

ON THE SUPPORT AND DETERMINANTAL STRUCTURE OF JUMPING LOCI OVER HILBERT SCHEMES

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ABSTRACT. Let Z be a smooth complex projective variety and let X be a closed parameter subscheme of a Hilbert scheme of Z . For a vector bundle on Z and a suitable twist on the universal family, we identify the first cohomological nonvanishing locus with the support of a first higher direct image. We distinguish the support scheme defined by the annihilator from the canonical determinantal scheme defined by the zeroth Fitting ideal, and, when the derived pushforward has amplitude $[0, 1]$, obtain the corresponding determinantal codimension bound. We also prove that if the nonempty jumping-line locus of a rank- r bundle on $\mathbb{P}^n$ has dimension $m \leq n-3$, then $r \geq n-m+1$; in particular, every nonuniform rank-two bundle on $\mathbb{P}^n$, $n \geq 3$, has a jumping-line locus of dimension at least $n-2$. For semistable rank-two bundles E on $\mathbb{P}^n$ with $c_2(E) = k$, we recover Barth's theorem that, when $c_1(E) = 0$, the nonempty jumping-line locus is a divisor of class $k\sigma_1$. For $c_1(E) = -1$, we characterize the natural determinantal jumping scheme and prove that each of its irreducible components has codimension at most two. When this scheme has no divisorial component, we prove that it is Cohen–Macaulay of pure codimension two and compute its class as $\frac{k^2}{2}\sigma_1^2 - \frac{k}{2}\sigma_{1,1}$. We also describe applications to the locus of lines of the second type on a smooth cubic threefold and to a family of plane sections of a quintic surface of general type. We further prove that the Fitting construction commutes with arbitrary base change, and compute the class and arithmetic genus of the second-type-line divisor. The quintic example extends to every smooth surface of degree $d \geq 5$ containing a line, with jumping ideal $\mathcal{I}_\Lambda^{d-2}$. Direct sums and infinitesimal base changes of this family distinguish the annihilator and Fitting scheme structures explicitly.

1. INTRODUCTION

The splitting type or cohomology of the restriction of a vector bundle to members of an algebraic family is generally constant only on an open subset of the parameter space. The exceptional members form a jumping locus. Classical examples include the jumping lines of vector bundles on projective space, but the same question arises for families parametrized by singular or non-reduced Hilbert schemes. In that generality it is important to separate three related objects: the set of jumping points, the support scheme defined by an annihilator, and the determinantal scheme defined by a Fitting ideal.

The use of restrictions to lines has a long history in the study of vector bundles on projective space. After the early work of Schwarzenberger on bundles on the projective plane [12], the generic splitting theorem of Grauert–Mülich made the restriction to a general line a basic tool for semistable bundles. The exceptional lines then retain global information that is invisible on a general member of the family. In rank two, Barth proved that for a normalized semistable bundle with $c_1 = 0$ the jumping lines form a divisor in the linear system $|\mathcal{O}_{G(2,n+1)}(c_2)|$ [1]; for odd first Chern class, Hulek introduced the related jumping lines of the second kind on $\mathbb{P}^2$ [10]. Barth also pointed out that for odd first Chern class the ordinary jumping locus may have components of codimension one or two [1, §0.3]. Thus no uniform purity statement is available in the odd case without an additional expected-codimension hypothesis. There are several standard ways to attach a scheme structure to a cohomological condition. Their comparison, including their behavior under base change and their multiplicities, is the issue considered here. In a complementary direction, the classification of uniform bundles—those having no jumping lines—shows how restrictive constant splitting type can be [11, 13]. These results established jumping loci as geometric invariants rather than merely exceptional sets.

[†] The author is sponsored by National Natural Science Foundation of China (Grant No. 12471040) and Science and Technology Commission of Shanghai Municipality (Grant No. 22DZ2229014).

A related determinantal viewpoint appears in the Brill–Noether theory of Costa and Miró-Roig [3]. They let the bundle vary in a moduli space and study the loci where h^0 jumps, realizing those loci as degeneracy schemes under suitable higher-cohomology vanishing assumptions and computing their expected dimensions. Here the roles of the parameters are different: the bundle E is fixed, while the line or, more generally, the subscheme on which E is restricted varies in a Hilbert parameter space. The common mechanism is a derived pushforward represented by a complex of vector bundles and the minors of its differential. Our emphasis is on this mechanism over possibly singular or non-reduced parameter schemes and on the distinction between the annihilator support and the canonical Fitting scheme.

This circle of ideas is also connected with the Hartshorne conjecture, which predicts that every rank-two vector bundle on $\mathbb{P}^n$ splits as a sum of line bundles when $n \geq 7$ [9]. Through the Serre construction, the conjecture is closely related to complete-intersection questions for smooth codimension-two subvarieties of projective space. On the other hand, for $n \geq 3$ a uniform rank-two bundle on $\mathbb{P}^n$ splits; hence any hypothetical indecomposable bundle in the range of the Hartshorne conjecture must have a nonempty jumping-line locus. Its dimension, scheme structure, and cycle class therefore provide concrete geometric constraints on a possible counterexample. The results of this paper do not settle the Hartshorne conjecture, but they place these constraints in a framework that keeps track of nilpotent and multiplicity information and that applies beyond the Grassmannian of lines.

Let $q : Y \rightarrow X$ be the universal family over a closed connected parameter subscheme of a Hilbert scheme of a smooth projective variety Z , and let $\mathcal{F} = p^*E \otimes \mathcal{L}$ such that the higher fiberwise cohomology vanishes, where p is a projective morphism from Y to Z (as in Section 2). The purpose of this paper is threefold. First, we express the first cohomological nonvanishing locus in terms of $R^1q_*\mathcal{F}$ over an arbitrary closed Hilbert parameter scheme, without smoothness or reducedness assumptions on the base. Second, we distinguish the annihilator support structure from the canonical Fitting determinantal structure and use a two-term representative of $\mathbf{R}q_*\mathcal{F}$ to obtain a codimension bound. Third, we compute determinantal cycle classes in the classical rank-two setting and test the formalism on Fano schemes of lines and on families of higher-degree sections. Thus the aim is both foundational and computational: to clarify what scheme is naturally attached to a jumping condition and to show what additional information that scheme retains. The main results can be stated as follows.

Theorem 1.1 (= Theorem 2.1). *The first cohomological nonvanishing set is closed. When it is endowed with the support scheme structure defined by the annihilator ideal, one has an equality of closed subschemes*

$$J_{\text{supp}}(E) = V(\text{Ann}_{\mathcal{O}_X}(R^1q_*\mathcal{F})).$$

Its underlying closed set is $\text{Supp}(R^1q_*\mathcal{F})$. This remains valid when X is singular or non-reduced.

Theorem 1.2 (= Theorem 2.2). *The zeroth Fitting ideal defines a canonical determinantal scheme*

$$J_{\text{det}}(E) := V(\text{Fitt}_0(R^1q_*\mathcal{F}))$$

with the same underlying closed set as $J_{\text{supp}}(E)$. If its underlying set is a proper subset of $|X|$, every irreducible component has codimension at most $\chi + 1$, where $\chi = \chi(V_x, \mathcal{F}|_{V_x})$ and $V_x = p(q^{-1}(x)) \subset Z$.

Theorem 1.3 (= Theorem 3.1). *Let E be a holomorphic vector bundle of rank r on $\mathbb{P}^n$, $n \geq 2$, and suppose that the set of jumping lines $J(E)$ is nonempty of dimension m . Then*

$$(1) \quad r \geq \begin{cases} n - m + 1, & m \leq n - 3, \\ 2, & m \geq n - 2. \end{cases}$$

Theorem 1.4 (= Theorem 3.5). *Let E be a semistable rank-two vector bundle on $\mathbb{P}^n$ with $c_2(E) = k$, and suppose that its jumping-line locus is nonempty. If $c_1(E) = 0$, Barth’s theorem states that the determinantal jumping scheme is a divisor of Plücker class*

$$[J_{\text{det}}(E)] = k\sigma_1.$$

If $c_1(E) = -1$, every irreducible component of its natural determinantal scheme has codimension at most two. If this scheme has no divisorial component, then it is Cohen–Macaulay of pure codimension two and

$$[J_{\det}(E)] = \frac{k^2}{2}\sigma_1^2 - \frac{k}{2}\sigma_{1,1} = \frac{k^2}{2}\sigma_2 + \frac{k(k-1)}{2}\sigma_{1,1}.$$

Section 4 contains two applications. First, let $Z \subset \mathbb{P}^4$ be a smooth cubic threefold and let $X = F(Z)$ be its Fano surface of lines. Then the locus $F_2(Z)$ of second-type lines satisfies

$$F_2(Z) = \text{Supp}(R^1 q_* p^*(T_Z(-1)))$$

as closed sets and has the natural Fitting determinantal structure. Second, for a smooth quintic surface $Z \subset \mathbb{P}^3$ containing a line, the family constructed in Section 4 has jumping support $\Lambda \cong \mathbb{P}^1$ and

$$\text{Ann}_{\mathcal{O}_X}(R^1 q_* p^* E) = \text{Fitt}_0(R^1 q_* p^* E) = \mathcal{I}_\Lambda^3.$$

These statements are Propositions 4.1 and 4.2, respectively.

The support and Fitting constructions above are standard; the distinction between them becomes particularly useful under a change of parameters. We keep the dependence on the chosen family and on the auxiliary twist explicit. In particular, a nonvanishing condition is a first jumping condition only when the generic value of h^1 is zero. Higher Fitting ideals handle a positive generic value as well.

For the cubic threefold, Proposition 4.3 identifies the cohomological Fitting scheme with the determinant of $Q \rightarrow \text{Sym}^2 U^\vee$ on the Fano surface. This recovers the classical divisor class $2K_{F(Z)}$ [2, 10.21], and gives Plücker degree 90 and arithmetic genus 136. We first give a detailed quintic calculation and then extend it as follows.

Theorem 1.5 (= Theorem 4.4). *Let $Z_d \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 5$ containing a line L , and put*

$$E_d = \mathcal{O}_{Z_d}(2L + (d-3)H).$$

For its universal family of plane sections over $X = |H|$, let $\Lambda \cong \mathbb{P}^1$ be the pencil of planes containing L . Then

$$\text{Ann}(R^1 q_* p^* E_d) = \text{Fitt}_0(R^1 q_* p^* E_d) = \mathcal{I}_\Lambda^{d-2}.$$

The common scheme is Cohen–Macaulay of pure codimension two and has fundamental cycle $\binom{d-1}{2}[\Lambda]$.

For the direct sum $E_d^{\oplus s}$, the annihilator remains $\mathcal{I}_\Lambda^{d-2}$, while the zeroth Fitting ideal becomes $\mathcal{I}_\Lambda^{s(d-2)}$. A transverse infinitesimal parameter gives a reduced annihilator support and a non-reduced Fitting scheme. These two calculations show separately how multiplicity and arbitrary base change distinguish the constructions.

The article is organized as follows. Section 2 develops the support and determinantal descriptions over a general Hilbert parameter scheme. Section 3 recalls splitting types on lines and proves the cycle-class formulas for semistable rank-two bundles on projective space, under the expected-codimension hypothesis in the odd-first-Chern-class case. Section 4 gives the cubic-threefold and quintic-surface examples, followed by the general degree- d construction, the direct-sum comparison, and the infinitesimal base-change example.

2. IDENTIFICATION OF JUMPING LOCI AS SHEAF SUPPORT

Let Z be a smooth complex projective manifold of dimension $d = \dim Z$. Let $P(t) \in \mathbb{Q}[t]$ be a Hilbert polynomial, and let $X \subseteq \text{Hilb}^P(Z)$ be a closed connected subscheme of the Hilbert scheme parameterizing a flat family of closed subschemes $\{V_x\}_{x \in X}$ of Z with Hilbert polynomial P . The base scheme X is a Noetherian projective scheme over $\mathbb{C}$. In general, X may be singular, non-reduced, or non-equidimensional.

Let $Y := \{(z, x) \in Z \times X \mid z \in V_x\} \subset Z \times X$ be the universal subscheme (incidence variety), equipped with the canonical projection morphisms:

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ \downarrow p & & \\ Z & & \end{array}$$

By the universal property of Hilbert schemes, $q : Y \rightarrow X$ is a proper and flat morphism whose fiber over a closed point $x \in X$ is canonically isomorphic to the subscheme $V_x = p(q^{-1}(x)) \subset Z$.

Let E be a vector bundle of rank r on Z . Let $\mathcal{L}$ be an invertible sheaf on Y , and define $\mathcal{F} := p^*E \otimes \mathcal{L}$. We assume that $\mathcal{L}$ is chosen so that the higher fiberwise cohomology vanishes:

$$H^i(V_x, \mathcal{F}|_{V_x}) = 0 \quad \text{for all } i \geq 2 \text{ and all closed points } x \in X.$$

Note that if $q : Y \rightarrow X$ has 1-dimensional fibers V_x , $H^i(V_x, \mathcal{F}|_{V_x}) \equiv 0$ ($i \geq 2$) holds automatically by Grothendieck's vanishing theorem.

For an arbitrary scheme point $x \in X$, write $V_x = Y_x$ and $\mathcal{F}|_{V_x} = \mathcal{F}_x$ for the fiber and its restricted sheaf over $\kappa(x)$. We define the *first cohomological nonvanishing set* $|J_{\text{supp}}(E)| \subseteq |X|$ by

$$|J_{\text{supp}}(E)| := \{x \in X \mid H^1(V_x, \mathcal{F}|_{V_x}) \neq 0\}.$$

For a coherent $\mathcal{O}_X$ -module $\mathcal{M}$, its annihilator ideal sheaf is

$$\text{Ann}_{\mathcal{O}_X}(\mathcal{M}) := \ker(\mathcal{O}_X \rightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M})).$$

We write $\text{Supp}(\mathcal{M})$ for the underlying closed set $\{x \in X \mid \mathcal{M}_x \neq 0\}$, and use $V(\text{Ann}_{\mathcal{O}_X}(\mathcal{M}))$ for the associated annihilator support scheme. For the cohomology sheaf in this family, we denote that scheme by $J_{\text{supp}}(E)$. The notation refers throughout to the fixed family q and the chosen twist $\mathcal{L}$; we also write $J_{\text{supp}}(\mathcal{F})$ when this dependence should be emphasized. If the generic value of h^1 is zero, the nonvanishing set above is the first jumping set in the usual sense. Proposition 2.4 describes jumps from a positive generic value.

Theorem 2.1. *Let Z be a smooth complex projective manifold, and let $X \subseteq \text{Hilb}^P(Z)$ be an arbitrary closed parameter subscheme (possibly singular or non-reduced), with $\mathcal{F}$ satisfying the fiberwise vanishing assumptions above. Then the nonvanishing set, equipped with the support scheme structure defined by the annihilator ideal sheaf $\text{Ann}_{\mathcal{O}_X}(R^1q_*\mathcal{F})$, is a closed algebraic subscheme of X satisfying*

$$J_{\text{supp}}(E) = V(\text{Ann}_{\mathcal{O}_X}(R^1q_*\mathcal{F})) \subseteq X.$$

Its underlying closed set is $\text{Supp}(R^1q_*\mathcal{F})$.

Proof. Because $q : Y \rightarrow X$ is a proper and flat morphism, $\mathcal{F}$ is flat over X . By assumption, the fiberwise higher cohomology vanishes:

$$H^2(q^{-1}(x), \mathcal{F}|_{q^{-1}(x)}) \cong H^2(V_x, \mathcal{F}|_{V_x}) = 0 \quad \text{for all closed points } x \in X.$$

By the cohomology and base-change theorem, the vanishing of the higher fiberwise cohomology implies that $R^i q_* \mathcal{F} = 0$ for $i \geq 2$, and the natural fiberwise evaluation morphism:

$$\phi_x : (R^1 q_* \mathcal{F}) \otimes_{\mathcal{O}_X} \kappa(x) \xrightarrow{\cong} H^1(V_x, \mathcal{F}|_{V_x})$$

is a canonical isomorphism of $\kappa(x)$ -vector spaces for every closed point $x \in X$, where $\kappa(x)$ is the residue field of x .

Now, let $x \in X$ be an arbitrary closed point. Applying Nakayama's Lemma to the finitely generated $\mathcal{O}_{X,x}$ -module $(R^1 q_* \mathcal{F})_x$:

$$(R^1 q_* \mathcal{F})_x \neq 0 \iff (R^1 q_* \mathcal{F}) \otimes_{\mathcal{O}_X} \kappa(x) \neq 0.$$

Substituting the base change isomorphism ϕ_x into this equivalency yields:

$$(R^1 q_* \mathcal{F})_x \neq 0 \iff H^1(V_x, \mathcal{F}|_{V_x}) \neq 0 \iff x \in |J_{\text{supp}}(E)|.$$

Thus the two sets of closed points are equal. Since q is proper and $\mathcal{F}$ is coherent, $R^1 q_* \mathcal{F}$ is a coherent $\mathcal{O}_X$ -module. Endowing this closed set with the scheme structure cut out by $\text{Ann}_{\mathcal{O}_X}(R^1 q_* \mathcal{F})$ gives the asserted scheme-theoretic equality by definition. The same base-change

and Nakayama argument applies at arbitrary scheme points: the fiberwise vanishing extends from closed points by semicontinuity. Thus the identification holds on the entire underlying topological space. $\square$

Definition 2.1. *Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$ -module of finite presentation. On an open subset $U \subseteq X$, choose a presentation*

$$\mathcal{O}_U^{\oplus m} \xrightarrow{A} \mathcal{O}_U^{\oplus s} \longrightarrow \mathcal{M}|_U \longrightarrow 0.$$

The zeroth Fitting ideal of $\mathcal{M}$ on U is

$$\mathrm{Fitt}_0(\mathcal{M})|_U := I_s(A),$$

where $I_s(A)$ is generated by the $s \times s$ maximal minors of a matrix representing A ; by convention, $I_s(A) = 0$ when $m < s$ and $I_0(A) = \mathcal{O}_U$. This ideal is independent of the chosen presentation and therefore glues to a coherent ideal sheaf on X . Moreover,

$$\sqrt{\mathrm{Fitt}_0(\mathcal{M})} = \sqrt{\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{M})},$$

so $V(\mathrm{Fitt}_0(\mathcal{M}))$ and $V(\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{M}))$ have the same underlying closed set, although their non-reduced scheme structures need not agree. In fact, $\mathrm{Fitt}_0(\mathcal{M}) \subseteq \mathrm{Ann}(\mathcal{M})$.

Theorem 2.2. *Under the hypotheses of Theorem 2.1, suppose also that X is connected, as in the setup of this section. Then the derived pushforward $\mathbf{R}q_*\mathcal{F}$ is Zariski-locally represented in the bounded derived category $D^b(X) = D^b(\mathrm{Coh} X)$ by a perfect two-term complex:*

$$(2) \quad \mathcal{A}^\bullet : \quad 0 \longrightarrow \mathcal{A}^0 \xrightarrow{\psi} \mathcal{A}^1 \longrightarrow 0,$$

where $\mathcal{A}^0$ and $\mathcal{A}^1$ are finite-rank locally free sheaves, placed in degrees 0 and 1, of ranks a and b , and $\chi := \chi(V_x, \mathcal{F}|_{V_x}) = a - b$ is the fiberwise Euler characteristic. (Here “perfect two-term” means locally quasi-isomorphic to such a complex.) The global determinantal scheme $J_{\det}(E) = J_{\det}(\mathcal{F})$ is cut out by $\mathrm{Fitt}_0(R^1q_*\mathcal{F})$; on an open set with the displayed representative, this is the ideal $\mathcal{I}_b(\psi)$ generated by the $b \times b$ maximal minors of ψ . Consequently, if its underlying set is a proper subset of $|X|$, every irreducible component $W \subseteq J_{\det}(E)$ satisfies

$$\mathrm{codim}_X(W) \leq \chi + 1.$$

Proof. Because $q : Y \rightarrow X$ is proper, flat, and of finite presentation and $\mathcal{F}$ is locally free, the complex $K = \mathbf{R}q_*\mathcal{F}$ is perfect and its formation commutes with arbitrary derived base change. For every point $x \in X$ this gives

$$K \otimes^{\mathbf{L}} \kappa(x) \simeq \mathbf{R}\Gamma(Y_x, \mathcal{F}_x).$$

Our fiberwise hypothesis implies that the right side has cohomology only in degrees zero and one. It suffices to impose the vanishing on closed fibers: semicontinuity and the fact that X is of finite type over $\mathbb{C}$ give it on all geometric fibers.

We spell out why this implies a two-term locally free representative. At a point of X , represent the perfect complex over the local ring by a bounded complex of finite free modules. Whenever a differential has a unit entry, elementary row and column operations isolate a contractible two-term summand. Removing such summands leaves a minimal free complex, whose differentials vanish after tensoring with the residue field. Since its derived fiber has cohomology only in degrees zero and one, its terms outside these degrees must vanish. These cancellations extend to a Zariski neighborhood. Thus K has Tor-amplitude contained in $[0, 1]$ and is represented there by $\mathcal{A}^0 \xrightarrow{\psi} \mathcal{A}^1$. This uses the vanishing on the derived fibers, not merely vanishing of the cohomology sheaves $R^iq_*\mathcal{F}$.

The fiberwise evaluation matches the homology of the complex:

$$\ker(\psi_x) \cong H^0(V_x, \mathcal{F}|_{V_x}), \quad \mathrm{coker}(\psi_x) \cong H^1(V_x, \mathcal{F}|_{V_x}).$$

By definition, a point $x \in |J_{\mathrm{supp}}(E)| \iff \mathrm{coker}(\psi_x) \neq 0 \iff \mathrm{rank}(\psi_x) \leq b - 1$. The locus where ψ fails to be surjective is defined by the ideal $\mathcal{I}_b(\psi)$ generated by its $b \times b$ maximal minors. These local ideals agree with $\mathrm{Fitt}_0(R^1q_*\mathcal{F})$ and hence glue globally. If the underlying locus is a proper subset of $|X|$, ψ is surjective at some point. Hence $a \geq b$ (equivalently, $\chi \geq 0$);

the determinantal height bound also applies to components on which all maximal minors vanish identically.

By the height theorem for determinantal ideals (see [5, Section 12] and [7, Section 14.4]), the codimension of every irreducible component W of a determinantal locus $D_{b-1}(\psi) = V(\mathcal{I}_b(\psi))$ is bounded above by its expected codimension:

$$\mathrm{codim}_X(W) \leq (a - (b - 1)) \times (b - (b - 1)) = a - b + 1 = \chi + 1.$$

□

For a disconnected parameter scheme the constructions are made componentwise. The Euler characteristic is constant on each connected component, and the codimension bound applies on every such component where the nonvanishing set is a proper subset.

2.1. Base change and comparison of the two structures. The preceding theorem gives a closed immersion

$$J_{\mathrm{supp}}(\mathcal{F}) \hookrightarrow J_{\mathrm{det}}(\mathcal{F})$$

with equality of underlying closed sets. The Fitting construction also has a useful functorial property which is not shared by annihilator support under arbitrary base change.

Proposition 2.3. *Let $g : T \rightarrow X$ be a morphism between locally Noetherian schemes, let $Y_T = Y \times_X T$, and let $q_T : Y_T \rightarrow T$ and $\mathcal{F}_T$ be the pulled-back family and sheaf. Then*

$$\mathbf{L}g^*(\mathbf{R}q_*\mathcal{F}) \simeq \mathbf{R}q_{T*}\mathcal{F}_T, \quad g^*R^1q_*\mathcal{F} \simeq R^1q_{T*}\mathcal{F}_T.$$

Consequently,

$$J_{\mathrm{det}}(\mathcal{F}_T) = J_{\mathrm{det}}(\mathcal{F}) \times_X T.$$

Proof. The derived base-change statement follows from the theorem used in the proof of Theorem 2.2. To pass to the first cohomology sheaf, work on an open set with the representative (2). Pulling back this complex gives

$$\mathbf{L}g^*(\mathbf{R}q_*\mathcal{F}) \simeq [g^*\mathcal{A}^0 \xrightarrow{g^*\psi} g^*\mathcal{A}^1].$$

There are no terms in degrees greater than one. Its first cohomology is therefore the cokernel, and right exactness of tensor product gives

$$R^1q_{T*}\mathcal{F}_T = \mathrm{coker}(g^*\psi) \simeq g^*\mathrm{coker}(\psi) = g^*R^1q_*\mathcal{F}.$$

These identifications are the natural base-change maps, so they glue. Finally, Fitting ideals commute with arbitrary base change. Applying this to the displayed isomorphism proves the scheme-theoretic equality. No flatness of g is required. □

Write $\mathcal{M} = R^1q_*\mathcal{F}$ and let a, b be the ranks in (2).

Proposition 2.4. *For $s \geq 1$, the scheme*

$$J_{\geq s}(\mathcal{F}) := V(\mathrm{Fitt}_{s-1}\mathcal{M})$$

has underlying set $\{x \mid h^1(Y_x, \mathcal{F}_x) \geq s\}$ and commutes with arbitrary base change. Locally in (2), it is cut out by the $(b - s + 1)$ -minors of ψ when $1 \leq s \leq b$, and is empty when $s > b$.

Proof. The dimension of the cokernel of ψ_x is $b - \mathrm{rank} \psi_x$. The stated condition is therefore exactly the rank condition defining these minors. Independence of presentation and base-change compatibility follow from the corresponding properties of Fitting ideals. □

If X is integral and the generic value of h^1 is h , the actual cohomology-jumping scheme is $J_{\geq h+1}(\mathcal{F})$. In particular, the first nonvanishing scheme is $J_{\geq 1}(\mathcal{F}) = J_{\mathrm{det}}(\mathcal{F})$. In this discussion $\mathcal{M} = R^1q_*\mathcal{F}$, and a, b refer to the ranks in (2).

Proposition 2.5. *The two constructions have the following elementary properties.*

- (1) *If $h^1(Y_x, \mathcal{F}_x) \leq 1$ at every point, then $J_{\mathrm{supp}}(\mathcal{F}) = J_{\mathrm{det}}(\mathcal{F})$.*
- (2) *For $s \geq 1$,*

$$\mathrm{Ann}(\mathcal{M}^{\oplus s}) = \mathrm{Ann} \mathcal{M}, \quad \mathrm{Fitt}_0(\mathcal{M}^{\oplus s}) = (\mathrm{Fitt}_0 \mathcal{M})^s.$$

Proof. For (1), Nakayama's lemma implies that every stalk of $\mathcal{M}$ is cyclic or zero. A cyclic module R/I has both annihilator and zeroth Fitting ideal equal to I . For (2), the annihilator assertion is immediate, while the Fitting assertion follows from the formula for a block diagonal presentation. $\square$

For a flat base change, annihilator ideals of finitely presented modules also commute with pullback: apply flat base change to the kernel defining the annihilator and to the corresponding Hom module. Example 2 shows explicitly what can happen for a nonflat base change, even when the two structures agree on the original smooth parameter space.

3. JUMPING LOCI OF VECTOR BUNDLES ON PROJECTIVE SPACES

We now specialize the general support and determinantal framework of Section 2 to the universal family of lines in projective space. In this setting, the jumping condition can be expressed concretely in terms of the splitting type of the restricted bundle, while the parameter space is the Grassmannian of lines. We first recall the relation between jumping lines, uniformity, and rank bounds for arbitrary vector bundles, and then study the natural determinantal structure and cycle class of the jumping locus for semistable rank-two bundles.

3.1. Jumping loci of rank r vector bundles on $\mathbb{P}^n$. By Grothendieck's theorem, for every line $L \subset \mathbb{P}^n$ there are uniquely determined integers $a_1(L) \geq \dots \geq a_r(L)$ such that

$$E|_L \cong \bigoplus_{i=1}^r \mathcal{O}_L(a_i(L)).$$

The ordered tuple $(a_1(L), \dots, a_r(L))$ is the *splitting type* of E along L . It is constant on a dense open subset of $\mathbb{G}(1, n)$; that constant tuple is the *generic splitting type*. A line is a *jumping line* if its splitting type differs from the generic one, and $J(E)$ denotes the set of all jumping lines. The bundle is *uniform* if $J(E) = \emptyset$. We call it *almost uniform* if $J(E)$ is nonempty and zero-dimensional.

Theorem 3.1. *Let E be a holomorphic vector bundle of rank r on $\mathbb{P}^n$, $n \geq 2$, and suppose that the set of jumping lines $J(E)$ is nonempty of dimension m . Then*

$$(3) \quad r \geq \begin{cases} n - m + 1, & m \leq n - 3, \\ 2, & m \geq n - 2. \end{cases}$$

Proof. Assume first that $m \leq n - 3$, and put $k = n - m \geq 3$. Choose a jumping line $L \subset \mathbb{P}^n$. We construct a flag

$$\mathbb{P}^n = P_0 \supset P_1 \supset \dots \supset P_m \cong \mathbb{P}^k, \quad L \subset P_i,$$

such that

$$\dim(J(E) \cap \mathbb{G}(1, P_i)) \leq m - i.$$

Suppose that P_{i-1} has been chosen, and set $W_{i-1} = J(E) \cap \mathbb{G}(1, P_{i-1})$. For every positive-dimensional irreducible component C of W_{i-1} , the lines parametrized by C span a linear subspace that is not contained in L . A general hyperplane $P_i \subset P_{i-1}$ containing L therefore does not contain this span. Hence no positive-dimensional component of W_{i-1} is contained in $\mathbb{G}(1, P_i)$, and intersection with $\mathbb{G}(1, P_i)$ lowers the dimension of every such component by at least one. This proves the asserted inequality by induction. Since $[L]$ belongs to every intersection, the final locus

$$W_m = J(E) \cap \mathbb{G}(1, P_m)$$

is nonempty and zero-dimensional.

A general line in P_m lies outside W_m and hence has the generic splitting type of E . It follows that the jumping locus of $E|_{P_m}$ is precisely W_m , so $E|_{P_m}$ is almost uniform. Ellia's theorem [6, Theorem 2] gives

$$r = \text{rank}(E|_{P_m}) \geq k + 1 = n - m + 1.$$

If $m \geq n - 2$, the nonemptiness of $J(E)$ rules out rank one, because every line bundle on projective space is uniform. Thus $r \geq 2$. $\square$

Corollary 3.2. *Every nonuniform rank-two vector bundle on $\mathbb{P}^n$, $n \geq 3$, satisfies*

$$\dim J(E) \geq n - 2.$$

Proof. Let $m = \dim J(E)$. If $m \leq n - 3$, Theorem 3.1 gives a contradiction. Hence $m \geq n - 2$. $\square$

For semistable rank-two bundles, the general lower bound in Corollary 3.2 can be substantially improved. After normalization, one has $c_1(E) \in \{0, -1\}$, and the Grauert–Mülich–Barth theorem determines the splitting type on a general line. As we shall prove below, whenever the jumping locus is nonempty, it has dimension $2n - 3$ if $c_1(E) = 0$, whereas for $c_1(E) = -1$ every irreducible component has dimension at least $2n - 4$. Thus, in the semistable rank-two case one always has

$$\dim J(E) = \dim J_{\det}(E) \geq 2n - 4,$$

with the stronger equality $\dim J_{\det}(E) = 2n - 3$ when $c_1(E) = 0$. In the case $c_1(E) = -1$, equality and scheme-theoretic purity hold when the jumping scheme has no divisorial component. We now turn to the determinantal description underlying these sharper estimates.

Let $Z = \mathbb{P}^n$ ($n \geq 2$) be a complex projective space of dimension n , and let $X = G(2, n + 1) = \mathbb{G}(1, n)$ be the Grassmannian of lines in $\mathbb{P}^n$, with $\dim X = 2n - 2$. Let $U \subset \mathbb{C}^{n+1} \otimes \mathcal{O}_X$ be the tautological rank-two subbundle, whose fiber at $[L]$ is the two-dimensional vector subspace defining L . We use the subspace convention for projectivization: $\mathbb{P}(U)$ parametrizes one-dimensional subspaces in the fibers of U , and $\mathcal{O}_{\mathbb{P}(U)}(-1)$ is tautological. Thus the incidence variety is the flag variety $Y = F(1, 2; n + 1) = \mathbb{P}(U)$. Its standard projection morphisms are:

$$\begin{array}{ccc} F(1, 2; n + 1) & \xrightarrow{q} & G(2, n + 1). \\ \downarrow p & & \\ \mathbb{P}^n & & \end{array}$$

We use the standard Schubert classes

$$\sigma_1 = c_1(U^\vee), \quad \sigma_{1,1} = c_2(U^\vee) = c_2(U), \quad \sigma_2 = \sigma_1^2 - \sigma_{1,1}.$$

Here σ_1 is the Plücker hyperplane class, and $A^i(X)$ will denote the codimension- i Chow group of X . We write $E(t) = E \otimes \mathcal{O}_{\mathbb{P}^n}(t)$ and $h^i(-) = \dim_{\mathbb{C}} H^i(-)$. As usual, $c_1(E) \in \{0, -1\}$ and $c_2(E) = k$ mean $c_1(E) \in \{0, -h\}$ and $c_2(E) = kh^2$, where $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. For any point $x \in X$, the fiber $q^{-1}(x) \cong \mathbb{P}^1$ corresponds to a line $L_x \subset \mathbb{P}^n$, and $(p^*E)|_{q^{-1}(x)} \cong E|_{L_x}$. Let E be a semistable rank 2 vector bundle on $\mathbb{P}^n$ normalized such that $c_1(E) \in \{0, -1\}$. The jumping locus of E is

$$J(E) = \left\{ x \in G(2, n + 1) \mid E|_{L_x} \not\cong \mathcal{O}_{L_x} \left(\left\lceil \frac{c_1(E)}{2} \right\rceil \right) \oplus \mathcal{O}_{L_x} \left(\left\lfloor \frac{c_1(E)}{2} \right\rfloor \right) \right\}.$$

Theorem 3.3 (Grauert–Mülich–Barth, see [11] Chapter 2). *Let E be a semistable rank 2 vector bundle on $\mathbb{P}^n$ with $c_1(E) \in \{0, -1\}$. For a general line $L \in G(2, n + 1)$, the restriction $E|_L$ splits as:*

$$E|_L \cong \begin{cases} \mathcal{O}_L \oplus \mathcal{O}_L, & \text{if } c_1(E) = 0, \\ \mathcal{O}_L \oplus \mathcal{O}_L(-1), & \text{if } c_1(E) = -1. \end{cases}$$

3.2. Degeneracy locus of the morphism between vector bundles. Let X be a Noetherian scheme (or a smooth complex projective variety). Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on X of ranks a and b respectively, with $a \geq b \geq 1$. Let $\phi : \mathcal{E} \rightarrow \mathcal{F}$ be an $\mathcal{O}_X$ -sheaf homomorphism (which locally corresponds to a $b \times a$ matrix with entries in $\mathcal{O}_X$).

For an integer k with $1 \leq k \leq b$, the k -th degeneracy locus of ϕ , denoted by $D_{k-1}(\phi)$, has underlying set:

$$D_{k-1}(\phi) := \{x \in X \mid \text{rank}(\phi_x) \leq k - 1\}.$$

Globally, $D_{k-1}(\phi)$ is cut out by the k -th determinantal ideal sheaf $\mathcal{I}_k(\phi) \subseteq \mathcal{O}_X$, which is locally generated by all $k \times k$ minors of the local representation matrix of ϕ .

Theorem 3.4 (Giambelli–Thom–Porteous formula, see [5] Section 12). *If $|D_{k-1}(\phi)| \neq |X|$, every irreducible component $W \subseteq D_{k-1}(\phi)$ satisfies*

$$\mathrm{codim}_X(W) \leq (a - k + 1)(b - k + 1).$$

The integer on the right is the expected codimension. If X is smooth and $D_{k-1}(\phi)$ has this pure codimension, then

$$[D_{k-1}(\phi)] = \Delta_{b-k+1}^{a-k+1}(c(\mathcal{F} - \mathcal{E})),$$

where

$$\Delta_q^p(c) := \det(c_{q+j-i})_{1 \leq i, j \leq p}, \quad c_0 = 1, \quad c_i = 0 \ (i < 0).$$

3.3. Jumping loci of semistable vector bundles on $\mathbb{P}^n$. Throughout this subsection the auxiliary twist is fixed by

$$\mathcal{F}_E = \begin{cases} p^*(E(-1)), & c_1(E) = 0, \\ p^*E, & c_1(E) = -1, \end{cases} \quad \mathcal{M}_E = R^1q_*\mathcal{F}_E.$$

We write $J_{\mathrm{supp}}(E) = V(\mathrm{Ann} \mathcal{M}_E)$ and $J_{\mathrm{det}}(E) = V(\mathrm{Fitt}_0 \mathcal{M}_E)$; their common underlying set is the ordinary jumping-line set $J(E)$.

Lemma 3.1. *Let E be a semistable rank-two vector bundle on $\mathbb{P}^n$ with $c_1(E) \in \{0, -1\}$.*

(1) *If $c_1(E) = 0$, then*

$$x \in |J_{\mathrm{det}}(E)| = J(E) \iff h^1(L_x, E|_{L_x}(-1)) \geq 1.$$

Moreover, $\mathbf{R}q_*(p^*(E(-1)))$ is represented by a two-term complex

$$0 \longrightarrow \mathcal{A}^0 \xrightarrow{\phi} \mathcal{A}^1 \longrightarrow 0, \quad \mathrm{rank} \mathcal{A}^0 = \mathrm{rank} \mathcal{A}^1.$$

(2) *If $c_1(E) = -1$, then*

$$x \in |J_{\mathrm{det}}(E)| = J(E) \iff h^1(L_x, E|_{L_x}) \geq 1.$$

Moreover, $\mathbf{R}q_*(p^*E)$ is represented by a two-term complex

$$0 \longrightarrow \mathcal{B}^0 \xrightarrow{\psi} \mathcal{B}^1 \longrightarrow 0, \quad \mathrm{rank} \mathcal{B}^0 - \mathrm{rank} \mathcal{B}^1 = 1.$$

Proof. If $c_1(E) = 0$, write

$$E|_{L_x} \cong \mathcal{O}_{L_x}(a) \oplus \mathcal{O}_{L_x}(-a), \quad a \geq 0.$$

The generic value is $a = 0$, and

$$h^1(L_x, E|_{L_x}(-1)) = h^1(\mathcal{O}_{L_x}(a-1)) + h^1(\mathcal{O}_{L_x}(-a-1)) = a.$$

Thus this group is nonzero exactly on the jumping lines. If $c_1(E) = -1$, similarly

$$E|_{L_x} \cong \mathcal{O}_{L_x}(a) \oplus \mathcal{O}_{L_x}(-a-1), \quad a \geq 0,$$

the generic value is $a = 0$, and

$$h^1(L_x, E|_{L_x}) = h^1(\mathcal{O}_{L_x}(a)) + h^1(\mathcal{O}_{L_x}(-a-1)) = a.$$

This proves both cohomological characterizations.

Apply Theorem 2.2 first to $p^*(E(-1))$ and then to p^*E . The higher direct images vanish in degrees at least two because the fibers of q are curves, so the two derived pushforwards have two-term locally free representatives. Their rank differences are the fiberwise Euler characteristics. On a general line these are

$$\chi(\mathcal{O}_L(-1)^{\oplus 2}) = 0 \quad \text{and} \quad \chi(\mathcal{O}_L \oplus \mathcal{O}_L(-1)) = 1,$$

respectively, giving the asserted rank formulas. Since the Grassmannian is smooth and projective, it has the resolution property: the perfect complexes admit bounded global locally free representatives. Their Tor-amplitude in $[0, 1]$ permits truncation to these two degrees. Thus the determinant and Porteous calculations below may be made with global representatives as well. $\square$

In 1977, Barth proved that, for a semistable rank-two bundle E on $\mathbb{P}^n$ with $c_1(E) = 0$, the nonempty locus of jumping lines is a pure codimension-one subscheme of $G(2, n+1)$ [1]. It is the support of $R^1 q_*(p^*(E(-1)))$, and its divisor class is $c_2(E)\sigma_1$. For $c_1(E) = -1$, Barth observed that the ordinary jumping locus may have components of codimension one and/or two [1, §0.3]. On $\mathbb{P}^2$, Hulek addressed related phenomena by introducing jumping lines of the second kind [10]. The determinantal framework above gives a uniform scheme structure for the ordinary jumping locus; in the odd case it gives a cycle-class formula when the locus has its expected codimension.

Theorem 3.5. *Let E be a semistable rank 2 vector bundle on $\mathbb{P}^n$ ($n \geq 2$) with $c_1(E) \in \{0, -1\}$ and $c_2(E) = k$. If the jumping locus $J_{\det}(E) \subset G(2, n+1)$ is non-empty, then*

- (1) (Barth) *If $c_1(E) = 0$, equip the jumping locus with the determinantal scheme structure*

$$J_{\det}(E) := V(\text{Fitt}_0(R^1 q_*(p^*(E(-1)))).$$

Its underlying closed set is $\text{Supp}(R^1 q_*(p^*(E(-1))))$. It is an effective divisor in $G(2, n+1)$ with $\dim J_{\det}(E) = 2n - 3$ and

$$[J_{\det}(E)] = c_2(E)\sigma_1 = k\sigma_1 \in A^1(G(2, n+1)).$$

- (2) *If $c_1(E) = -1$, equip the jumping locus with its natural determinantal scheme structure*

$$J_{\det}(E) := V(\text{Fitt}_0(R^1 q_*(p^*E))).$$

Its underlying closed set is $\text{Supp}(R^1 q_*(p^*E))$, and every irreducible component has codimension at most 2. If $J_{\det}(E)$ has no divisorial component, then it is Cohen–Macaulay and pure of codimension 2, with

$$\dim J_{\det}(E) = 2n - 4.$$

Under this hypothesis its Chow class $[J_{\det}(E)] \in A^2(G(2, n+1))$ is given by

$$[J_{\det}(E)] = \frac{k^2}{2}\sigma_1^2 - \frac{k}{2}\sigma_{1,1}.$$

Proof. (1) Case $c_1(E) = 0$: By Lemma 3.1, $\mathbf{R}q_*(p^*(E(-1)))$ is represented by $\phi : \mathcal{A}^0 \rightarrow \mathcal{A}^1$ with $\text{rank}(\mathcal{A}^0) = \text{rank}(\mathcal{A}^1) = m$. A line L_x is a jumping line if and only if $h^1(L_x, E|_{L_x}(-1)) \geq 1 \iff \text{rank}(\phi_x) \leq m - 1$. The locus where ϕ drops rank is defined globally by the vanishing of the top determinant:

$$J_{\det}(E) = \{x \in X \mid \det(\phi_x) = 0\} = V(\det \phi).$$

The general line is nonjumping, so this determinant is not the zero section. Since $\det(\phi)$ is a section of the line bundle $\det(\mathcal{A}^1) \otimes \det(\mathcal{A}^0)^\vee$, $J_{\det}(E)$ is an effective divisor in the smooth $(2n - 2)$ -dimensional manifold $G(2, n+1)$. Thus, $\dim J_{\det}(E) = 2n - 3$.

The fundamental class of the divisor $J_{\det}(E)$ in the Chow ring $A^1(X)$ is given by the first Chern class of the determinant line bundle

$$[J_{\det}(E)] = c_1(\det(\mathcal{A}^1) \otimes (\det \mathcal{A}^0)^\vee) = -c_1(\mathbf{R}q_*(p^*(E(-1)))).$$

Applying relative Grothendieck–Riemann–Roch to the smooth morphism $q : Y \rightarrow X$ of relative dimension one gives

$$\text{ch}(\mathbf{R}q_*(p^*(E(-1)))) = q_*(\text{ch}(p^*(E(-1))) \text{Td}(T_q)).$$

Here ch denotes the Chern character, Td the Todd class, and T_q the relative tangent line bundle. Let $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ and $\xi = p^*h = c_1(\mathcal{O}_{\mathbb{P}(U)}(1))$. Since $c_1(E) = 0$ and $c_2(E) = kh^2$,

$$\text{ch}(p^*(E(-1))) = 2 - 2\xi + (1 - k)\xi^2 + \cdots.$$

The relative Euler sequence gives

$$t := c_1(T_q) = 2\xi - \sigma_1, \quad \text{Td}(T_q) = 1 + \frac{t}{2} + \frac{t^2}{12} + \cdots.$$

Using $q_*(\xi) = 1$, $q_*(\xi^2) = \sigma_1$, and $q_*(q^*\sigma_1^2) = 0$, the codimension-two part of the integrand yields

$$\text{ch}_1(\mathbf{R}q_*(p^*(E(-1)))) = q_* \left[(1 - k)\xi^2 - \xi t + \frac{t^2}{6} \right]$$

$$\begin{aligned}
&= q_* \left[\left(-\frac{1}{3} - k \right) \xi^2 + \frac{1}{3} \sigma_1 \xi + \frac{1}{6} \sigma_1^2 \right] \\
&= -k \sigma_1.
\end{aligned}$$

Consequently,

$$[J_{\det}(E)] = -\text{ch}_1(\mathbf{R}q_*(p^*(E(-1)))) = k\sigma_1.$$

Thus the jumping divisor belongs to $|\mathcal{O}_X(k)|$. Its degree as a projective subvariety in the Plücker embedding is $k \deg X$; this should be distinguished from the coefficient k of its divisor class.

- (2) Case $c_1(E) = -1$: Put $X = G(2, n+1)$ and $\mathcal{M} = R^1q_*(p^*E)$. We give $J_{\det}(E)$ its natural determinantal scheme structure

$$(4) \quad J_{\det}(E) := V(\text{Fitt}_0(\mathcal{M})).$$

The underlying closed set of this scheme is $\text{Supp}(\mathcal{M})$. We first prove the unconditional codimension bound. Under the additional hypothesis that there is no divisorial component, we then prove purity and compute the fundamental class.

Step 1: the determinantal ideal. By Lemma 3.1, locally on X the perfect complex $\mathbf{R}q_*(p^*E)$ is represented by a morphism

$$\psi : \mathcal{B}^0 \longrightarrow \mathcal{B}^1, \quad \text{rank}(\mathcal{B}^0) = r+1, \quad \text{rank}(\mathcal{B}^1) = r.$$

Its first cohomology sheaf is $\mathcal{M} = \text{coker}(\psi)$. Consequently,

$$\text{Fitt}_0(\mathcal{M}) = I_r(\psi),$$

where $I_r(\psi)$ is generated locally by the $r \times r$ maximal minors of a matrix representing ψ . Cohomology and base change gives, for every closed point $x \in X$,

$$x \in J_{\det}(E) \iff H^1(L_x, E|_{L_x}) \neq 0 \iff \text{rank}(\psi_x) \leq r-1.$$

Thus (4) is precisely the degeneracy locus $D_{r-1}(\psi)$, including its scheme structure. The generalized principal ideal theorem for determinantal ideals gives

$$(5) \quad \text{codim}_X W \leq ((r+1) - (r-1))(r - (r-1)) = 2$$

for every irreducible component W of $J_{\det}(E)$.

Step 2: the expected-codimension hypothesis. The general line has splitting type $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ by Grauert–Müllich–Barth, so $J_{\det}(E) \neq X$. Consequently every irreducible component has positive codimension. Together with (5), this shows unconditionally that every irreducible component has codimension one or two.

Barth explicitly noted that in the odd-first-Chern-class case both possibilities may occur [1, §0.3]; hence a divisorial component cannot be excluded in general. Assume now, as in the second assertion of the theorem, that $J_{\det}(E)$ has no divisorial component. Then

$$(6) \quad \text{codim}_X W \geq 2$$

for every irreducible component $W \subseteq J_{\det}(E)$. Combining (5) and (6), and using the hypothesis that $J_{\det}(E)$ is non-empty, we obtain

$$(7) \quad \text{codim}_X W = 2$$

for every irreducible component W of $J_{\det}(E)$.

Step 3: scheme-theoretic purity. Equality (7) proves equidimensionality of the underlying closed set, but one must still exclude embedded components. This follows from the determinantal structure. At any point $x \in J_{\det}(E)$, the local ring $R = \mathcal{O}_{X,x}$ is regular, hence Cohen–Macaulay. The ideal $I_r(\psi)_x$ is the ideal of maximal minors of an $r \times (r+1)$ matrix and has height two. Therefore it has the expected grade two. Since R is Cohen–Macaulay and $\text{ht } I_r(\psi)_x = 2$, one has $\text{grade } I_r(\psi)_x = 2$. The acyclicity theorem for the Eagon–Northcott complex therefore applies. In the present $r \times (r+1)$ maximal-minor

case, this complex is precisely the Hilbert–Burch complex (see [4, Section 20]), and it gives an exact free resolution

$$0 \longrightarrow R^r \xrightarrow{\psi_x^t} R^{r+1} \longrightarrow R \longrightarrow R/I_r(\psi)_x \longrightarrow 0.$$

Consequently, $R/I_r(\psi)_x$ is a perfect R -module of projective dimension two. Since R is regular, the Auslander–Buchsbaum formula shows that it is Cohen–Macaulay of codimension two. In particular, all its associated primes are minimal and have height two. Hence the determinantal scheme $J_{\det}(E)$ has no embedded components and is pure of codimension two. Since $\dim X = 2n - 2$, it follows that

$$\dim J_{\det}(E) = 2n - 4.$$

Step 4: The class of jumping loci in the Chow ring. Let

$$\mathcal{K} := \mathbf{R}q_*(p^*E) = [\mathcal{B}^0] - [\mathcal{B}^1] \quad \text{and} \quad \mathcal{V} := -\mathcal{K} = [\mathcal{B}^1] - [\mathcal{B}^0] \quad \text{in } K^0(X).$$

Here $K^0(X)$ is the Grothendieck group of vector bundles on X , and the brackets denote their classes. The virtual rank of $\mathcal{V}$ is -1 . Since the degeneracy locus has its expected codimension two, the Giambelli–Thom–Porteous formula applies. Here $e = r + 1$, $f = r$, and the rank bound is $r - 1$. Therefore the relevant determinant is

$$[J_{\det}(E)] = \Delta_1^2(c(\mathcal{V})) = \det \begin{pmatrix} c_1(\mathcal{V}) & c_2(\mathcal{V}) \\ 1 & c_1(\mathcal{V}) \end{pmatrix} = c_1(\mathcal{V})^2 - c_2(\mathcal{V}).$$

Step 5: computation by relative GRR. Let U be the tautological rank-two subbundle on X , and identify Y with $\mathbb{P}(U)$. Put

$$\xi = p^*c_1(\mathcal{O}_{\mathbb{P}^n}(1)), \quad c_1(U) = -\sigma_1, \quad c_2(U) = \sigma_{1,1}.$$

The projective-bundle relation and its pushforward consequences are

$$\begin{aligned} \xi^2 - \sigma_1\xi + \sigma_{1,1} &= 0, \\ q_*(\xi) &= 1, \quad q_*(\xi^2) = \sigma_1, \quad q_*(\xi^3) = \sigma_1^2 - \sigma_{1,1}. \end{aligned}$$

The relative tangent bundle is a line bundle with

$$t := c_1(T_q) = 2\xi - \sigma_1, \quad \mathrm{Td}(T_q) = 1 + \frac{t}{2} + \frac{t^2}{12} + \cdots.$$

There is no term of degree three in the Todd series of a line bundle. Since $c_1(E) = -h$ and $c_2(E) = kh^2$, where $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$, we have

$$\mathrm{ch}(p^*E) = 2 - \xi + \left(\frac{1}{2} - k\right)\xi^2 + \frac{3k-1}{6}\xi^3 + \cdots.$$

Relative Grothendieck–Riemann–Roch gives

$$\mathrm{ch}(\mathcal{K}) = q_*(\mathrm{ch}(p^*E) \mathrm{Td}(T_q)).$$

The codimension-two part of the integrand gives

$$\begin{aligned} \mathrm{ch}_1(\mathcal{K}) &= q_* \left[\left(\frac{1}{2} - k\right)\xi^2 - \frac{\xi t}{2} + \frac{t^2}{6} \right] \\ &= q_* \left[\left(\frac{1}{6} - k\right)\xi^2 - \frac{1}{6}\sigma_1\xi + \frac{1}{6}\sigma_1^2 \right] \\ &= -k\sigma_1. \end{aligned}$$

Here $q_*(q^*\sigma_1^2) = 0$, since q has relative dimension one. Similarly, the codimension-three part gives

$$\begin{aligned} \mathrm{ch}_2(\mathcal{K}) &= q_* \left[\frac{3k-1}{6}\xi^3 + \left(\frac{1}{2} - k\right)\frac{\xi^2 t}{2} - \frac{\xi t^2}{12} \right] \\ &= q_* \left[-\frac{k}{2}\xi^3 + \left(\frac{k}{2} + \frac{1}{12}\right)\sigma_1\xi^2 - \frac{1}{12}\sigma_1^2\xi \right] \\ &= \frac{k}{2}\sigma_{1,1}. \end{aligned}$$

It follows that

$$c_1(\mathcal{V}) = \text{ch}_1(\mathcal{V}) = k\sigma_1, \quad \text{ch}_2(\mathcal{V}) = -\frac{k}{2}\sigma_{1,1}.$$

For a virtual bundle one still has $\text{ch}_2 = (c_1^2 - 2c_2)/2$; hence

$$c_2(\mathcal{V}) = \frac{1}{2}c_1(\mathcal{V})^2 - \text{ch}_2(\mathcal{V}) = \frac{k^2}{2}\sigma_1^2 + \frac{k}{2}\sigma_{1,1}.$$

Substitution into the Porteous determinant now yields

$$[J_{\det}(E)] = c_1(\mathcal{V})^2 - c_2(\mathcal{V}) = \frac{k^2}{2}\sigma_1^2 - \frac{k}{2}\sigma_{1,1}.$$

Equivalently, using $\sigma_1^2 = \sigma_2 + \sigma_{1,1}$,

$$[J_{\det}(E)] = \frac{k^2}{2}\sigma_2 + \frac{k(k-1)}{2}\sigma_{1,1}.$$

In particular, when $n = 2$ one has $\sigma_2 = 0$, and the length of the zero-dimensional jumping-line scheme is $k(k-1)/2$. □

Remark 1. Although the formula in Theorem 3.5(2) is written as

$$[J_{\det}(E)] = \frac{k^2}{2}\sigma_2 + \frac{k(k-1)}{2}\sigma_{1,1},$$

it always defines an integral Chow class. If $n = 2$, then $G(2, 3) \cong (\mathbb{P}^2)^\vee$ and $\sigma_2 = 0$, so the remaining coefficient $k(k-1)/2$ is an integer. If $n \geq 3$, restrict E to a linear subspace $\mathbb{P}^3 \subset \mathbb{P}^n$. The Schwarzenberger integrality condition for a rank-two vector bundle F on $\mathbb{P}^3$ states that

$$c_1(F)c_2(F) \equiv 0 \pmod{2};$$

see [11]. For $F = E|_{\mathbb{P}^3}$ one has $c_1(F) = -H$ and $c_2(F) = kH^2$, and hence k is even. Equivalently, since F is locally free of rank two and therefore $c_3(F) = 0$, this parity condition follows directly from Hirzebruch–Riemann–Roch:

$$\chi(\mathbb{P}^3, F) = 1 - \frac{3k}{2} \in \mathbb{Z}.$$

Thus odd k can occur here only when $n = 2$, where the σ_2 term vanishes; for $n \geq 3$ both coefficients in the displayed Chow class are integers.

3.4. The almost-uniform boundary and an explicit plane example.

Remark 2. The first bound applies only when the final linear section has dimension at least three. On $\mathbb{P}^2$ there are stable rank-two bundles with one jumping line, as shown by Ellia [6, Lemma 3] and by the explicit construction in Example 1 below. Thus the almost-uniform rank estimate used in the proof cannot be extended to dimension two. For $m = 0$ and $n \geq 3$, the theorem reproduces Ellia’s lower bound; we do not assert that it is sharp for every pair (n, m) .

Example 1 (A single jumping line). *Fix $k \geq 2$ and homogeneous coordinates $[x : y : z]$ on $\mathbb{P}^2$. Define a rank-two bundle E_k by*

$$(8) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-k-1) \xrightarrow{(x^k, y^k, z)} \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-k) \longrightarrow E_k \longrightarrow 0.$$

The three entries have no common zero, so the cokernel is locally free. Its Chern classes are $c_1(E_k) = -1$ and $c_2(E_k) = k$. Moreover $H^0(E_k) = 0$, since line bundles on $\mathbb{P}^2$ have no first cohomology. The usual normalized rank-two stability criterion therefore shows that E_k is stable; equivalently, a saturated line subsheaf of nonnegative degree would give a nonzero section of $E_k(-a)$ for some $a \geq 0$, which is excluded by the same resolution.

Let $D = V(z)$. Restricting (8) to a line ℓ gives

$$H^1(\ell, E_k|_{\ell}) \simeq \text{coker} \left(H^1(\mathcal{O}_{\ell}(-k-1)) \xrightarrow{z|_{\ell}} H^1(\mathcal{O}_{\ell}(-k)) \right).$$

For $\ell \neq D$, the dual multiplication map on sections of $\mathcal{O}_\ell(k-2)$ is injective, so this cokernel is zero. On D , the map is zero and the cokernel has dimension $k-1$. In fact

$$E_k|_D \simeq \mathcal{O}_D(k-1) \oplus \mathcal{O}_D(-k).$$

Thus $J(E_k) = \{[D]\}$.

The presentation also identifies its scheme structure. On the affine chart of the dual plane parametrizing $\ell_{u,v} = V(z + ux + vy)$, use x, y as coordinates on each line. In the corresponding cohomology bases, multiplication by $z|_{\ell_{u,v}} = -ux - vy$ gives the bidiagonal matrix of Lemma 4.1 with $m = k-1$. Therefore

$$\text{Ann } \mathcal{M}_{E_k} = \text{Fitt}_0 \mathcal{M}_{E_k} = \mathcal{I}_{[D]}^{k-1}, \quad \text{length } J_{\det}(E_k) = \frac{k(k-1)}{2}.$$

This realizes the no-divisorial-component hypothesis for every $k \geq 2$ on $\mathbb{P}^2$ and agrees with Theorem 3.5. The existence of rank-two bundles with one jumping line is classical in this context; compare [6, Lemma 3]. The explicit presentation records the thickness as well as the jumping set.

4. MORE EXAMPLES

The construction of Theorem 2.1 is not restricted to projective spaces. It also applies naturally when the parameter space is a Fano scheme of rational curves on a Fano manifold. We illustrate this with the Fano surface of lines on a cubic threefold.

Let $Z \subset \mathbb{P}^4$ be a smooth cubic threefold, and let $X = F(Z)$ be its Fano scheme of lines. The scheme X is a smooth projective surface. Let

$$Y = \{(z, [L]) \in Z \times X \mid z \in L\}$$

be the universal line, with its natural projections

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ \downarrow p & & \\ Z & & \end{array}$$

The morphism q is a $\mathbb{P}^1$ -bundle, and its fiber over $[L]$ is the line L itself.

Proposition 4.1. *Let $Z \subset \mathbb{P}^4$ be a smooth cubic threefold and let $X = F(Z)$. A line $L \subset Z$ is called a line of the first type if*

$$N_{L/Z} \cong \mathcal{O}_L \oplus \mathcal{O}_L,$$

and a line of the second type if

$$N_{L/Z} \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1).$$

Then

$$H^1(L, T_Z(-1)|_L) \neq 0 \iff L \text{ is a line of the second type.}$$

Consequently, if $F_2(Z) \subseteq F(Z)$ denotes the locus of lines of the second type, then

$$(9) \quad F_2(Z) = \text{Supp}(R^1 q_* p^*(T_Z(-1)))$$

as closed subsets of $F(Z)$. Its natural determinantal scheme structure, in the sense of Definition 2.1, is

$$(10) \quad F_2(Z)_{\det} := V(\text{Fitt}_0(R^1 q_* p^*(T_Z(-1)))) .$$

In particular, the locus of second-type lines is algebraic and closed.

Proof. Let $L \subset Z$ be a line. Since $N_{L/\mathbb{P}^4} \cong \mathcal{O}_L(1)^{\oplus 3}$ and $N_{Z/\mathbb{P}^4}|_L \cong \mathcal{O}_L(3)$, the normal bundle sequence is

$$(11) \quad 0 \longrightarrow N_{L/Z} \longrightarrow \mathcal{O}_L(1)^{\oplus 3} \longrightarrow \mathcal{O}_L(3) \longrightarrow 0.$$

It follows that $N_{L/Z}$ has rank two and degree zero. By the Birkhoff–Grothendieck theorem, after ordering the two summands we may write

$$N_{L/Z} \cong \mathcal{O}_L(a) \oplus \mathcal{O}_L(-a) \quad \text{with } a \geq 0.$$

The injection in (11) implies $a \leq 1$. Thus $a = 0$ or $a = 1$, which gives precisely the two splitting types in the statement.

Next consider the tangent-normal sequence

$$(12) \quad 0 \longrightarrow T_L \cong \mathcal{O}_L(2) \longrightarrow T_Z|_L \longrightarrow N_{L/Z} \longrightarrow 0.$$

This sequence splits in both cases. In fact, its extension class lies in

$$\mathrm{Ext}^1(N_{L/Z}, \mathcal{O}_L(2)) \cong H^1(L, N_{L/Z}^\vee(2)).$$

For a line of the first type, the group on the right is $H^1(\mathcal{O}_L(2)^{\oplus 2}) = 0$. For a line of the second type it is

$$H^1(\mathcal{O}_L(1) \oplus \mathcal{O}_L(3)) = 0.$$

Consequently,

$$T_Z|_L \cong \begin{cases} \mathcal{O}_L(2) \oplus \mathcal{O}_L \oplus \mathcal{O}_L, & L \text{ of the first type,} \\ \mathcal{O}_L(2) \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1), & L \text{ of the second type.} \end{cases}$$

After tensoring by $\mathcal{O}_L(-1)$, this becomes

$$T_Z(-1)|_L \cong \begin{cases} \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1)^{\oplus 2}, & L \text{ of the first type,} \\ \mathcal{O}_L(1) \oplus \mathcal{O}_L \oplus \mathcal{O}_L(-2), & L \text{ of the second type.} \end{cases}$$

Using

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = 0 \iff d \geq -1,$$

we obtain

$$h^1(L, T_Z(-1)|_L) = \begin{cases} 0, & L \text{ of the first type,} \\ 1, & L \text{ of the second type.} \end{cases}$$

This proves the asserted cohomological characterization.

Finally, apply Theorem 2.1 to the vector bundle $T_Z(-1)$ and to the universal family $q : Y \rightarrow F(Z)$. All fibers of q are one-dimensional, so their cohomology vanishes in degrees at least two. Cohomology and base change identifies the fiber of the first higher direct image at $[L]$ with

$$(R^1 q_* p^*(T_Z(-1))) \otimes \kappa([L]) \cong H^1(L, T_Z(-1)|_L).$$

The preceding computation therefore gives (9). The zeroth Fitting ideal of this coherent sheaf gives the canonical determinantal structure (10), completing the proof. $\square$

This example also illustrates how Theorem 2.1 can be combined with the determinantal description and relative Grothendieck–Riemann–Roch. Once a locally free complex representing $\mathbf{R}q_* p^*(T_Z(-1))$ is chosen, one may study the dimension, scheme-theoretic multiplicities, and cycle class of $F_2(Z)$ on the Fano surface by means of its minors and the Chern character of the derived pushforward. The same construction applies to Fano schemes of lines on other smooth Fano hypersurfaces, provided the relevant generic and exceptional splitting types are known. Proposition 4.3 below carries out the determinant and cycle-class calculation in this example after the quintic computation.

4.1. A quintic surface of general type. We give an example on a surface of general type for which the parameter space is smooth. Let $Z \subset \mathbb{P}^3$ be a smooth quintic surface containing a line L , and write $H = \mathcal{O}_Z(1)$. For instance, the Fermat quintic

$$Z_F = V(x_0^5 + x_1^5 + x_2^5 + x_3^5) \subset \mathbb{P}^3$$

is smooth and contains the line

$$V(x_0 - \alpha x_1, x_2 - \beta x_3)$$

whenever $\alpha^5 = \beta^5 = -1$. More generally, if $L = V(x_0, x_1)$, a general equation $x_0 F_4 + x_1 G_4 = 0$ defines a smooth quintic containing L . Bertini's theorem gives smoothness away from L , and the restrictions of the general quartics F_4 and G_4 to L have no common zero, which gives smoothness along L . By adjunction,

$$K_Z = (K_{\mathbb{P}^3} + Z)|_Z \cong \mathcal{O}_Z(1) = H.$$

Thus K_Z is ample and Z is a minimal surface of general type.

Let

$$X = |H| \cong (\mathbb{P}^3)^\vee.$$

A point $[\Pi] \in X$ corresponds to the plane section $C_\Pi = Z \cap \Pi$. These sections have a fixed Hilbert polynomial and form the universal divisor

$$Y = \{(z, [\Pi]) \in Z \times X \mid z \in \Pi\},$$

with projections

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ \downarrow p & & \\ Z & & \end{array}$$

In particular, X is a smooth closed parameter subscheme of the Hilbert scheme. In the notation of Theorem 2.1, we take the auxiliary line bundle $\mathcal{L} = \mathcal{O}_Y$.

Proposition 4.2. *Let $Z \subset \mathbb{P}^3$ be a smooth quintic surface containing a line L , and set*

$$E = \mathcal{O}_Z(2L + 2H).$$

For a plane $\Pi \subset \mathbb{P}^3$, one has

$$(13) \quad H^1(C_\Pi, E|_{C_\Pi}) \neq 0 \iff \Pi \supset L.$$

Consequently, if

$$\Lambda = \{[\Pi] \in (\mathbb{P}^3)^\vee \mid \Pi \supset L\} \cong \mathbb{P}^1,$$

then

$$|J_{\text{supp}}(E)| = \text{Supp}(R^1 q_* p^* E) = \Lambda$$

as closed sets. Moreover, the annihilator structure of Theorem 2.1 and the zeroth-Fitting determinantal structure agree and are equal to the thickening of Λ defined by the cube of its ideal:

$$(14) \quad \text{Ann}_{\mathcal{O}_X}(R^1 q_* p^* E) = \text{Fitt}_0(R^1 q_* p^* E) = \mathcal{I}_\Lambda^3.$$

Proof. We first compute the cohomology groups on Z . Adjunction for $L \cong \mathbb{P}^1$ gives

$$-2 = \deg K_L = (K_Z + L) \cdot L = H \cdot L + L^2 = 1 + L^2,$$

so

$$(15) \quad L^2 = -3.$$

For $k = 1, 2$, restriction to L gives

$$(16) \quad 0 \longrightarrow \mathcal{O}_Z(kH) \longrightarrow \mathcal{O}_Z(L + kH) \longrightarrow \mathcal{O}_L(k - 3) \longrightarrow 0.$$

The hypersurface sequence of $Z \subset \mathbb{P}^3$ shows that $H^1(Z, \mathcal{O}_Z(kH)) = 0$ for all k . For $k = 2$, both $H^0(\mathcal{O}_L(-1))$ and $H^1(\mathcal{O}_L(-1))$ vanish. Serre duality also gives

$$H^2(Z, \mathcal{O}_Z(2H)) = H^0(Z, \mathcal{O}_Z(-H))^\vee = 0.$$

It follows from (16) that

$$H^1(Z, \mathcal{O}_Z(L + 2H)) = H^2(Z, \mathcal{O}_Z(L + 2H)) = 0.$$

For $k = 1$, we verify the same assertion separately. Since $H^0(\mathcal{O}_L(-2)) = 0$, sequence (16) gives

$$h^0(Z, \mathcal{O}_Z(L + H)) = h^0(Z, \mathcal{O}_Z(H)) = 4.$$

The Riemann–Roch formula on the quintic surface, for which $\chi(\mathcal{O}_Z) = 1 + p_g(Z) = 5$, gives

$$\begin{aligned} \chi(\mathcal{O}_Z(L + H)) &= 5 + \frac{1}{2}(L + H) \cdot ((L + H) - K_Z) \\ &= 5 + \frac{1}{2}(L + H) \cdot L = 5 + \frac{L^2 + H \cdot L}{2} = 4. \end{aligned}$$

Furthermore, by Serre duality,

$$h^2(Z, \mathcal{O}_Z(L + H)) = h^0(Z, \mathcal{O}_Z(-L)) = 0.$$

Therefore

$$(17) \quad H^i(Z, \mathcal{O}_Z(L + kH)) = 0 \quad (i = 1, 2; k = 1, 2).$$

A second restriction sequence is

$$(18) \quad 0 \longrightarrow \mathcal{O}_Z(L + kH) \longrightarrow \mathcal{O}_Z(2L + kH) \longrightarrow \mathcal{O}_L(k - 6) \longrightarrow 0, \quad k = 1, 2.$$

Using (17), its cohomology sequence yields

$$(19) \quad H^1(Z, \mathcal{O}_Z(2L + H)) \cong H^1(L, \mathcal{O}_L(-5)), \quad h^1(Z, \mathcal{O}_Z(2L + H)) = 4,$$

$$(20) \quad H^1(Z, \mathcal{O}_Z(2L + 2H)) \cong H^1(L, \mathcal{O}_L(-4)), \quad h^1(Z, \mathcal{O}_Z(2L + 2H)) = 3.$$

The same sequence also gives

$$(21) \quad H^2(Z, \mathcal{O}_Z(2L + H)) = H^2(Z, \mathcal{O}_Z(2L + 2H)) = 0.$$

We now use the universal plane section. Let $\pi : Z \times X \rightarrow X$ be the projection and let $i : Y \hookrightarrow Z \times X$ be the inclusion. The universal equation gives an exact sequence on $Z \times X$:

$$(22) \quad 0 \longrightarrow \mathrm{pr}_Z^* E(-H) \otimes \pi^* \mathcal{O}_X(-1) \xrightarrow{\ell_{\mathrm{univ}}} \mathrm{pr}_Z^* E \longrightarrow i_* p^* E \longrightarrow 0,$$

where the last term is supported on Y . Applying $\mathbf{R}\pi_*$ and using (21), we obtain a presentation

$$(23) \quad H^1(Z, E(-H)) \otimes \mathcal{O}_X(-1) \xrightarrow{\psi} H^1(Z, E) \otimes \mathcal{O}_X \longrightarrow R^1 q_* p^* E \longrightarrow 0.$$

Under the identifications (19)–(20), the fiber of ψ at $[\Pi]$ is multiplication by the restricted linear form:

$$H^1(L, \mathcal{O}_L(-5)) \xrightarrow{\ell_{\Pi|L}} H^1(L, \mathcal{O}_L(-4)).$$

If Π does not contain L , then $\ell_{\Pi|L}$ is nonzero. The dual map under Serre duality on $L \cong \mathbb{P}^1$ is

$$H^0(L, \mathcal{O}_L(2)) \xrightarrow{\ell_{\Pi|L}} H^0(L, \mathcal{O}_L(3)),$$

which is injective. Hence the original map is surjective. If $\Pi \supset L$, then $\ell_{\Pi|L} = 0$, so the original map is zero. Cohomology and base change applied to (22) therefore gives

$$h^1(C_{\Pi}, E|_{C_{\Pi}}) = \begin{cases} 0, & \Pi \not\supset L, \\ 3, & \Pi \supset L. \end{cases}$$

This proves (13). Since the fibers of q are curves, their cohomology vanishes in degrees at least two, so Theorem 2.1 gives the asserted equality of closed supports.

It remains to compute the scheme structure. Choose homogeneous coordinates on $L \cong \mathbb{P}^1$. The restriction of the universal linear form can be written as $as + bt$, where a, b are independent linear forms on X and

$$\Lambda = V(a, b) \subset X.$$

With the monomial bases of $H^1(\mathcal{O}_L(-5))$ and $H^1(\mathcal{O}_L(-4))$ dual to the standard bases of $H^0(\mathcal{O}_L(3))$ and $H^0(\mathcal{O}_L(2))$, respectively, the map ψ in (23) is represented, up to invertible changes of bases, by

$$(24) \quad \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & b \end{pmatrix}.$$

Indeed, with respect to the monomial bases $\{s^2, st, t^2\}$ and $\{s^3, s^2t, st^2, t^3\}$, multiplication by $as + bt$ is represented by

$$\begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & b & a \\ 0 & 0 & b \end{pmatrix}.$$

The map ψ is its dual under Serre duality, and is therefore represented by the transpose matrix in (24).

Its four maximal minors are

$$a^3, \quad a^2b, \quad ab^2, \quad b^3.$$

Consequently,

$$\mathrm{Fitt}_0(R^1 q_* p^* E) = (a, b)^3 = \mathcal{I}_{\Lambda}^3.$$

For completeness, let e_1, e_2, e_3 denote the generators of the cokernel of (24). Its relations are

$$ae_1 = 0, \quad be_1 + ae_2 = 0, \quad be_2 + ae_3 = 0, \quad be_3 = 0.$$

They show that every cubic in (a, b) annihilates the cokernel. To see that there are no annihilators of smaller degree, note first that ae_2 and ae_3 form a basis in degree one. Thus a linear form which annihilates all e_i must be zero. In degree two, put $q = \alpha a^2 + \beta ab + \gamma b^2$. The relations give

$$qe_1 = \gamma a^2 e_3, \quad qe_2 = -\beta a^2 e_3, \quad qe_3 = \alpha a^2 e_3,$$

where $a^2 e_3 = b^2 e_1 = -abe_2$ is nonzero. Hence q annihilates all three generators only if $\alpha = \beta = \gamma = 0$. It follows that

$$\text{Ann}_{\mathcal{O}_X}(R^1 q_* p^* E) = (a, b)^3.$$

This proves (14) and completes the proof. $\square$

Remark 3. The exponent three in $\mathcal{I}_\Lambda^3$ is not the generic cycle multiplicity. A transverse two-dimensional local ring has quotient with basis $1, a, b, a^2, ab, b^2$, so the transverse length is six. Thus the common support and Fitting scheme of the quintic example has fundamental cycle $6[\Lambda]$. The general form of this calculation appears in Theorem 4.4 below.

4.2. The determinantal divisor of second-type lines. We return to the cubic threefold of Proposition 4.1. Write $F = F(Z)$ and let U and Q be the restrictions to F of the tautological subbundle and quotient bundle on $G(2, 5)$. In this subsection $Y = \mathbb{P}(U)$ again denotes the universal line, with projections $q : Y \rightarrow F$ and $p : Y \rightarrow Z$. The Fano surface is smooth and integral, and a general line is of the first type; see [2] and [8, §3].

Proposition 4.3. *There is a natural map of rank-three bundles on F ,*

$$(25) \quad \alpha : Q \longrightarrow \text{Sym}^2 U^\vee,$$

induced by the derivatives of the cubic equation, with

$$R^1 q_* p^*(T_Z(-1)) \simeq \text{coker } \alpha.$$

Its determinant defines the Fitting scheme $D = J_{\det}(T_Z(-1))$ of second-type lines. This is a nonempty effective Cartier divisor satisfying

$$(26) \quad D \in |2K_F|, \quad \deg_{\mathbb{P}^1} D = 90, \quad p_a(D) = 136.$$

The annihilator support scheme equals D .

Proof. The splitting calculation in Proposition 4.1 gives $h^1(L, T_Z(-1)|_L) = 0$ or 1 according to the type of L . Write $\mathbb{P}^4 = \mathbb{P}(V)$, where $\dim V = 5$. The tangent sequence of the cubic, twisted by -1 , is

$$0 \longrightarrow T_Z(-1) \longrightarrow T_{\mathbb{P}^4}(-1)|_Z \longrightarrow \mathcal{O}_Z(2) \longrightarrow 0.$$

Pulling it back to Y and applying q_* gives a presentation

$$(27) \quad V \otimes \mathcal{O}_F \longrightarrow \text{Sym}^2 U^\vee \longrightarrow R^1 q_* p^*(T_Z(-1)) \longrightarrow 0.$$

Indeed, the Euler sequence and $\mathbf{R}q_* \mathcal{O}_Y(-1) = 0$ identify $q_* p^*(T_{\mathbb{P}^4}(-1)|_Z)$ with $V \otimes \mathcal{O}_F$ and give zero first higher direct image. Also $q_* \mathcal{O}_Y(2) = \text{Sym}^2 U^\vee$.

If $f \in \text{Sym}^3 V^\vee$ is the cubic equation, the first map in (27) sends v to the restriction of the directional derivative $\partial_v f$ to the line. It vanishes on U : the restriction of f to the universal two-dimensional subspace is identically zero, so all its derivatives in directions in that subspace vanish there as well. Thus the map factors through $Q = (V \otimes \mathcal{O}_F)/U$, yielding (25) with the same cokernel. This also shows directly that the Fitting scheme is the classical determinantal locus of the polar map.

A general line has the first type, so $\det \alpha$ is not the zero section. Since F is smooth and integral, its vanishing scheme is an effective Cartier divisor. Put $H_F = c_1(U^\vee)|_F$. The determinant line is

$$\det(\text{Sym}^2 U^\vee) \otimes (\det Q)^{-1} \simeq (\det U^\vee)^2,$$

because $\det(\text{Sym}^2 U^\vee) = (\det U^\vee)^3$ and $\det Q = \det U^\vee$. Consequently $D \in |\mathcal{O}_F(2)|$.

The surface F is the regular zero locus of the section of $\mathrm{Sym}^3 U^\vee$ defined by f on $G(2, 5)$. Adjunction gives

$$K_F = (K_{G(2,5)} + c_1(\mathrm{Sym}^3 U^\vee))|_F = (-5H_F + 6H_F) = H_F.$$

The standard Plücker intersection number is $H_F^2 = 45$ [8, §3]. It can also be computed from

$$\int_{G(2,5)} c_1(U^\vee)^2 c_4(\mathrm{Sym}^3 U^\vee) = 45.$$

The nontrivial ample line bundle $\mathcal{O}_F(2)$ has no nowhere-vanishing section, so D is nonempty. Its degree is $2H_F^2 = 90$, and adjunction for an effective Cartier divisor on a smooth surface gives

$$p_a(D) = 1 + \frac{D \cdot (D + K_F)}{2} = 1 + 3K_F^2 = 136.$$

Finally, the fiberwise first cohomology has dimension at most one. Proposition 2.5 therefore implies that the annihilator and Fitting ideals coincide. $\square$

The linear equivalence $D \sim 2K_F$ is classical [2, 10.21]. The proposition identifies the cohomological construction with that divisor and gives its invariants in the same notation. Smoothness of D requires an additional generality assumption on Z ; it holds for a general cubic threefold, as recalled in [8, §3]. Formula (26) concerns arithmetic genus and applies without assuming that D is smooth or irreducible.

Remark 4. For $T_Z(-1)|_L$, the Euler characteristic is two, so the bound from the full derived pushforward is $\mathrm{codim} D \leq 3$. The rank-three map (25) gives the stronger description as a divisor because it removes the rank-two subbundle U in the kernel of (27). A presentation of the first cohomology sheaf can thus provide a sharper codimension estimate than the rank difference of the full derived pushforward.

4.3. Plane sections of surfaces of degree $d \geq 5$. We first record the local algebra needed to determine annihilators as well as maximal minors.

Lemma 4.1. *Let $A = \mathbb{C}[a, b]$, let $m \geq 1$, and let C_m be the cokernel of*

$$(28) \quad T_m = \begin{pmatrix} a & b & 0 & \cdots & 0 \\ 0 & a & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a & b \end{pmatrix} : A^{m+1} \longrightarrow A^m.$$

Then

$$(29) \quad \mathrm{Fitt}_0(C_m) = \mathrm{Ann}_A(C_m) = (a, b)^m.$$

These identities persist under polynomial extension and localization. The vector-space length of $A/(a, b)^m$ is $m(m+1)/2$.

Proof. The maximal minors, up to sign, are $a^m, a^{m-1}b, \dots, b^m$, proving the Fitting assertion. For the annihilator, let $e_1, \dots, e_m$ be the standard generators of C_m . Its relations are

$$ae_1 = 0, \quad be_i + ae_{i+1} = 0 \quad (1 \leq i < m), \quad be_m = 0.$$

Give each e_i degree zero. In degree $j \geq 0$, the relations reduce every monomial multiple of a generator to a linear combination of

$$a^j e_{j+1}, \dots, a^j e_m.$$

Indeed, powers of b can be moved to powers of a using $be_i = -ae_{i+1}$, while $a^i e_i = 0$. Thus $\dim_{\mathbb{C}}(C_m)_j \leq m - j$ for $0 \leq j < m$, and all higher degrees vanish.

Set $B = A/(a, b)^m$ and let $B^\vee = \mathrm{Hom}_{\mathbb{C}}(B, \mathbb{C})$, with the A -action $(r\lambda)(u) = \lambda(ru)$. For a monomial $a^i b^j$ of degree less than m , write $(a^i b^j)^\vee$ for its coefficient functional on B . Sending

$$e_i \longmapsto (-1)^{i-1} (a^{i-1} b^{m-i})^\vee \quad (1 \leq i \leq m)$$

respects all the displayed relations. The images generate $B^\vee$, since multiplication by a and b on the dual lowers the corresponding monomial exponents. Hence there is a surjection $C_m \rightarrow B^\vee$. The dimension estimate gives

$$\dim_{\mathbb{C}} C_m \leq \sum_{j=0}^{m-1} (m-j) = \dim_{\mathbb{C}} B^\vee = \frac{m(m+1)}{2},$$

so this surjection is an isomorphism. The dual module $B^\vee$ is faithful over B : if r annihilates it, then $\lambda(r) = 0$ for every coefficient functional, forcing $r = 0$ in B . It follows that $\text{Ann}_A C_m = (a, b)^m$.

Fitting ideals commute with base change. Annihilators of finitely presented modules commute with flat base change, as follows by applying flat base change to the kernel of $A \rightarrow \text{Hom}_A(C_m, C_m)$. Polynomial extensions and localizations are flat, proving the remaining assertion. $\square$

Let $Z_d \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 5$ containing a line L , and put $H = \mathcal{O}_{Z_d}(1)$. Such surfaces exist for every d : for example, the Fermat surface contains $V(x_0 - \alpha x_1, x_2 - \beta x_3)$ whenever $\alpha^d = \beta^d = -1$. Adjunction gives $K_{Z_d} = (d-4)H$, which is ample. Thus these are minimal surfaces of general type.

Let $X = |H| \cong (\mathbb{P}^3)^\vee$ and let $q : Y \rightarrow X$ be the universal family of plane sections $C_\Pi = Z_d \cap \Pi$, with evaluation $p : Y \rightarrow Z_d$. The complete linear system is a closed Hilbert parameter scheme, and the universal divisor is flat over it. Let

$$\Lambda = \{[\Pi] \in X \mid \Pi \supset L\} \cong \mathbb{P}^1, \quad E_d = \mathcal{O}_{Z_d}(2L + (d-3)H).$$

We take $\mathcal{L} = \mathcal{O}_Y$, so $\mathcal{F} = p^*E_d$.

Theorem 4.4. *With this notation, put $m = d-2$ and $\mathcal{M}_d = R^1 q_* p^* E_d$. Then*

(1) *The fiberwise cohomology is*

$$h^1(C_\Pi, E_d|_{C_\Pi}) = \begin{cases} 0, & \Pi \not\supset L, \\ m, & \Pi \supset L. \end{cases}$$

(2) *The cohomology sheaf admits a presentation*

$$(30) \quad \mathcal{O}_X(-1)^{m+1} \xrightarrow{\psi} \mathcal{O}_X^m \longrightarrow \mathcal{M}_d \longrightarrow 0,$$

whose matrix is $T_m(a, b)$, where a, b are independent linear equations for Λ .

(3) *Its annihilator and zeroth Fitting ideals are*

$$(31) \quad \text{Ann } \mathcal{M}_d = \text{Fitt}_0 \mathcal{M}_d = \mathcal{I}_\Lambda^m.$$

The common scheme is Cohen–Macaulay of pure codimension two, and its fundamental cycle is

$$(32) \quad [J_{\text{supp}}(E_d)] = [J_{\text{det}}(E_d)] = \binom{d-1}{2} [\Lambda].$$

Proof. Write $K = K_{Z_d}$ in the cohomology calculations. Adjunction on $L \cong \mathbb{P}^1$ gives

$$-2 = (K + L) \cdot L = (d-4) + L^2, \quad L^2 = 2 - d.$$

We first show

$$(33) \quad H^i(\mathcal{O}_{Z_d}(K + L)) = H^i(\mathcal{O}_{Z_d}(K + H + L)) = 0 \quad (i = 1, 2).$$

For the first sheaf, use

$$0 \longrightarrow \mathcal{O}_{Z_d}(K) \longrightarrow \mathcal{O}_{Z_d}(K + L) \longrightarrow \mathcal{O}_L(-2) \longrightarrow 0.$$

The hypersurface sequence gives $H^1(\mathcal{O}_{Z_d}) = 0$, hence $H^1(\mathcal{O}_{Z_d}(K)) = 0$. By Serre duality, $H^2(\mathcal{O}_{Z_d}(K)) \cong \mathbb{C}$, whereas $H^2(\mathcal{O}_{Z_d}(K+L)) = 0$ because $H^0(\mathcal{O}_{Z_d}(-L)) = 0$. Thus the boundary map from $H^1(\mathcal{O}_L(-2)) \cong \mathbb{C}$ to $H^2(\mathcal{O}_{Z_d}(K))$ is an isomorphism, proving the first vanishing. For the second sheaf, the analogous restriction quotient is $\mathcal{O}_L(-1)$, which has no cohomology, and Kodaira vanishing gives $H^i(\mathcal{O}_{Z_d}(K + H)) = 0$ for $i > 0$. This proves (33).

Now restrict the divisors with coefficient two along L :

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{Z_d}(K+L) \longrightarrow \mathcal{O}_{Z_d}(K+2L) \longrightarrow \mathcal{O}_L(-d) \longrightarrow 0, \\ 0 &\longrightarrow \mathcal{O}_{Z_d}(K+H+L) \longrightarrow \mathcal{O}_{Z_d}(K+H+2L) \longrightarrow \mathcal{O}_L(1-d) \longrightarrow 0. \end{aligned}$$

Since $E_d(-H) = \mathcal{O}_{Z_d}(K+2L)$ and $E_d = \mathcal{O}_{Z_d}(K+H+2L)$, we obtain functorial isomorphisms

$$\begin{aligned} (34) \quad H^1(Z_d, E_d(-H)) &\simeq H^1(L, \mathcal{O}_L(-d)), & h^1(Z_d, E_d(-H)) &= d-1, \\ H^1(Z_d, E_d) &\simeq H^1(L, \mathcal{O}_L(1-d)), & h^1(Z_d, E_d) &= d-2, \\ H^2(Z_d, E_d(-H)) &= H^2(Z_d, E_d) = 0. \end{aligned}$$

Let $\pi : Z_d \times X \rightarrow X$ be the projection and $i : Y \hookrightarrow Z_d \times X$ the inclusion. The universal section gives

$$0 \longrightarrow \mathrm{pr}_{Z_d}^* E_d(-H) \otimes \pi^* \mathcal{O}_X(-1) \xrightarrow{\ell_{\mathrm{univ}}} \mathrm{pr}_{Z_d}^* E_d \longrightarrow i_* p^* E_d \longrightarrow 0.$$

Applying $R\pi_*$ and using (34) yields

$$H^1(Z_d, E_d(-H)) \otimes \mathcal{O}_X(-1) \xrightarrow{\psi} H^1(Z_d, E_d) \otimes \mathcal{O}_X \longrightarrow \mathcal{M}_d \longrightarrow 0.$$

The cohomology identifications are induced by restriction to L , hence intertwine multiplication by the universal plane equation with multiplication by its restriction to L .

Choose coordinates s, t on L and write this restriction as $as+bt$, so $\Lambda = V(a, b)$. Serre duality on L identifies the dual of ψ with multiplication

$$H^0(\mathcal{O}_L(d-3)) \otimes \mathcal{O}_X \xrightarrow{as+bt} H^0(\mathcal{O}_L(d-2)) \otimes \mathcal{O}_X(1).$$

In monomial bases its transpose is the matrix $T_m(a, b)$ in (28). This proves (2). If $(a, b) \neq (0, 0)$, multiplication by $as+bt$ is injective, so ψ is surjective on the fiber. If $(a, b) = (0, 0)$, its fiber is zero. Proposition 2.3 now gives (1), including for reducible or singular plane sections.

On either of the affine coordinate charts covering Λ , the coordinate ring is a polynomial extension of $\mathbb{C}[a, b]$. Lemma 4.1 therefore gives (31) near Λ . Away from Λ the cohomology sheaf vanishes, so both ideals are the unit ideal. This proves (3) globally.

Locally along Λ , the quotient by $(a, b)^m$ is a polynomial extension of the Artinian ring $\mathbb{C}[a, b]/(a, b)^m$. It is Cohen–Macaulay, and remains so after localization. At the generic point of Λ , the normal directions contribute the length

$$\sum_{j=0}^{m-1} (j+1) = \frac{m(m+1)}{2} = \binom{d-1}{2},$$

which proves the cycle formula. $\square$

The case $d = 5$ recovers Proposition 4.2, including its explicit 3×4 matrix. The separate calculation there explains concretely how the general exponent $d-2$ arises.

4.4. Direct sums and non-reduced base change. The same family gives two explicit comparisons of the constructions.

Corollary 4.5. *Let $s \geq 1$ and set $E_{d,s} = E_d^{\oplus s}$. The jumping set is still Λ , and*

$$(35) \quad \mathrm{Ann}(R^1 q_* p^* E_{d,s}) = \mathcal{I}_\Lambda^{d-2}, \quad \mathrm{Fitt}_0(R^1 q_* p^* E_{d,s}) = \mathcal{I}_\Lambda^{s(d-2)}.$$

For $s > 1$, the annihilator support is a proper closed subscheme of the Fitting scheme. Their fundamental cycles are, respectively,

$$\binom{d-1}{2}[\Lambda], \quad \binom{s(d-2)+1}{2}[\Lambda].$$

Proof. Higher direct image commutes with finite direct sums. Apply Proposition 2.5 and Theorem 4.4. The powers of the regular ideal (a, b) are strictly decreasing, and their generic transverse lengths are given by Lemma 4.1. $\square$

Example 2 (A transverse infinitesimal parameter). *Fix a point of Λ and an affine chart of X with coordinates a, b, c , centered at that point, such that $\Lambda = V(a, b)$. Let*

$$R = \mathbb{C}[\varepsilon]/(\varepsilon^2), \quad T = \operatorname{Spec} R,$$

and define a closed immersion $g : T \hookrightarrow X$ by

$$a \mapsto \varepsilon, \quad b \mapsto 0, \quad c \mapsto 0.$$

This is a length-two subscheme of the Hilbert parameter space and therefore itself parametrizes a flat family of plane sections. Put $m = d - 2 \geq 3$. The base change of (30) has matrix

$$T_m(\varepsilon, 0) = \begin{pmatrix} \varepsilon & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \varepsilon & 0 \end{pmatrix}.$$

Consequently, with $\mathcal{M}_T = R^1 q_{T*} p_T^* E_d$,

$$\mathcal{M}_T \simeq (R/(\varepsilon))^{\oplus m}, \quad \operatorname{Ann}_R \mathcal{M}_T = (\varepsilon), \quad \operatorname{Fitt}_0^R \mathcal{M}_T = (\varepsilon^m) = 0.$$

Thus

$$J_{\operatorname{supp}}(p_T^* E_d) = T_{\operatorname{red}}, \quad J_{\operatorname{det}}(p_T^* E_d) = T.$$

On the other hand, the base change of the original annihilator ideal $\mathcal{I}_\Lambda^m$ is zero. Therefore

$$J_{\operatorname{supp}}(p_T^* E_d) \neq J_{\operatorname{supp}}(p^* E_d) \times_X T, \quad J_{\operatorname{det}}(p_T^* E_d) = J_{\operatorname{det}}(p^* E_d) \times_X T.$$

The two structures agree on the original smooth parameter space but become different on this non-reduced base. This supplies a geometric instance of the distinction made in Proposition 2.3.

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