

JUMPING LOCI IN DEFORMATIONS OF VECTOR BUNDLES ON PROJECTIVE SPACES

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ABSTRACT. Let $\pi : Y = \mathbb{P}(\mathcal{V}) \rightarrow X$ be a projective bundle over an n -dimensional smooth projective manifold X , where $\mathcal{V}$ is a vector bundle of rank $m + 1$ on X . For a holomorphic vector bundle E on Y , we investigate the jumping loci of global sections along the projective fibers $F_t \cong \mathbb{P}^m$. We approach this problem by translating the fiberwise cohomological jumping condition into a rank-drop problem of morphisms of vector bundles on the base manifold. Specifically, to overcome the obstruction of infinite homological resolutions, we leverage relative Castelnuovo-Mumford regularity. By applying an appropriate twist, we canonically compress the derived direct image complex $\mathbb{R}\pi_* E(k)$ in the derived category $D^b(\text{Coh}(X))$ into a two-term complex of locally free sheaves. Applying the Giambelli-Thom-Porteous formula to the resulting determinantal degeneracy locus, and combining this with the Grothendieck-Riemann-Roch theorem and Gysin pushforward integrations, we explicitly compute the cycle classes, dimensions, and geometric degrees of these jumping loci. Ultimately, we demonstrate that these geometric invariants are rigidly and completely determined by the Chern classes of E , the base bundle $\mathcal{V}$, and the tangent bundle of X .

Keywords: vector bundle; jumping locus; Chern class

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1. INTRODUCTION

A central theme in modern algebraic geometry and vector bundle theory pioneered by Barth, Hartshorne, Horrocks, Hulek, etc. is understanding the degeneration and stratification of vector bundles along parametrized families of subvarieties. Over the past few decades, this field has been extensively developed across multiple frontiers. A systematic framework for global cohomological jumping loci was established by Green-Lazarsfeld ([10]), who connected the deformation of local systems and Picard varieties to topological invariants of algebraic varieties. In the realm of moduli spaces, researchers such as Atiyah, Drinfel'd, Hitchin, Friedman, etc., demonstrated that the jumping behavior of vector bundles restricted to lines, planes, or rational curves serves as a fundamental detector for the geometric and boundary structures of moduli spaces. More recently, the study of jumping loci has found profound applications in the Higgs bundle and non-abelian Hodge theory by Simpson ([15]), as well as in the degeneration of vector bundles on higher-dimensional projective varieties and Fano manifolds. Despite these extensive developments, explicitly computing the algebraic cycles and numerical degrees of these jumping loci has remained a challenging problem.

Let X be a smooth projective manifold of dimension $\dim(X) = n$ defined over the complex numbers $\mathbb{C}$. Let $\mathcal{V}$ be a vector bundle or locally free sheaf of rank $m + 1$ on X , and consider the projective bundle projection

$$\pi : Y = \mathbb{P}(\mathcal{V}) \longrightarrow X.$$

For any closed point $t \in X$, the fiber $F_t = \pi^{-1}(t)$ is isomorphic to the projective space $\mathbb{P}^m$. Let E be a holomorphic vector bundle of rank r on the total space Y .

While the restriction $E_t := E|_{F_t}$ exhibits a generic splitting behavior (or generic cohomology dimensions) for a general point $t \in X$, there exists a Zariski-closed subscheme of X where the

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topology of E_t mutates dramatically. We define the s -th jumping locus of global sections as

$$J_s = \{t \in X \mid h^0(F_t, E_t(k)) \geq s\},$$

where $E(k) := E \otimes \mathcal{O}_Y(k)$ is a suitable twist by the tautological line bundle $\mathcal{O}_Y(1)$. Our objective is to compute the cycle class $[J_s]$ in the Chow ring $A^*(X)$ and its corresponding degree with respect to a fixed polarization, using only the Chern classes of E and $\mathcal{V}$.

Theorem 1.1. *Let X be a connected, smooth projective manifold of dimension $\dim(X) = n$ over $\mathbb{C}$, endowed with a fixed polarization $H = c_1(\mathcal{O}_X(1)) \in A^1(X)$. Let $\mathcal{V}$ be a vector bundle of rank $m+1$ on X , and consider the projective bundle projection $\pi : Y = \mathbb{P}(\mathcal{V}) \rightarrow X$ with fibers $F_t = \pi^{-1}(t) \cong \mathbb{P}^m$. Let E be a holomorphic vector bundle of rank r on Y .*

- *Then there exists some $k \in \mathbb{Z}$ such that the higher fiberwise cohomology vanishes:*

$$h^i(F_t, E_t(k)) = 0 \quad \text{for all } i > 1 \text{ and for all } t \in X.$$

- *Let $\chi := \chi(F_t, E_t(k)) = h^0(F_t, E_t(k)) - h^1(F_t, E_t(k))$ denote the constant fiberwise Euler characteristic. Then there exists a globally defined two-term complex of vector bundles $A \rightarrow B$ on X such that:*

$$\mathbb{R}\pi_* E(k) \cong_{D^b(\text{Coh}(X))} [0 \rightarrow A \xrightarrow{\phi} B \rightarrow 0].$$

Moreover, the associated degeneracy locus J_s is the determinantal degeneracy locus of the vector bundle homomorphism $\phi : A \rightarrow B$.

- *Assuming that the associated degeneracy locus J_s achieves its expected codimension (e.g., via generic transversality of the defining morphism), then:*
 - (1) *The expected codimension of J_s in X is $s(s-\chi)$. Specifically, if $s(s-\chi) \leq n$, its actual dimension is $\dim(J_s) = n - s(s-\chi)$; if $s(s-\chi) > n$, then $J_s = \emptyset$.*
 - (2) *In the Chow ring $A^{s(s-\chi)}(X)$, the fundamental cycle class $[J_s]$ is given by the $s \times s$ determinant:*

$$[J_s] = \det(c_{s-\chi+j-i}(W))_{1 \leq i, j \leq s} \in A^{s(s-\chi)}(X),$$

where $W := -\mathbb{R}\pi_*(E(k)) = -\pi_1(E(k)) \in K(X)$ is the virtual vector bundle representing the negative of the derived pushforward.

- (3) *If $s(s-\chi) \leq n$, the degree of $[J_s]$ with respect to the polarization H is computed by the intersection number*

$$\deg_H([J_s]) = \int_X [J_s] \cdot H^{n-s(s-\chi)} = \int_X \det(c_{s-\chi+j-i}(W))_{1 \leq i, j \leq s} \cdot H^{n-s(s-\chi)}.$$

The core idea of the proof of our main theorem is to translate a fiberwise cohomological jumping condition into a rank-drop problem on the morphisms of vector bundles, and then apply classical intersection theory. By leveraging Grauert's base change theorem and the vanishing of higher cohomology, the derived direct image complex $\mathbb{R}\pi_* E(k)$ in the derived category $D^b(\text{Coh}(X))$ can be represented globally on the base manifold X by a two-term complex of locally free sheaves $A \xrightarrow{\phi} B$. Using an elementary theorem in linear algebra, the condition that the space of global sections has dimension at least s is recast as a constraint on the rank of the bundle homomorphism ϕ , thereby identifying the jumping locus J_s scheme-theoretically with a determinantal degeneracy locus $D_{a-s}(\phi)$, where a is the rank of A . Geometrically, this generic condition ensures that the bundle morphism ϕ varies sufficiently generally over X such that the minor determinants defining the rank-drop impose independent conditions. Under generic transversality, this locus achieves its expected codimension $s(s-\chi)$, which allows the Giambelli-Thom-Porteous formula to explicitly express the Chow cycle class $[J_s]$ as an $s \times s$ Schur determinant of Chern classes of the virtual difference bundle $W = -\mathbb{R}\pi_* E(k)$. Finally, the numerical degree of the locus relative to a fixed polarization is computed by taking the intersection product of this determinantal cycle class with the complementary power of the hyperplane section class $H^{n-s(s-\chi)}$ and integrating over the fundamental class of X .

The study of jumping loci is not merely a technical pursuit of exceptional fibers; it is a fundamental gateway to understanding the stratification of moduli spaces of vector bundles and the geometry of parameter spaces. Historically, computing the exact algebraic cycles and numerical degrees of these jumping loci has been a formidable challenge due to the infinite homological

resolutions required to track fiberwise cohomological mutations. The main contribution of this paper lies in establishing a deterministic and computationally rigid bridge between relative sheaf cohomology and classical intersection theory. By introducing the technique of homological compression via Castelnuovo-Mumford regularity, we bypass the need for infinite resolutions. Our main result demonstrates a striking geometric rigidity: despite the wild local topological variations of the bundle restricted to individual fibers, the global Chow cycles and degree invariants of these jumping stratification loci are completely, universally, and explicitly governed by the Chern classes of the initial bundle and the base manifold. This provides algebraic geometers with a powerful and explicit computational machinery, which we successfully apply to classical mysteries such as the Horrocks-Mumford bundle on $\mathbb{P}^4$ ([11]). Moreover, this theorem can also be used to assist in classifying uniform bundles on higher dimensional projective spaces or general rational homogeneous spaces (see [13], [5] or [4] for the introduction of the uniform bundles) such as [7] Proposition 3.4 and [6] Lemma 3.4 without complicated arguments.

The remainder of this article is organized as follows. In Section 2, we establish the theoretical foundation of homological compression. We show that by enforcing a suitable twist governed by relative Castelnuovo-Mumford regularity along the fibers, the higher direct image sheaves vanish identically. Using Grothendieck's derived base change theorem and the perfection of complexes, we canonically compress the derived pushforward complex $\mathbb{R}\pi_*E(k)$ in $D^b(\text{Coh}(X))$ into a globally defined two-term complex of locally free sheaves, translating cohomological jumps into linear algebraic rank-drop conditions. In Section 3, we synthesize our two-term complex representation with classical intersection theory. By identifying the jumping loci scheme-theoretically with determinantal degeneracy loci, we apply the Giambelli-Thom-Porteous formula. Combining this with the Grothendieck-Riemann-Roch theorem and Gysin pushforward integrations, we derive the explicit determinantal formulas for the Chow cycle classes $[J_s]$ and their numerical degrees for proving Theorem 1.1. Finally, in Section 4, we demonstrate the power and computational effectiveness of our machinery through three concrete, progressively complex examples: a rank-2 extension bundle over $\mathbb{P}^1 \times \mathbb{P}^1$, the pullback of instanton bundles over $\mathbb{P}^3$ parameterized by the Grassmannian $\mathbb{G}(1, 3)$, and notably, the pullback of the celebrated indecomposable Horrocks-Mumford bundle over $\mathbb{P}^4$ onto $\mathbb{G}(2, 4)$, where we rigorously prove the vanishing of global sections on generic 2-planes via monad cohomology and conormal filtrations.

2. HOMOLOGICAL COMPRESSION VIA CASTELNUOVO-MUMFORD REGULARITY AND DERIVED BASE CHANGE

Let X be a smooth projective manifold of dimension n over $\mathbb{C}$, and let $\pi : Y = \mathbb{P}(\mathcal{V}) \rightarrow X$ be a projective bundle associated to a vector bundle $\mathcal{V}$ of rank $m + 1$ on X . For any closed point $x \in X$, the fiber is a projective space $F_x := \pi^{-1}(x) \cong \mathbb{P}^m$.

Let E be a holomorphic vector bundle on the total space Y . To analyze the geometric loci where the fiberwise cohomology jumps without becoming entangled in infinite free resolutions, we must compress the derived direct image complex $\mathbb{R}\pi_*E(k)$ into a finite, highly manageable algebraic form. This section establishes how Castelnuovo-Mumford regularity along the fibers allows us to reduce this complex to a simple two-term homomorphism of vector bundles on X .

2.1. Relative Castelnuovo-Mumford Regularity. On a single projective space $\mathbb{P}^m$, a coherent sheaf $\mathcal{F}$ is said to be p -regular in the sense of Castelnuovo-Mumford if its cohomology groups satisfy:

$$H^i(\mathbb{P}^m, \mathcal{F}(p - i)) = 0 \quad \text{for all } i \geq 1.$$

A celebrated property of Castelnuovo-Mumford regularity is that if $\mathcal{F}$ is p -regular, then it is also $(p + 1)$ -regular, and more generally, $H^i(\mathbb{P}^m, \mathcal{F}(k)) = 0$ for all $i \geq 1$ and $k \geq p - i$.

We extend this notion relatively to the fibration $\pi : Y \rightarrow X$. For each point $x \in X$, let $E_x := E|_{F_x}$ denote the restriction of E to the projective fiber $F_x \cong \mathbb{P}^m$. Because the base manifold X is compact and π is a proper flat morphism, the families of vector bundles $\{E_x\}_{x \in X}$ are bounded. Consequently, by Semicontinuity Theorem, there exists an integer k_0 such that for every fiber F_x , the restricted sheaf E_x is k_0 -regular.

Now, choose an integer $k \geq k_0 - 1$. By the definition of p -regularity (with $p = k_0$), for any index $i \geq 2$, we have:

$$k \geq k_0 - 1 \geq k_0 - i \implies H^i(F_x, E_x(k)) = 0 \quad \text{for all } x \in X \text{ and all } i \geq 2.$$

In other words, by selecting an appropriate twist k by the relative tautological line bundle $\mathcal{O}_Y(1)$, we force the vanishing of all fiberwise cohomology groups of degree 2 and higher, across every fiber of the bundle. Thus, we have proved the first part of Theorem 1.1.

2.2. Vanishing of Higher Direct Image Sheaves. We now transition from fiberwise cohomology to global sheaves on the base X . By definition, the higher direct image sheaves $R^i\pi_*E(k)$ are the right derived functors of the pushforward π_* , associated to the presheaf $U \mapsto H^i(\pi^{-1}(U), E(k))$, where U is an open set on X .

Because $\pi : Y \rightarrow X$ is a proper morphism of Noetherian schemes and $E(k)$ is flat over X (as it is locally free), the Base Change Theorem asserts that for any point $x \in X$, there is a canonical base change morphism:

$$\phi^i(x) : R^i\pi_*E(k) \otimes_{\mathcal{O}_{X,x}} \kappa(x) \longrightarrow H^i(F_x, E_x(k)),$$

where $\kappa(x) \cong \mathbb{C}$ is the residue field at x . Because we chose k such that $H^i(F_x, E_x(k)) = 0$ for all $i \geq 2$ and all $x \in X$, the fiber cohomology vanishes identically. By Nakayama's Lemma applied to the stalk $(R^i\pi_*E(k))_x$, the vanishing of the fiber modulo the maximal ideal implies the vanishing of the stalk itself:

$$(R^i\pi_*E(k))_x = 0 \quad \text{for all } x \in X \text{ and all } i \geq 2.$$

Since a coherent sheaf with zero stalks everywhere is the zero sheaf, we conclude:

$$R^i\pi_*E(k) = 0 \quad \text{for all } i \geq 2.$$

2.3. The Derived Direct Image Complex and Perfectness. We recall the classical definition of the higher direct image functors $R^i\pi_*$ within the framework of abelian categories of sheaves. Let $\text{Mod}(\mathcal{O}_Y)$ and $\text{Mod}(\mathcal{O}_X)$ be the categories of $\mathcal{O}_Y$ -modules and $\mathcal{O}_X$ -modules, respectively. The pushforward functor $\pi_* : \text{Mod}(\mathcal{O}_Y) \rightarrow \text{Mod}(\mathcal{O}_X)$ is left-exact but generally fails to be right-exact. The higher direct image sheaves $R^i\pi_*E(k)$ are defined as the right derived functors of π_* , evaluated through an injective resolution of the sheaf $E(k)$. Specifically, we choose an acyclic resolution of $E(k)$ by injective $\mathcal{O}_Y$ -modules, denoted by $I^\bullet$:

$$0 \longrightarrow E(k) \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \dots$$

Applying the left-exact pushforward functor π_* to the complex $I^\bullet$ (omitting the initial term $E(k)$) yields a complex of sheaves on the base manifold X :

$$\pi_*I^\bullet : \quad 0 \longrightarrow \pi_*I^0 \xrightarrow{\pi_*d^0} \pi_*I^1 \xrightarrow{\pi_*d^1} \pi_*I^2 \longrightarrow \dots$$

By definition, the classical i -th higher direct image sheaf is the i -th cohomology sheaf of this pushed-forward injective complex:

$$R^i\pi_*E(k) := \mathcal{H}^i(\pi_*I^\bullet) = \frac{\ker(\pi_*d^i)}{\text{im}(\pi_*d^{i-1})}.$$

In modern algebraic geometry, rather than tracking individual derived sheaves $R^i\pi_*E(k)$ scattered across different degrees, Grothendieck's derived category $D(X) := D(\text{Mod}(\mathcal{O}_X))$ embeds the entire homological history into a single object. The derived pushforward functor $\mathbb{R}\pi_* : D(Y) \rightarrow D(X)$ is constructed by mapping a complex to the pushforward of its injectively resolved counterpart. For the single sheaf $E(k)$, viewed as a complex concentrated in degree 0, its image under $\mathbb{R}\pi_*$ is defined precisely as the complex $\pi_*I^\bullet$ itself:

$$\mathbb{R}\pi_*E(k) := \pi_*I^\bullet \in D(X).$$

By returning to the definition established above, taking the standard i -th cohomology sheaf functor $\mathcal{H}^i(-)$ of this derived object directly recovers the classical higher direct image sheaf. This yields the fundamental tautological identification:

$$(1) \quad \mathcal{H}^i(\mathbb{R}\pi_*E(k)) = \mathcal{H}^i(\pi_*I^\bullet) = R^i\pi_*E(k).$$

The injective complex $\pi_*I^\bullet$ is a purely theoretical tool; its components are infinitely large, non-coherent, and utterly incompatible with intersection-theoretic operations like Chern class assignments. To bridge the gap between theory and computation, we invoke the property of “perfectness”.

Because the projection $\pi : Y \rightarrow X$ is a proper, flat morphism and $E(k)$ is locally free, it follows from the relative finiteness theorems in SGA 6 ([3] Exposé III, Cor. 4.8.1) that the derived direct image $\mathbb{R}\pi_*E(k)$ is a perfect complex (meaning it is locally quasi-isomorphic to a bounded complex of locally free sheaves of finite rank) in $D^b(\text{Coh}(X))$. Consequently, there exists a bounded complex of finite-rank locally free sheaves $\mathcal{L}^\bullet$ on X that is isomorphic to $\mathbb{R}\pi_*E(k)$ within the derived category $D^b(\text{Coh}(X))$:

$$\mathcal{L}^\bullet \cong_{D^b(\text{Coh}(X))} \mathbb{R}\pi_*E(k).$$

Therefore, we obtain:

$$\mathcal{H}^i(\mathcal{L}^\bullet) \cong \mathcal{H}^i(\mathbb{R}\pi_*E(k)) \quad \text{for all } i \in \mathbb{Z}.$$

Combining with (1), we have

$$\mathcal{H}^i(\mathcal{L}^\bullet) \cong R^i\pi_*E(k).$$

Let us examine the cohomology sheaves $\mathcal{H}^i(\mathcal{L}^\bullet)$ of this complex. For $i < 0$, $\mathcal{H}^i(\mathcal{L}^\bullet) = R^i\pi_*E(k) = 0$ because $E(k)$ is a sheaf whose cohomology vanishes in negative degrees. For $i \geq 2$, $\mathcal{H}^i(\mathcal{L}^\bullet) = R^i\pi_*E(k) = 0$ by our Castelnuovo-Mumford regularity construction. Thus, the entire homological content of the complex $\mathbb{R}\pi_*E(k)$ is strictly concentrated in degrees 0 and 1.

2.4. Geometric Consequences of the Two-Term Representation. Next, we claim that a perfect complex whose derived fiber cohomology is concentrated in degrees 0 and 1 can be globally represented by a two-term complex of vector bundles. To execute this rigorously, we rely on the fundamental results from [3] (or the local version in [12], Chapter II, Section 5, Lemma 1) regarding strictly perfect complexes.

Theorem 2.1. (cf. [3], Exposé II, Lem. 2.2.8) *Let X be a quasi-compact scheme equipped with an ample family of invertible sheaves (e.g., a smooth projective manifold), and let $\mathcal{K}^\bullet$ be a perfect complex of $\mathcal{O}_X$ -modules. Assume that for every closed point $x \in X$, the cohomology of the derived fiber satisfies:*

$$H^i(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X}^{\mathbb{L}} \kappa(x)) = 0 \quad \text{for all } i \notin [a, b],$$

where $\kappa(x)$ is the residue field at x . Then there exists a globally defined strictly perfect complex of finite-rank locally free sheaves $\mathcal{E}^\bullet$ concentrated in degrees $a \leq i \leq b$ (i.e., $\mathcal{E}^i = 0$ for $i < a$ and $i > b$) such that $\mathcal{E}^\bullet$ is isomorphic to $\mathcal{K}^\bullet$ in the derived category $\mathcal{D}^b(\text{Coh}(X))$.

In our setting, we take $\mathcal{K}^\bullet = \mathbb{R}\pi_*E(k)$. As demonstrated by Lemma 2.1 below, Grothendieck’s derived base change theorem ensures that the derived fiber of our complex computes exactly the sheaf cohomology of the geometric fiber: $H^i(\mathbb{R}\pi_*E(k) \otimes_{\mathcal{O}_X}^{\mathbb{L}} \kappa(x)) \cong H^i(F_x, E_x(k))$. Due to our Castelnuovo-Mumford regularity twist, these cohomology groups vanish for all $i \notin [0, 1]$. Therefore, setting $[a, b] = [0, 1]$ in Theorem 2.1, we conclude that there exists a globally defined two-term complex of locally free sheaves $A \rightarrow B$ (where $A := \mathcal{E}^0$ and $B := \mathcal{E}^1$) such that:

$$\mathbb{R}\pi_*E(k) \cong_{D^b(\text{Coh}(X))} [0 \rightarrow A \xrightarrow{\phi} B \rightarrow 0],$$

with A sitting in degree 0 and B sitting in degree 1. Thus, the second part of the Theorem 1.1 has been proved.

By replacing the abstract derived functor $\mathbb{R}\pi_*E(k)$ with the concrete vector bundle homomorphism $\phi : A \rightarrow B$, we translate topological deformation phenomena into elementary linear

algebra. For any point $x \in X$, restricting our two-term complex to the fiber over x (which corresponds to taking the derived tensor product $\otimes^{\mathbb{L}} \kappa(x)$) yields a correspondence between the linear map of finite-dimensional vector spaces $\phi(x) : A_x \otimes \kappa(x) \rightarrow B_x \otimes \kappa(x)$ and the sheaf cohomology of the fiber F_x .

Lemma 2.1. *Let $\pi : Y \rightarrow X$ be a proper, flat morphism of smooth algebraic varieties, and let E be a locally free sheaf on Y . Suppose the derived direct image $\mathbb{R}\pi_* E(k)$ is represented in $D^b(\text{Coh}(X))$ by a perfect complex $\mathcal{L}^\bullet$ of locally free $\mathcal{O}_X$ -modules. Then, for any closed point $x \in X$, there is a canonical isomorphism of $\mathbb{C}$ -vector spaces:*

$$H^i(F_x, E_x(k)) \cong H^i(\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \kappa(x)),$$

where $F_x = \pi^{-1}(x)$ is the geometric fiber over x , $E_x(k) = E(k)|_{F_x}$, and $\kappa(x)$ is the residue field at x .

Proof. We proceed by analyzing the derived functors via the Cartesian fiber square. Let $x \in X$ be a closed point. We construct the following Cartesian diagram:

$$\begin{array}{ccc} F_x & \xrightarrow{j_x} & Y \\ \pi_x \downarrow & & \downarrow \pi \\ \text{Spec } \kappa(x) & \xrightarrow{i_x} & X \end{array}$$

where i_x is the natural closed immersion of the point x , j_x is the induced inclusion of the fiber F_x , and π_x is the structure morphism of the fiber over the point. By Grothendieck's Derived Base Change Theorem, (see [3], Exposé IV, Proposition 3.1.1]) since π is proper and flat, and the base extensions are Tor-independent, there exists a canonical quasi-isomorphism in the derived category $D^b(\text{Spec } \kappa(x))$:

$$(2) \quad \mathbb{L}i_x^*(\mathbb{R}\pi_* E(k)) \cong \mathbb{R}\pi_{x*}(\mathbb{L}j_x^* E(k)).$$

We now evaluate both sides of the isomorphism (2) independently. Since $E(k)$ is a locally free sheaf on Y , and thus it is flat over $\mathcal{O}_Y$. Because $E(k)$ is flat, its derived pullback along j_x requires no higher homological corrections (i.e., all higher Tor sheaves vanish). Consequently, the derived pullback coincides with the ordinary geometric restriction:

$$\mathbb{L}j_x^* E(k) \simeq j_x^* E(k) = E_x(k).$$

Furthermore, since the target of π_x is simply $\text{Spec } \kappa(x)$, the derived pushforward $\mathbb{R}\pi_{x*}$ corresponds exactly to the derived global section functor $\mathbb{R}\Gamma(F_x, -)$. Thus, the right-hand side of (2) simplifies to:

$$(3) \quad \mathbb{R}\pi_{x*}(\mathbb{L}j_x^* E(k)) \cong \mathbb{R}\Gamma(F_x, E_x(k)).$$

By our initial hypothesis, the derived direct image $\mathbb{R}\pi_* E(k)$ is quasi-isomorphic in $D^b(\text{Coh}(X))$ to the bounded complex $\mathcal{L}^\bullet$. Therefore, applying the derived pullback yields:

$$\mathbb{L}i_x^*(\mathbb{R}\pi_* E(k)) \cong \mathbb{L}i_x^* \mathcal{L}^\bullet.$$

Crucially, because $\mathcal{L}^\bullet$ is a perfect complex chosen such that each of its terms $\mathcal{L}^i$ is a locally free $\mathcal{O}_X$ -module, the complex itself is a complex of flat $\mathcal{O}_X$ -modules. The derived pullback of a flat complex is simply computed by the ordinary term-wise tensor product. Taking the pullback along i_x is algebraically equivalent to tensoring with the residue field $\kappa(x)$:

$$(4) \quad \mathbb{L}i_x^* \mathcal{L}^\bullet \simeq \mathcal{L}^\bullet \otimes_{\mathcal{O}_X}^{\mathbb{L}} \kappa(x) \simeq \mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \kappa(x).$$

Note that $\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \kappa(x)$ is now a bounded chain complex of finite-dimensional $\mathbb{C}$ -vector spaces. Substituting the simplified results (3) and (4) back into the base change quasi-isomorphism (2), we obtain an isomorphism of complexes in the derived category of vector spaces:

$$\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \kappa(x) \cong \mathbb{R}\Gamma(F_x, E_x(k)).$$

Since two complexes are isomorphic in the derived category if and only if they are quasi-isomorphic, their cohomology groups in every degree i must be strictly isomorphic. Applying the i -th cohomology functor $H^i(-)$ to both sides, we yield the final result:

$$H^i(\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \kappa(x)) \cong H^i(F_x, E_x(k)).$$

□

In our setting, $\mathbb{R}\pi_*E(k)$ is represented in $D^b(\text{Coh}(X))$ by $[0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow 0]$, so

$$H^0(F_x, E_x(k)) \cong H^0(A \otimes_{\mathcal{O}_X} \kappa(x) \xrightarrow{\phi(x)} B \otimes_{\mathcal{O}_X} \kappa(x)) = \ker(\phi(x))$$

and

$$H^1(F_x, E_x(k)) \cong H^1(A \otimes_{\mathcal{O}_X} \kappa(x) \xrightarrow{\phi(x)} B \otimes_{\mathcal{O}_X} \kappa(x)) = \text{coker}(\phi(x)) = \frac{B \otimes_{\mathcal{O}_X} \kappa(x)}{\text{im}(\phi(x))}.$$

Let $a = \text{rank}(A)$ and $b = \text{rank}(B)$. Then

$$\chi := \chi(F_x, E_x(k)) = h^0(F_x, E_x(k)) - h^1(F_x, E_x(k)) = a - b.$$

So the dimension of the space of global sections is tied directly to the rank of the linear map $\phi(x)$:

$$h^0(F_x, E_x(k)) = \dim \ker(\phi(x)) = a - \text{rank}(\phi(x)).$$

Therefore, the condition that the jumping locus occurs is equivalent to a rank-drop condition on the bundle morphism ϕ :

$$t \in J_s \iff a - \text{rank}(\phi(t)) \geq s \iff \text{rank}(\phi(t)) \leq a - s.$$

This recognizes J_s as the determinantal degeneracy locus of the vector bundle homomorphism $\phi : A \rightarrow B$.

3. JUMPING LOCUS PROBLEM

From last section, any “jump” in the dimension of the fiberwise sections $h^0(F_x, E_x(k))$ corresponds strictly and deterministically to a drop in the rank of the matrix $\phi(x)$, paving the way for the exact calculation of jumping loci via the Giambelli-Thom-Porteous degeneracy formulas.

3.1. Giambelli-Thom-Porteous formula. Giambelli-Thom-Porteous formula (see [8], Chapter 12) is a foundational result in algebraic and intersection theory that computes the fundamental class of a degeneracy locus in the Chow ring of a smooth manifold.

Given a morphism of vector bundles $\phi : \mathcal{E}^0 \rightarrow \mathcal{E}^1$ over a smooth manifold X of ranks $\text{rank}(\mathcal{E}^0) = a$ and $\text{rank}(\mathcal{E}^1) = b$, its p -th degeneracy locus is defined scheme-theoretically as the closed subscheme where the rank of the bundle map drops to at most p :

$$D_p(\phi) = \{x \in X \mid \text{rank}(\phi(x)) \leq p\}.$$

When $0 \leq p < \min(a, b)$ and the degeneracy locus $D_p(\phi)$ either is empty or has the expected codimension $d = (a - p)(b - p)$ in X in general case, the Giambelli-Thom-Porteous formula expresses its cycle class $[D_p(\phi)] \in A^d(X)$ as an $(a - p) \times (a - p)$ Schur determinant in the Chern classes of the virtual difference bundle $\mathcal{E}^1 - \mathcal{E}^0$:

$$[D_p(\phi)] = \det \begin{pmatrix} c_{b-p}(\mathcal{E}^1 - \mathcal{E}^0) & c_{b-p+1}(\mathcal{E}^1 - \mathcal{E}^0) & \cdots & c_{b+a-2p-1}(\mathcal{E}^1 - \mathcal{E}^0) \\ c_{b-p-1}(\mathcal{E}^1 - \mathcal{E}^0) & c_{b-p}(\mathcal{E}^1 - \mathcal{E}^0) & \cdots & c_{b+a-2p-2}(\mathcal{E}^1 - \mathcal{E}^0) \\ \vdots & \vdots & \ddots & \vdots \\ c_{b-2a+p+1}(\mathcal{E}^1 - \mathcal{E}^0) & c_{b-2a+p+2}(\mathcal{E}^1 - \mathcal{E}^0) & \cdots & c_{b-p}(\mathcal{E}^1 - \mathcal{E}^0) \end{pmatrix}.$$

In condensed notation, the (i, j) -th entry of this matrix is $c_{b-p+j-i}(\mathcal{E}^1 - \mathcal{E}^0)$ for $1 \leq i, j \leq a - p$, where the total Chern class of the formal difference is defined by the ratio

$$c(\mathcal{E}^1 - \mathcal{E}^0) := c(\mathcal{E}^1)/c(\mathcal{E}^0).$$

3.2. Jumping Locus Problem. To compute the cycle class $[J_s]$ of the s -th jumping locus and its degree with respect to a fixed polarization, we apply the Giambelli-Thom-Porteous formula to the derived pushforward of the twisted bundle $E(k)$.

We have proved that J_s is precisely the degeneracy locus $D_{a-s}(\phi)$ of the vector bundle homomorphism ϕ . Setting the rank bound to $p = a - s$, the size of the Porteous matrix becomes $a - p = s$. The subscript index for the matrix entries evaluates to:

$$b - p + j - i = b - (a - s) + j - i = s - (a - b) + j - i = s - \chi + j - i.$$

The expected codimension of J_s in X is $d = s(b - a + s) = s(s - \chi)$. Let W be the virtual vector bundle in the Grothendieck ring $K(X)$ given by the negative of the derived pushforward:

$$W := [\mathcal{E}^1] - [\mathcal{E}^0] = -\mathbb{R}\pi_*(E(k)) = -\pi_!(E(k)).$$

By the Giambelli-Thom-Porteous formula, the cycle class $[J_s] \in A^{s(s-\chi)}(X)$ is given by:

$$[J_s] = \det (c_{s-\chi+j-i}(W))_{1 \leq i, j \leq s} = \det \begin{pmatrix} c_{s-\chi}(W) & c_{s-\chi+1}(W) & \cdots & c_{2s-\chi-1}(W) \\ c_{s-\chi-1}(W) & c_{s-\chi}(W) & \cdots & c_{2s-\chi-2}(W) \\ \vdots & \vdots & \ddots & \vdots \\ c_{1-\chi}(W) & c_{2-\chi}(W) & \cdots & c_{s-\chi}(W) \end{pmatrix}.$$

So we have finished the proof of Theorem 1.1.

Note: If $s \leq \chi$, then $h^0 \geq \chi \geq s$ holds trivially everywhere, yielding $J_s = X$ and $[J_s] = [X] \in A^0(X)$.

3.3. Explicit Chern Classes via Grothendieck-Riemann-Roch. To express the Chern classes $c_i(W)$ purely in terms of the initial data E and $\mathcal{V}$, we apply the Grothendieck-Riemann-Roch (GRR) theorem to the projection π :

$$\text{ch}(\pi_! E(k)) = \pi_* \left(\text{ch}(E(k)) \cdot \text{td}(T_\pi) \right).$$

Let $\xi = c_1(\mathcal{O}_Y(1)) \in A^1(Y)$. Using Hartshorne's convention where $\mathbb{P}(\mathcal{V}) = \text{Proj}(\text{Sym } \mathcal{V}^\vee)$, let $\alpha_1, \dots, \alpha_{m+1}$ denote the Chern roots of $\mathcal{V}$.

Proposition 3.1. *Let X be a smooth projective manifold and $\mathcal{V}$ be a vector bundle of rank $m+1$ on X . Consider the projective bundle $\pi : Y = \mathbb{P}(\mathcal{V}) \rightarrow X$. Then the Todd class of the relative tangent bundle T_π is given by:*

$$\text{td}(T_\pi) = \prod_{j=1}^{m+1} \frac{\xi + \alpha_j}{1 - e^{-(\xi + \alpha_j)}}.$$

Proof. On the total space Y , the tangent bundle along the fibers is canonically described by the relative Euler exact sequence:

$$0 \rightarrow \mathcal{O}_Y \rightarrow \pi^* \mathcal{V} \otimes \mathcal{O}_Y(1) \rightarrow T_\pi \rightarrow 0.$$

By the Whitney sum formula, the total Chern class is strictly multiplicative over short exact sequences. Thus, we have:

$$c(\pi^* \mathcal{V} \otimes \mathcal{O}_Y(1)) = c(\mathcal{O}_Y) \cdot c(T_\pi).$$

Since $\mathcal{O}_Y$ is the trivial line bundle, its total Chern class is $c(\mathcal{O}_Y) = 1$. Consequently, the total Chern class and by extension, the set of non-zero Chern roots of T_π precisely coincides with that of the middle bundle $\pi^* \mathcal{V} \otimes \mathcal{O}_Y(1)$.

To find the Chern roots of $\pi^* \mathcal{V} \otimes \mathcal{O}_Y(1)$, we start with the Chern roots of $\mathcal{V}$, which are given as $\alpha_1, \dots, \alpha_{m+1}$. The pullback bundle $\pi^* \mathcal{V}$ retains these identical roots over Y . Then the Chern roots of $\pi^* \mathcal{V} \otimes \mathcal{O}_Y(1)$ are explicitly:

$$\beta_j = \xi + \alpha_j \quad \text{for } j = 1, 2, \dots, m+1.$$

By the formal definition of the Todd class evaluated on a bundle with roots x_i , $td(E) = \prod \frac{x_i}{1-e^{-x_i}}$. Substituting our derived roots β_j into this definition directly yields the desired relative Todd class:

$$td(T_\pi) = \prod_{j=1}^{m+1} \frac{\xi + \alpha_j}{1 - e^{-(\xi + \alpha_j)}}.$$

□

Substituting $\text{ch}(E(k)) = \text{ch}(E) \cdot e^{k\xi}$, the Chern character of our virtual bundle $W = -\pi_!(E(k))$ is computed in $A^*(X) \otimes \mathbb{Q}$ as:

$$\text{ch}(W) = -\pi_* \left[\text{ch}(E) \cdot e^{k\xi} \cdot \prod_{j=1}^{m+1} \frac{\xi + \alpha_j}{1 - e^{-(\xi + \alpha_j)}} \right].$$

The pushforward $\pi_* : A^*(Y) \rightarrow A^{*-m}(X)$ is evaluated via fiber integration: for any base class $\beta \in A^*(X)$ and exponent N , we have $\pi_*(\pi^*(\beta) \cdot \xi^N) = \beta \cdot s_{N-m}(\mathcal{V})$ ([9], Chapter 3), where $s_l(\mathcal{V})$ denotes the l -th Segre class of $\mathcal{V}$ (with $s_0 = 1$ and $s_l = 0$ for $l < 0$), defined via the inverse relation $s(\mathcal{V}) = c(\mathcal{V})^{-1}$.

Once each degree- p component $\text{ch}_p(W)$ is calculated, the total Chern class $c(W) = 1 + c_1(W) + c_2(W) + \dots$ is recovered via Newton's exponential identities:

$$c(W) = \exp \left(\sum_{p=1}^{\infty} (-1)^{p-1} (p-1)! \text{ch}_p(W) \right).$$

Extracting the degree- l term from this expansion yields $c_l(W)$, which is substituted directly into the Porteous determinant.

4. EXAMPLES

To see the immense power of this machinery in action, we need some concrete, computable examples where we can see both the geometric reality and the algebraic machinery working in perfect parallel.

Before proceeding with the explicit computations, we make a crucial remark regarding the actual dimension of the jumping loci in our examples. In all subsequent cases, we strategically apply twists such that the fiberwise Euler characteristic vanishes ($\chi = 0$), and we investigate the first jumping locus J_1 ($s = 1$). Under these parameters, the expected codimension of J_1 given by Theorem 1.1 is $s(s - \chi) = 1$.

Scheme-theoretically, the locus J_1 is the degeneracy locus of a map between vector bundles of the same rank, which means it is cut out by a single equation, $\det(\phi) = 0$, on the smooth base manifold X . By Krull's Principal Ideal Theorem, provided that the locus does not occupy the entire space X , which we rigorously verify in each example by demonstrating that the generic fiber does not jump, the actual codimension of J_1 strictly achieves the expected codimension 1. This geometric transversality ensures that the virtual cycle computed via the Giambelli-Thom-Porteous formula coincides exactly with the true fundamental geometric class $[J_1]$.

First, let's drop down to one of the easiest and classic examples in algebraic geometry: the trivial $\mathbb{P}^1$ -bundle over a $\mathbb{P}^1$ base.

4.1. Extension bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. Let $X = \mathbb{P}^1$ and $Y = \mathbb{P}^1 \times \mathbb{P}^1$. The Projection $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ (projection onto the first factor). The fiber F_x over any point $x \in X$ is just the second $\mathbb{P}^1$.

First, we construct a rank 2 jumping vector bundle E on Y via a global non-trivial extension of line bundles. Let E be defined by the short exact sequence:

$$0 \rightarrow \mathcal{O}_Y(-1, 1) \rightarrow E \rightarrow \mathcal{O}_Y(1, -1) \rightarrow 0.$$

By the Künneth formula, the space of such extensions is

$$\text{Ext}^1(\mathcal{O}(1, -1), \mathcal{O}(-1, 1)) \cong H^1(Y, \mathcal{O}(-2, 2)) \cong H^0(X, \mathcal{O}(2)) \otimes H^1(F, \mathcal{O}(-2)) \cong \mathbb{C}^3.$$

This means any global extension class e uniquely corresponds to a global section $s \in H^0(X, \mathcal{O}_X(2))$. A section of $\mathcal{O}(2)$ on $\mathbb{P}^1$ is a quadratic polynomial, which geometrically has exactly 2 zeros.

Let's look at the fibers $E_x = E|_{F_x}$. At generic points where $s(x) \neq 0$, the extension remains non-trivial on the fiber. The only non-trivial extension of $\mathcal{O}(-1)$ by $\mathcal{O}(1)$ is the trivial bundle. Thus, the generic splitting type is $E_t \cong \mathcal{O} \oplus \mathcal{O}$. At the 2 roots where $s(x) = 0$, the extension class vanishes. The sequence splits, meaning the fiber “jumps” to $E_x \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$.

Now $h^0(\mathcal{O} \oplus \mathcal{O}) = 2$ and $h^0(\mathcal{O}(1) \oplus \mathcal{O}(-1)) = 2$. The global sections haven't jumped in number, only in their distribution. To feed this into our formula, we apply the twist $k = -1$ along the fibers. Let $E' = E \otimes \mathcal{O}_Y(0, -1)$. For generic fiber,

$$E'_t \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1) \implies h^0(E'_t) = 0, h^1(E'_t) = 0.$$

For jumping fiber (2 points),

$$E'_x \cong \mathcal{O} \oplus \mathcal{O}(-2) \implies h^0(E'_t) = 1, h^1(E'_t) = 1.$$

Now we have a generic $h^0(E'_t) = 0$, a jump of $s = 1$ and $\chi = 0$.

The Chow ring of Y is given by $A^*(Y) = \mathbb{Z}[h, f]/(h^2, f^2)$, where h is the pullback of the point class h_x from the base space X , and $f = c_1(\mathcal{O}_Y(0, 1))$ is the pullback of the point class f_y from the second factor. The intersection product hf represents the class of a point in Y , satisfying $h^2 = 0$, $f^2 = 0$, and hf is a point class.

The Gysin Pushforward π_* : Integration along the fiber simply extracts the coefficient of f . Thus

$$\pi_*(h) = 0, \quad \pi_*(f) = 1, \quad \pi_*(hf) = h_x.$$

Since the total space is the trivial product $Y = \mathbb{P}^1 \times \mathbb{P}^1$, the relative tangent bundle

$$\mathcal{T}_\pi \cong \mathcal{O}_Y(0, 2)$$

whose first Chern class is

$$c_1(\mathcal{T}_\pi) = 2f.$$

Using the power series expansion of the Todd class, we have

$$\text{Td}(\mathcal{T}_\pi) = 1 + f.$$

Recall the construction of our rank 2 vector bundle E' :

$$0 \longrightarrow \mathcal{O}_Y(-1, 0) \longrightarrow E' \longrightarrow \mathcal{O}_Y(1, -2) \longrightarrow 0.$$

The Chern character is strictly additive over short exact sequences:

$$\text{ch}(E') = \text{ch}(\mathcal{O}(-1, 0)) + \text{ch}(\mathcal{O}(1, -2)).$$

Using the exponential expansion $\text{ch}(\mathcal{L}) = e^{c_1(\mathcal{L})} = 1 + c_1 + \frac{1}{2}c_1^2 + \dots$, we get

$$\text{ch}(\mathcal{O}(-1, 0)) = e^{-h} = 1 - h$$

and

$$\text{ch}(\mathcal{O}(1, -2)) = 1 + h - 2f - 2hf.$$

Summing these two components, we get

$$\text{ch}(E') = 2 - 2f - 2hf.$$

We are now ready to compute the Chern character of the virtual difference bundle $W = -\pi_! E'$ on the base manifold X . By the Grothendieck-Riemann-Roch theorem:

$$\text{ch}(W) = -\pi_* \left(\text{ch}(E') \cdot \text{Td}(\mathcal{T}_\pi) \right).$$

First, we execute the algebraic multiplication inside the parenthesis:

$$\text{ch}(E') \cdot \text{Td}(\mathcal{T}_\pi) = (2 - 2f - 2hf) \cdot (1 + f) = 2 - 2hf.$$

Now, applying the Gysin pushforward π_* , we arrive at the final result:

$$\text{ch}(W) = 2h_x.$$

By our main theorem,

$$\text{codim}(J_1) = s(s - \chi) = 1,$$

which means the jumping locus consists of a set of discrete points. The cycle class of this jumping locus in $A^*(X)$ is precisely given by

$$[J_1] = 2h_x.$$

So the jumping locus is just two points, which matches the above calculation.

Having verified our determinantal machinery on a baseline low-dimensional model where the jumping locus consists merely of discrete points, we now advance to a higher-dimensional parameter space of fundamental importance in algebraic geometry : the parameter space of lines in $\mathbb{P}^3$. While the projection still forms a $\mathbb{P}^1$ -bundle, moving the base manifold to the 4-dimensional Grassmannian $G(2, 4)$ elevates the expected jumping locus from zero-dimensional points to a codimension-one hypersurface. In this second example, we investigate the pullback of a classical instanton bundle of charge k , demonstrating how our homological compression effortlessly captures the fundamental class of jumping lines without tracking individual local resolutions.

4.2. Pull back the instanton bundle over $\mathbb{P}^3$. Let base space $X = G(2, 4)$ be the parameter space of all lines in $\mathbb{P}^3$, which is a manifold of dimension 4. Its Chow ring is generated by Schubert cycles. Let σ_1 be the most fundamental Schubert class representing the set of lines intersecting a given line. For the tautological rank 2 subbundle S on $G(2, 4)$, its first Chern class is $c_1(S) = -\sigma_1$.

Let total space Y be the incidence variety $Y = \{(x, L) \in \mathbb{P}^3 \times G(2, 4) \mid x \in L\}$. Since there is a $\mathbb{P}^1$ over each point in X , $q : Y \rightarrow X$ strictly forms a $\mathbb{P}^1$ -projective bundle $Y = \mathbb{P}(S)$. Let the relative hyperplane class be ξ corresponding to the pullback of the hyperplane class p^*h from the projection to $\mathbb{P}^3$, where h is the hyperplane class in $\mathbb{P}^3$.

Let vector bundle $E = p^*\mathcal{E}$, where $\mathcal{E}$ is an instanton bundle of charge k on $\mathbb{P}^3$ (see [1], [2]). Its Chern classes are $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = k$. For the projective bundle $q : \mathbb{P}(S) \rightarrow X$, the dimensional reduction integration satisfies the relation $\xi^2 + c_1(S)\xi + c_2(S) = 0$. Therefore

$$q_*(1) = 0, \quad q_*(\xi) = 1, \quad q_*(\xi^2) = -c_1(S) = \sigma_1$$

For an instanton bundle on $\mathbb{P}^3$, its splitting type on a generic line L is trivial:

$$E|_L \cong \mathcal{O} \oplus \mathcal{O}.$$

We apply a twist of $k = -1$ along the fiber direction:

$$E(-\xi) := E \otimes \mathcal{O}_Y(-\xi).$$

For generic fiber,

$$E(-\xi)|_L \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1) \implies h^0(E(-\xi)|_L) = h^1(E(-\xi)|_L) = 0 \implies \chi = 0.$$

For jumping fiber (1st-order), the splitting type mutates to $\mathcal{O}(1) \oplus \mathcal{O}(-1)$. After the twist, it becomes

$$\mathcal{O} \oplus \mathcal{O}(-2) \implies h^0(E(-\xi)|_L) = h^1(E(-\xi)|_L) = 1.$$

Thus $s = 1$ and $\chi = 0$.

The expected codimension of the jumping locus J_1 is $s(s - \chi) = 1$, which means the jumping lines form a hypersurface.

Since the base space is high-dimensional, we must retain terms up to degree 2 (codimension 2 in Y), because the integration q_* will reduce it by one degree, perfectly matching $c_1(W)$ of the virtual bundle.

Since $c(E) = 1 + k\xi^2$, we have $\text{ch}(E) = 2 - k\xi^2$.

So

$$\text{ch}(E(-\xi)) = \text{ch}(E) \cdot e^{-\xi} = (2 - k\xi^2) \left(1 - \xi + \frac{1}{2}\xi^2\right).$$

Expanding and truncating up to degree 2 yields

$$\text{ch}(E(-\xi)) = 2 - 2\xi + (1 - k)\xi^2 + \dots$$

For the projective bundle $\mathbb{P}(S)$, the Chern class of the relative tangent bundle satisfies $c_1(T_q) = c_1(S) + 2\xi$. Substituting this into the Taylor expansion of the Todd class $\text{Td}(x) = 1 + \frac{1}{2}x + \frac{1}{12}x^2$:

$$\text{Td}(T_q) = 1 + \frac{1}{2}(c_1(S) + 2\xi) + \frac{1}{12}(c_1(S) + 2\xi)^2.$$

Extracting each degree component:

$$\mathrm{Td}_0 = 1;$$

$$\mathrm{Td}_1 = \frac{1}{2}c_1(S) + \xi;$$

$$\mathrm{Td}_2 = \frac{1}{12}c_1(S)^2 + \frac{1}{3}c_1(S)\xi + \frac{1}{3}\xi^2.$$

By Grothendieck-Riemann-Roch theorem, we need to compute the degree 2 term of $\mathrm{ch}(E(-\xi)) \cdot \mathrm{Td}(T_q)$ in the total space Y , denoted as $[\mathrm{ch} \cdot \mathrm{Td}]_2$. Then

$$\begin{aligned} [\mathrm{ch} \cdot \mathrm{Td}]_2 &= \mathrm{ch}_2 \mathrm{Td}_0 + \mathrm{ch}_1 \mathrm{Td}_1 + \mathrm{ch}_0 \mathrm{Td}_2 \\ &= (1-k)\xi^2 \cdot 1 - 2\xi \left(\frac{1}{2}c_1(S) + \xi \right) + 2 \left(\frac{1}{12}c_1(S)^2 + \frac{1}{3}c_1(S)\xi + \frac{1}{3}\xi^2 \right) \\ &= \left(-\frac{1}{3} - k \right) \xi^2 - \frac{1}{3}c_1(S)\xi + \frac{1}{6}c_1(S)^2. \end{aligned}$$

Now, we apply q_* to push this complex second-order form forward to the 4-dimensional base space $X = G(2, 4)$. According to the pushforward rules, the class $c_1(S)$ from the base space can be factored out as a constant:

$$\begin{aligned} q_*([\mathrm{ch} \cdot \mathrm{Td}]_2) &= \left(-\frac{1}{3} - k \right) q_*(\xi^2) - \frac{1}{3}c_1(S)q_*(\xi) + \frac{1}{6}c_1(S)^2q_*(1) \\ &= \left(-\frac{1}{3} - k \right) (-c_1(S)) - \frac{1}{3}c_1(S)(1) + 0 \\ &= k \cdot c_1(S) \\ &= -k\sigma_1. \end{aligned}$$

Since the virtual bundle is $W = -\mathbb{R}q_*E(-\xi) = -q_!E(-\xi)$, we have

$$c_1(W) = \mathrm{ch}_1(W) = -q_*([\mathrm{ch} \cdot \mathrm{Td}]_2) = k\sigma_1.$$

By our main theorem, the cycle class of the jumping locus is

$$[J_1] = c_1(W) = k\sigma_1.$$

While the previous example illustrated the computational effectiveness of our formula on a higher-dimensional base, the integration along the $\mathbb{P}^1$ -fibers only reduced the homological degree by one. To fully test the analytical limits and algorithmic power of our derived two-term representation, we now escalate both the fiber dimension and the structural complexity of the underlying vector bundle. In our final example, we examine the pullback of the celebrated Horrocks-Mumford bundle—the only known indecomposable rank 2 vector bundle on $\mathbb{P}^4$ —onto the 6-dimensional Grassmannian $\mathbb{G}(2, 4) \cong G(3, 5)$. Here, the total space forms a $\mathbb{P}^2$ -projective bundle, meaning the Gysin pushforward q_* drops the homological degree by exactly two. Consequently, determining the first Chern class of the virtual difference bundle requires tracking higher-order Todd and Chern character expansions up to degree three, combined with a rigorous verification of section vanishing via monad cohomology and conormal filtrations.

4.3. Pull back the Horrocks-Mumford bundle over $\mathbb{P}^4$. The Horrocks-Mumford bundle $\mathcal{E}$ is famously the only known indecomposable rank 2 vector bundle on $\mathbb{P}^4$ (see [11] or [14], Chapter 1, Section 5 and Chapter 2, Section 3) with $c_1 = 5, c_2 = 10$.

Let the base space $X = \mathbb{G}(2, 4)$ be the Grassmannian of 2-planes in $\mathbb{P}^4$. Algebraically, this is $G(3, 5)$ (3-dimensional vector subspaces of $\mathbb{C}^5$) of dimension 6. Let S be the tautological rank 3 subbundle on X . Its first Chern class is $c_1(S) = -\sigma_1$, where σ_1 is the positive Schubert class (the hyperplane class in the Plücker embedding).

Let the total space Y be the incidence variety $Y = \{(x, P) \in \mathbb{P}^4 \times \mathbb{G}(2, 4) \mid x \in P \subset \mathbb{P}^4\}$. The projection $q : Y \rightarrow X$ has fibers isomorphic to $P \cong \mathbb{P}^2$. Thus, $Y = \mathbb{P}(S)$ is a $\mathbb{P}^2$ -projective bundle over X of dimension 8. Let $\xi = c_1(\mathcal{O}_{\mathbb{P}(S)}(1))$ be the relative hyperplane class.

Let the vector bundle E be the pullback of the Horrocks-Mumford bundle, $E = p^*\mathcal{E}$. Its Chern classes are completely determined by ξ :

$$c_1(E) = 5\xi, \quad c_2(E) = 10\xi^2.$$

Because the fiber is 2-dimensional, the integration q_* reduces the homological degree by exactly 2. The governing relation is $\xi^3 + c_1(S)\xi^2 + \dots = 0$. Thus,

$$q_*(1) = 0, \quad q_*(\xi) = 0, \quad q_*(\xi^2) = 1, \quad q_*(\xi^3) = -c_1(S).$$

By Hirzebruch-Riemann-Roch on $\mathbb{P}^2$, the Euler characteristic for $E(k) := E \otimes \mathcal{O}(k\xi)$ restricting to P is

$$\chi(\mathcal{E}(k)|_P) = 2 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{3}{2}c_1.$$

Substituting $c_1 = 5 + 2k$ and $c_2 = 10 + 5k + k^2$, it simplifies to

$$\chi = k^2 + 8k + 12.$$

For simplicity, we demand $\chi = 0$. Solving $k^2 + 8k + 12 = 0$ gives roots at -6 and -2 . We twist the bundle to $E_{tw} = E(-2\xi)$.

Proposition 4.1. *Let $\mathcal{E}$ be the Horrocks-Mumford vector bundle on $\mathbb{P}_{\mathbb{C}}^4$, defined by its standard monad. For any generic 2-plane $P \cong \mathbb{P}^2 \subset \mathbb{P}^4$, $H^0(P, \mathcal{E}(-2)|_P) = 0$.*

Proof. By the classical construction of the Horrocks-Mumford bundle, $\mathcal{E}$ is the cohomology bundle of the following monad on $\mathbb{P}^4$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 5} \xrightarrow{\alpha} \Omega_{\mathbb{P}^4}^2(2)^{\oplus 2} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^4}^{\oplus 5} \longrightarrow 0.$$

To study $\mathcal{E}(-2)$, we tensor this entire monad complex with the line bundle $\mathcal{O}_{\mathbb{P}^4}(-2)$ and get a monad for $\mathcal{E}(-2)$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 5} \longrightarrow \Omega_{\mathbb{P}^4}^2 \xrightarrow{\beta(-2)} \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 5} \longrightarrow 0.$$

Let $P \cong \mathbb{P}^2$ be a generic 2-plane in $\mathbb{P}^4$. Let us define the kernel subbundle on P as $\mathcal{K}_P := \ker(\beta(-2))|_P$.

This decomposes the restricted monad into two fundamental short exact sequences of locally free sheaves on P :

$$(5) \quad 0 \longrightarrow \mathcal{K}_P \longrightarrow \left(\Omega_{\mathbb{P}^4}^2|_P \right)^{\oplus 2} \longrightarrow \mathcal{O}_P(-2)^{\oplus 5} \longrightarrow 0$$

and

$$(6) \quad 0 \longrightarrow \mathcal{O}_P(-3)^{\oplus 5} \longrightarrow \mathcal{K}_P \longrightarrow \mathcal{E}(-2)|_P \longrightarrow 0.$$

Taking the long exact sequence of cohomology for sequence (6) on P , we obtain

$$H^0(P, \mathcal{K}_P) \longrightarrow H^0\left(P, \mathcal{E}(-2)|_P\right) \longrightarrow 0.$$

Next, taking the long exact sequence of cohomology for sequence (5):

$$0 \longrightarrow H^0(P, \mathcal{K}_P) \longrightarrow H^0\left(P, \Omega_{\mathbb{P}^4}^2|_P\right)^{\oplus 2}.$$

Thus, the proof thus strictly reduces to showing $H^0\left(P, \Omega_{\mathbb{P}^4}^2|_P\right) = 0$.

To evaluate the global sections of $\Omega_{\mathbb{P}^4}^2|_P$, we use the conormal exact sequence. The plane $P \subset \mathbb{P}^4$ is defined by two generic linear forms, meaning its conormal bundle is exactly $\mathcal{N}_{P/\mathbb{P}^4}^{\vee} \cong \mathcal{O}_P(-1)^{\oplus 2}$. The restriction of the cotangent bundle yields the second fundamental exact sequence:

$$0 \longrightarrow \mathcal{O}_P(-1)^{\oplus 2} \longrightarrow \Omega_{\mathbb{P}^4}^1|_P \longrightarrow \Omega_P^1 \longrightarrow 0.$$

Taking the second exterior power $\bigwedge^2$ of the central bundle $\Omega_{\mathbb{P}^4}^1|_P$ produces a canonical filtration $0 \subset F^2 \subset F^1 \subset F^0 = \Omega_{\mathbb{P}^4}^2|_P$. By multilinear algebra, the successive quotients $Q_p = F^p/F^{p+1}$ (for $p = 0, 1, 2$) are computed as $Q_p \cong \bigwedge^p \mathcal{N}^{\vee} \otimes \bigwedge^{2-p} \Omega_P^1$.

- (1) $p = 0$: $Q_0 \cong \bigwedge^0 \mathcal{N}^\vee \otimes \bigwedge^2 \Omega_P^1 \cong \mathcal{O}_P \otimes \omega_P \cong \mathcal{O}_P(-3)$.
- (2) $p = 1$: $Q_1 \cong \bigwedge^1 \mathcal{N}^\vee \otimes \bigwedge^1 \Omega_P^1 \cong \mathcal{O}_P(-1)^{\oplus 2} \otimes \Omega_P^1 \cong \Omega_P^1(-1)^{\oplus 2}$.
- (3) $p = 2$: $Q_2 \cong \bigwedge^2 \mathcal{N}^\vee \otimes \bigwedge^0 \Omega_P^1 \cong \bigwedge^2 (\mathcal{O}_P(-1)^{\oplus 2}) \otimes \mathcal{O}_P \cong \mathcal{O}_P(-2)$.

This filtration induces two short exact sequences:

$$(7) \quad 0 \longrightarrow F^1 \longrightarrow \Omega_{\mathbb{P}^4}^2|_P \longrightarrow \mathcal{O}_P(-3) \longrightarrow 0,$$

and

$$(8) \quad 0 \longrightarrow \mathcal{O}_P(-2) \longrightarrow F^1 \longrightarrow \Omega_P^1(-1)^{\oplus 2} \longrightarrow 0.$$

Taking the long exact sequence of (8), we have

$$0 \rightarrow H^0(\mathcal{O}_P(-2)) \rightarrow H^0(F^1) \rightarrow H^0(\Omega_P^1(-1)^{\oplus 2}).$$

By Bott formula, $H^0(P, \Omega_P^1(-1)) = 0$. Thus $H^0(F^1) = 0$.

Taking the long exact sequence of (7), we have

$$0 \rightarrow H^0(F^1) \rightarrow H^0(\Omega_{\mathbb{P}^4}^2|_P) \rightarrow H^0(\mathcal{O}_P(-3)) \implies H^0(\Omega_{\mathbb{P}^4}^2|_P) = 0.$$

Therefore, $H^0(P, \mathcal{E}(-2)|_P) = 0$.

□

On a generic plane P , by Serre duality

$$h^2(P, \mathcal{E}(-2)|_P) = h^0(P, (\mathcal{E}(-2)|_P)^\vee \otimes \mathcal{O}(-3)) = h^0(P, \mathcal{E}(-6)|_P) = 0.$$

Thus $h^1(P, \mathcal{E}(-2)|_P) = 0$. If h^0 jumps to 1 at a special plane ($s = 1$), the expected codimension in the base space X is $s(s - \chi) = 1$. Thus, the jumping planes will form a hypersurface in $G(3, 5)$.

The twisted bundle E_{tw} has new Chern classes $c_1(E_{tw}) = c_1 - 4\xi = \xi$ and $c_2(E_{tw}) = c_2 - 2c_1\xi + 4\xi^2 = 4\xi^2$. Thus, the Chern Character $\text{ch}(E_{tw})$:

$$\begin{aligned} \text{ch}_0 &= 2, \text{ch}_1 = \xi, \text{ch}_2 = \frac{1}{2}(c_1^2 - 2c_2) = \frac{1}{2}(\xi^2 - 8\xi^2) = -\frac{7}{2}\xi^2, \\ \text{ch}_3 &= \frac{1}{6}(c_1^3 - 3c_1c_2) = \frac{1}{6}(\xi^3 - 12\xi^3) = -\frac{11}{6}\xi^3. \end{aligned}$$

From the Euler sequence

$$0 \rightarrow \mathcal{O} \rightarrow q^*S \otimes \mathcal{O}(\xi) \rightarrow T_q \rightarrow 0,$$

we extract the Chern classes of the relative tangent bundle: $c_1(T_q) = 3\xi + c_1(S)$ and $c_2(T_q) = 3\xi^2 + 2c_1(S)\xi + c_2(S)$. Expanding the Todd polynomial yields:

$$\begin{aligned} \text{Td}_0 &= 1, \text{Td}_1 = \frac{3}{2}\xi + \frac{1}{2}c_1(S), \text{Td}_2 = \xi^2 + \frac{2}{3}c_1(S)\xi + \dots, \\ \text{Td}_3 &= \frac{3}{8}\xi^3 + \frac{3}{8}c_1(S)\xi^2 + \dots \end{aligned}$$

Because integration q_* drops the degree by 2, computing $c_1(W)$ in the base space X requires isolating exactly the degree 3 cross-product $I_3 = [\text{ch} \cdot \text{Td}]_3$ in the total space Y :

$$I_3 = \text{ch}_3\text{Td}_0 + \text{ch}_2\text{Td}_1 + \text{ch}_1\text{Td}_2 + \text{ch}_0\text{Td}_3.$$

We multiply the coefficients, gathering only the ξ^3 and $c_1(S)\xi^2$ terms (any term containing $c_2(S)\xi$ will be mercilessly annihilated because $q_*(\xi) = 0$):

$$q_*(I_3) = -q_*\left(\frac{16}{3}\xi^3 + \frac{1}{3}c_1(S)\xi^2\right).$$

By the GRR formula, the first Chern class of the virtual difference bundle $W = -\mathbb{R}q_*E_{tw} = -q!E_{tw}$ is

$$[J_1] = c_1(W) = -q_*(I_3).$$

Injecting the Gysin integration rules $q_*(\xi^2) = 1$ and $q_*(\xi^3) = -c_1(S)$:

$$[J_1] = c_1(W) = - \left[-\frac{16}{3}(-c_1(S)) - \frac{1}{3}c_1(S)(1) \right] = -5c_1(S) = 5\sigma_1.$$

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