

THE NON-SPLITTING LOCUS OF A HORROCKS–MUMFORD ABELIAN SURFACE

RONG DU

ABSTRACT. Let $X \subset \mathbb{P}^4$ be a smooth abelian surface embedded by the complete linear system of the polarization $H = \mathcal{O}_X(1)$. Let $\sigma : S^2 T_X \rightarrow N$, where $N = N_{X/\mathbb{P}^4}$, be its second fundamental form. It defines the rank-four bundle homomorphism $\varphi : T_X \otimes N^* \rightarrow (S^2 T_X^* \otimes T_X)_0$, where the subscript denotes projection onto the trace-free part. We prove that φ is an isomorphism at a general point. Thus Mok’s question has an affirmative answer for every smooth embedded abelian surface. The non-splitting locus is consequently a divisor $\Sigma_f \in |10H|$. The rank-one degeneracy scheme $D_1(\sigma)$ is zero-dimensional of length 150, and the local equation of Σ_f lies in the square of its ideal. All these data descend to a three-dimensional subsystem of a $(1, 5)$ polarization on $X/K(H) \simeq \widehat{X}$, with base scheme of length 6. We also give a local node criterion. The stronger statement that Σ_f has exactly 150 ordinary nodes and normalization genus 351 is explicitly conditional on reducedness, nondegeneracy, transversality, and integrality hypotheses.

Keywords. Horrocks–Mumford bundle; abelian surface; second fundamental form; non-splitting locus; theta heat equation; degeneracy locus.

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1. INTRODUCTION AND STATEMENTS

The problem considered here originates in Mok’s study of holomorphic immersions into Kähler manifolds of constant holomorphic sectional curvature [4]. For such an immersion $f : X \rightarrow M$, the tangent sequence

$$0 \longrightarrow T_X \longrightarrow f^* T_M \longrightarrow N_f \longrightarrow 0$$

measures how the intrinsic geometry of X sits inside the ambient geometry. Mok proves that, when X is compact, this sequence splits holomorphically if and only if the immersion is totally geodesic. More generally, he relates the intrinsic obstruction to a holomorphic projective connection on X to the extrinsic obstruction to splitting this sequence, through a map induced by the second fundamental form. Thus the second fundamental form serves as

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a bridge between intrinsic projective geometry and the extrinsic geometry of the immersion.

The case of a complex surface in a four-dimensional complex space form is especially sharp. Here Mok combines the second fundamental form with the projective connection to obtain a homomorphism of rank-four vector bundles

$$\varphi : T_X \otimes N^* \longrightarrow E := (S^2 T_X^* \otimes T_X)_0,$$

where the subscript denotes the trace-free part. If φ is generically invertible, the zero divisor of $\det \varphi$ is called the non-splitting locus. The equality of the two ranks is crucial: it turns the pointwise comparison into a determinantal divisor that records where Mok's splitting criterion fails.

Abelian surfaces provide a particularly revealing test of this mechanism. Their tangent bundles are trivial, and their universal covers give them flat holomorphic projective connections, so the intrinsic side of Mok's comparison is as simple as possible. At the same time, an abelian surface embedded in $\mathbb{P}^4$ is far from a totally geodesic linear subspace, and its projective geometry is governed by the highly symmetric Horrocks–Mumford construction. Mok's question therefore asks whether his intrinsic–extrinsic comparison produces a concrete projective invariant in this distinguished family, and whether that invariant reflects the Horrocks–Mumford symmetry. Against this background, in [4, Question 3] Mok asks whether φ is generically invertible for an abelian surface in $\mathbb{P}^4$ and, if so, how the non-splitting locus is related to the geometry of the Horrocks–Mumford bundle.

We answer the first part for every smooth embedded abelian surface and give a quotient-theoretic description for the second part. Throughout, all varieties are over $\mathbb{C}$. We write $H = \mathcal{O}_X(1)$ and $N = N_{X/\mathbb{P}^4}$.

The following theorem collects the principal conclusions.

Theorem 1.1. *Let $X \subset \mathbb{P}^4$ be a smooth abelian surface embedded by the complete linear system of a polarization H . Then:*

- (i) *The homomorphism $\varphi : T_X \otimes N^* \rightarrow E$ is an isomorphism at a general point of X ;*
- (ii) *its non-splitting locus*

$$\Sigma_f = V(\det \varphi)$$

is an effective divisor in $|10H|$, and hence

$$H \cdot \Sigma_f = 100, \quad \Sigma_f^2 = 1000, \quad p_a(\Sigma_f) = 501;$$

- (iii) *the degeneracy scheme*

$$Z = D_1(\sigma) = \{x \in X : \text{rank } \sigma_x \leq 1\}$$

defined by the 2×2 minors of σ is zero-dimensional of length 150. Moreover, a local equation of Σ_f belongs to I_Z^2 , and therefore

$$Z \subset \text{Sing}(\Sigma_f)$$

scheme-theoretically.

Write

$$K(H) := \{a \in X : t_a^* H \simeq H\}$$

for the polarization kernel, where $t_a : X \rightarrow X$, $x \mapsto x+a$, denotes translation by a . The translations in $K(H)$ are induced by projective transformations of $\mathbb{P}^4$ in the Horrocks–Mumford construction; the symmetry group, the abelian-surface zero loci, and the associated moduli interpretation are developed in [2, Sections 1, 5, and 6]. More precisely, after a projective change of coordinates, [2, Theorem 5.2] realizes every such X as the zero scheme of a regular section $s \in H^0(\mathbb{P}^4, F)$, where F is the twist of the Horrocks–Mumford bundle with $\det F \simeq \mathcal{O}_{\mathbb{P}^4}(5)$. For a regular section of a rank-two bundle, the normal bundle of its zero scheme is the restriction of that bundle. Hence

$$N_{X/\mathbb{P}^4} \simeq F|_X$$

in the Horrocks–Mumford construction (See also [5, Chapter 1, Section 5]).

There is also a precise description on the quotient by the polarization kernel. Via the polarization isogeny, this quotient is the dual abelian surface:

$$\pi : X \longrightarrow Y := X/K(H) \simeq \hat{X}.$$

The next statement records how the non-splitting divisor and the degeneracy scheme descend through the polarization kernel.

Theorem 1.2. *Under the assumptions of Theorem 1.1, there are a polarization M of type $(1, 5)$ on Y and a three-dimensional subspace $W \subset H^0(Y, M)$ such that:*

- (i) $\pi^* M \simeq \mathcal{O}_X(5H)$;
- (ii) *the base scheme $\bar{Z}$ of W is the closed subscheme cut out by the evaluation map*

$$W \otimes \mathcal{O}_Y \longrightarrow M$$

(equivalently, by the three sections in any basis of W). It is zero-dimensional of length 6, and $Z = X \times_Y \bar{Z}$;

- (iii) *the incomplete linear system $\mathbb{P}(W) \subsetneq |M|$ defines the rational map $\bar{\kappa} : Y \dashrightarrow \mathbb{P}(W^*) \simeq \mathbb{P}^2$. If $\Delta \subset \mathbb{P}^2$ is the smooth conic of rank-one binary quadratic forms, then*

$$\bar{\Sigma} = \overline{\bar{\kappa}^{-1}(\Delta)}^{\text{sch}} \in |2M|, \quad \Sigma_f = X \times_Y \bar{\Sigma}.$$

Here the bar denotes scheme-theoretic closure. In particular, $p_a(\bar{\Sigma}) = 21$, and $\bar{\Sigma}$ is at least double along $\bar{Z}$.

The following result isolates the additional local hypotheses needed to identify the singularities. Integrality is required only for the normalization genus.

Theorem 1.3. *Retain the assumptions of Theorem 1.1, and suppose in addition that:*

- (a) $\bar{Z}$ is reduced, and for every $\bar{p} \in \bar{Z}$, or equivalently for any lift $p \in \pi^{-1}(\bar{p})$, the one-dimensional image of the dual map

$$\sigma_p^* : N_p^* \longrightarrow S^2 T_{X,p}^*$$

is generated by a nondegenerate binary quadratic form;

- (b) *on $U = Y \setminus \bar{Z}$, the rational map $\bar{\kappa}$ is a morphism transverse to Δ .*

Then Σ_f has exactly 150 ordinary nodes and no other singularities. If, in addition, Σ_f is integral, its normalization has genus

$$g(\tilde{\Sigma}_f) = 351.$$

The order of the argument is as follows. First, a pointwise calculation identifies $\det \varphi$ with the resultant of the two scalar second fundamental forms. Second, the heat equation supplies three globally independent normal sections. The Horrocks–Mumford symmetry identifies their span with the invariant normal sections obtained by varying the defining section of the Horrocks–Mumford bundle. The base locus of that bundle then shows that σ is surjective away from a finite set. Only after this fact is known can the assumption $\det \varphi \equiv 0$ be converted into a square-root contradiction in $\text{NS}(X)$. Finally, the absence of divisorial degeneracy and a separate Chern-class argument show that $D_1(\sigma)$ has the expected codimension; only at that stage is the Thom–Porteous formula applicable.

2. THE POINTWISE RESULTANT AND THE NORMAL BUNDLE

We begin with the numerical consequences of the complete embedding.

Lemma 2.1. *Under the assumptions of Theorem 1.1, the polarization H has type $(1, 5)$. In particular,*

$$H^2 = 10, \quad K(H) \simeq (\mathbb{Z}/5)^2,$$

and the class of H is primitive in $\text{NS}(X)$.

Proof. Because the embedding is defined by the complete linear system $|H|$ and has target $\mathbb{P}^4$, one has $h^0(X, H) = 5$. Kodaira vanishing, together with

$K_X \simeq \mathcal{O}_X$, gives $H^i(X, H) = 0$ for $i > 0$. Riemann–Roch on the abelian surface therefore yields

$$5 = h^0(X, H) = \chi(X, H) = \frac{H^2}{2}.$$

If (d_1, d_2) is the type of H , where $d_1 \mid d_2$, then $\chi(X, H) = d_1 d_2$. Hence $d_1 d_2 = 5$, and consequently $(d_1, d_2) = (1, 5)$ and $H^2 = 10$.

For the polarization isogeny

$$\phi_H : X \longrightarrow \widehat{X} = \text{Pic}^0(X), \quad a \longmapsto t_a^* H \otimes H^{-1},$$

the structure theorem gives

$$\ker \phi_H \simeq (\mathbb{Z}/d_1)^2 \times (\mathbb{Z}/d_2)^2.$$

Since $\phi_H(a) = 0$ precisely when $t_a^* H \simeq H$, this proves the assertion about $K(H)$. Finally, if $H = mL$ in $\text{NS}(X)$ for an integer $m > 1$, then $H^2 = m^2 L^2$, which is impossible because $H^2 = 10$. Thus H is primitive; in particular, it is not divisible by 2 in $\text{NS}(X)$. $\square$

Following Mok’s description [4, Lemma 3], we now recall the pointwise linear-algebra construction underlying the homomorphism φ .

Let V and W be two-dimensional complex vector spaces, and let $s : S^2 V \rightarrow W$. After choosing a basis of W^* , the map s is represented by two binary quadratic forms q_0, q_1 . The pointwise construction gives

$$\varphi_s : V \otimes W^* \longrightarrow (S^2 V^* \otimes V)_0, \quad \varphi_s(\xi \otimes \eta) = [(u, v) \longmapsto \eta(s(u, v))\xi]_0.$$

Here $[\]_0$ denotes the canonical projection onto the trace-free summand of $S^2 V^* \otimes V$.

The following pointwise criterion converts the invertibility problem for φ into the vanishing of a classical resultant.

Lemma 2.2. *The map φ_s is invertible if and only if there is no $0 \neq v \in V$ such that $s(v^2) = 0$. Equivalently,*

$$\det \varphi_s = c \text{Res}(q_0, q_1), \quad c \in \mathbb{C}^*.$$

Proof. The first assertion is Mok’s criterion [4, Lemma 3]. If s is not surjective, then $\mathbb{P}(\ker s)$ contains a projective line, which meets the Veronese conic $\mathbb{P}(V) \hookrightarrow \mathbb{P}(S^2 V)$ over $\mathbb{C}$. If s is surjective, $\mathbb{P}(\ker s)$ is a point, and Mok’s criterion says that singularity occurs precisely when this point lies on the same conic. In dual language this is the existence of a common projective zero of q_0 and q_1 , hence the vanishing of their resultant. Both determinants are degree-four polynomials on the affine space $\text{Hom}(S^2 V, W)$. The resultant hypersurface is irreducible, and neither polynomial is zero. They therefore differ by a nonzero scalar. This is a polynomial identity on

$\text{Hom}(S^2V, W)$, not merely an equality of zero sets after specializing to a particular surface. $\square$

For the abelian surface, $T_X \simeq \mathcal{O}_X^2$. Put $V = H^0(X, T_X)$ and write the second fundamental form as

$$\sigma : S^2V \otimes \mathcal{O}_X \longrightarrow N, \quad N = N_{X/\mathbb{P}^4}.$$

Adjunction gives

$$\det N \simeq K_X \otimes K_{\mathbb{P}^4}^{-1}|_X \simeq 5H.$$

Its three maximal minors are sections

$$s_0, s_1, s_2 \in H^0(X, \det N) = H^0(X, 5H).$$

To make the relation with the pointwise resultant explicit, choose a basis v_0^2, v_0v_1, v_1^2 of S^2V and a local frame of N . Write

$$\sigma = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

and choose the signs of the maximal minors as

$$s_0 = bf - ce, \quad s_1 = cd - af, \quad s_2 = ae - bd.$$

The two scalar second fundamental forms are

$$q_0(x, y) = ax^2 + 2bxy + cy^2, \quad q_1(x, y) = dx^2 + 2exy + fy^2.$$

With respect to the coordinates z_0, z_1, z_2 associated with v_0^2, v_0v_1, v_1^2 , the Veronese conic has equation

$$Q(z_0, z_1, z_2) = z_1^2 - 4z_0z_2.$$

A direct calculation gives the universal identity

$$\text{Res}(q_0, q_1) = (cd - af)^2 - 4(bf - ce)(ae - bd) = Q(s_0, s_1, s_2).$$

When σ_x has rank two, the vector $(s_0(x), s_1(x), s_2(x))$ spans its kernel; when its rank is at most one, all three minors vanish and the same polynomial identity remains valid. Combining this calculation with Lemma 2.2 gives globally

$$(1) \quad \det \varphi = c Q(s_0, s_1, s_2), \quad c \in \mathbb{C}^*.$$

The line bundle of this determinant can also be computed directly. Since T_X has rank two and trivial determinant,

$$\det(T_X \otimes N^*) \simeq (\det T_X)^{\otimes 2} \otimes (\det N^*)^{\otimes 2} \simeq (\det N)^{-2}.$$

The bundle $E = (S^2T_X^* \otimes T_X)_0$ is trivial, because T_X is trivial, and hence $\det E \simeq \mathcal{O}_X$. Taking the determinant of a map from $T_X \otimes N^*$ to E therefore produces a section of

$$\det E \otimes \det(T_X \otimes N^*)^{-1} \simeq (\det N)^2 \simeq \mathcal{O}_X(10H).$$

Consequently

$$(2) \quad \det \varphi \in H^0(X, 10H).$$

We now record the cohomology and positivity properties of the normal bundle needed below.

The Euler and normal sequences are

$$0 \rightarrow \mathcal{O}_X \rightarrow H^{\oplus 5} \rightarrow T_{\mathbb{P}^4}|_X \rightarrow 0, \quad 0 \rightarrow T_X \rightarrow T_{\mathbb{P}^4}|_X \rightarrow N \rightarrow 0.$$

By Lemma 2.1, $h^0(X, H) = 5$, and Kodaira vanishing gives $H^i(X, H) = 0$ for $i > 0$. Since $h^0(X, \mathcal{O}_X) = 1$, $h^1(X, \mathcal{O}_X) = 2$, and $h^2(X, \mathcal{O}_X) = 1$, the cohomology sequence of the Euler sequence gives

$$0 \rightarrow \mathbb{C} \rightarrow \mathbb{C}^{25} \rightarrow H^0(T_{\mathbb{P}^4}|_X) \rightarrow \mathbb{C}^2 \rightarrow 0,$$

as well as

$$H^1(T_{\mathbb{P}^4}|_X) \simeq H^2(\mathcal{O}_X) \simeq \mathbb{C}, \quad H^2(T_{\mathbb{P}^4}|_X) = 0.$$

Thus

$$h^0(T_{\mathbb{P}^4}|_X) = 26, \quad h^1(T_{\mathbb{P}^4}|_X) = 1, \quad h^2(T_{\mathbb{P}^4}|_X) = 0.$$

Put $V = H^0(X, T_X)$ and $U = H^1(X, \mathcal{O}_X)$, so that $H^1(T_X) = V \otimes U$. Under the preceding identification, the map $H^1(T_X) \rightarrow H^1(T_{\mathbb{P}^4}|_X)$ in the normal sequence is

$$\mu_H : V \otimes U \rightarrow H^2(\mathcal{O}_X) = \bigwedge^2 U, \quad v \otimes \alpha \mapsto \alpha \smile \iota_v c_1(H).$$

The map $v \mapsto \iota_v c_1(H)$ is the differential of the polarization isogeny $\phi_H : X \rightarrow \widehat{X}$ and hence is an isomorphism $V \simeq U$. It follows that μ_H is surjective and has a three-dimensional kernel. The cohomology sequence of the normal sequence therefore becomes

$$0 \rightarrow \mathbb{C}^2 \rightarrow \mathbb{C}^{26} \rightarrow H^0(N) \rightarrow \mathbb{C}^4 \xrightarrow{\mu_H} \mathbb{C} \rightarrow H^1(N) \rightarrow \mathbb{C}^2 \rightarrow 0,$$

and $H^2(N) = 0$. Exactness now gives

$$(3) \quad h^0(N) = 27, \quad h^1(N) = 2, \quad h^2(N) = 0.$$

This agrees with Manolache's representation-theoretic computation [3, Proposition 1, p. 117]. The Euler sequence also exhibits $T_{\mathbb{P}^4}$ as a quotient of $\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5}$, so its restriction is ample and globally generated. Hence the quotient bundle N is ample and globally generated as well.

3. THE HEAT EQUATION AND GLOBAL NORMAL SECTIONS

We first establish the global injectivity of the second fundamental form; this analytic input is needed for the rank arguments below.

Proposition 3.1. *For a complete $(1, 5)$ theta embedding*

$$X \hookrightarrow \mathbb{P}(H^0(X, H)^*) \simeq \mathbb{P}^4,$$

the second fundamental form induces an injection

$$H^0(\sigma) : S^2 H^0(X, T_X) \hookrightarrow H^0(X, N).$$

Proof. Pass, if necessary, to a finite level cover so that polarized abelian surfaces with theta structure admit a local fine moduli space B . Let $\mathcal{X} \rightarrow B$ be the local universal family through X and let $\mathcal{H}$ be the relative polarization. Welters identifies first-order polarized deformations with $H^0(S^2 T_X)$ [6, (2.1)], gives an étale-local flat trivialization of the projective theta spaces [6, (2.7)], and identifies its horizontal lift by the heat equation [6, (2.12)].

Write $p : \mathcal{X} \rightarrow B$. After shrinking B , cohomology and base change identify $\mathcal{E} := p_* \mathcal{H}$ as a rank-five vector bundle, and the relative polarization remains very ample on every fibre. The relative evaluation homomorphism is

$$\text{ev} : p^* \mathcal{E} \longrightarrow \mathcal{H}, \quad (x, s) \longmapsto s(x).$$

For $b \in B$, put $H_b = \mathcal{H}|_{X_b}$. The restriction of this map to X_b gives

$$\iota_b : X_b \longrightarrow \mathbb{P}(H^0(X_b, H_b)^*), \quad x \longmapsto [\text{ev}_x], \quad \text{ev}_x(s) = s(x).$$

Here $s(x)$ belongs to the one-dimensional vector space $(H_b)_x$; choosing an identification $(H_b)_x \simeq \mathbb{C}$ changes ev_x only by a nonzero scalar, so its projective class is well defined. Since H_b is very ample, ι_b is the embedding defined by the complete linear system $|H_b|$. These fibrewise maps assemble into the relative embedding

$$\iota : \mathcal{X} \longrightarrow \mathbb{P}_B(\mathcal{E}^*).$$

Welters' projectively flat trivialization identifies $\mathbb{P}_B(\mathcal{E}^*)$, étale-locally near $0 \in B$, with $\mathbb{P}(\mathcal{E}_0^*) \times B \simeq \mathbb{P}^4 \times B$. Thus ι may be viewed as a family of embeddings $\iota_b : X_b \hookrightarrow \mathbb{P}^4$ with $\iota_0 : X \hookrightarrow \mathbb{P}^4$ the given embedding.

For $q \in T_{B,0}$, pull this family back to $D = \text{Spec } \mathbb{C}[\varepsilon]/(\varepsilon^2)$ along the tangent vector q . Choose an open cover $\{U_i\}$ of X and local trivializations of the resulting first-order deformation over D . Differentiating the corresponding local maps to $\mathbb{P}^4$ in the ε -direction gives sections $\xi_i \in H^0(U_i, T_{\mathbb{P}^4}|_X)$. On overlaps, the change of trivialization is tangent to X , so the images of the ξ_i in N agree. They therefore glue to a global normal section, denoted

$$\rho(q) \in H^0(X, N).$$

This construction is linear in q and defines the embedded Kodaira–Spencer map

$$\rho : T_{B,0} \longrightarrow H^0(X, N).$$

Let $\delta : H^0(N) \rightarrow H^1(T_X)$ be the connecting map in the normal sequence, and let

$$\text{KS}_B : T_{B,0} \longrightarrow H^1(X, T_X)$$

denote the Kodaira–Spencer map of the family $\mathcal{X} \rightarrow B$: it sends a tangent vector $q \in T_{B,0}$ to the class of the corresponding first-order deformation of the fibre X . To see directly that this map is injective, retain the symplectic level structure used to construct the fine moduli space. Analytically, a neighborhood of 0 in B is an open subset of the Siegel upper half-space $\mathfrak{H}_2$, whose points are symmetric period matrices τ with positive-definite imaginary part. Thus $T_{B,0}$ is the space of symmetric matrices $q = (q_{ij})$. The vector q represents the first-order period matrix $\tau + \varepsilon q$. Because the level structure fixes the symplectic marking, a trivial first-order deformation has constant period matrix; hence $\text{KS}_B(q) = 0$ implies $q = 0$.

There is also an intrinsic description of the image. For $\kappa \in H^1(X, T_X)$, the obstruction to extending H over the corresponding first-order deformation is the contraction of κ with $c_1(H)$, namely the map $\mu_H : H^1(T_X) \rightarrow H^2(\mathcal{O}_X)$ computed above. Consequently $\text{Im } \text{KS}_B \subset \ker \mu_H$. Both spaces have dimension three: $\dim B = \dim \mathfrak{H}_2 = 3$, while $\dim \ker \mu_H = 3$. Therefore KS_B identifies $T_{B,0}$ with the subspace of first-order deformations that preserve the polarization. The definition of the connecting homomorphism now gives

$$(4) \quad \delta \circ \rho = \text{KS}_B.$$

Indeed, on $U_i \cap U_j$ the tangent vector field $\xi_i - \xi_j$ is the Čech cocycle obtained by differentiating the transition functions of the first-order family. It represents both $\delta(\rho(q))$ and the Kodaira–Spencer class of the underlying deformation.

We now compare the embedded Kodaira–Spencer section with the second fundamental form. Work analytically on a simply connected neighborhood of $0 \in B$, and use the theta structure to choose homogeneous theta coordinates. A local homogeneous lift of the complete theta embedding is

$$\Theta_\tau(z) = (\theta_0(z, \tau), \dots, \theta_4(z, \tau)).$$

After identifying $T_{B,0}$ with $S^2 H^0(T_X)$ as in [6, (2.1)], write $q = (q_{ij})$. The heat equations for the horizontal theta frame have the form

$$\frac{\partial \theta_\alpha}{\partial \tau_{ij}} = c_{ij} \frac{\partial^2 \theta_\alpha}{\partial z_i \partial z_j}, \quad c_{ij} \neq 0.$$

It follows that the derivative of the homogeneous lift in the direction q is

$$(5) \quad \dot{\Theta}_q = \sum_{i \leq j} c_{ij} q_{ij} \frac{\partial^2 \Theta}{\partial z_i \partial z_j}.$$

At a point $x \in X$, the normal fibre is obtained from the homogeneous coordinate space by quotienting first by the position vector $\Theta(x)$ and then by the first derivatives $\partial\Theta/\partial z_i(x)$. The Gauss formula therefore identifies the normal class of $\partial^2\Theta/\partial z_i\partial z_j$ with $\sigma(\partial/\partial z_i, \partial/\partial z_j)$. Changes of the local coordinates z or of the homogeneous lift add only position and first-derivative terms, so the resulting normal class is intrinsic. Multiplication of the symmetric coordinates q_{ij} by the nonzero constants c_{ij} defines an automorphism

$$A : S^2 H^0(T_X) \longrightarrow S^2 H^0(T_X).$$

For clarity, one must also compare the theta frame used in (5) with the projective trivialization used to define ρ . Their ratio is a holomorphic map from the base to PGL_5 ; after making the two frames agree at 0, its derivative in the direction q is an ambient projective vector field $\eta_q \in H^0(\mathbb{P}^4, T_{\mathbb{P}^4})$. Thus the preceding calculation gives the following equality in $H^0(X, N)$:

$$(6) \quad \rho(q) = H^0(\sigma)(Aq) + [\eta_q|_X]_N.$$

The last summand lifts globally to $T_{\mathbb{P}^4}|_X$ and is killed by the connecting map δ . Combining (6) with (4) yields the choice-independent identity

$$(7) \quad \delta(H^0(\sigma)(Aq)) = \mathrm{KS}_B(q).$$

In particular,

$$H^0(\sigma)(Aq) = 0 \implies \mathrm{KS}_B(q) = 0.$$

The Kodaira–Spencer map of a local fine moduli space is injective, and A is invertible. Therefore $q = 0$, proving that $H^0(\sigma)$ is injective. $\square$

Remark 3.2. Conceptually, the heat equation acts as a dictionary: it translates a first-order motion of the period matrix into second derivatives of the theta embedding. The Gauss formula retains their normal component as the second fundamental form, while δ discards motion caused only by changing projective coordinates. Thus Proposition 3.1 says that none of the three infinitesimal polarized-deformation directions becomes invisible in the normal bundle. Equation (6) makes this translation an equality of global normal sections, rather than a merely pointwise analogy.

4. INVARIANT NORMAL SECTIONS AND GENERIC INVERTIBILITY

We now identify the heat-equation sections with the normal sections coming from the defining Horrocks–Mumford bundle.

Proposition 4.1. *Let F be the Horrocks–Mumford bundle in the convention in which a regular section $s \in H^0(\mathbb{P}^4, F)$ cuts out X , and put $G = K(H)$. Under the identification $N \simeq F|_X$, one has*

$$\mathrm{Im} H^0(\sigma) = H^0(X, N)^G = \mathrm{Im}(H^0(\mathbb{P}^4, F)/\mathbb{C}s \longrightarrow H^0(X, N)).$$

This common space has dimension three. Its evaluation map generates N away from the finite set obtained by intersecting X with the twenty-five base lines of the Horrocks–Mumford bundle. Consequently $\sigma : S^2T_X \rightarrow N$ is surjective away from a finite set.

Proof. We first describe the group action precisely. For each $a \in K(H)$, choose an isomorphism $\alpha_a : t_a^*H \xrightarrow{\sim} H$. Pullback of sections followed by α_a induces a linear automorphism of $H^0(X, H)$. Replacing α_a by a scalar multiple changes this automorphism by the same scalar, so its projective class is independent of the choice. We therefore obtain a well-defined projective representation

$$\bar{\rho} : K(H) \longrightarrow \mathrm{PGL}(H^0(X, H)^*)$$

for which the complete theta embedding $\iota_H : X \hookrightarrow \mathbb{P}(H^0(X, H)^*)$ is equivariant:

$$\bar{\rho}(a) \circ \iota_H = \iota_H \circ t_a.$$

This projective representation can be written explicitly after choosing a level-five theta structure. Let ζ be a primitive fifth root of unity and choose a basis $e_0, \dots, e_4$ of $V = H^0(X, H)^*$, with indices modulo 5. The two standard lifts of generators of $K(H)$ act by

$$S(e_j) = e_{j+1}, \quad T(e_j) = \zeta^j e_j.$$

They satisfy $TS = \zeta ST$. Hence the finite Heisenberg group $\mathcal{H}_5 = \langle S, T \rangle \subset \mathrm{SL}(V)$ has center $\mu_5 = \langle \zeta I_V \rangle$ and fits into the central extension

$$1 \longrightarrow \mu_5 \longrightarrow \mathcal{H}_5 \longrightarrow (\mathbb{Z}/5)^2 \longrightarrow 1$$

[2, Section 1]. The chosen theta structure identifies $\mathcal{H}_5/\mu_5$ with $K(H) = G$. Since the center acts on V by scalars, it acts trivially on $\mathbb{P}(V)$; thus the linear Schrödinger representation of $\mathcal{H}_5$ induces precisely the projective action $\bar{\rho}$ of G on $\mathbb{P}(V) \simeq \mathbb{P}^4$. The equivariance identity above shows both that this action preserves X and that its restriction to X is the translation action of G [2, Theorem 5.2].

We next compare the linearizations on $F|_X$ and on the normal bundle. The Horrocks–Mumford construction is $\mathcal{H}_5$ -equivariant and gives F an $\mathcal{H}_5$ -linearization. Horrocks and Mumford compute that all four global sections of F are fixed by $\mathcal{H}_5$ [2, Section 4]. The centre μ_5 acts trivially on these sections. Since they generate F away from the union of the twenty-five base

lines [2, Sections 3–5], they generate every fibre on a dense open set. It follows that the centre acts trivially on F itself. Thus the $\mathcal{H}_5$ -linearization descends to a genuine G -linearization of F , and $H^0(\mathbb{P}^4, F)$ is a four-dimensional trivial G -module. In particular, the section s defining X is G -invariant.

The differential of this invariant regular section is a G -equivariant map

$$ds : T_{\mathbb{P}^4}|_X \longrightarrow F|_X.$$

Regularity says that it is surjective and that its kernel is T_X . Consequently it induces a G -equivariant isomorphism

$$(8) \quad N_{X/\mathbb{P}^4} \xrightarrow{\sim} F|_X.$$

Here the action on the left is the canonical action obtained by differentiating the projective G -action. This verifies that the two G -actions used below are the same, rather than linearizations differing by an unspecified character.

It remains to determine the kernel of restriction to X . The Koszul resolution of the zero scheme of the regular section s is

$$0 \longrightarrow \det F^* \longrightarrow F^* \longrightarrow I_X \longrightarrow 0.$$

Since $\det F \simeq \mathcal{O}_{\mathbb{P}^4}(5)$, tensoring this sequence by F gives

$$0 \longrightarrow F(-5) \longrightarrow \operatorname{End} F \longrightarrow F \otimes I_X \longrightarrow 0.$$

The simplicity of F , together with $H^0(F(-5)) = H^1(F(-5)) = 0$, gives $H^0(F \otimes I_X) \simeq H^0(\operatorname{End} F) = \mathbb{C} \operatorname{id}_F$; under the last arrow, id_F maps to s . These cohomology facts are part of the Horrocks–Mumford calculation [2, Section 4]. Hence the kernel of $H^0(\mathbb{P}^4, F) \rightarrow H^0(X, F|_X)$ is exactly $\mathbb{C}s$, and its image is a three-dimensional subspace of $H^0(X, N)^G$.

We compute the invariant part of $H^0(X, N)$ with the same canonical linearization. Recall that $V = H^0(X, H)^*$ is the Schrödinger representation. The restricted Euler sequence is

$$0 \longrightarrow \mathcal{O}_X \longrightarrow H \otimes V \longrightarrow T_{\mathbb{P}^4}|_X \longrightarrow 0.$$

Although the centre acts nontrivially on H and on V , the two central characters cancel in $H \otimes V$. On global sections the middle term is

$$H^0(X, H) \otimes V = V^* \otimes V = \operatorname{End}(V),$$

with G acting by conjugation. The matrices $S^i T^j$, for $0 \leq i, j \leq 4$, are eigenvectors with the twenty-five distinct characters of G . Hence $\operatorname{End}(V)$ is the regular representation of G .

Translations act trivially on $H^i(X, \mathcal{O}_X)$, and $H^i(X, H) = 0$ for $i > 0$. Taking cohomology of the restricted Euler sequence gives an exact sequence of G -modules

$$0 \longrightarrow \mathbb{C} \longrightarrow \operatorname{End}(V) \longrightarrow H^0(T_{\mathbb{P}^4}|_X) \longrightarrow \mathbb{C}^2 \longrightarrow 0.$$

The first map sends 1 to id_V . Since $\text{End}(V)$ is the regular representation of G , its quotient by this trivial summand is

$$\text{End}(V)/\mathbb{C}\text{id}_V \simeq \bigoplus_{\substack{\chi \in \widehat{G} \\ \chi \neq 1}} \mathbb{C}_\chi.$$

Here $\mathbb{C}_\chi$ denotes the one-dimensional representation with character χ . The final term $\mathbb{C}^2 \simeq H^1(X, \mathcal{O}_X)$ is also a trivial G -module. By Maschke's theorem, the resulting short exact sequence splits G -equivariantly. Hence

$$H^0(T_{\mathbb{P}^4}|_X) \simeq \mathbb{C}^{\oplus 2} \oplus \bigoplus_{\substack{\chi \in \widehat{G} \\ \chi \neq 1}} \mathbb{C}_\chi.$$

The equivariant injection $H^0(T_X) \simeq \mathbb{C}^2 \rightarrow H^0(T_{\mathbb{P}^4}|_X)$ identifies its source with the entire trivial isotypic summand. Moreover, the equivariant cohomology sequence of the normal sequence maps $H^0(N)$ onto $\ker \mu_H \subset H^1(T_X)$, which is a three-dimensional trivial G -module by the calculation preceding Proposition 3.1. It follows, again by semisimplicity, that

$$H^0(X, N) \simeq \mathbb{C}^{\oplus 3} \oplus \bigoplus_{\substack{\chi \in \widehat{G} \\ \chi \neq 1}} \mathbb{C}_\chi.$$

This also recovers the representation-theoretic computation in [3, Proposition 1, p. 117]. In particular, $h^0(X, N)^G = 3$, and the preceding image equals the full invariant subspace.

Translations preserve every translation-invariant vector field, so they act trivially on $H^0(X, T_X)$ and hence on its symmetric square. The second fundamental form is natural under projective transformations preserving X , and is therefore G -equivariant. It follows that $\text{Im } H^0(\sigma) \subset H^0(X, N)^G$. The source $S^2 H^0(X, T_X)$ has dimension three, and Proposition 3.1 says that $H^0(\sigma)$ is injective. Its image consequently has dimension three and must equal the invariant subspace, which already equals the image of $H^0(\mathbb{P}^4, F)/\mathbb{C}s$.

Horrocks and Mumford prove that F is generated by its four global sections away from the union L of twenty-five pairwise disjoint lines [2, Sections 3–5]. At a point $x \in X \setminus L$, the evaluation $H^0(F) \rightarrow F_x$ is surjective. Since $s(x) = 0$, this map factors through $H^0(F)/\mathbb{C}s$, and its image still equals $F_x = N_x$. Finally, every morphism from $\mathbb{P}^1$ to an abelian variety is constant, so an abelian surface contains no projective line. Thus it contains no component of L , and $X \cap L$ is finite. The common three-dimensional space generates N away from that finite set. As T_X is trivial and $H^0(\sigma)$ is the map on constant sections, this is precisely the pointwise surjectivity of σ there. $\square$

Remark 4.2. The G -action may be viewed as a symmetry filter on $H^0(X, N)$. The four Horrocks–Mumford sections, modulo the defining section s , produce a three-dimensional space of invariant normal motions. Proposition 3.1 produces three independent invariant normal sections from the second fundamental form. Since the full invariant subspace also has dimension three, the two constructions describe the same normal motions. Away from the finite set $X \cap L$, this common space spans every normal fibre; equivalently, the second fundamental form fills the normal plane at a general point.

It remains to show that the pointwise resultant is not identically zero; this gives generic invertibility of the homomorphism φ .

Proposition 4.3. *Under the assumptions of Theorem 1.1, $\det \varphi \neq 0$.*

Proof. Let

$$U = \{x \in X : \text{rank } \sigma_x = 2\}.$$

Proposition 4.1 says that $X \setminus U$ is finite. On U there is an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{R} \longrightarrow S^2 T_X|_U \xrightarrow{\sigma} N|_U \longrightarrow 0, \quad \text{rank } \mathcal{R} = 1.$$

Suppose, for contradiction, that $\det \varphi \equiv 0$. By Lemma 2.2, for every $x \in U$ the point $[\mathcal{R}_x] \in \mathbb{P}(S^2 T_{X,x})$ lies on the relative Veronese conic. The relative Veronese morphism

$$\nu_2 : \mathbb{P}(T_X|_U) \longrightarrow \mathbb{P}(S^2 T_X|_U), \quad [\ell] \longmapsto [\ell^{\otimes 2}],$$

is a closed immersion and an isomorphism onto that conic. The section $x \mapsto [\mathcal{R}_x]$ therefore has a unique inverse through ν_2 . Pulling back the tautological line in $T_X|_U$ gives a line subbundle $A_U \subset T_X|_U$ for which multiplication induces an isomorphism

$$(9) \quad A_U^{\otimes 2} \simeq \mathcal{R}.$$

Taking determinants in the exact sequence on U and using $\det(S^2 T_X) \simeq (\det T_X)^{\otimes 3}$ gives

$$\mathcal{R} \simeq \det(S^2 T_X|_U) \otimes (\det N|_U)^{-1} \simeq (\det T_X)^{\otimes 3}|_U \otimes (\det N)^{-1}|_U.$$

Since $T_X \simeq \mathcal{O}_X^{\oplus 2}$ and $\det N \simeq \mathcal{O}_X(5H)$, this reduces to $\mathcal{R} \simeq \mathcal{O}_X(-5H)|_U$. Together with (9), this yields

$$(10) \quad A_U^{\otimes 2} \simeq \mathcal{O}_X(-5H)|_U.$$

We now justify passage across $X \setminus U$. On a smooth variety, removal of a subset of codimension at least two induces an isomorphism $\text{Pic}(X) \xrightarrow{\sim} \text{Pic}(U)$. Equivalently, the reflexive hull of the extension by zero of A_U is a rank-one reflexive sheaf, hence a line bundle A on the smooth surface X .

Both sides of (10) extend to line bundles on X , and an isomorphism between line bundles over U extends uniquely across the finite complement. Thus

$$A^{\otimes 2} \simeq \mathcal{O}_X(-5H)$$

on X . Taking first Chern classes gives

$$2c_1(A) = -5H \quad \text{in } \text{NS}(X).$$

Since 5 is odd, this equality would make H divisible by 2 in $\text{NS}(X)$, contradicting the primitivity of H . Thus the assumption $\det \varphi \equiv 0$ is false.

Therefore $\det \varphi$ is a nonzero section. Since the source and target of φ both have rank four, φ is an isomorphism on the nonempty dense open set $X \setminus V(\det \varphi)$. This proves Theorem 1.1(i). $\square$

We now complete the proof of the remaining parts of the main theorem. By Proposition 4.3 and (2), $\det \varphi$ is a nonzero section of $\mathcal{O}_X(10H)$. This ample, nontrivial line bundle has no nowhere-vanishing section, since such a section would trivialize it. Therefore its zero scheme Σ_f is an effective divisor with numerical class $10H$. Using $H^2 = 10$, one obtains

$$H \cdot \Sigma_f = H \cdot (10H) = 100, \quad \Sigma_f^2 = (10H)^2 = 1000.$$

Adjunction on the smooth surface X , together with $K_X \simeq \mathcal{O}_X$, then gives

$$p_a(\Sigma_f) = 1 + \frac{\Sigma_f \cdot (\Sigma_f + K_X)}{2} = 1 + \frac{1000}{2} = 501.$$

This proves Theorem 1.1(ii).

We now analyze the rank-one degeneracy scheme of σ in order to prove Theorem 1.1(iii). Proposition 4.1 says that σ has rank two off a finite set. Hence the scheme $Z = D_1(\sigma)$ defined by σ whose rank is at most one, is either empty or zero-dimensional; in the latter case it has the expected codimension

$$(3-1)(2-1) = 2$$

for the degeneracy locus of a map of ranks three to two. We first show that it is not empty. Otherwise σ would be everywhere surjective, with exact sequence

$$0 \longrightarrow (\det N)^{-1} \longrightarrow \mathcal{O}_X^3 \longrightarrow N \longrightarrow 0.$$

Indeed, the kernel is a line bundle and taking determinants identifies it with $(\det N)^{-1}$. The Whitney formula would give

$$c(N) = c((\det N)^{-1})^{-1} = \frac{1}{1-5H} = 1 + 5H + 25H^2,$$

and hence $c_2(N) = 25H^2$. On the other hand, the Euler sequence gives $c(T_{\mathbb{P}^4}|_X) = (1+H)^5$, while $c(T_X) = 1$ because T_X is trivial. The normal

sequence therefore gives

$$c(N) = \frac{c(T_{\mathbb{P}^4}|_X)}{c(T_X)} = (1 + H)^5 = 1 + 5H + 10H^2,$$

after discarding terms of degree greater than two on the surface. This is incompatible with $c_2(N) = 25H^2$. Thus Z is nonempty and has the expected codimension.

The Thom–Porteous formula [1, Chapter 14] applies to the bundle homomorphism $\sigma : S^2T_X \rightarrow N$ of ranks $e = 3$ and $f = 2$. For the degeneracy scheme $Z = D_1(\sigma)$, the source bundle is trivial, so the relevant Chern class is $c(N - \mathcal{O}_X^3) = c(N)$, and the Porteous determinant is

$$[Z] = \det \begin{pmatrix} c_1(N) & c_2(N) \\ 1 & c_1(N) \end{pmatrix} = c_1(N)^2 - c_2(N).$$

Since $H^2 = 10$,

$$(11) \quad \text{length } Z = \int_X (c_1(N)^2 - c_2(N)) = (25 - 10)H^2 = 15 \cdot 10 = 150.$$

Equation (1) also shows that a local equation of Σ_f satisfies

$$Q(s_0, s_1, s_2) \in (s_0, s_1, s_2)^2 = I_Z^2.$$

Here (s_0, s_1, s_2) is exactly the local determinantal ideal defining $D_1(\sigma)$. Since $d(I_Z^2) \subset I_Z \Omega_X^1$, both the local equation and all its first derivatives lie in I_Z . Thus the ideal of the singular scheme of Σ_f is contained in I_Z . Reversing ideals, this is the scheme-theoretic inclusion

$$Z \subset \text{Sing}(\Sigma_f)$$

This proves part (iii) of Theorem 1.1.

5. THE DESCENDED THETA LINEAR SUBSYSTEM

Put $G = K(H)$. It acts on X by translations,

$$G \times X \longrightarrow X, \quad (a, x) \longmapsto x + a.$$

This action is free: if $x + a = x$, then $a = 0$. Hence the orbit map

$$\pi : X \longrightarrow Y := X/G, \quad x \longmapsto G \cdot x,$$

is a finite étale morphism of degree $|G| = 25$. Since smoothness descends under finite étale morphisms, Y is smooth; moreover, the group law on X descends and makes Y an abelian surface.

The polarization isogeny

$$\phi_H : X \longrightarrow \widehat{X}, \quad x \longmapsto t_x^* H \otimes H^{-1},$$

has kernel G . It is therefore constant on the G -orbits and factors uniquely as

$$\phi_H = \bar{\phi}_H \circ \pi, \quad \bar{\phi}_H : Y \longrightarrow \hat{X}.$$

Since $\deg \phi_H = 25 = \deg \pi$, one has $\deg \bar{\phi}_H = 1$. Thus $\bar{\phi}_H$ is an isomorphism, which gives

$$Y = X/K(H) \simeq \hat{X}.$$

Each element of G is induced by a projective automorphism of $\mathbb{P}^4$ preserving X . Its differential acts linearly on $T_{\mathbb{P}^4}|_X$ and preserves the subbundle T_X ; hence it acts on the quotient $N = T_{\mathbb{P}^4}|_X/T_X$ and on $\det N$. These actions satisfy the cocycle condition and give genuine G -linearizations, even though the original action on homogeneous coordinates was defined only projectively. This is the canonical normal-bundle linearization already used in (8). Finite Étale descent for the free G -action therefore produces a line bundle M on Y such that

$$\pi^* M \simeq \det N \simeq \mathcal{O}_X(5H).$$

The pullback is ample, so M is ample. Intersection numbers under the degree-25 map satisfy $(\pi^* M)^2 = 25M^2$; therefore

$$25M^2 = (5H)^2 = 25H^2 = 250, \quad M^2 = 10.$$

If (d_1, d_2) is the polarization type of M , then $d_1 d_2 = M^2/2 = 5$ and $d_1 \mid d_2$. Thus M has type $(1, 5)$.

We next descend the maximal minors scheme-theoretically. Put $V = H^0(X, T_X)$. The exterior square of the second fundamental form is the G -equivariant homomorphism

$$\bigwedge^2 \sigma : \bigwedge^2 (S^2 V) \otimes \mathcal{O}_X \longrightarrow \det N.$$

Translations act trivially on V , so the rank-three source has the trivial G -linearization. The three coordinate sections of this homomorphism are precisely the maximal minors s_0, s_1, s_2 ; each is G -invariant and hence descends uniquely to a section $\bar{s}_i \in H^0(Y, M)$. Let W_0 be their span and let $\bar{Z}$ be their common zero scheme. Pulling back its ideal recovers the determinantal ideal (s_0, s_1, s_2) , so

$$Z = X \times_Y \bar{Z}$$

as schemes. Because π is finite and surjective, a positive-dimensional component of $\bar{Z}$ would have a positive-dimensional inverse image in X . Since Z is zero-dimensional, the displayed equality shows that $\bar{Z}$ is zero-dimensional as well. No complete-intersection assertion is used here: its ideal is generated by the three descended maximal minors. Since π is finite flat of degree

25, lengths multiply by 25, and (11) gives

$$150 = \text{length } Z = 25 \text{ length } \bar{Z}, \quad \text{length } \bar{Z} = 6.$$

It remains to check that W_0 really has dimension three. The minors are not all zero because σ has rank two on a dense open set. If their span had dimension one, their common zero scheme would be the divisor of a single nonzero section of the ample line bundle M , contradicting the fact that $\bar{Z}$ is zero-dimensional. If their span had dimension two, choose a basis $\bar{s}_0, \bar{s}_1$. These two sections have no common divisorial component, again because $\bar{Z}$ is zero-dimensional. Their common zero scheme is therefore a proper complete intersection, whose length is

$$c_1(M)^2 = M^2 = 10.$$

This contradicts $\text{length } \bar{Z} = 6$. Hence $\dim W_0 = 3$; write $W = W_0 \subset H^0(Y, M)$.

The three sections define the rational map

$$\bar{\kappa} : Y \dashrightarrow \mathbb{P}(W^*) \simeq \mathbb{P}^2, \quad y \longmapsto [\bar{s}_0(y) : \bar{s}_1(y) : \bar{s}_2(y)],$$

whose indeterminacy scheme is $\bar{Z}$. Let $\Delta \subset \mathbb{P}^2$ be the Veronese conic $Q = 0$. Since Q is homogeneous of degree two, $Q(\bar{s}_0, \bar{s}_1, \bar{s}_2)$ is a section of $M^{\otimes 2}$. Its pullback is, up to the nonzero scalar in (1), $\det \varphi$, so it is not the zero section. Therefore

$$\bar{\Sigma} = V(Q(\bar{s}_0, \bar{s}_1, \bar{s}_2)) \in |2M|$$

is the scheme-theoretic closure of $\bar{\kappa}^{-1}(\Delta)$ from $Y \setminus \bar{Z}$ to Y , and

$$\Sigma_f = X \times_Y \bar{\Sigma} = \pi^{-1}(\bar{\Sigma})$$

scheme-theoretically. Locally $Q(\bar{s}_0, \bar{s}_1, \bar{s}_2)$ belongs to $(\bar{s}_0, \bar{s}_1, \bar{s}_2)^2 = I_{\bar{Z}}^2$, which proves that $\bar{\Sigma}$ is at least double along $\bar{Z}$. Finally, adjunction on the abelian surface Y gives

$$p_a(\bar{\Sigma}) = 1 + \frac{(2M)^2}{2} = 1 + 2M^2 = 21.$$

This proves every assertion of Theorem 1.2.

Finally, we record the local analysis needed for the conditional node statement. We prove Theorem 1.3. Let $p \in Z$ be a reduced point. If $\text{rank } \sigma_p = 0$, all six entries of a local 2×3 matrix for σ belong to the maximal ideal $\mathfrak{m}_p$. Its maximal minors then belong to $\mathfrak{m}_p^2$, so they cannot generate $\mathfrak{m}_p$, as would be necessary for the zero-dimensional scheme Z to be reduced at p . Hence $\text{rank } \sigma_p = 1$.

The image of $\sigma_p : S^2 T_{X,p} \rightarrow N_p$ is now a line, and choosing a nonzero linear functional on this line gives, up to scalar, one binary quadratic form on $T_{X,p}$. Assume that it is nondegenerate. Over $\mathbb{C}$ we may choose tangent

coordinates x, y in which this form is xy , and then choose a normal frame whose first vector spans $\text{Im } \sigma_p$. Relative to the basis x^2, xy, y^2 of S^2T_X , the matrix of σ at p is therefore

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The upper middle entry remains a unit near p ; rescale the first row to make it 1, and subtract a multiple of that row from the second row to make the lower middle entry zero. Thus the local matrix takes the form

$$\begin{pmatrix} A & 1 & C \\ a & 0 & c \end{pmatrix}, \quad A, C, a, c \in \mathfrak{m}_p.$$

Its signed maximal minors are $-a$, $Ac - aC$, and c , so their ideal is

$$(-a, Ac - aC, c) = (a, c).$$

Because Z is reduced at p , this ideal equals $\mathfrak{m}_p$. Consequently the linear terms a_1, c_1 form a basis of $\mathfrak{m}_p/\mathfrak{m}_p^2$. Up to an overall sign, the Plücker coordinates of the kernel are the cross product of the two rows, namely

$$(c, aC - Ac, -a).$$

In coordinates (z_0, z_1, z_2) on $\mathbb{P}(S^2T_{X,p})$, the Veronese conic has equation $z_1^2 - 4z_0z_2 = 0$. Substitution gives

$$(aC - Ac)^2 + 4ac.$$

For $h \in \mathfrak{m}_p^2$, write

$$\text{in}_2(h) := h \bmod \mathfrak{m}_p^3 \in \mathfrak{m}_p^2/\mathfrak{m}_p^3 \simeq \text{Sym}^2(\mathfrak{m}_p/\mathfrak{m}_p^2)$$

for its quadratic initial form. The first summand has order at least four, whereas the quadratic initial form of the second is $4a_1c_1$. After absorbing the nonzero normalization constant from (1), there is a scalar $\lambda \in \mathbb{C}^*$ such that

$$\text{in}_2(\det \varphi) = \lambda a_1c_1.$$

Since a_1 and c_1 are independent, the tangent cone is the union of two distinct lines. Thus the multiplicity is two and the quadratic tangent cone is nondegenerate, which is precisely an ordinary node. If the nonzero scalar quadratic form has rank one, the analogous normal form has quadratic initial term equal to the square of a linear form; this explains the nondegeneracy assumption in Theorem 1.3(a).

Under hypothesis (a), the reduced scheme $\bar{Z}$ has length 6 and therefore consists of six distinct points. Each has exactly 25 distinct preimages because π is étale. Thus Z consists of $6 \cdot 25 = 150$ reduced points, and the local calculation makes each one an ordinary node of Σ_f .

On $U = Y \setminus \bar{Z}$, the rational map $\bar{\kappa}$ is a morphism, and $\bar{\Sigma} \cap U$ is the inverse image of the smooth conic Δ . Let $y \in U$ satisfy $\bar{\kappa}(y) \in \Delta$, and let q be a local defining equation of Δ near $\bar{\kappa}(y)$. Hypothesis (b) means that

$$d\bar{\kappa}_y(T_y U) + T_{\bar{\kappa}(y)} \Delta = T_{\bar{\kappa}(y)} \mathbb{P}^2.$$

Because the normal space of Δ in $\mathbb{P}^2$ is one-dimensional, the transversality condition in (b) is equivalent to

$$d(q \circ \bar{\kappa})_y \neq 0.$$

Thus $q \circ \bar{\kappa}$ is a local equation of $\bar{\Sigma}$ with nonzero differential at y . The implicit function theorem shows that $\bar{\Sigma}$ is smooth away from $\bar{Z}$. Since π is étale, its pullback Σ_f is also smooth away from Z . The preceding 150 nodes are therefore all the singularities.

Finally, assume in addition that Σ_f is integral. The normalization of an integral curve with δ ordinary nodes has genus $p_a - \delta$. Here $\delta = 150$, so

$$\begin{aligned} g(\tilde{\Sigma}_f) &= p_a(\Sigma_f) - 150 \\ &= \left(1 + \frac{(10H)^2}{2}\right) - 150 \\ &= 501 - 150 = 351. \end{aligned}$$

For completeness, the same genus can be checked after resolving the net on the quotient. If the six base points are reduced, let $\beta : \tilde{Y} \rightarrow Y$ be their blow-up, with exceptional curves $E_1, \dots, E_6$. The resolved map to $\mathbb{P}^2$ is defined by the base-point-free divisor

$$L = M - E_1 - \dots - E_6, \quad L^2 = M^2 - 6 = 4.$$

The strict transform D of the discriminant belongs to $|2L|$, so $D^2 = 16$. Since $K_{\tilde{Y}} = E_1 + \dots + E_6$, one has $D \cdot K_{\tilde{Y}} = 12$, and adjunction gives

$$p_a(D) = 1 + \frac{D^2 + D \cdot K_{\tilde{Y}}}{2} = 1 + \frac{16 + 12}{2} = 15.$$

In the transverse integral case, the induced cover of normalizations is étale of degree 25. Riemann–Hurwitz then reads

$$351 - 1 = 25(15 - 1),$$

as expected.

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School of Mathematical Sciences, Key Laboratory of MEA (Ministry of Education) and Shanghai Key Laboratory of PMMP, East China Normal University, Rm. 312, Math. Bldg, No. 500, Dongchuan Road, Shanghai, 200241, P. R. China.

E-mail address: `rdu@math.ecnu.edu.cn`