u > 0$, $T \geq 1$, we have*

$$\begin{aligned} &\mathrm{Sp}(\mathcal{B}_{u,T}) = \mathrm{Sp}(u^2 \mathcal{B}_{1,T}) = \mathrm{Sp}(u^2 D^{\mathcal{E},2}_{Z,T}), \quad \mathrm{Sp}(\mathcal{B}_{r,u,T}) = \mathrm{Sp}(T^{2(r-1)} u^2 D^{\mathcal{E},2}_{Z,T}), \\ &\mathrm{Sp}(\mathcal{B}_1) = \mathrm{Sp}(\widetilde{D}^2_Y), \quad \mathrm{Sp}(\mathcal{B}_r) = \mathrm{Sp}(D^2_r), \quad r \geq 2. \end{aligned} \tag{5.27}$$

Set

$$
(5.28)\quad
\begin{aligned}
F_{r,u,T} &:= u^{-2}\psi_{S\times\mathbb{R}}\delta_{u^2}\widetilde{\mathrm{Tr}}\left[g\cdot\frac{1}{2\pi\sqrt{-1}}\int_{\Delta_0}\mathrm{e}^{-u^2\lambda}(\lambda-\mathcal{B}_{r,1,T})^{-1}d\lambda\right], \quad r\geq 2;\\
F_{r,u,\infty} &:= u^{-2}\psi_{S\times\mathbb{R}}\delta_{u^2}\widetilde{\mathrm{Tr}}\left[g\cdot\frac{1}{2\pi\sqrt{-1}}\int_{\Delta_0}\mathrm{e}^{-u^2\lambda}(\lambda-\mathcal{B}_{r})^{-1}d\lambda\right], \quad r\geq 2;\\
G_{r,u,T} &:= u^{-2}\psi_{S\times\mathbb{R}}\delta_{u^2}\widetilde{\mathrm{Tr}}\left[g\cdot\frac{1}{2\pi\sqrt{-1}}\int_{\delta_0}\mathrm{e}^{-u^2\lambda}(\lambda-\mathcal{B}_{r,1,T})^{-1}d\lambda\right], \quad r\geq 1;\\
G_{r,u,\infty} &:= u^{-2}\psi_{S\times\mathbb{R}}\delta_{u^2}\widetilde{\mathrm{Tr}}\left[g\cdot\frac{1}{2\pi\sqrt{-1}}\int_{\delta_0}\mathrm{e}^{-u^2\lambda}(\lambda-\mathcal{B}_{r})^{-1}d\lambda\right], \quad r\geq 1,
\end{aligned}
$$

and

$$
(5.29)\quad
\begin{aligned}
f_{r,u,T} &:= \{F_{r,u,T}\}^{du}, \quad f_{r,u,\infty} := \{F_{r,u,\infty}\}^{du}, \quad r\geq 2,\\
g_{r,u,T} &:= \{G_{r,u,T}\}^{du}, \quad g_{r,u,\infty} := \{G_{r,u,\infty}\}^{du}, \quad r\geq 1.
\end{aligned}
$$

Note that $\mathcal{B}_{1,1,T}=\mathcal{B}_{1,T}$. Set

$$
(5.30)\quad
\begin{aligned}
f_{1,u,T} &= \left\{\psi_{S\times\mathbb{R}}\widetilde{\mathrm{Tr}}[g\exp(-\mathcal{B}_{u,T})]\right\}^{du} - g_{1,u,T},\\
f_{1,u,\infty} &= \left\{\psi_{S\times\mathbb{R}}\widetilde{\mathrm{Tr}}[g\exp(-\mathcal{B}_{1,u^2})]\right\}^{du} - g_{1,u,\infty}.
\end{aligned}
$$

By (5.26)-(5.30), for $T\gg 1$,

$$
(5.31)\quad
\begin{aligned}
\gamma^u(T,u) &= f_{1,u,T}+g_{1,u,T},\\
\gamma_r(u) &= f_{r,u,\infty}+g_{r,u,\infty}, \quad r\geq 1.
\end{aligned}
$$

When $T\gg 1$, by (5.17), (5.27) and (5.28) (see also [20, (2.106)]), we have

$$
(5.32)\quad
\begin{aligned}
g_{1,u,T} = \sum_{r=2}^{r_0}&\left\{u^{-2}\psi_{S\times\mathbb{R}}\delta_{u^2}\widetilde{\mathrm{Tr}}\left[g\cdot\frac{1}{2\pi\sqrt{-1}}\int_{\frac{\Delta_0}{T^{2(r-1)}}}\mathrm{e}^{-u^2\lambda}(\lambda-\mathcal{B}_{1,T})^{-1}d\lambda\right]\right\}^{du}\\
&+\left\{u^{-2}\psi_{S\times\mathbb{R}}\delta_{u^2}\widetilde{\mathrm{Tr}}\left[g\cdot\frac{1}{2\pi\sqrt{-1}}\int_{\frac{\delta_0}{T^{2(r_0-1)}}}\mathrm{e}^{-u^2\lambda}(\lambda-\mathcal{B}_{1,T})^{-1}d\lambda\right]\right\}^{du}.
\end{aligned}
$$

By (5.24) and (5.25), for $r\geq 2$,

$$
(5.33)\quad \mathcal{B}_{r,1,T} = T^{2(r-1)}\delta_{T^{2(r-1)}}\left(B_T^2+T^{1-r}du\wedge\left(D_{Z,T}+\frac{c(\mathcal{T}_{Z,T})}{4}\right)\right)\delta^{-1}_{T^{2(r-1)}}.
$$

So taking $\lambda'=T^{2(1-r)}\lambda$, by (5.28) and (5.29),

(5.34)

$$
\begin{aligned}
f_{r,T^{1-r}u,T} &= \left\{u^{-2}T^{2(r-1)}\psi_{S\times\mathbb{R}}\delta_{u^2T^{2(1-r)}}\widetilde{\mathrm{Tr}}\left[g\cdot\frac{1}{2\pi\sqrt{-1}}\int_{\Delta_0}e^{-u^2T^{2(1-r)}\lambda}(\lambda-\mathcal{B}_{r,1,T})^{-1}d\lambda\right]\right\}^{du}\\
&= \left\{u^{-2}T^{2(r-1)}\psi_{S\times\mathbb{R}}\delta_{u^2}\widetilde{\mathrm{Tr}}\left[g\cdot\frac{1}{2\pi\sqrt{-1}}\int_{\frac{\Delta_0}{T^{2(r-1)}}}e^{-u^2\lambda'}\right.\right.\\
&\qquad\left.\left.\times\left(\lambda'-\left(B_T^2+T^{1-r}du\wedge\left(D_{Z,T}+\frac{c(\mathcal{T}_{Z,T})}{4}\right)\right)\right)^{-1}d\lambda'\right]\right\}^{du}\\
&= \left\{u^{-2}T^{r-1}\psi_{S\times\mathbb{R}}\delta_{u^2}\widetilde{\mathrm{Tr}}\left[g\cdot\frac{1}{2\pi\sqrt{-1}}\int_{\frac{\Delta_0}{T^{2(r-1)}}}\mathrm{e}^{-u^2\lambda}(\lambda-\mathcal{B}_{1,T})^{-1}d\lambda\right]\right\}^{du}.
\end{aligned}
$$

By (5.32) and (5.34),

$$
(5.35)\quad g_{1,u,T} = \sum_{r=2}^{r_0} f_{r,T^{1-r}u,T}\cdot T^{1-r} + g_{r_0,T^{1-r_0}u,T}\cdot T^{1-r_0}.
$$

From (3.29), [15, Proposition 5.14 and Theorem 5.15] hold in our case. So there exist $\delta, c, C, T_0 > 0$ such that for any $u \geq 1, T \geq T_0$,

$$|f_{1,u,T}| \leq C e^{-cu}, \quad |f_{1,u,\infty}| \leq C e^{-cu}, \quad |f_{1,u,T} - f_{1,u,\infty}| \leq \frac{C}{T^{\delta}} e^{-cu}. \tag{5.36}$$

Following the same proof, there exist $\delta, C, T_0 > 0$ such that for any $u \geq 1, T \geq T_0$,

$$|G_{1,u,T} - G_{1,u,\infty}| \leq \frac{C}{T^{\delta}}. \tag{5.37}$$

Replacing u by su, by (5.37), for $su \geq 1$,

$$|\{G_{1,su,T} - G_{1,su,\infty}\}^{ds}| \leq \frac{C}{T^{\delta}}. \tag{5.38}$$

Note that for $T \in [T_0, +\infty]$,

$$g_{1,u,T} = \{G_{1,su,T}\}^{d(su)}|_{s=1} = u^{-1}\{G_{1,su,T}\}^{ds}|_{s=1}. \tag{5.39}$$

By (5.38) and (5.39), we have

$$|g_{1,u,T} - g_{1,u,\infty}| \leq \frac{C}{uT^{\delta}}. \tag{5.40}$$

For $r \geq 2$, $T \gg 1$, we set

$$\tilde{p}_{r,T} := \frac{1}{2\pi\sqrt{-1}} \int_{|\lambda|=\sqrt{c_2}} (\lambda - T^{r-1} D^{\mathcal{E}}_{Z,T})^{-1} d\lambda \tag{5.41}$$

and $\tilde{p}^{\perp}_{r,T} := 1 - \tilde{p}_{r,T}$. Let $\widetilde{E}_{r,T}$ be the image of $\tilde{p}_{r,T}$. Then $\tilde{p}_{r,T}$ is the orthogonal projection onto $\widetilde{E}_{r,T}$. From the correspondence in (5.16), for $T \gg 1$, $\widetilde{E}_{r,T}$ are vector bundles over S and $\dim \widetilde{E}_{r,T} = \dim E_r$. Let $\tilde{p}_{r,T}(x, x')$ and $p_r(x, x')$ $(x, x' \in Z_b, b \in S)$ be the kernels of $\tilde{p}_{r,T}$ and p_r respectively.

Proposition 5.5. For $k \in \mathbb{N}$, there exist $\delta > 0$ and $C > 0$ such that for $r \geq 2$, $T \gg 1$, $x, x' \in Z_b$, $b \in S$, the C^k norm

$$|\tilde{p}_{r,T}(x, x') - p_r(x, x')|_{C^k} \leq \frac{C}{T^{\delta}}. \tag{5.42}$$

Proof. The proof of this proposition is the same as [20, Proposition 2.12]. We only need to notice that in our case $p_{k,T} = p_k$, $Q_{k,T} = \mathrm{Id}$ and $D_{k,T} = D_k$. So although the Dirac operator here cannot be decomposed into the sum of two nilpotent operators in general, (5.42) also holds. □

Let $\mathcal{B}^{(0)}_{r,1,T}$ be the 0-degree part of $\mathcal{B}_{r,1,T}$ in $\Lambda T^*(S \times \mathbb{R})$. Then by (5.33), $\mathcal{B}^{(0)}_{r,1,T} = T^{2(r-1)} D^{\mathcal{E},2}_{Z,T}$. Set $R_{r,1,T} := \mathcal{B}_{r,1,T} - \mathcal{B}^{(0)}_{r,1,T}$. Recall that p^T is the orthogonal projection onto $\ker D^{\mathcal{E}}_{Z,T}$. Let $p^{T,\perp} := 1 - p^T$. As in [20, Definition 2.13], for $2 \leq r \leq n$, $T \gg 1$, $s \in E_0$, we define

$$|s|^2_{T,1} := \|p^T s\|^2 + T^2 \|p^{T,\perp} s\|^2 + \sum_p \|{}^0\nabla^{\mathcal{E}}_{f^H_{p,X}} s\|^2 + T^2 \sum_i \|{}^0\nabla^{\mathcal{E}}_{e_i} p^{T,\perp} s\|^2, \tag{5.43}$$

and

$$|s|^2_{r,T,1} := \|\tilde{p}_{r,T} s\|^2 + \sum_{k=2}^{r-1} T^{2(r-k)} \|(\tilde{p}_{k,T} - \tilde{p}_{k+1,T}) s\|^2 + T^{2(r-1)} |\tilde{p}^{\perp}_{2,T} s|^2_{T,1}. \tag{5.44}$$

Proposition 5.6. There exist $C_1, C_2, C_3 > 0$, $T_0 > 0$ such that for $T \geq T_0$, $s, s' \in \Lambda T^* S \widehat{\otimes} E_0$,

$$\begin{aligned} &\langle \mathcal{B}^{(0)}_{r,1,T} s, s \rangle_E \geq C_1 |s|^2_{r,T,1} - C_2 \|s\|^2, \\ &|\langle \mathcal{B}^{(0)}_{r,1,T} s, s' \rangle_E| \leq C_3 |s|^2_{r,T,1} |s'|_{r,T,1}, \\ &|\langle R_{r,1,T} s, s' \rangle_E| \leq C_3 (|s|_{r,T,1} \|s'\| + \|s\| |s'|_{r,T,1}). \end{aligned} \tag{5.45}$$

Proof. Set $\bar{R}_{r,1,T} := \mathcal{B}_{r,1,T} - T^{2(r-1)}\delta_{T^{2(r-1)}} B_T^2 \delta^{-1}_{T^{2(r-1)}}$. From (5.33), (5.43) and (5.44), we have

$$|\langle \bar{R}_{r,1,T} s, s' \rangle_E| \le C_3 (|s|_{r,T,1} \|s'\| + \|s\| |s'|_{r,T,1}). \tag{5.46}$$

The proof of other parts are the same as that of [20, Theorem 2.14]. □

Recall that we assume S is compact. There exists a family of C^∞ sections of TY (resp. TX), $U_1, \cdots, U_r$ (resp. $U'_1, \cdots, U'_{r'}$), such that for any $y \in V$ (resp. $x \in W$), $U_1(y), \cdots, U_r(y)$ (resp. $U'_1(x), \cdots, U_{r'}(x)$) span T_yY (resp. T_xX).

As in [20, Definition 2.15], let $\mathcal{D}_T$ be a family of operators on E_0,

$$\mathcal{D}_T = \left\{ p^T\, {}^0\nabla^{\mathcal{E}}_{U^H_{p,X}} p^T + \; p^{T,\perp}\, {}^0\nabla^{\mathcal{E}}_{U^H_{p,X}} p^{T,\perp}, \; p^{T,\perp}\, {}^0\nabla^{\mathcal{E}}_{U'_i} p^{T,\perp} \right\}. \tag{5.47}$$

For $T > 0$, Let $\mathcal{D}'_T$ be a family of operators on E_0,

$$\mathcal{D}'_T = \left\{ \tilde{p}^{\perp}_{2,T} Q \tilde{p}^{\perp}_{2,T} : \; Q \in \mathcal{D}_T \right\}. \tag{5.48}$$

Note that in the formula above [20, Definition 2.15], the corresponding set of operators is stated as

$$\{ p_T\, {}^0\nabla^{\Lambda(T^{*(0,1)}Z)\otimes\xi}_{U^H_{l,1}} p_T, \; p^\perp_T\, {}^0\nabla^{\Lambda(T^{*(0,1)}Z)\otimes\xi}_{U^H_{l,1}} p^\perp_T, \; p^\perp_T\, {}^0\nabla^{\Lambda(T^{*(0,1)}Z)\otimes\xi}_{U'_i} p^\perp_T \}.$$

We need to read it as

$$\mathcal{D}_T = \{ p_T\, {}^0\nabla^{\Lambda(T^{*(0,1)}Z)\otimes\xi}_{U^H_{l,1}} p_T + \; p^\perp_T\, {}^0\nabla^{\Lambda(T^{*(0,1)}Z)\otimes\xi}_{U^H_{l,1}} p^\perp_T, \; p^\perp_T\, {}^0\nabla^{\Lambda(T^{*(0,1)}Z)\otimes\xi}_{U'_i} p^\perp_T \}.$$

Proposition 5.7. For any $k \in \mathbb{N}$ fixed, there exists $C_k > 0$, $T_0 \ge 1$ such that for $T \ge T_0$, $Q_1, \cdots, Q_k \in \mathcal{D}'_T$ and $s, s' \in \Lambda T^*S \widehat{\otimes} E_0$, we have

$$|\langle [Q_1, [Q_2, \cdots [Q_k, \mathcal{B}_{r,1,T}], \cdots]] s, s' \rangle_E| \le C_k |s|_{r,T,1} |s'|_{r,T,1}. \tag{5.49}$$

Proof. Note that the corresponding commutator $[Q_1, [Q_2, \cdots [Q_k, \bar{R}_{r,1,T}], \cdots]]$ has the same structure as $\bar{R}_{r,1,T}$. So we have

$$|\langle [Q_1, [Q_2, \cdots [Q_k, \bar{R}_{r,1,T}], \cdots]] s, s' \rangle_E| \le C_k |s|_{r,T,1} |s'|_{r,T,1}. \tag{5.50}$$

The proof of other parts are the same as that of [20, Theorem 2.16]. □

For $r \ge 2$, $T \in [T_0, \infty]$, set

$$\overline{F}_{r,u,T} = \frac{1}{2\pi\sqrt{-1}} \int_{\Delta_0} e^{-u^2\lambda} (\lambda - \mathcal{B}_{r,1,T})^{-1} d\lambda. \tag{5.51}$$

Let $\overline{F}_{r,u,T}(x, x')$ $(x, x' \in Z_b)$ be the kernel of the operator $\overline{F}_{r,u,T}$.

The proof of the following proposition is the same as that of [20, Theorem 2.17] (see also [15, Proposition 5.14]).

Proposition 5.8. (i) For $u_0 > 0$ fixed, for $m \in \mathbb{N}$, $b \in S$, there exist $C, C' > 0$, $T_0 \ge 1$, such that for $x, x' \in Z_b$, $u \ge u_0$, $T \ge T_0$,

$$\sup_{|\alpha|, |\alpha'| \le m} \left| \frac{\partial^{|\alpha| + |\alpha'|}}{\partial x^\alpha \partial x'^{\alpha'}} \overline{F}_{r,u,T}(x, x') \right| \le C \exp(-C' u^2). \tag{5.52}$$

(ii) For $u_0 > 0$ fixed, for $m \in \mathbb{N}$, $b \in S$, there exist $C, C' > 0$, such that for $x, x' \in Z_b$, $u \ge u_0$,

$$\sup_{|\alpha|, |\alpha'| \le m} \left| \frac{\partial^{|\alpha| + |\alpha'|}}{\partial x^\alpha \partial x'^{\alpha'}} \overline{F}_{r,u,\infty}(x, x') \right| \le C \exp(-C' u^2). \tag{5.53}$$

Let $E^\perp_{r,T}$ be the orthogonal space of $\widetilde{E}_{r,T}$ in E_0. we write $\mathcal{B}_{r,1,T}$ in matrix form with respect to the splitting $E_0 = \widetilde{E}_{r,T} \oplus E^\perp_{r,T}$,

$$\mathcal{B}_{r,1,T} = \begin{pmatrix} \mathcal{B}_{r,1,T,1} & \mathcal{B}_{r,1,T,2} \\ \mathcal{B}_{r,1,T,3} & \mathcal{B}_{r,1,T,4} \end{pmatrix}. \tag{5.54}$$

For $s \in E_0$, set

$$|s|_{r,T,-1} = \sup_{0 \ne s'} \frac{|\langle s, s' \rangle_E|}{|s'|_{T,1}}. \tag{5.55}$$

For $T \gg 1$, $r \geq 2$, let $Q'_{r,T} : E_r \to \widetilde{E}_{r,T}$ be the linear operator defined by $Q'_{r,T} s := \tilde{p}_{r,T} s$. From Theorem 5.2, for $T \gg 1$, $Q'_{r,T}$ is an isomorphism. Set

$$J_{r,T} := Q'_{r,T}((Q'_{r,T})^* Q'_{r,T})^{-1/2}. \tag{5.56}$$

Then $J_{r,T} : E_r \to \widetilde{E}_{r,T}$ is a linear isometry.

The proof of the following proposition is the same as that of [20, Theorem 2.18].

Proposition 5.9. There exist $\delta, C > 0$, $T_0 > 0$ such that for $\lambda \in U_0$, $T \geq T_0$, $r \geq 2$,

$$\left|\left(\mathcal{B}_{r,1,T,1} + \mathcal{B}_{r,1,T,2}(\lambda^2 - \mathcal{B}_{r,1,T,4})^{-1}\mathcal{B}_{r,1,T,3} - \tilde{p}_{r,T} J_{r,T} \mathcal{B}_r J_{r,T}^{-1} \tilde{p}_{r,T}\right) s\right|_{r,T,-1} \leq \frac{C}{T^\delta} |s|_{r,T,1}. \tag{5.57}$$

Comparing with [20, (2.98), (2.105)], we have the following lemma.

Lemma 5.10. (1) *There exist $\delta, c, C, T_0 > 0$ such that for any $u \geq 1, T \geq T_0, r \geq 2$,*

$$\begin{aligned} |f_{r,u,T}| \leq C\mathrm{e}^{-cu}, \quad |f_{r,u,\infty}| \leq C\mathrm{e}^{-cu}, \\ |f_{r,u,T} - f_{r,u,\infty}| \leq \frac{C}{T^\delta}\mathrm{e}^{-cu}, \quad |g_{r,u,T} - g_{r,u,\infty}| \leq \frac{C}{uT^\delta}. \end{aligned} \tag{5.58}$$

(2) *There exist $C, \delta > 0$, such that for any $u \in \mathbb{C}, |u| \leq 1, T \geq T_0, r \geq 1$, we have*

$$|u^2\delta_{u^2}^{-1} F_{r,u,T} - u^2\delta_{u^2}^{-1} F_{r,u,\infty}| \leq \frac{C}{T^\delta}, \quad |u^2\delta_{u^2}^{-1} G_{r,u,T} - u^2\delta_{u^2}^{-1} G_{r,u,\infty}| \leq \frac{C}{T^\delta}. \tag{5.59}$$

Proof. (1) From Proposition 5.8, we have $|f_{r,u,T}| \leq C\mathrm{e}^{-cu}$, and $|f_{r,u,\infty}| \leq C\mathrm{e}^{-cu}$. Using Propositions 5.8, 5.9 and the methods in [8, §11 (p), §13 (o)-(q)], we have $|f_{r,u,T} - f_{r,u,\infty}| \leq \frac{C}{T^\delta}\mathrm{e}^{-cu}$. Following the same argument as in (5.37)-(5.40), we have $|g_{r,u,T} - g_{r,u,\infty}| \leq \frac{C}{uT^\delta}$.

(2) By (5.28), using Propositions 5.6-5.9 and the methods in [8, §11 (p), §13 (o)-(q)], we obtain (5.59). □

Since

$$\mathrm{e}^{-u^2\lambda} = \sum_{k=0}^{\infty} (-1)^k \frac{(u^2\lambda)^k}{k!}, \tag{5.60}$$

by (5.28) and (5.29), there exists $N \in \mathbb{N}$ and there exist $a_{r,i,T}$, $b_{r,i,T}$, $b_{1,i,T} \in \Omega^\bullet(S)$, depending smoothly on $T \in [T_0, +\infty]$, $r \geq 2$, $-N \leq i \leq 0$, such that

$$\begin{aligned} f_{r,u,T} &= \sum_{i=-N}^{0} a_{r,i,T} u^i + \mathrm{O}(u), \quad r \geq 2, \\ g_{r,u,T} &= \sum_{i=-N}^{0} b_{r,i,T} u^i + \mathrm{O}(u), \quad r \geq 1. \end{aligned} \tag{5.61}$$

Now we collect the properties of $a_{r,i,T}$ and $b_{r,i,T}$ for $T \in [T_0, +\infty]$. From (5.59), when $T \to +\infty$, the functions $\{u^2\delta_{u^2}^{-1} F_{r,u,T}, u^2\delta_{u^2}^{-1} G_{r,u,T}\}$ are uniformly bounded holomorphic functions on $\{u \in \mathbb{C} : |u| \leq 1\}$. Hence they have uniform expansions in the domain of u. By (5.59) and Cauchy formula, the coefficients of expansions of $u^2\delta_{u^2}^{-1} F_{r,u,T}$ and $u^2\delta_{u^2}^{-1} G_{r,u,T}$ in u are convergent in the sense of $\mathrm{O}\left(\frac{1}{T^\delta}\right)$ when $T \to +\infty$. Therefore, for $T \to +\infty$,

$$a_{r,i,T} = a_{r,i,\infty} + \mathrm{O}\left(\frac{1}{T^\delta}\right), \quad b_{r,i,T} = b_{r,i,\infty} + \mathrm{O}\left(\frac{1}{T^\delta}\right). \tag{5.62}$$

By (5.35) and (5.61), we have

$$b_{1,i,T} = \sum_{r=2}^{r_0} a_{r,i,T} T^{(1-r)(i+1)} + b_{r_0,i,T} T^{(1-r_0)(i+1)}. \tag{5.63}$$

By Theorem 5.2, 0 is the unique eigenvalue of $\mathcal{B}_{r_0,1,T}$ in δ_0. For $u \in \mathbb{C}$, $|u| \geq 1$,

$$g_{r_0,u,T} = \left\{\psi_{S\times\mathbb{R}} \widetilde{\mathrm{Tr}}\left[g \cdot \frac{1}{2\pi\sqrt{-1}} \int_{\delta_0} e^{-\lambda}(\lambda - \mathcal{B}_{r_0,u,T})^{-1}\right]\right\}^{du}. \tag{5.64}$$

Then following the same argument as in [5, Theorem 9.29], we have

$$g_{r_0,u,T} = \sum_{i=-N}^{0} b_{r_0,i,T} u^i. \tag{5.65}$$

As in (5.37)-(5.40), we have

$$b_{r_0,0,T} = 0. \tag{5.66}$$

For $r \geq 2$, from (4.13) and (5.31), we have

$$f_{r,u,\infty} + g_{r,u,\infty} = \mathrm{O}(1) \quad \text{when } u \to 0. \tag{5.67}$$

So for $i < 0$,

$$a_{r,i,\infty} + b_{r,i,\infty} = 0. \tag{5.68}$$

By (5.58), $|f_{r,u,\infty}| \leq C\mathrm{e}^{-cu}$ when $u \to +\infty$. For $u \in \mathbb{C}$, $|u| \geq 1$,

$$g_{r,u,\infty} = \left\{ \psi_{S\times\mathbb{R}} \widetilde{\mathrm{Tr}} \left[g \cdot \frac{1}{2\pi\sqrt{-1}} \int_{\delta_0} e^{-\lambda} (\lambda - \mathcal{B}_{r,u^2})^{-1} \right] \right\}^{du}, \tag{5.69}$$

where $\mathcal{B}_{r,u^2} = B^2_{r,u^2} + du \wedge D_r$. Since 0 is the unique eigenvalue of $\mathcal{B}_r$ in δ_0, following the same argument as in [5, Theorem 9.29], we have

$$g_{r,u,\infty} = \sum_{i=-N}^{0} b_{r,i,\infty} u^i. \tag{5.70}$$

In fact, by counting the powers of u in [5, (9.152)], we have

$$b_{r,-1,\infty} = b_{r,0,\infty} = 0. \tag{5.71}$$

By (5.68) and (5.71),

$$a_{r,-1,\infty} = 0. \tag{5.72}$$

Now, we will prove Theorem 4.5 as follows.

By (4.16), (4.9), (5.30), (5.36), (5.63) and the dominated convergence theorem,

$$\begin{aligned} \lim_{T\to+\infty} \int_1^{+\infty} \gamma^u(T,u)du &= \lim_{T\to+\infty} \int_1^{+\infty} f_{1,u,T} du + \lim_{T\to+\infty} \int_1^{+\infty} g_{1,u,T} du \\ &= \int_1^{+\infty} f_{1,u,\infty} du + \lim_{T\to+\infty} \int_1^{+\infty} g_{1,u,T} du \\ = \int_1^{+\infty} \{\gamma_1(u) - g_{1,u,\infty}\} du &+ \lim_{T\to+\infty} \sum_{r=2}^{r_0} \int_1^{+\infty} T^{1-r} f_{r,T^{1-r}u,T} du + \lim_{T\to+\infty} \int_1^{+\infty} T^{1-r_0} g_{r_0,T^{1-r_0}u,T} du \\ &= \int_1^{+\infty} \gamma_1(u)du - \int_1^{+\infty} g_{1,u,\infty} du + \lim_{T\to+\infty} \sum_{r=2}^{r_0} \int_{T^{1-r}}^{+\infty} f_{r,u,T} du + \lim_{T\to+\infty} \int_{T^{1-r_0}}^{+\infty} g_{r_0,u,T} du \\ &= \int_1^{+\infty} \gamma_1(u)du - \int_1^{+\infty} g_{1,u,\infty} du + \lim_{T\to+\infty} Q_{1,T} + \lim_{T\to+\infty} Q_{2,T}, \end{aligned} \tag{5.73}$$

where

$$\begin{aligned} Q_{1,T} &:= \sum_{r=2}^{r_0} \int_1^{+\infty} f_{r,u,T} du + \int_1^{+\infty} g_{r_0,u,T} du, \\ Q_{2,T} &:= \sum_{r=2}^{r_0} \int_{T^{1-r}}^{1} f_{r,u,T} du + \int_{T^{1-r_0}}^{1} g_{r_0,u,T} du. \end{aligned} \tag{5.74}$$

By (5.58), (5.62), (5.65) and the dominated convergence theorem, when $T \to +\infty$,

$$\lim_{T\to+\infty} Q_{1,T} = \sum_{r=2}^{r_0} \int_1^{+\infty} f_{r,u,\infty} du + \int_1^{+\infty} g_{r_0,u,\infty} du. \tag{5.75}$$

By (5.61), (5.65), (5.66) and (5.74),

$$
\begin{aligned}
Q_{2,T} =& \sum_{r=2}^{r_0} \int_{T^{1-r}}^{1} \left\{ f_{r,u,T} - \sum_{i=-N}^{-1} a_{r,i,T} u^i \right\} du + \sum_{r=2}^{r_0} \sum_{i=-N}^{-1} a_{r,i,T} \int_{T^{1-r}}^{1} u^i du \\
& + \int_{T^{1-r_0}}^{1} \sum_{i=-N}^{-1} b_{r_0,i,T} u^i du \\
=& \sum_{r=2}^{r_0} \int_{T^{1-r}}^{1} \left\{ f_{r,u,T} - \sum_{i=-N}^{-1} a_{r,i,T} u^i \right\} du + \sum_{i=-N}^{-2} \frac{1}{i+1} \sum_{r=2}^{r_0} \left(a_{r,i,T} - a_{r,i,T} T^{(1-r)(i+1)} \right) \\
& + \sum_{r=2}^{r_0} (r-1) a_{r,-1,T} \log T + \sum_{i=-N}^{-2} \frac{1}{i+1} \left(b_{r_0,i,T} - b_{r_0,i,T} T^{(1-r_0)(i+1)} \right) \\
& + (r_0 - 1) b_{r_0,-1,T} \log T.
\end{aligned} \tag{5.76}
$$

From (5.63), we have

$$
\begin{aligned}
Q_{2,T} =& \sum_{r=2}^{r_0} \int_{T^{1-r}}^{1} \left\{ f_{r,u,T} - \sum_{i=-N}^{-1} a_{r,i,T} u^i \right\} du + \sum_{r=2}^{r_0} (r-1) a_{r,-1,T} \log T \\
& + \sum_{i=-N}^{-2} \frac{1}{i+1} \left(b_{r_0,i,T} - b_{1,i,T} + \sum_{r=2}^{r_0} a_{r,i,T} \right) + (r_0 - 1) b_{r_0,-1,T} \log T.
\end{aligned} \tag{5.77}
$$

From (5.71) and (5.72), we have $a_{r,-1,\infty} = 0$ for $r \geq 2$ and $b_{r_0,-1,\infty} = 0$. Then by (5.62), when $T \to +\infty$,

$$
a_{r,-1,T} \log T = \mathrm{O}\left(\frac{\log T}{T^{\delta}} \right), \quad b_{r_0,-1,T} \log T = \mathrm{O}\left(\frac{\log T}{T^{\delta}} \right). \tag{5.78}
$$

From (5.58), (5.62), (5.78) and the dominated convergence theorem,

$$
\begin{aligned}
\lim_{T \to +\infty} Q_{2,T} =& \sum_{r=2}^{r_0} \int_{0}^{1} \left\{ f_{r,u,\infty} - \sum_{i=-N}^{-1} a_{r,i,\infty} u^i \right\} du \\
& + \sum_{i=-N}^{-2} \frac{1}{i+1} \left(b_{r_0,i,\infty} - b_{1,i,\infty} + \sum_{r=2}^{r_0} a_{r,i,\infty} \right).
\end{aligned} \tag{5.79}
$$

From (5.68)-(5.71) and (5.79), we have

$$
\begin{aligned}
\lim_{T \to +\infty} Q_{2,T} =& \sum_{r=2}^{r_0} \int_{0}^{1} \left\{ f_{r,u,\infty} + \sum_{i=-N}^{-1} b_{r,i,\infty} u^i \right\} du - \sum_{i=-N}^{-2} \frac{1}{i+1} \sum_{r=1}^{r_0-1} b_{r,i,\infty} \\
=& \sum_{r=2}^{r_0} \int_{0}^{1} \left\{ f_{r,u,\infty} + g_{r,u,\infty} \right\} du + \sum_{r=1}^{r_0-1} \int_{1}^{+\infty} \left\{ \sum_{i=-N}^{-2} b_{r,i,\infty} u^i \right\} du \\
=& \sum_{r=2}^{r_0} \int_{0}^{1} \gamma_r(u) du + \sum_{r=1}^{r_0-1} \int_{1}^{+\infty} g_{r,u,\infty} du.
\end{aligned} \tag{5.80}
$$

From (4.14), (5.73), (5.75) and (5.80), we have
(5.81)

$$\begin{aligned}\lim_{T\to+\infty}\int_1^{+\infty}\gamma^u(T,u)du &= \int_1^{+\infty}\gamma_1(u)du - \int_1^{+\infty} g_{1,u,\infty}du + \sum_{r=2}^{r_0}\int_1^{+\infty} f_{r,u,\infty}du \\ &\quad + \int_1^{+\infty} g_{r_0,u,\infty}du + \sum_{r=2}^{r_0}\int_0^1\gamma_r(u)du + \sum_{r=1}^{r_0-1}\int_1^{+\infty} g_{r,u,\infty}du \\ &= \int_1^{+\infty}\gamma_1(u)du + \sum_{r=2}^{r_0}\int_0^1\gamma_r(u)du + \sum_{r=2}^{r_0}\int_1^{+\infty}(f_{r,u,\infty}+g_{r,u,\infty})du \\ &= \int_1^{+\infty}\gamma_1(u)du + \sum_{r=2}^{r_0}\int_0^{+\infty}\gamma_r(u)du \\ &= \int_1^{+\infty}\gamma_1(u)du - \sum_{r=2}^{r_0}\widetilde{\eta}_g(E_r,B_r).\end{aligned}$$

The proof of Theorem 4.5 is complete.

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