

HIGHER CENTERED CHERN INEQUALITIES FROM HOLOMORPHIC PINCHING ON KÄHLER–EINSTEIN MANIFOLDS

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ABSTRACT. We study compact Kähler–Einstein manifolds with nonzero Einstein constant and holomorphically pinched curvature. We introduce centered Chern classes by adjoining a zero Chern root and subtracting the mean of all roots. Their Chern–Weil representatives are shown to be homogeneous polynomials in the Bochner curvature. This reduces higher reverse Chern number inequalities to finite-dimensional algebraic optimization and gives uniform estimates in every degree, for both positive and negative pinching. Using the exact spherical mean from the author’s earlier variance formula, the Bhatia–Davis inequality yields a mean-sensitive refinement of the author’s degree-two reverse Yau inequalities. In complex dimension three, we determine the sharp maximal-torus-invariant constant, prove a sharp unitary-invariant sum-of-squares estimate for the cubic density, and obtain an explicit uniform bound on the full algebraic Bochner curvature space. A complex quadric supplies the equality example. In complex dimension three we also prove that the corresponding full-space constant is sharp, using a pointwise Bernstein inequality for degree-four spherical harmonics.

1. INTRODUCTION

The characteristic number

$$\left\langle \left(2(n+1)c_2 - nc_1^2 \right) c_1^{n-2}, [M] \right\rangle$$

is governed, on a Kähler–Einstein manifold M of dimension n , by the squared norm of the Bochner curvature. Here $c_j = c_j(T^{1,0}M) \in H^{2j}(M; \mathbb{Z})$ is the j -th Chern class of the holomorphic tangent bundle, juxtaposition denotes the cup product, and $[M] \in H_{2n}(M; \mathbb{Z})$ is the fundamental class. This observation underlies Yau’s Chern number inequality and its rigidity statement [10], as well as recent reverse inequalities obtained from bounds on the holomorphic sectional curvature. From an algebro-geometric viewpoint, Du and Sun [6] obtained inequalities involving all Chern classes for nonsingular projective n -folds with ample canonical or anti-canonical line bundle. In a related positivity setting, Li and Zheng [7] established Chern class inequalities for polarized manifolds and nef vector bundles. In earlier work, the author [5] gave a formula expressing the second centered Chern number as the integral of the variance of the holomorphic sectional curvature.

The purpose of this article is to isolate the higher-order mechanism. The correct characteristic classes are not the individual c_k , but the classes obtained after adjoining one zero Chern root and subtracting the mean of all $n+1$ roots. Namely, with $c_0 = 1$, set

$$\widehat{c}_k := \sum_{j=0}^k \binom{n+1-j}{k-j} c_j \left(-\frac{c_1}{n+1} \right)^{k-j}, \quad 0 \leq k \leq n. \quad (1.1)$$

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For $0 \leq k \leq n$, the associated *centered Chern number* is

$$N_k(M) := \left\langle \widehat{c}_k c_1^{n-k}, [M] \right\rangle. \quad (1.2)$$

Here $\widehat{c}_k \in H^{2k}(M; \mathbb{Q})$ is given by (1.1), and the brackets denote evaluation on the fundamental class, equivalently integration:

$$\langle \alpha, [M] \rangle = \int_M \alpha, \quad \alpha \in H^{2n}(M; \mathbb{Q}).$$

Thus $\widehat{c}_k c_1^{n-k}$ has total degree $2n$, and $N_k(M)$ is a rational characteristic number. In particular, the usual Yau discriminant is exactly

$$2(n+1)N_2(M) = \left\langle (2(n+1)c_2 - nc_1^2)c_1^{n-2}, [M] \right\rangle. \quad (1.3)$$

The first four centered classes are

$$\widehat{c}_0 = 1, \quad \widehat{c}_1 = 0, \quad (1.4)$$

$$\widehat{c}_2 = c_2 - \frac{n}{2(n+1)}c_1^2, \quad (1.5)$$

$$\widehat{c}_3 = c_3 - \frac{n-1}{n+1}c_1c_2 + \frac{n(n-1)}{3(n+1)^2}c_1^3. \quad (1.6)$$

Our first point is that, for a Kähler–Einstein metric, the Chern–Weil form representing $\widehat{c}_k$ is a pure degree- k polynomial in the Bochner tensor. This is most naturally seen from the standard c-projective tractor connection. Here a c-projective structure is an equivalence class of torsion-free complex connections having the same J -planar curves, namely curves γ for which $\nabla_{\dot{\gamma}}\dot{\gamma}$ lies in the real span of $\dot{\gamma}$ and $J\dot{\gamma}$. Its standard tractor bundle is a canonical rank- $(n+1)$ complex vector bundle with a natural connection encoding the c-projectively invariant curvature. For a Kähler–Einstein metric, the curvature of this connection reduces to the Bochner-curvature block; see Section 3. This converts the global curvature question into an optimization problem on the finite-dimensional space $\mathcal{B}_n$ of algebraic Kähler Bochner tensors.

For the normalization fixed in Section 4, let $q_{k,n}(B)$ denote the scalar density associated with $\widehat{c}_k$, let

$$h_B(\xi) = B_{\xi\bar{\xi}\xi\bar{\xi}}, \quad \|h_B\|_{\infty} = \max_{|\xi|=1} |h_B(\xi)|,$$

and define

$$K_{n,k} := \sup_{0 \neq B \in \mathcal{B}_n} \frac{|q_{k,n}(B)|}{\|h_B\|_{\infty}^{k-2} q_{2,n}(B)}. \quad (1.7)$$

The denominator is positive for $B \neq 0$, and the supremum is finite. To state an effective bound, let

$$\mathfrak{d}_k := \#\{\sigma \in \mathcal{S}_k : \sigma(r) \neq r \text{ for every } r\} = k! \sum_{j=0}^k \frac{(-1)^j}{j!}$$

be the number of derangements of k letters, set $S_n := n(n+1)(n+2)(n+3)$, and put

$$U_{n,k} := 2^{3-k} \frac{(n-k)!}{k!(n-2)!} \mathfrak{d}_k^2 S_n^{(k-2)/2}. \quad (1.8)$$

Our main result is the following.

Theorem 1.1. *Let (M, ω) be a compact Kähler–Einstein manifold of dimension n ($n \geq 2$) with $\text{Ric}(g) = \mu g$ and $\mu \neq 0$. Set*

$$\Lambda_B := \sup_{x \in M} \sup_{|\xi|=1} |B_{\xi\bar{\xi}\xi\bar{\xi}}|.$$

Then, for every $2 \leq k \leq n$,

$$|N_k(M)| \leq K_{n,k} \left(\frac{\Lambda_B}{|\mu|} \right)^{k-2} |N_2(M)| \leq U_{n,k} \left(\frac{\Lambda_B}{|\mu|} \right)^{k-2} |N_2(M)|. \quad (1.9)$$

If, in addition, the holomorphic sectional curvature is positively or negatively δ -pinched for some $0 < \delta \leq 1$, then

$$|N_k(M)| \leq K_{n,k} \left(\frac{2(1-\delta)}{(n+1)\delta} \right)^{k-2} |N_2(M)| \leq U_{n,k} \left(\frac{2(1-\delta)}{(n+1)\delta} \right)^{k-2} |N_2(M)|. \quad (1.10)$$

Here $K_{n,k} \leq U_{n,k}$. For $k = 2$ the statement is an identity with $K_{n,2} = U_{n,2} = 1$. For $k \geq 3$ and $\delta = 1$, all displayed upper bounds vanish.

The constant $K_{n,k}$ retains the sharp local optimization problem, while $U_{n,k}$ is a closed effective upper bound. For fixed k , the latter grows like a constant times n^{k-2} ; this growth is canceled by the factor $(n+1)^{-(k-2)}$ in the pinched global estimate. Determining $K_{n,k}$ exactly remains a separate problem. In degree three, Corollary 4.9 gives a strict spectral improvement of $U_{n,3}$ in every dimension. In complex dimension three, let $T^3 \subset U(3)$ be the diagonal maximal torus, let $\mathcal{B}_3^{T^3}$ be its fixed subspace, and denote by $K_{3,3}^{T^3}$ the supremum in (1.7) restricted to $0 \neq B \in \mathcal{B}_3^{T^3}$. The sum-of-squares argument is substantially stronger: we prove

$$K_{3,3}^{T^3} = \frac{6}{7}$$

on the maximal-torus-invariant subspace, while the Bochner tensor of the standard complex quadric gives

$$K_{3,3} \geq 2.$$

The invariant sum-of-squares decomposition below determines the Euclidean cubic constant exactly. A pointwise Bernstein–Duffin–Schaeffer estimate on great circles supplies the missing sharp sectional norm bound, and hence we prove the full-space equality $K_{3,3} = 2$. The quadric tensor attains equality.

The resulting dimension-three estimate is the main concrete consequence of the sharp local calculation.

Corollary 1.2. *For every algebraic Kähler Bochner tensor $B \in \mathcal{B}_3$,*

$$|q_{3,3}(B)| \leq 2 \|h_B\|_\infty q_{2,3}(B). \quad (1.11)$$

Consequently, $K_{3,3} = 2$. If (M^3, ω) is a compact Kähler–Einstein threefold with $\text{Ric}(g) = \mu g$, $\mu \neq 0$, then

$$|N_3(M)| \leq 2 \frac{\Lambda_B}{|\mu|} |N_2(M)|. \quad (1.12)$$

If its holomorphic sectional curvature is positively or negatively δ -pinched, $0 < \delta \leq 1$, this becomes

$$|N_3(M)| \leq \frac{1-\delta}{\delta} |N_2(M)|. \quad (1.13)$$

At degree two, the spherical mean is known exactly from the Einstein condition. Applying the Bhatia–Davis inequality before discarding this mean gives Theorem 5.3, a scale-invariant improvement of both the one-sided and two-sided reverse Yau estimates in [5, Theorems 1.2 and 1.4].

The organization is as follows. Sections 2 and 3 establish the curvature conventions and the pure Bochner Chern–Weil formula. Section 4 defines the local constants, computes the exact sectional second moment, and proves the explicit bound $K_{n,k} \leq U_{n,k}$ in every degree. Section 5 proves Theorem 1.1 and the Bhatia–Davis refinement of the author’s degree-two

estimate; a closing remark records the pointwise obstruction to reversing the higher-degree estimate. Sections 6 and 7 contain the third-order calculations and quadric examples.

2. CURVATURE CONVENTIONS AND CENTERED CLASSES

Let (M^n, J, g, ω) be a Kähler manifold of complex dimension n . We use the convention

$$\rho = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad c_1(M) = \left[\frac{\rho}{2\pi} \right].$$

If g is Kähler–Einstein, then

$$R_{i\bar{j}} = \mu g_{i\bar{j}}, \quad \rho = \mu \omega.$$

For a unit vector $\xi \in T_x^{1,0}M$, the holomorphic sectional curvature is

$$H_x(\xi) = R_{\xi\bar{\xi}\xi\bar{\xi}}.$$

Definition 2.1. The metric is *positively δ -pinched* if there is a constant $L > 0$ such that

$$\delta L \leq H_x(\xi) \leq L$$

for all $x \in M$ and all unit $\xi \in T_x^{1,0}M$. It is *negatively δ -pinched* if

$$-L \leq H_x(\xi) \leq -\delta L$$

under the same quantifiers.

The following average identity fixes the factor appearing in our main estimate.

Lemma 2.2 (Berger’s lemma). *At every point of a Kähler–Einstein manifold,*

$$\frac{1}{\text{vol}(S^{2n-1})} \int_{S^{2n-1}} H_x(\xi) d\xi = \frac{2\mu}{n+1}. \quad (2.1)$$

Proof. In unitary coordinates, the standard complex spherical moment identity is

$$\frac{1}{\text{vol}(S^{2n-1})} \int_{S^{2n-1}} \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l d\xi = \frac{\delta_{ij} \delta_{kl} + \delta_{il} \delta_{kj}}{n(n+1)}.$$

Contracting with $R_{i\bar{j}k\bar{l}}$ and using both Kähler symmetries gives $2S/(n(n+1))$, where $S = n\mu$ is the complex scalar curvature. This is (2.1). $\square$

Under the unitary group, an algebraic Kähler curvature tensor decomposes orthogonally into its scalar-curvature, trace-free Ricci, and Bochner components; see [1, Chapter 2, Section D]. General background on Kähler curvature and holomorphic sectional curvature may also be found in [11]. For a Kähler–Einstein metric the trace-free Ricci component vanishes, so the decomposition reduces to

$$R_{i\bar{j}k\bar{l}} = \frac{\mu}{n+1} (g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}) + B_{i\bar{j}k\bar{l}}, \quad (2.2)$$

where B is the Bochner tensor and

$$g^{k\bar{l}} B_{i\bar{j}k\bar{l}} = 0. \quad (2.3)$$

Indeed, contracting the two metric terms in (2.2) gives $(n+1)g_{i\bar{j}}$; comparison with $\text{Ric}_{i\bar{j}} = \mu g_{i\bar{j}}$ therefore fixes the coefficient $\mu/(n+1)$. The remaining tensor is totally trace-free. Since the inner product with the scalar-curvature component is a trace contraction, this also explains the asserted orthogonality. Consequently,

$$h_B(\xi) := B_{\xi\bar{\xi}\xi\bar{\xi}} = H_x(\xi) - \frac{2\mu}{n+1}. \quad (2.4)$$

To interpret (1.1), let $x_1, \dots, x_n$ be the formal Chern roots of $T^{1,0}M$ and put $s = c_1/(n+1)$. Then $\widehat{c}_k$ is the k -th elementary symmetric polynomial in

$$x_1 - s, \dots, x_n - s, -s. \quad (2.5)$$

We spell out the calculation. Let $y_1, \dots, y_r$ be the formal Chern roots of a rank- r bundle E , and write $\ell = c_1(L)$. By the splitting principle,

$$\begin{aligned} c_k(E \otimes L) &= e_k(y_1 + \ell, \dots, y_r + \ell) \\ &= \sum_{\substack{I \subset \{1, \dots, r\} \\ |I|=k}} \prod_{i \in I} (y_i + \ell). \end{aligned}$$

In a term of the last expansion, suppose that the factors y_i are chosen precisely for the indices in a fixed set J of size j . There are $\binom{r-j}{k-j}$ sets I of size k containing J . Summing first over J therefore gives the twisting identity

$$c_k(E \otimes L) = \sum_{j=0}^k \binom{r-j}{k-j} c_j(E) \ell^{k-j}. \quad (2.6)$$

Now take the rank- $(n+1)$ formal bundle $E_0 = T^{1,0}M \oplus \mathcal{O}$, whose roots are $x_1, \dots, x_n, 0$ and whose Chern classes satisfy $c_j(E_0) = c_j(T^{1,0}M) = c_j$. Set $\ell = -s = -c_1/(n+1)$. Substitution into (2.6) gives, term by term,

$$\begin{aligned} c_k(E_0 \otimes L) &= \sum_{j=0}^k \binom{n+1-j}{k-j} c_j(E_0) \left(-\frac{c_1}{n+1}\right)^{k-j} \\ &= \sum_{j=0}^k \binom{n+1-j}{k-j} c_j \left(-\frac{c_1}{n+1}\right)^{k-j} = \widehat{c}_k. \end{aligned}$$

The roots of $E_0 \otimes L$ are exactly $x_1 - s, \dots, x_n - s, -s$, which also verifies (2.5). Although ℓ need not be the integral first Chern class of an actual line bundle, the calculation is a symmetric polynomial identity over rational cohomology and is therefore intrinsic.

3. THE CENTERED TRACTOR AND THE PURE BOCHNER FORMULA

The formal roots (2.5) arise naturally from the standard c-projective tractor construction. A standard vector tractor requires a choice of a density root, while its projectivization exists without that choice. We record both objects; see [3, Section 3.1] for the Cartan-geometric construction and the full tractor calculus.

Definition 3.1. Let $K_M = \Lambda^{n,0}T^*M$ be the canonical line bundle. On an open set $U \subset M$ on which it exists, choose a c-projective density line $\mathcal{E}(1)$ characterized by

$$\mathcal{E}(1)^{\otimes(n+1)} \cong K_M^*.$$

Write $\mathcal{E}(-1) = \mathcal{E}(1)^*$. The *standard cotractor bundle* and the *standard tractor bundle* are, respectively,

$$\mathcal{T}^* := J^{1,0}\mathcal{E}(1), \quad \mathcal{T} := (J^{1,0}\mathcal{E}(1))^*,$$

where $J^{1,0}\mathcal{E}(1)$ is the bundle of complex-linear first jets of local sections of $\mathcal{E}(1)$. The first-jet exact sequence is

$$0 \longrightarrow T^{*1,0}M \otimes \mathcal{E}(1) \longrightarrow \mathcal{T}^* \longrightarrow \mathcal{E}(1) \longrightarrow 0.$$

After dualizing, one obtains the intrinsic filtration

$$0 \longrightarrow \mathcal{E}(-1) \longrightarrow \mathcal{T} \longrightarrow T^{1,0}M \otimes \mathcal{E}(-1) \longrightarrow 0. \quad (3.1)$$

A torsion-free complex connection ∇ in the c-projective class splits the jet sequence by sending a first jet to its value and its $(1,0)$ covariant derivative. Equivalently, it identifies $\mathcal{T}$ with the direct sum of the two graded terms in (3.1). Changing ∇ changes this identification but not the local tractor bundle or its normal tractor connection.

Thus the local bundle $\mathcal{T}$ has complex rank $n+1$ and, in any connection splitting,

$$\mathrm{gr}(\mathcal{T}) \cong \mathcal{E}(-1) \oplus (T^{1,0}M \otimes \mathcal{E}(-1)).$$

Since

$$c_1(\mathcal{E}(-1)) = \frac{c_1(K_M)}{n+1} = -\frac{c_1(M)}{n+1},$$

the roots of the associated graded bundle are precisely $x_1 - s, \dots, x_n - s, -s$. The Whitney product formula applied to (3.1), followed by the calculation in (2.6), therefore gives, whenever the density root exists globally,

$$c_k(\mathcal{T}) = \widehat{c}_k. \quad (3.2)$$

The local choices of $\mathcal{E}(1)$ differ only by scalar twisting. Their projectivizations and normal connections therefore glue to a global $PGL(n+1, \mathbb{C})$ tractor bundle with connection, denoted by $\mathbb{P}\mathcal{T}$. Moreover,

$$\mathbb{P}(\mathrm{gr}(\mathcal{T})) \cong \mathbb{P}(T^{1,0}M \oplus \mathcal{O}),$$

because tensoring a vector bundle by a line bundle does not change its projectivization. This global projective object, rather than a nonexistent choice of density root, is what will be used below. The Lie algebra $\mathfrak{pgl}(n+1, \mathbb{C})$ is canonically identified with $\mathfrak{sl}(n+1, \mathbb{C})$ by taking the trace-free representative. Hence every conjugation-invariant polynomial on $\mathfrak{sl}(n+1, \mathbb{C})$ defines a global Chern–Weil form for $\mathbb{P}\mathcal{T}$.

We next recall the curvature tensors of the normal tractor connection.

Definition 3.2. In the Kähler setting of this article, define the c-projective Schouten tensor by

$$P_{i\bar{j}} := \frac{1}{n+1} \mathrm{Ric}_{i\bar{j}}, \quad P_{ij} = P_{\bar{i}\bar{j}} := 0. \quad (3.3)$$

The *c-projective Weyl curvature* is the trace-free remainder $W_{i\bar{j}k}{}^l$ in the decomposition

$$R_{i\bar{j}k}{}^l = W_{i\bar{j}k}{}^l + \delta_i^l P_{k\bar{j}} + \delta_k^l P_{i\bar{j}}. \quad (3.4)$$

Indeed, contraction in k, l and (3.3) give $W_{i\bar{j}k}{}^k = 0$. This trace-free curvature is unchanged when the torsion-free complex connection is replaced by a c-projectively equivalent one.

The *c-projective Cotton tensor* is the covariant exterior derivative of the Schouten tensor,

$$C_{ABC} := 2\nabla_{[A}P_{B]C} = \nabla_A P_{BC} - \nabla_B P_{AC}, \quad (3.5)$$

where A, B, C are complexified tangent indices. Since $P_{ik} = 0$ for a Kähler connection, its relevant mixed component is

$$C_{i\bar{j}k} = \nabla_i P_{k\bar{j}}; \quad (3.6)$$

the conjugate component contains the same information. In the normal tractor connection, W is the diagonal curvature component and C is the off-diagonal curvature component; see [3].

If the metric is Kähler–Einstein, then

$$P_{i\bar{j}} = \frac{\mu}{n+1} g_{i\bar{j}}, \quad \nabla P = 0, \quad C = 0. \quad (3.7)$$

Comparing (3.4) with the curvature decomposition (2.2) also gives

$$W_{i\bar{j}k}{}^l = B_{i\bar{j}k}{}^l. \quad (3.8)$$

These identifications give a pure Bochner Chern–Weil formula.

Proposition 3.3. *Let (M^n, ω) be Kähler–Einstein. The normal projective tractor connection on $\mathbb{PT}$ has, in a metric splitting, the $\mathfrak{sl}(n+1, \mathbb{C})$ -valued curvature*

$$\Omega_{i\bar{j}}^{\mathcal{T}} = \begin{pmatrix} 0 & 0 \\ 0 & B_{i\bar{j}k}{}^l \end{pmatrix}. \quad (3.9)$$

For every $0 \leq k \leq n$, the coefficient of t^k in $\det(I_{n+1} + \sqrt{-1}t\Omega^{\mathcal{T}}/(2\pi))$ is a globally defined closed form representing $\hat{c}_k \in H^{2k}(M, \mathbb{Q})$. In particular, $\hat{c}_k$ has a Chern–Weil representative that is a homogeneous degree- k invariant polynomial in B , without any assumption that K_M^* has an $(n+1)$ -st root.

Proof. By Definition 3.2, the curvature of the normal standard tractor connection consists of the c-projective Weyl block and the c-projective Cotton block. Equations (3.7) and (3.8) identify these blocks with B and 0, respectively. This gives (3.9). We now write the Chern determinant explicitly. Put

$$\alpha_0 := \frac{\sqrt{-1}}{2\pi}, \quad B_b{}^a := B_{i\bar{j}b}{}^a dz^i \wedge d\bar{z}^j.$$

Equation (3.9) gives

$$\det(I_{n+1} + \alpha_0 t \Omega^{\mathcal{T}}) = \det \begin{pmatrix} 1 & 0 \\ 0 & I_n + \alpha_0 t B \end{pmatrix} = \det(I_n + \alpha_0 t B). \quad (3.10)$$

If

$$\delta_{a_1 \dots a_k}^{b_1 \dots b_k} := \det(\delta_{a_q}^{b_p})_{p,q=1}^k$$

is the generalized Kronecker symbol, then the coefficient of t^k in (3.10) is the concrete $2k$ -form

$$\mathfrak{c}_k(B) = \frac{\alpha_0^k}{k!} \delta_{a_1 \dots a_k}^{b_1 \dots b_k} B_{b_1}{}^{a_1} \wedge \dots \wedge B_{b_k}{}^{a_k}. \quad (3.11)$$

Every summand in (3.11) contains exactly k factors of B , so the asserted homogeneity is immediate. The polynomial

$$P_k(X) := [t^k] \det(I_{n+1} + \alpha_0 t X), \quad X \in \mathfrak{sl}(n+1, \mathbb{C}),$$

is invariant under $PGL(n+1, \mathbb{C})$. Consequently, its local expressions in (3.10) glue, and the Chern–Weil argument shows that $\mathfrak{c}_k(B)$ is globally defined and closed. If $\mathcal{E}(1)$ exists globally, this is the usual form $c_k(\nabla^{\mathcal{T}})$. We next identify its cohomology class without assuming the existence of that root.

Let

$$\Theta_k{}^l = R_{i\bar{j}k}{}^l dz^i \wedge d\bar{z}^j$$

be the curvature matrix of $T^{1,0}M$. The direct-sum connection on $E_0 = T^{1,0}M \oplus \mathcal{O}$ induces a connection on the global projective bundle $\mathbb{P}(E_0)$. The trace-free representative of its projective curvature is

$$\Theta^{\text{cen}} = \begin{pmatrix} \Theta & 0 \\ 0 & 0 \end{pmatrix} - \frac{\text{tr } \Theta}{n+1} I_{n+1}. \quad (3.12)$$

Define

$$\mathfrak{c}_k^{\text{cen}} := P_k(\Theta^{\text{cen}}).$$

Expanding the determinant exactly as in (2.6) gives the identity of differential forms

$$\mathfrak{c}_k^{\text{cen}} = \sum_{j=0}^k \binom{n+1-j}{k-j} c_j(\nabla) \left(-\frac{c_1(\nabla)}{n+1} \right)^{k-j}. \quad (3.13)$$

It follows directly from (1.1) that $[\mathbf{c}_k^{\text{cen}}] = \widehat{c}_k$ in rational de Rham cohomology. Using $\text{Ric}_{i\bar{j}} = \mu g_{i\bar{j}}$ and (2.2), the component form of (3.12) is

$$(\Theta^{\text{cen}})_{i\bar{j}} = \begin{pmatrix} B_{i\bar{j}k}{}^l + \frac{\mu}{n+1} \delta_i^l g_{k\bar{j}} & 0 \\ 0 & -\frac{\mu}{n+1} g_{i\bar{j}} \end{pmatrix}. \quad (3.14)$$

The constant and mixed summands visible in (3.14) need not vanish separately as differential forms. A connection splitting identifies $\mathbb{P}\mathcal{T}$ with $\mathbb{P}(E_0)$, so the centered projective connection ∇^{cen} and the normal projective tractor connection $\nabla^{\mathcal{T}}$ are connections on the same principal $PGL(n+1, \mathbb{C})$ -bundle. Their difference is therefore a globally defined $\mathfrak{pgl}(n+1, \mathbb{C})$ -valued one-form. Set

$$A := \nabla^{\mathcal{T}} - \nabla^{\text{cen}}, \quad \nabla^s := \nabla^{\text{cen}} + sA, \quad F_s := (\nabla^s)^2.$$

Let Φ_k be the symmetric polarization of the normalized k -th Chern invariant; thus

$$\Phi_k(F, \dots, F) = P_k(F).$$

The usual Chern–Weil transgression computation is recalled here to fix the exact statement; compare [4, Section 3, pp. 52–54]. Since

$$\frac{dF_s}{ds} = d_{\nabla^s} A, \quad d_{\nabla^s} F_s = 0,$$

the invariance of Φ_k and the Bianchi identity give

$$\frac{d}{ds} P_k(F_s) = k \Phi_k(d_{\nabla^s} A, F_s, \dots, F_s) = d(k \Phi_k(A, F_s, \dots, F_s)).$$

Integrating this identity from $s = 0$ to $s = 1$ gives

$$\mathbf{c}_k(B) - \mathbf{c}_k^{\text{cen}} = d \left(k \int_0^1 \Phi_k(A, F_s, \dots, F_s) ds \right). \quad (3.15)$$

Thus the constant-curvature and mixed terms in the centered determinant are absorbed into the exact form on the right of (3.15); the pure representative is precisely (3.11). Together with (3.13), this proves $[\mathbf{c}_k(B)] = \widehat{c}_k$ without a density-root hypothesis. In particular, for any closed form γ representing $c_1(M)$, compactness of M and Stokes' theorem yield

$$\int_M \mathbf{c}_k^{\text{cen}} \wedge \gamma^{n-k} = \int_M \mathbf{c}_k(B) \wedge \gamma^{n-k}. \quad (3.16)$$

□

Remark 3.4. The tractor argument is also the conceptual explanation for the otherwise non-obvious binomial coefficients in (1.1). Merely subtracting a scalar from the n tangent Chern roots would produce different coefficients; the additional root $-c_1/(n+1)$ is essential. For example, put $s = c_1/(n+1)$. The $n+1$ centered roots first satisfy

$$\sum_{i=1}^n (x_i - s) - s = c_1 - (n+1)s = 0.$$

In degree two, choosing two of these roots gives

$$\begin{aligned} e_2(x_1 - s, \dots, x_n - s, -s) &= c_2 - ns c_1 + \binom{n+1}{2} s^2 \\ &= c_2 - \frac{n}{2(n+1)} c_1^2 = \widehat{c}_2. \end{aligned}$$

By contrast, if one centers only the n tangent roots, their mean is $\bar{s} = c_1/n$, and the same calculation gives

$$e_2(x_1 - \bar{s}, \dots, x_n - \bar{s}) = c_2 - \frac{n-1}{2n}c_1^2,$$

which is different. Thus, already for a complex surface,

$$\hat{c}_2 = c_2 - \frac{1}{3}c_1^2, \quad e_2\left(x_1 - \frac{c_1}{2}, x_2 - \frac{c_1}{2}\right) = c_2 - \frac{1}{4}c_1^2.$$

4. CHERN-COMPATIBLE LOCAL DENSITIES AND SHARP CONSTANTS

Fix a Hermitian vector space (V, ω) of complex dimension n . Let $\mathcal{B}_n$ be the real vector space of algebraic Kähler curvature tensors B satisfying (2.3). If $\mathfrak{c}_k(B)$ denotes the degree $2k$ Chern–Weil form obtained from (3.9), define the real scalar $q_{k,n}(B)$ by

$$n(n-1)(2\pi)^k \mathfrak{c}_k(B) \wedge \omega^{n-k} = q_{k,n}(B) \omega^n. \quad (4.1)$$

This is the *Chern-compatible normalization*. The common factor $n(n-1)$ is independent of k . We now compute the quadratic density and fix all norm and sign conventions explicitly.

We first define the operator notation used below, in every complex dimension. For $u, v \in V$, put

$$u \odot v := \frac{1}{2}(u \otimes v + v \otimes u),$$

and give $\text{Sym}^2 V$ the Hermitian metric induced from $V \otimes V$. Thus, if $e_1, \dots, e_n$ is a unitary basis of V , then

$$E_{ii} := e_i \odot e_i \quad (1 \leq i \leq n), \quad E_{ij} := \sqrt{2} e_i \odot e_j \quad (1 \leq i < j \leq n) \quad (4.2)$$

is an orthonormal basis of $\text{Sym}^2 V$. The Kähler symmetries of B imply that there is a unique Hermitian endomorphism

$$A_B \in \text{End}_{\text{Herm}}(\text{Sym}^2 V) \quad (4.3)$$

such that, for all $u, v, x, y \in V$,

$$\langle A_B(u \odot v), x \odot y \rangle = B_{u\bar{x}v\bar{y}}. \quad (4.4)$$

The right-hand side is symmetric in u, v and in x, y , so (4.4) is well defined on the symmetric square; the reality symmetry $B_{u\bar{x}v\bar{y}} = \overline{B_{x\bar{u}y\bar{v}}}$ makes A_B Hermitian. Conversely, every Hermitian endomorphism of $\text{Sym}^2 V$ defines an algebraic Kähler curvature tensor by this formula. More explicitly, enumerate the basis (4.2) as $E_1, \dots, E_m$, where $m = n(n+1)/2$, and write $A_B = (A_{\alpha\bar{\beta}})$. Then

$$B_{i\bar{j}k\bar{l}} = \sum_{\alpha, \beta=1}^m A_{\alpha\bar{\beta}} (E_{\alpha})_{ik} \overline{(E_{\beta})_{jl}}. \quad (4.5)$$

It follows from the orthonormality of the E_{α} that, for any two algebraic Kähler tensors B, C ,

$$\sum_{i,j,k,l} B_{i\bar{j}k\bar{l}} \overline{C_{i\bar{j}k\bar{l}}} = \text{tr}_{\text{Sym}^2 V}(A_B A_C). \quad (4.6)$$

Here $A_B A_C$ is ordinary composition of endomorphisms and $\text{tr}_{\text{Sym}^2 V}$ is the operator trace on the m -dimensional space $\text{Sym}^2 V$. In particular, this definition is independent of the chosen unitary basis.

In a unitary frame, regard

$$\mathbf{B} = (B_b^a), \quad B_b^a = B_{i\bar{j}b}^a dz^i \wedge d\bar{z}^j,$$

as an endomorphism-valued $(1, 1)$ -form, and define its Hilbert–Schmidt norm by

$$|B|_{\text{HS}}^2 := \sum_{a,b} \langle B_b^a, B_b^a \rangle = \sum_{i,j,k,l} |B_{i\bar{j}k\bar{l}}|^2. \quad (4.7)$$

By (4.6), this is also $\text{tr}_{\text{Sym}^2 V}(A_B^2)$.

Put $\alpha_0 = \sqrt{-1}/(2\pi)$ as above. Since $\text{tr } \mathbf{B} = 0$, the second coefficient of the Chern determinant is

$$\begin{aligned} \mathfrak{c}_2(B) &= \frac{1}{2} \left((\alpha_0 \text{tr } \mathbf{B})^2 - \text{tr}((\alpha_0 \mathbf{B})^2) \right) \\ &= \frac{1}{8\pi^2} \text{tr}(\mathbf{B} \wedge \mathbf{B}). \end{aligned} \quad (4.8)$$

The trace-free identity (2.3) and the Kähler symmetries also give

$$\Lambda B_b^a = g^{i\bar{j}} B_{i\bar{j}b}^a = 0,$$

so each entry is a primitive $(1, 1)$ -form. For such a form η ,

$$\eta \wedge \bar{\eta} \wedge \omega^{n-2} = -\frac{|\eta|^2}{n(n-1)} \omega^n. \quad (4.9)$$

Moreover, the curvature matrix is skew-Hermitian, so $B_a^b = -\overline{B_b^a}$. Summing (4.9) over a, b therefore yields

$$\text{tr}(\mathbf{B} \wedge \mathbf{B}) \wedge \omega^{n-2} = \frac{|B|_{\text{HS}}^2}{n(n-1)} \omega^n. \quad (4.10)$$

Substitution of (4.8) and (4.10) into (4.1) gives, without an unspecified constant,

$$\begin{aligned} q_{2,n}(B) \omega^n &= n(n-1)(2\pi)^2 \mathfrak{c}_2(B) \wedge \omega^{n-2} \\ &= \frac{1}{2} |B|_{\text{HS}}^2 \omega^n. \end{aligned}$$

Consequently,

$$q_{2,n}(B) = \frac{1}{2} |B|_{\text{HS}}^2 = \frac{1}{2} \text{tr}_{\text{Sym}^2 V}(A_B^2). \quad (4.11)$$

Thus the coefficient relating $q_{2,n}(B)$ to $|B|_{\text{HS}}^2$ is $1/2$, independently of n , in the conventions fixed here. The common factor $n(n-1)$ in (4.1), together with the degree-dependent powers of 2π , cancels when Chern numbers are compared.

We next compute the exact second sectional moment.

Lemma 4.1. *Let $d\sigma$ be the normalized probability measure on the unit sphere in $V \cong \mathbb{C}^n$. For every $B \in \mathcal{B}_n$,*

$$\int_{|\xi|=1} h_B(\xi)^2 d\sigma(\xi) = \frac{4}{n(n+1)(n+2)(n+3)} |B|_{\text{HS}}^2 = \frac{8}{n(n+1)(n+2)(n+3)} q_{2,n}(B). \quad (4.12)$$

Consequently,

$$\frac{|B|_{\text{HS}}}{\|h_B\|_\infty} \leq \frac{1}{2} \sqrt{n(n+1)(n+2)(n+3)} \quad (B \neq 0). \quad (4.13)$$

Proof. The fourth complex spherical moment is

$$\int_{|\xi|=1} \xi_{i_1} \cdots \xi_{i_4} \bar{\xi}_{j_1} \cdots \bar{\xi}_{j_4} d\sigma(\xi) = \frac{1}{n(n+1)(n+2)(n+3)} \sum_{\tau \in S_4} \prod_{a=1}^4 \delta_{i_a j_{\tau(a)}}.$$

Contract this identity with $B_{i_1 \bar{j}_1 i_2 \bar{j}_2} B_{i_3 \bar{j}_3 i_4 \bar{j}_4}$. Every summand containing a contraction within one copy of B vanishes by the Ricci trace condition (2.3). Exactly four permutations remain:

they pair the two holomorphic indices of each copy with the two antiholomorphic indices of the other copy. By the Kähler symmetries, each of these four terms equals $|B|_{\text{HS}}^2$. This proves the first equality in (4.12); the second follows from (4.11). Since $d\sigma$ is a probability measure, $\int h_B^2 d\sigma \leq \|h_B\|_\infty^2$, which gives (4.13). $\square$

The sectional supremum used below is a norm.

Lemma 4.2. *The map*

$$B \mapsto \|h_B\|_\infty = \max_{|\xi|=1} |B_{\xi\bar{\xi}\xi\bar{\xi}}|$$

is a norm on $\mathcal{B}_n$.

Proof. The maximum exists because the unit sphere is compact. Linearity of $B \mapsto h_B$ immediately gives, for $\lambda \in \mathbb{R}$,

$$\|h_{\lambda B}\|_\infty = |\lambda| \|h_B\|_\infty, \quad \|h_{B+C}\|_\infty \leq \|h_B\|_\infty + \|h_C\|_\infty.$$

It remains only to prove definiteness. Suppose $\|h_B\|_\infty = 0$. In unitary coordinates write

$$h_B(z) = B_{p\bar{q}r\bar{s}} z_p z_r \bar{z}_q \bar{z}_s. \quad (4.14)$$

For $z \neq 0$, the fourth-order homogeneity gives

$$h_B(z) = |z|^4 h_B\left(\frac{z}{|z|}\right) = 0;$$

the value at $z = 0$ also vanishes. Thus the polynomial (4.14) is identically zero on V .

To recover an individual curvature component, use the Wirtinger derivatives $\partial_i = \partial/\partial z_i$ and $\bar{\partial}_j = \partial/\partial \bar{z}_j$. The Kähler symmetries $B_{p\bar{q}r\bar{s}} = B_{r\bar{q}p\bar{s}} = B_{p\bar{s}r\bar{q}}$ give

$$\begin{aligned} \partial_i \partial_k \bar{\partial}_j \bar{\partial}_l h_B \Big|_{z=0} &= B_{p\bar{q}r\bar{s}} (\delta_{ip} \delta_{kr} + \delta_{ir} \delta_{kp}) (\delta_{jq} \delta_{ls} + \delta_{js} \delta_{lq}) \\ &= 4B_{i\bar{j}k\bar{l}}. \end{aligned}$$

The left-hand side is zero because $h_B \equiv 0$. Hence every $B_{i\bar{j}k\bar{l}} = 0$, and therefore $B = 0$. $\square$

Definition 4.3. For $2 \leq k \leq n$, define

$$K_{n,k} := \sup_{0 \neq B \in \mathcal{B}_n} \frac{|q_{k,n}(B)|}{\|h_B\|_\infty^{k-2} q_{2,n}(B)}. \quad (4.15)$$

We set $K_{n,2} = 1$.

Proposition 4.4. *For every fixed n and k , the constant $K_{n,k}$ is finite and is attained. Consequently,*

$$|q_{k,n}(B)| \leq K_{n,k} \|h_B\|_\infty^{k-2} q_{2,n}(B) \quad (4.16)$$

for every $B \in \mathcal{B}_n$.

Proof. The quotient in (4.15) is homogeneous of degree zero. By (4.11), normalize to $q_{2,n}(B) = 1$. This is a compact sphere in $\mathcal{B}_n$. Lemma 4.2 and compactness imply that $\|h_B\|_\infty$ has a positive minimum on that sphere. The numerator and denominator are continuous, so the quotient attains a finite maximum. $\square$

We now compute the derangement numbers that occur in the contraction count.

Lemma 4.5. For $k \geq 0$, let

$$\mathfrak{d}_k := \#\{\sigma \in \mathcal{S}_k : \sigma(r) \neq r \text{ for every } r\}. \quad (4.17)$$

Then

$$\mathfrak{d}_k = \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)! = k! \sum_{j=0}^k \frac{(-1)^j}{j!}. \quad (4.18)$$

Moreover,

$$\mathfrak{d}_0 = 1, \quad \mathfrak{d}_1 = 0, \quad \mathfrak{d}_k = (k-1)(\mathfrak{d}_{k-1} + \mathfrak{d}_{k-2}) \quad (k \geq 2). \quad (4.19)$$

In particular, $\mathfrak{d}_2 = 1$, $\mathfrak{d}_3 = 2$, and $\mathfrak{d}_4 = 9$.

Proof. For $1 \leq r \leq k$, put $A_r := \{\sigma \in \mathcal{S}_k : \sigma(r) = r\}$. A derangement is an element of $\mathcal{S}_k \setminus \bigcup_{r=1}^k A_r$. If $J \subseteq \{1, \dots, k\}$ has $|J| = j$, then every permutation in $\bigcap_{r \in J} A_r$ fixes the elements of J and permutes the other $k-j$ elements freely; hence

$$\left| \bigcap_{r \in J} A_r \right| = (k-j)!.$$

The inclusion–exclusion principle therefore gives

$$\mathfrak{d}_k = \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)! = k! \sum_{j=0}^k \frac{(-1)^j}{j!},$$

which proves (4.18).

For the recurrence, choose the value $s = \sigma(1)$, with $s \neq 1$. If $\sigma(s) = 1$, then $(1 \ s)$ is a two-cycle; deleting it leaves a derangement of the other $k-2$ letters. This gives $(k-1)\mathfrak{d}_{k-2}$ possibilities. If 1 lies in a cycle of length at least three, delete 1 from that cycle. The result is a derangement of the other $k-1$ letters together with the choice of s , and conversely one recovers σ by inserting 1 immediately before s . This gives $(k-1)\mathfrak{d}_{k-1}$ possibilities. Adding the two cases proves (4.19); the displayed low-degree values follow immediately. $\square$

We also need an iterated Cauchy–Schwarz inequality.

Lemma 4.6. Let $x_1, \dots, x_N$ be independent variables on finite probability spaces, and let $f_1, \dots, f_r$ be nonnegative functions, each depending on some of the variables. Suppose that each variable occurs in at most two distinct functions. Then

$$\mathbb{E} \prod_{s=1}^r f_s \leq \prod_{s=1}^r \left(\mathbb{E} f_s^2 \right)^{1/2}, \quad (4.20)$$

where the expectation on the right is over the variables on which f_s depends.

Proof. Induct on the number of variables. If a variable x occurs in two functions, say f_1 and f_2 , condition on all other variables and apply Cauchy–Schwarz in x :

$$\mathbb{E}_x(f_1 f_2) \leq (\mathbb{E}_x f_1^2)^{1/2} (\mathbb{E}_x f_2^2)^{1/2}.$$

Replace f_1, f_2 by the two functions on the right and apply the induction hypothesis to the remaining variables. Their squared L^2 norms are unchanged by Fubini’s theorem. If x occurs in only one function, use $\mathbb{E}_x f \leq (\mathbb{E}_x f^2)^{1/2}$ instead. Variables occurring in no function may be discarded. This proves (4.20). $\square$

These estimates give explicit constants in every degree.

Theorem 4.7. *Let $\mathfrak{d}_k$ be the derangement number computed in Lemma 4.5. For every $2 \leq k \leq n$ and every $B \in \mathcal{B}_n$,*

$$|q_{k,n}(B)| \leq \frac{(n-k)!}{k!(n-2)!} \mathfrak{d}_k^2 |B|_{\text{HS}}^k. \quad (4.21)$$

Consequently,

$$K_{n,k} \leq U_{n,k} = 2^{3-k} \frac{(n-k)!}{k!(n-2)!} \mathfrak{d}_k^2 [n(n+1)(n+2)(n+3)]^{(k-2)/2}. \quad (4.22)$$

Proof. Write $\mathbf{i} = \sqrt{-1}$ and $(n)_k = n(n-1) \cdots (n-k+1) = n!/(n-k)!$. For scalar $(1,1)$ -forms $\eta_1, \dots, \eta_k$, direct expansion against a unitary volume form gives

$$\frac{\eta_1 \wedge \cdots \wedge \eta_k \wedge \omega^{n-k}}{\omega^n} = \frac{1}{\mathbf{i}^k (n)_k} \delta_{j_1 \cdots j_k}^{i_1 \cdots i_k} \prod_{r=1}^k (\eta_r)_{i_r \bar{j}_r}. \quad (4.23)$$

The coefficient in (4.23) is checked by taking every $\eta_r = \omega$, when the generalized delta contracts to $(n)_k$. Apply (4.23) to the component formula (3.11). Since $(2\pi)^k \alpha_0^k = \mathbf{i}^k$, the normalization (4.1) yields

$$q_{k,n}(B) = \frac{(n-k)!}{k!(n-2)!} \mathcal{D}_k(B), \quad (4.24)$$

where

$$\mathcal{D}_k(B) := \delta_{j_1 \cdots j_k}^{i_1 \cdots i_k} \delta_{b_1 \cdots b_k}^{a_1 \cdots a_k} \prod_{r=1}^k B_{i_r \bar{j}_r a_r \bar{b}_r}. \quad (4.25)$$

Expand both generalized deltas. After replacing permutations by their inverses if necessary, this gives

$$\mathcal{D}_k(B) = \sum_{\sigma, \tau \in \mathcal{S}_k} \text{sgn}(\sigma) \text{sgn}(\tau) C_{\sigma, \tau}(B), \quad (4.26)$$

with

$$C_{\sigma, \tau}(B) := \sum_{\substack{i_1, \dots, i_k \\ a_1, \dots, a_k}} \prod_{r=1}^k B_{i_r \bar{i}_{\sigma(r)} a_r \bar{a}_{\tau(r)}}. \quad (4.27)$$

If $\sigma(r) = r$, summing over i_r produces $\sum_i B_{i \bar{i} a \bar{b}} = 0$. If $\tau(r) = r$, summing over a_r produces $\sum_a B_{i \bar{j} a \bar{a}} = 0$. Both identities are the Ricci trace condition (2.3). Hence a nonzero term requires both σ and τ to be derangements, leaving at most $\mathfrak{d}_k^2$ contractions.

Fix such a pair. Give each of the $2k$ indices in (4.27) the uniform probability measure on $\{1, \dots, n\}$. Each index i_s occurs in exactly the two distinct factors indexed by s and $\sigma^{-1}(s)$; similarly, a_s occurs in the factors s and $\tau^{-1}(s)$. Lemma 4.6, applied to the absolute values of the k tensor entries, therefore gives

$$\begin{aligned} n^{-2k} |C_{\sigma, \tau}(B)| &\leq \prod_{r=1}^k \left(n^{-4} \sum_{i, j, a, b} |B_{i \bar{j} a \bar{b}}|^2 \right)^{1/2} \\ &= n^{-2k} |B|_{\text{HS}}^k. \end{aligned}$$

Thus $|C_{\sigma, \tau}(B)| \leq |B|_{\text{HS}}^k$. Equations (4.24)–(4.26) now prove (4.21).

Finally, (4.11) and (4.13) imply

$$\begin{aligned} \frac{|q_{k,n}(B)|}{\|h_B\|_{\infty}^{k-2} q_{2,n}(B)} &\leq 2 \frac{(n-k)!}{k!(n-2)!} \mathfrak{d}_k^2 \left(\frac{|B|_{\text{HS}}}{\|h_B\|_{\infty}} \right)^{k-2} \\ &\leq U_{n,k}. \end{aligned}$$

Taking the supremum proves (4.22). $\square$

Remark 4.8. Since $\mathfrak{d}_2 = 1$, $\mathfrak{d}_3 = 2$, and $\mathfrak{d}_4 = 9$, Theorem 4.7 gives

$$U_{n,2} = 1, \quad U_{n,3} = \frac{2\sqrt{n(n+1)(n+2)(n+3)}}{3(n-2)},$$

and, for $n \geq 4$,

$$U_{n,4} = \frac{27n(n+1)(n+2)(n+3)}{16(n-2)(n-3)}.$$

For $k = 2$, the only surviving pair in (4.26) consists of the two transpositions, and $\mathcal{D}_2(B) = |B|_{\text{HS}}^2$. Thus (4.24) also recovers the exact identity (4.11), not merely an upper bound. For fixed k and $n \rightarrow \infty$,

$$U_{n,k} = \frac{2^{3-k} \mathfrak{d}_k^2}{k!} n^{k-2} (1 + O(n^{-1})).$$

The bound counts all potentially nonzero contraction diagrams separately and therefore is not expected to be sharp once $k \geq 3$.

In degree three, the spectral norm gives a sharper estimate.

Corollary 4.9. *For each integer $d \geq 2$, define*

$$\gamma_d := \frac{d-2}{\sqrt{d(d-1)}}.$$

For $n \geq 3$, set

$$m := \frac{n(n+1)}{2} = \dim_{\mathbb{C}} \text{Sym}^2 V.$$

Here d denotes the dimension of the trace-free Hermitian matrix in the spectral estimate below; the two applications are $d = m$ for A_B and $d = n^2$ for $\mathcal{R}_B$. Then

$$K_{n,3} \leq \frac{\sqrt{n(n+1)(n+2)(n+3)}}{3(n-2)} (\gamma_m + \gamma_{n^2}) < U_{n,3}. \quad (4.28)$$

Proof. Besides A_B on $\text{Sym}^2 V$, define a Hermitian operator $\mathcal{R}_B$ on $V \otimes V$ by

$$(\mathcal{R}_B)_{(i,c),(\bar{j},\bar{a})} := B_{i\bar{j}a\bar{c}}.$$

The reality symmetry of B makes $\mathcal{R}_B$ Hermitian. The Ricci trace condition and (4.6) give

$$\text{tr } A_B = \text{tr } \mathcal{R}_B = 0, \quad \text{tr}(A_B^2) = \text{tr}(\mathcal{R}_B^2) = |B|_{\text{HS}}^2. \quad (4.29)$$

Indeed, both traces in the first equality are the double Ricci contraction, and reshuffling the four indices does not change the Hilbert–Schmidt norm.

For $k = 3$, the two derangements are the two 3-cycles. In (4.26), the two pairs of cycles with the same orientation give $\text{tr}(A_B^3)$, whereas the two pairs with opposite orientations give $\text{tr}(\mathcal{R}_B^3)$. Explicitly,

$$\begin{aligned} \text{tr}(A_B^3) &= B_{i\bar{j}a\bar{b}} B_{j\bar{k}b\bar{c}} B_{k\bar{i}c\bar{a}}, \\ \text{tr}(\mathcal{R}_B^3) &= B_{i\bar{j}a\bar{c}} B_{j\bar{k}b\bar{a}} B_{k\bar{i}c\bar{b}}. \end{aligned}$$

Consequently,

$$q_{3,n}(B) = \frac{\text{tr}(A_B^3) + \text{tr}(\mathcal{R}_B^3)}{3(n-2)}. \quad (4.30)$$

We use the elementary sharp spectral inequality

$$|\text{tr}(X^3)| \leq \gamma_d (\text{tr}(X^2))^{3/2} \quad (4.31)$$

for every trace-free Hermitian $d \times d$ matrix X . To verify it, diagonalize X and normalize its eigenvalues by $\sum \lambda_i^2 = 1$ and $\sum \lambda_i = 0$. Lagrange multipliers show that an extremizer of $|\sum \lambda_i^3|$ has at most two distinct eigenvalues. If their multiplicities are p and $d - p$, the absolute cubic sum is

$$\frac{|d - 2p|}{\sqrt{dp(d - p)}}.$$

Its maximum occurs at $p = 1$ or $p = d - 1$ and equals γ_d , proving (4.31).

Apply (4.31) to the two operators in (4.29). Equation (4.30) gives

$$|q_{3,n}(B)| \leq \frac{\gamma_m + \gamma_{n^2}}{3(n-2)} |B|_{\text{HS}}^3.$$

Divide by $\|h_B\|_\infty q_{2,n}(B)$ and use (4.11) and (4.13). This proves the first inequality in (4.28). The second follows from $\gamma_m < 1$, $\gamma_{n^2} < 1$, and the formula for $U_{n,3}$ in Remark 4.8. $\square$

5. PROOF OF THE HIGHER PINCHING INEQUALITY

We first separate the analytic pinching estimate from the Chern–Weil integration.

Lemma 5.1. *Let (M^n, ω) be compact Kähler–Einstein with $\mu \neq 0$, and assume that its holomorphic sectional curvature is positively or negatively δ -pinched, where $0 < \delta \leq 1$. Then*

$$\frac{\|h_B\|_{L^\infty(M)}}{|\mu|} \leq \frac{2(1 - \delta)}{(n + 1)\delta}. \quad (5.1)$$

Proof. In the positive case, write $\delta L \leq H \leq L$. By Lemma 2.2,

$$\delta L \leq \frac{2\mu}{n + 1} \leq L, \quad L \leq \frac{2\mu}{(n + 1)\delta}.$$

The average lies in the same interval, and therefore

$$\left| H - \frac{2\mu}{n + 1} \right| \leq L - \delta L \leq \frac{2(1 - \delta)}{(n + 1)\delta} \mu.$$

Use (2.4). In the negative case, apply the identical argument to $-H$ and $-\mu$. $\square$

Proof of Theorem 1.1. Since $c_1(\nabla) = \mu\omega/(2\pi)$, Proposition 3.3 and (4.1) give

$$N_k(M) = \frac{(2\pi)^{-n} \mu^{n-k}}{n(n-1)} \int_M q_{k,n}(B) \omega^n. \quad (5.2)$$

Using (4.16),

$$\begin{aligned} |N_k(M)| &\leq \frac{(2\pi)^{-n} |\mu|^{n-k}}{n(n-1)} \int_M |q_{k,n}(B)| \omega^n \\ &\leq \frac{(2\pi)^{-n} |\mu|^{n-k}}{n(n-1)} K_{n,k} \Lambda_B^{k-2} \int_M q_{2,n}(B) \omega^n. \end{aligned}$$

By (4.11), the last integral is nonnegative. Applying (5.2) with $k = 2$ yields

$$\frac{(2\pi)^{-n} |\mu|^{n-2}}{n(n-1)} \int_M q_{2,n}(B) \omega^n = |N_2(M)|.$$

Dividing the powers of $|\mu|$ proves (1.9). Under the additional δ -pinching hypothesis, Lemma 5.1 gives

$$\frac{\Lambda_B}{|\mu|} \leq \frac{2(1 - \delta)}{(n + 1)\delta}.$$

Substitution into (1.9) proves (1.10). The second inequalities in (1.9) and (1.10) follow from Theorem 4.7, which proves $K_{n,k} \leq U_{n,k}$. $\square$

Corollary 5.2. *If*

$$N_2(M) = 0,$$

then $B \equiv 0$, every $\widehat{c}_k$ has zero Chern–Weil representative, and the metric has constant holomorphic sectional curvature.

Proof. Equation (5.2) for $k = 2$ and positivity (4.11) imply $B \equiv 0$. Proposition 3.3 then gives the vanishing of all centered Chern forms, and (2.2) is the constant-curvature formula. $\square$

We next use the Bhatia–Davis inequality to refine the degree-two estimates.

Theorem 5.3. *Let (M^n, ω) be a compact Kähler–Einstein manifold, $n \geq 2$, with $\text{Ric}(g) = \mu g$ and $\mu \neq 0$. Define the nonnegative Chern numbers*

$$\mathcal{I}_1(M) := |\langle c_1^n, [M] \rangle|, \quad \mathcal{I}_2(M) := |\langle c_2 c_1^{n-2}, [M] \rangle|.$$

Suppose that two constants $H_- \leq H_+$ satisfy

$$H_- \leq H_x(\xi) \leq H_+$$

for every $x \in M$ and every unit $\xi \in T_x^{1,0}M$. Put

$$\overline{H} := \frac{2\mu}{n+1}, \quad A_n := \frac{(n+1)^2(n+2)(n+3)}{4(n-1)}.$$

Then

$$\begin{aligned} 0 \leq 2(n+1)\mathcal{I}_2(M) - n\mathcal{I}_1(M) &= 2(n+1) |N_2(M)| \\ &\leq A_n \frac{(H_+ - \overline{H})(\overline{H} - H_-)}{\mu^2} \mathcal{I}_1(M). \end{aligned} \quad (5.3)$$

In particular, suppose that the holomorphic sectional curvature has one sign and set

$$H_m := \max_{x \in M, |\xi|=1} |H_x(\xi)|, \quad \eta := \frac{H_m}{|\mu|}.$$

Necessarily $\eta \geq 2/(n+1)$. Then

$$2(n+1)\mathcal{I}_2(M) \leq \left[n + \frac{(n+1)(n+2)(n+3)}{2(n-1)} \left(\eta - \frac{2}{n+1} \right) \right] \mathcal{I}_1(M). \quad (5.4)$$

More generally, if $-H_m \leq H_x(\xi) \leq H_m$, then

$$2(n+1)\mathcal{I}_2(M) \leq \left[n + A_n \left(\eta^2 - \frac{4}{(n+1)^2} \right) \right] \mathcal{I}_1(M). \quad (5.5)$$

Proof. Let $d\sigma$ be the probability measure on the unit sphere in $T_x^{1,0}M$ and define

$$f(x) := \int_{|\xi|=1} \left(H_x(\xi) - \overline{H} \right)^2 d\sigma(\xi). \quad (5.6)$$

Since $H_x(\xi) - \overline{H} = h_B(\xi)$, the exact second-moment calculation in Lemma 4.1 gives $f(x)$ with the coefficient in the author’s variance identity; explicitly, it is the quantity on the right-hand side of (4.12). Substitution into (5.2) with $k = 2$, followed by $\omega^n = n! dV_g$, gives the exact identity

$$2(n+1) |N_2(M)| = \frac{(n+1)(n+3)! |\mu|^{n-2}}{2^{n+2} \pi^n (n-1)} \int_M f dV_g. \quad (5.7)$$

This is the author's earlier variance identity [5, Theorem 1.1], translated to the curvature and volume normalizations used here. For clarity, in that paper the author writes the left-hand side before centering as

$$\left| \left\langle \left(2(n+1)c_2 - nc_1^2 \right) c_1^{n-2}, [M] \right\rangle \right|,$$

which is exactly $2(n+1)|N_2(M)|$ by (1.3).

Fix $x \in M$ and regard $X = H_x(\xi)$ as a random variable on the unit sphere. Lemma 2.2 gives $\mathbb{E}X = \overline{H}$. The Bhatia–Davis inequality [2] therefore gives

$$f(x) = \text{Var}(X) \leq (H_+ - \overline{H})(\overline{H} - H_-). \quad (5.8)$$

Moreover, $c_1 = [\mu\omega/(2\pi)]$, so

$$\mathcal{I}_1(M) = \frac{n!|\mu|^n}{(2\pi)^n} \text{Vol}_g(M). \quad (5.9)$$

Substituting (5.8) and (5.9) into (5.7) yields the upper estimate in (5.3), because

$$\frac{(n+1)(n+3)!}{4n!(n-1)} = A_n.$$

It remains to justify the first equality in (5.3). Formula (5.2) with $k=2$ and (4.11) show that $N_2(M)$ has the sign of μ^{n-2} . The number $\langle c_1^n, [M] \rangle$ has the sign of μ^n , which is the same sign. Since

$$\langle c_2 c_1^{n-2}, [M] \rangle = N_2(M) + \frac{n}{2(n+1)} \langle c_1^n, [M] \rangle,$$

all three terms have that sign. Taking absolute values proves

$$2(n+1)\mathcal{I}_2(M) - n\mathcal{I}_1(M) = 2(n+1)|N_2(M)| \geq 0.$$

For the one-sided case, use $(H_-, H_+) = (0, H_m)$ when $H \geq 0$ and $(H_-, H_+) = (-H_m, 0)$ when $H \leq 0$. In either case,

$$\frac{(H_+ - \overline{H})(\overline{H} - H_-)}{\mu^2} = \frac{2}{n+1} \left(\eta - \frac{2}{n+1} \right),$$

which gives (5.4). The comparison with the author's earlier estimate is the elementary identity

$$\frac{\eta^2}{4} - \frac{2}{n+1} \left(\eta - \frac{2}{n+1} \right) = \left(\frac{\eta}{2} - \frac{2}{n+1} \right)^2 \geq 0.$$

For the symmetric interval $[-H_m, H_m]$, the normalized Bhatia–Davis factor is $\eta^2 - 4/(n+1)^2$, proving (5.5).

Finally, equality in the Bhatia–Davis inequality requires X to take only the two endpoint values almost surely. The function $\xi \mapsto H_x(\xi)$ is continuous and S^{2n-1} is connected, so it must then be constant. If the bounds are the actual global extrema, equality in the integrated estimate is therefore equivalent to $H_x(\xi) = \overline{H}$ everywhere, hence to $B \equiv 0$ by (2.4). The converse is immediate. $\square$

Remark 5.4. The inequality (5.4) improves the scale-invariant form of the author's earlier result [5, Theorem 1.2], whose additional coefficient is

$$\frac{(n+1)^2(n+2)(n+3)}{16(n-1)}\eta^2.$$

The new coefficient is no larger and is strictly smaller unless $\eta = 4/(n+1)$. The inequality (5.5) improves the scale-invariant form of the author's earlier result [5, Theorem 1.4] by subtracting $4A_n/(n+1)^2 = (n+2)(n+3)/(n-1)$ from its additional coefficient. If H_- and H_+ are the

actual global minimum and maximum, respectively, then equality in the upper bound of (5.3) holds if and only if $B \equiv 0$, equivalently, g has constant holomorphic sectional curvature.

The degree-two variance refinement has the following limitation.

Remark 5.5. The gain in Theorem 5.3 is specific to degree two. A variance bound does not control the independent cubic contractions occurring in $q_{3,n}$, and in fact no positive pointwise reverse constant exists in degree three. Indeed, use the orthonormal basis

$$e_1^2, \sqrt{2}e_1e_2, \sqrt{2}e_1e_3, e_2^2, \sqrt{2}e_2e_3, e_3^2$$

of $\text{Sym}^2 \mathbb{C}^3$. The diagonal Hermitian matrix

$$A = \text{diag} \left(-\frac{1}{2}, 1, 0, -\frac{1}{2}, 0, 0 \right)$$

defines an algebraic Kähler curvature tensor with zero Ricci contraction. The direct formulas of Proposition 6.1 below give

$$q_{2,3}(B) = \frac{3}{4}, \quad q_{3,3}(B) = 0.$$

Moreover $h_B(e_1) = -1/2$, so $\|h_B\|_\infty > 0$. Consequently no $c > 0$ can make

$$|q_{3,3}(B)| \geq c \|h_B\|_\infty q_{2,3}(B)$$

hold for every $B \in \mathcal{B}_3$.

This obstruction is compatible with arbitrarily strong algebraic pinching: adding tB to a positive complex space-form tensor leaves the Ricci contraction unchanged, and for each fixed $0 < \delta < 1$ the result is positively δ -pinched when $|t|$ is sufficiently small. Its centered cubic density still vanishes while its quadratic density does not.

The same pointwise obstruction holds in every odd degree $3 \leq k \leq n$. Indeed, on the connected sphere $S = \{B \in \mathcal{B}_n : q_{2,n}(B) = 1\}$, the continuous odd function $q_{k,n}$ either vanishes identically or changes sign between B_0 and $-B_0$; in either case it vanishes at some $\tilde{B} \in S$. Lemma 4.2 gives $\|\tilde{h}_{\tilde{B}}\|_\infty > 0$, so a positive reverse constant is impossible. This is only a local Chern–Weil obstruction. A global lower bound on a restricted class of compact manifolds would require additional topology, classification, or a nondegeneracy hypothesis. Even degrees require separate analysis. Thus the local constants (4.15) remain necessary for the higher-degree upper estimates, but they cannot be reversed in general.

6. THE CUBIC PROBLEM IN COMPLEX DIMENSION THREE

We now specialize the general construction (4.3)–(4.5) to $n = 3$ and abbreviate

$$A := A_B, \quad q_2 := q_{2,3}, \quad q_3 := q_{3,3}$$

throughout this section. In particular, taking $u = v = z$ and $x = y = w$ in (4.4) gives

$$\langle A(z^2), w^2 \rangle = B_{z\bar{w}z\bar{w}}, \quad z, w \in \mathbb{C}^3, \quad (6.1)$$

and polarization recovers all components of B . Thus $h_B(z) = \langle A(z^2), z^2 \rangle$. The Ricci contraction is a surjective map from the 36-dimensional real space of such Hermitian forms to the 9-dimensional space of Hermitian 3×3 matrices. Indeed, for a Hermitian form H , the algebraic Kähler tensor

$$C(H)_{i\bar{j}k\bar{l}} = H_{i\bar{j}}\delta_{kl} + H_{k\bar{j}}\delta_{il} + H_{i\bar{l}}\delta_{kj} + H_{k\bar{l}}\delta_{ij}$$

has Ricci contraction $5H + (\text{tr } H)I$; the map $H \mapsto 5H + (\text{tr } H)I$ is invertible. Hence

$$\dim_{\mathbb{R}} \mathcal{B}_3 = 36 - 9 = 27. \quad (6.2)$$

We first calculate the quadratic and cubic contractions.

Proposition 6.1. *For $B \in \mathcal{B}_3$ and its Hermitian operator A on $\text{Sym}^2 \mathbb{C}^3$, the Chern-compatible normalization (4.1) gives*

$$q_{2,3}(B) = \frac{1}{2} \text{tr}(A^2) \quad (6.3)$$

and

$$3q_{3,3}(B) = T_1(B) + T_2(B), \quad (6.4)$$

where

$$T_1(B) = \text{tr}(A^3), \quad (6.5)$$

$$T_2(B) = B_{i\bar{j}a\bar{c}} B_{j\bar{k}b\bar{a}} B_{k\bar{i}c\bar{b}}. \quad (6.6)$$

The two cubic contractions are linearly independent on $\mathcal{B}_3$.

Proof. Let $E_1, \dots, E_6$ denote the orthonormal basis of $\text{Sym}^2 \mathbb{C}^3$ displayed above and write $A = (A_{\alpha\bar{\beta}})$ in this basis. Polarization of (6.1) gives the component relation

$$B_{i\bar{j}k\bar{l}} = \sum_{\alpha, \beta=1}^6 A_{\alpha\bar{\beta}} (E_\alpha)_{ik} \overline{(E_\beta)_{jl}}. \quad (6.7)$$

Because the E_α are orthonormal, this identification preserves the Hilbert–Schmidt inner product. Hence

$$|B|_{\text{HS}}^2 = \sum_{i,j,k,l} |B_{i\bar{j}k\bar{l}}|^2 = \sum_{\alpha, \beta} |A_{\alpha\bar{\beta}}|^2 = \text{tr}(A^2). \quad (6.8)$$

We first calculate the quadratic coefficient. Since $\text{tr} \mathbf{B} = 0$, Newton’s identity and (4.10) with $n = 3$ give

$$\mathfrak{c}_2(B) = \frac{1}{8\pi^2} \text{tr}(\mathbf{B} \wedge \mathbf{B}), \quad \text{tr}(\mathbf{B} \wedge \mathbf{B}) \wedge \omega = \frac{|B|_{\text{HS}}^2}{6} \omega^3.$$

Using the defining normalization (4.1), we therefore obtain

$$\begin{aligned} q_{2,3}(B) \omega^3 &= 6(2\pi)^2 \mathfrak{c}_2(B) \wedge \omega \\ &= 6(2\pi)^2 \frac{1}{8\pi^2} \frac{|B|_{\text{HS}}^2}{6} \omega^3 = \frac{1}{2} \text{tr}(A^2) \omega^3, \end{aligned}$$

which proves (6.3) with every numerical factor displayed.

For the cubic coefficient, we use the following top-degree wedge identity. If $\eta_r = (\eta_r)_{i\bar{j}} dz^i \wedge d\bar{z}^j$ are three scalar $(1, 1)$ -forms in a unitary frame and $\sqrt{-1}$ is denoted by $\mathbf{i}$ in this calculation, then

$$\frac{\eta_1 \wedge \eta_2 \wedge \eta_3}{\omega^3} = \frac{1}{6\mathbf{i}^3} \delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} (\eta_1)_{i_1 \bar{j}_1} (\eta_2)_{i_2 \bar{j}_2} (\eta_3)_{i_3 \bar{j}_3}. \quad (6.9)$$

The factor is fixed, for example, by taking all three forms equal to $\omega = \mathbf{i} \delta_{ij} dz^i \wedge d\bar{z}^j$; the generalized delta then contracts to $3! = 6$.

Lower the endomorphism index of B in the same unitary frame and set

$$\mathcal{D}(B) := \delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} \delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} \prod_{r=1}^3 B_{i_r \bar{j}_r a_r \bar{b}_r}. \quad (6.10)$$

Equations (3.11), (4.1), and (6.9) give

$$q_{3,3}(B) \omega^3 = 6(2\pi)^3 \frac{\alpha_0^3}{3!} \frac{\mathcal{D}(B)}{6\mathbf{i}^3} \omega^3 = \frac{1}{6} \mathcal{D}(B) \omega^3, \quad (6.11)$$

because $(2\pi)^3 \alpha_0^3 = \mathbf{i}^3$.

It remains to expand the two generalized deltas. Write

$$\delta_{j_1 j_2 j_3}^{i_1 i_2 i_3} = \sum_{\sigma \in S_3} \text{sgn}(\sigma) \prod_{r=1}^3 \delta_{j_{\sigma(r)}}^{i_r},$$

and use the identical expansion for the a, b indices. If the first permutation σ has a fixed point, the corresponding term contains a factor $\sum_i B_{ii\bar{a}\bar{b}}$ and vanishes. If the second permutation τ has a fixed point, it contains $\sum_a B_{i\bar{j}a\bar{a}}$ and also vanishes. These are the two equivalent Ricci traces of the Bochner tensor. The only fixed-point-free permutations of three letters are (123) and (132), both of positive sign. Thus only four of the original $6 \cdot 6 = 36$ terms survive.

After relabeling dummy indices, the two terms with the same orientation are both

$$B_{i\bar{j}a\bar{b}} B_{j\bar{k}b\bar{c}} B_{k\bar{i}c\bar{a}} = \text{tr}(A^3) = T_1(B),$$

where the equality follows directly from (6.7). The two terms with opposite orientations are both

$$B_{i\bar{j}a\bar{c}} B_{j\bar{k}b\bar{a}} B_{k\bar{i}c\bar{b}} = T_2(B).$$

Consequently,

$$\mathcal{D}(B) = 2T_1(B) + 2T_2(B). \quad (6.12)$$

Combining (6.11) and (6.12) gives $3q_{3,3} = T_1 + T_2$, including the factor 3 in (6.4).

Finally, we verify linear independence without appealing to a dimension count. For a diagonal matrix $A = \text{diag}(d_1, \dots, d_6)$ in the ordered basis above, its Ricci diagonal is

$$\left(d_1 + \frac{d_2 + d_3}{2}, \quad d_4 + \frac{d_2 + d_5}{2}, \quad d_6 + \frac{d_3 + d_5}{2} \right).$$

Hence the following two diagonal matrices lie in $\mathcal{B}_3$. Direct substitution in (6.7), (6.5), and (6.6) gives

A	T_1	T_2
$\text{diag}(-\frac{1}{2}, 1, 0, -\frac{1}{2}, 0, 0)$	$\frac{3}{4}$	$-\frac{3}{4}$
$\text{diag}(-1, 1, 1, -\frac{1}{2}, 0, -\frac{1}{2})$	$\frac{3}{4}$	-3

The determinant of these two value vectors is $-27/16 \neq 0$, proving that T_1 and T_2 are linearly independent on $\mathcal{B}_3$. $\square$

6.1. The maximal-torus-invariant subspace. A torus-invariant tensor is represented by

$$A(u, v, w) = \text{diag} \left(-\frac{u+v}{2}, u, v, -\frac{u+w}{2}, w, -\frac{v+w}{2} \right). \quad (6.13)$$

Direct substitution gives

$$q_2 = \frac{3}{4}(u^2 + v^2 + w^2) + \frac{1}{4}(uv + uw + vw), \quad (6.14)$$

$$q_3 = -\frac{3}{8} \sum_{\text{sym}} u^2 v + \frac{1}{4} uvw, \quad (6.15)$$

where

$$\sum_{\text{sym}} u^2 v = u^2 v + u^2 w + v^2 u + v^2 w + w^2 u + w^2 v.$$

Theorem 6.2. *On the maximal-torus-invariant subspace of $\mathcal{B}_3$,*

$$|q_3(B)| \leq \frac{6}{7} \|h_B\|_{\infty} q_2(B). \quad (6.16)$$

The constant is sharp. Equality occurs, up to nonzero scaling and permutation, at $(u, v, w) = (1, 1, -3)$.

Proof. Set

$$x = u + v, \quad y = u + w, \quad z = v + w, \quad M = \max\{|x|, |y|, |z|\}.$$

Evaluation of h_B at the three coordinate vectors gives $\|h_B\|_\infty \geq M/2$. A direct change of variables yields

$$q_2 = \frac{1}{4}D, \quad q_3 = \frac{1}{8}N,$$

where

$$\begin{aligned} D &= 2(x^2 + y^2 + z^2) - (xy + xz + yz), \\ N &= -(x^3 + y^3 + z^3) + \sum_{\text{sym}} x^2y - 5xyz. \end{aligned}$$

Thus it suffices to prove

$$7|N| \leq 6MD. \tag{6.17}$$

By homogeneity normalize $M = 1$. Permutation symmetry and simultaneous sign change allow us to take $x = 1$. Put $s = y + z$ and $p = yz$. Since $y, z \in [-1, 1]$, the admissible interval for p is

$$\begin{cases} -1 - s \leq p \leq s^2/4, & -2 \leq s \leq 0, \\ s - 1 \leq p \leq s^2/4, & 0 \leq s \leq 2. \end{cases}$$

Furthermore,

$$\begin{aligned} D &= 2 + 2s^2 - s - 5p, \\ N &= -1 + s + s^2 - s^3 + (4s - 7)p. \end{aligned}$$

For $6D + 7N$, the coefficient of p is $28s - 79 < 0$. Its minimum is therefore at $p = s^2/4$, where it equals

$$5 + s - \frac{3}{4}s^2 \geq 0 \quad (-2 \leq s \leq 2).$$

For $6D - 7N$, the coefficient of p is $19 - 28s$. Taking the relevant endpoint of the admissible interval gives the following three lower bounds:

range of s	minimum of $6D - 7N$
$[-2, 0]$	$s(7s^2 + 33s - 4)$
$[0, 19/28]$	$s(7s^2 - 23s + 34)$
$[19/28, 2]$	$19 - 13s + \frac{39}{4}s^2$.

For $6D + 7N$, equality requires $p = s^2/4$ and $5 + s - 3s^2/4 = 0$. The only root in $[-2, 2]$ is $s = -2$, which gives $(x, y, z) = (1, -1, -1)$. For $6D - 7N$, the quadratic factor $7s^2 + 33s - 4$ has no root in $[-2, 0]$, the discriminant of $7s^2 - 23s + 34$ is -423 , and the discriminant of $19 - 13s + 39s^2/4$ is -572 . Hence equality in these three ranges occurs only at $s = 0, p = -1$, giving $(x, y, z) = (1, 1, -1)$ or $(1, -1, 1)$. This proves (6.17). Inverting the change of variables,

$$u = \frac{x + y - z}{2}, \quad v = \frac{x + z - y}{2}, \quad w = \frac{y + z - x}{2},$$

shows that all equality triples form the scaling, sign, and permutation orbit of $(1, 1, -3)$.

It remains to check that the preliminary estimate $\|h_B\|_\infty \geq M/2$ is also an equality there. For the representative $(u, v, w) = (1, 1, -3)$, put $r_i = |z_i|^2$ for a unit vector $z \in \mathbb{C}^3$. The diagonal operator (6.13) gives

$$h_B(z) = -r_1^2 + r_2^2 + r_3^2 + 2r_1r_2 + 2r_1r_3 - 6r_2r_3.$$

If $s = r_2 + r_3$ and $t = r_2 r_3$, this is $-1 + 4s - 2s^2 - 8t$. Since $0 \leq s \leq 1$ and $0 \leq t \leq s^2/4$, one has $-1 \leq h_B(z) \leq 1$; both endpoints occur at coordinate vectors. Thus $\|h_B\|_\infty = 1 = M/2$. At this tensor,

$$q_2 = 7, \quad q_3 = -6, \quad \|h_B\|_\infty = 1,$$

so the constant is exactly $6/7$. $\square$

6.2. The quadric tensor.

Definition 6.3. Let

$$v = \frac{e_1^2 + e_2^2 + e_3^2}{\sqrt{3}} \in \text{Sym}^2 \mathbb{C}^3$$

and let P_v be the rank-one orthogonal projection onto $\mathbb{C}v$, explicitly

$$P_v(u) = \langle u, v \rangle v, \quad u \in \text{Sym}^2 \mathbb{C}^3.$$

Here $I_6 = \text{Id}_{\text{Sym}^2 \mathbb{C}^3}$. Define the Hermitian operator

$$A_* := \frac{1}{\sqrt{15}}(I_6 - 6P_v). \quad (6.18)$$

The *normalized quadric tensor* is the unique algebraic Kähler curvature tensor B_* whose associated operator in (4.4) is $A_{B_*} = A_*$. Equivalently, its curvature components are

$$(B_*)_{i\bar{j}k\bar{l}} = \frac{1}{\sqrt{15}} \left[\frac{1}{2}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}) - 2\delta_{ik}\delta_{jl} \right]. \quad (6.19)$$

Indeed, in a unitary frame its Ricci contraction is

$$\sum_{k=1}^3 (B_*)_{i\bar{j}k\bar{k}} = \frac{1}{\sqrt{15}} \left[\frac{1}{2}(3\delta_{ij} + \delta_{ij}) - 2\delta_{ij} \right] = 0.$$

Thus $B_* \in \mathcal{B}_3$. The factor $1/\sqrt{15}$ is the normalization for which $q_{2,3}(B_*) = 1$, as verified in Proposition 6.4.

Proposition 6.4. *The normalized quadric tensor of Definition 6.3 satisfies*

$$q_{2,3}(B_*) = 1, \quad q_{3,3}(B_*) = -\frac{2}{\sqrt{15}}, \quad \|h_{B_*}\|_\infty = \frac{1}{\sqrt{15}}.$$

Consequently,

$$K_{3,3} \geq 2. \quad (6.20)$$

Proof. For a unit vector $z \in \mathbb{C}^3$,

$$h_{B_*}(z) = \frac{1}{\sqrt{15}} \left(1 - 2|z_1^2 + z_2^2 + z_3^2|^2 \right).$$

The squared absolute value ranges from 0 to 1, proving the norm formula. The eigenvalues of A_* are $-5/\sqrt{15}$ once and $1/\sqrt{15}$ five times, so (6.3) gives $q_2 = 1$. Put $a = 1/\sqrt{15}$. The same eigenvalue list gives

$$T_1 = \text{tr}(A_*^3) = (-5a)^3 + 5a^3 = -120a^3 = -\frac{8}{\sqrt{15}}.$$

For $\mathcal{R}_{B_*}$ from (4.29), the scalar, trace-free symmetric, and skew-symmetric subspaces of 3×3 matrices have respective eigenvalues $0, -3a/2, 5a/2$ and dimensions $1, 5, 3$. Therefore

$$T_2 = \text{tr}(\mathcal{R}_{B_*}^3) = 5 \left(-\frac{3a}{2} \right)^3 + 3 \left(\frac{5a}{2} \right)^3 = 30a^3 = \frac{2}{\sqrt{15}}.$$

Equation (6.4) gives $q_3 = -2/\sqrt{15}$ and hence (6.20). $\square$

Proposition 7.1 below shows that, up to scale, B_* is the Bochner tensor of the standard Kähler–Einstein metric on the complex quadric Q^3 . It is important that Theorem 6.2 does not control this tensor: the toric constant $6/7$ is not the full-space constant.

Theorem 6.5. *The sharp full-space constant is*

$$K_{3,3} = 2,$$

In particular, the normalized quadric tensor B_* and its sign and $U(3)$ -translates attain equality.

For comparison, define the Euclidean cubic constant

$$A_{3,3} := \sup_{B \neq 0} \frac{|q_{3,3}(B)|}{q_{2,3}(B)^{3/2}}. \quad (6.21)$$

We next determine this constant by an exact invariant sum-of-squares calculation. Give $V := \mathcal{B}_3$ the inner product

$$\langle B, C \rangle = \text{tr}_{\text{Sym}^2 \mathbb{C}^3}(A_B A_C), \quad \|B\|^2 = 2q_{2,3}(B). \quad (6.22)$$

Here A_B and A_C are the Hermitian operators defined for general n by (4.4); their juxtaposition means ordinary composition, not a tensor contraction left implicit. Formula (4.6) shows that this is precisely the Hilbert–Schmidt inner product of the curvature tensors.

We next record the group action and our highest-weight notation. For $g \in SU(3)$, define

$$\begin{aligned} (g \cdot B)(u, \bar{x}, v, \bar{y}) \\ := B(g^{-1}u, \overline{g^{-1}x}, g^{-1}v, \overline{g^{-1}y}). \end{aligned} \quad (6.23)$$

This action preserves the Kähler symmetries and the zero Ricci contraction, so it preserves $V = \mathcal{B}_3$. If $S^2 g$ denotes the induced unitary operator on $\text{Sym}^2 \mathbb{C}^3$, then (4.4) gives

$$A_{g \cdot B} = (S^2 g) A_B (S^2 g)^{-1}. \quad (6.24)$$

In particular, the inner product (6.22) is $SU(3)$ -invariant.

Let $\alpha_1^\vee, \alpha_2^\vee$ be the simple coroots of $SU(3)$, and let ϖ_1, ϖ_2 be the fundamental weights characterized by $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$. For integers $p, q \geq 0$, we write $V_{(p,q)}$ for the unique irreducible complex $SU(3)$ -module of highest weight $p\varpi_1 + q\varpi_2$; these are the standard highest-weight conventions [9]. Thus

$$\mathbb{C}^3 = V_{(1,0)}, \quad (\mathbb{C}^3)^* = V_{(0,1)}, \quad \text{Sym}^2 \mathbb{C}^3 = V_{(2,0)}, \quad (6.25)$$

and the Weyl dimension formula is

$$\dim_{\mathbb{C}} V_{(p,q)} = \frac{1}{2}(p+1)(q+1)(p+q+2). \quad (6.26)$$

Here and below $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ denotes the complexification of the real curvature module $V = \mathcal{B}_3$.

We now justify the type of this complex module. Under the correspondence $B \leftrightarrow A_B$, the complexification of the 36-dimensional real space of all algebraic Kähler curvature tensors is

$$\begin{aligned} \text{End}(\text{Sym}^2 \mathbb{C}^3) &\cong V_{(2,0)} \otimes V_{(0,2)} \\ &\cong V_{(2,2)} \oplus V_{(1,1)} \oplus V_{(0,0)}, \end{aligned} \quad (6.27)$$

where the second line is the Littlewood–Richardson decomposition. The complexified Ricci contraction is the $SU(3)$ -equivariant surjection

$$\text{Ric} : \text{End}(\text{Sym}^2 \mathbb{C}^3) \longrightarrow \text{End}(\mathbb{C}^3) \cong V_{(1,1)} \oplus V_{(0,0)}. \quad (6.28)$$

Its surjectivity was proved above by the explicit tensor $C(H)$. Each irreducible summand in (6.27) has multiplicity one. Schur’s lemma and surjectivity therefore show that Ric is an

isomorphism on the $V_{(1,1)}$ and $V_{(0,0)}$ summands, while its kernel is the remaining summand. Consequently,

$$V_{\mathbb{C}} = \ker(\text{Ric}) \cong V_{(2,2)}. \quad (6.29)$$

This proves, rather than assumes, that the complexified Bochner-curvature module is irreducible of highest weight $(2, 2)$; its dimension is $\frac{1}{2}(3)(3)(6) = 27$, in agreement with (6.2).

The induced norm on $\text{Sym}^2 V$ is normalized by $\|B \odot B\|^2 = \|B\|^4$. Let $X_1, \dots, X_8$ be an orthonormal basis of $\mathfrak{su}(3)$ for $-\text{tr}(XY)$. If ρ denotes the infinitesimal $\mathfrak{su}(3)$ -action on $V = \mathcal{B}_3$, its induced action on $\text{Sym}^2 V$ is

$$\rho_2(X)(B \odot C) = \rho(X)B \odot C + B \odot \rho(X)C, \quad X \in \mathfrak{su}(3). \quad (6.30)$$

The corresponding Casimir operator is

$$\mathcal{C} = - \sum_{\alpha=1}^8 \rho_2(X_{\alpha})^2. \quad (6.31)$$

The inner product $-\text{tr}(XY)$ is $\text{Ad}(SU(3))$ -invariant, so $\mathcal{C}$ is independent of the chosen orthonormal basis $X_1, \dots, X_8$. Each $\rho_2(X_{\alpha})$ is skew-adjoint; consequently $\mathcal{C}$ is self-adjoint and nonnegative. It also commutes with the $SU(3)$ -action.

By a *Casimir eigenvalue* we mean a real number $c \geq 0$ for which

$$E_c := \ker(\mathcal{C} - c \text{Id}_{\text{Sym}^2 V}) = \{U \in \text{Sym}^2 V : \mathcal{C}U = cU\} \quad (6.32)$$

is nonzero. We write $\Pi_c : \text{Sym}^2 V \rightarrow E_c$ for the orthogonal projection. Equivalently, by Schur's lemma, $\mathcal{C}$ acts by one scalar c on each irreducible $SU(3)$ -summand of $\text{Sym}^2 V$; that scalar is its Casimir eigenvalue. Thus the symbols Π_{30} , Π_{24} , and Π_6 in Theorem 6.6 are now completely specified. Gradients below are taken with respect to (6.22). For later exact calculations, if

$$\Sigma = \{48, 36, 30, 24, 16, 6, 0\},$$

then the spectral projection is the explicit polynomial

$$\Pi_c = \prod_{d \in \Sigma \setminus \{c\}} \frac{\mathcal{C} - d \text{Id}}{c - d}, \quad c \in \Sigma. \quad (6.33)$$

We now state the invariant quartic sum-of-squares identity.

Theorem 6.6. *For every $B \in \mathcal{B}_3$,*

$$\begin{aligned} \frac{6}{5} q_{2,3}(B)^2 - \|\nabla q_{3,3}(B)\|^2 &= \frac{63}{320} \|\Pi_{30}(B \odot B)\|^2 + \frac{141}{200} \|\Pi_{24}(B \odot B)\|^2 \\ &+ \frac{4179}{4000} \|\Pi_6(B \odot B)\|^2 + \frac{1}{360} \|\nabla T_1(B) + 4\nabla T_2(B)\|^2. \end{aligned} \quad (6.34)$$

Consequently,

$$\|\nabla q_{3,3}(B)\|^2 \leq \frac{6}{5} q_{2,3}(B)^2, \quad (6.35)$$

$$q_{3,3}(B)^2 \leq \frac{4}{15} q_{2,3}(B)^3, \quad (6.36)$$

and

$$A_{3,3} = \frac{2}{\sqrt{15}}. \quad (6.37)$$

The quadric tensor B_* is an equality case in all three statements.

Proof. By (6.29), $V_{\mathbb{C}} \cong V_{(2,2)}$. We first prove the decomposition used below. If $g \in SU(3)$ and W is any finite-dimensional complex representation with character χ_W , then

$$\chi_{\text{Sym}^2 W}(g) = \frac{1}{2} \left(\chi_W(g)^2 + \chi_W(g^2) \right). \quad (6.38)$$

Indeed, $\text{Sym}^2 W$ is the $+1$ eigenspace of the flip $\tau(u \otimes v) = v \otimes u$ on $W \otimes W$, so its projector is $(I + \tau)/2$; taking the trace of $g \otimes g$ on this projector gives (6.38).

We evaluate this identity exactly using Schur functions. Under restriction from $GL(3, \mathbb{C})$ to $SU(3)$, the module $V_{(2,2)}$ is represented by the Schur function $s_{(4,2)}$. Since $h_2 = (p_1^2 + p_2)/2$, formula (6.38) is the plethysm

$$\text{ch}(\text{Sym}^2 V_{(2,2)}) = h_2[s_{(4,2)}] = \frac{1}{2} \left(s_{(4,2)}^2 + p_2[s_{(4,2)}] \right). \quad (6.39)$$

Expanding the product by the Littlewood–Richardson rule and the second term by the power-sum substitution gives the exact Schur identity [8, Chapter I, Section 8]

$$\begin{aligned} h_2[s_{(4,2)}] &= s_{(8,4)} + s_{(8,2,2)} + s_{(6,6)} + s_{(7,4,1)} + s_{(7,3,2)} \\ &\quad + s_{(6,5,1)} + 2s_{(6,4,2)} + s_{(5,4,3)} + s_{(4,4,4)}. \end{aligned} \quad (6.40)$$

For clarity, this calculation is reproducible directly from $s_\lambda = \det(h_{\lambda_i - i + j})$ and $p_2[f](x_1, x_2, x_3) = f(x_1^2, x_2^2, x_3^2)$: insert these two formulas into (6.39), expand, and collect the Schur determinants. Thus (6.40) is an exact character identity, not a dimension count or a numerical decomposition.

On restriction to $SU(3)$, the Schur module indexed by $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ has highest weight $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3)$. Applying this rule term by term to (6.40) gives

$$\begin{aligned} \text{Sym}^2(V_{\mathbb{C}}) &\cong V_{(4,4)} \oplus V_{(6,0)} \oplus V_{(0,6)} \oplus V_{(3,3)} \oplus V_{(4,1)} \oplus V_{(1,4)} \\ &\quad \oplus 2V_{(2,2)} \oplus V_{(1,1)} \oplus V_{(0,0)}. \end{aligned} \quad (6.41)$$

The nine partitions in (6.40) yield these summands in exactly the displayed order. Since irreducible characters are linearly independent, the character identity proves the direct-sum decomposition. As a check, the dimensions on the right are respectively

$$125, 28, 28, 64, 35, 35, 27, 27, 8, 1,$$

and sum to $378 = \dim \text{Sym}^2 V$. In the normalization (6.31), the Casimir eigenvalue on $V_{(p,q)}$ is

$$c(p, q) = \frac{2}{3} \left(p^2 + pq + q^2 + 3p + 3q \right).$$

Thus Π_{30} projects onto $V_{(3,3)}$, Π_{24} onto the real form of $V_{(4,1)} \oplus V_{(1,4)}$, and Π_6 onto $V_{(1,1)}$.

For completeness, we describe an exact finite verification of the coefficients in (6.34). If the weights of $V_{(2,2)}$ are ν with multiplicities m_ν , then

$$\sum_{r \geq 0} \chi_{\text{Sym}^r V_{\mathbb{C}}} t^r = \prod_{\nu} (1 - te^\nu)^{-m_\nu}.$$

Extracting the coefficient of t^4 and subtracting highest-weight characters shows that

$$\dim_{\mathbb{C}} \left(\text{Sym}^4 V_{\mathbb{C}} \right)^{SU(3)} = 4. \quad (6.42)$$

This extraction uses only integer arithmetic: for each weight ν , truncate

$$(1 - te^\nu)^{-m_\nu} = \sum_{r=0}^4 \binom{m_\nu + r - 1}{r} t^r e^{r\nu} + O(t^5),$$

multiply the truncated factors, and successively subtract the character of the irreducible module generated by each remaining dominant highest weight. The zero-weight coefficient left after these subtractions is 4. To specify the space whose basis we need, define

$$\mathcal{I}_4(V) := \{P : V \rightarrow \mathbb{R} : P \text{ is a homogeneous polynomial of degree 4,} \\ P(g \cdot B) = P(B) \text{ for all } g \in SU(3), B \in V\}. \quad (6.43)$$

Equivalently, $\mathcal{I}_4(V) = (\text{Sym}^4 V^*)^{SU(3)}$. The invariant inner product (6.22) identifies V equivariantly with V^* , and this identification commutes with complexification. Therefore

$$\mathcal{I}_4(V) \otimes_{\mathbb{R}} \mathbb{C} \cong (\text{Sym}^4 V_{\mathbb{C}}^*)^{SU(3)} \cong (\text{Sym}^4 V_{\mathbb{C}})^{SU(3)}, \quad \dim_{\mathbb{R}} \mathcal{I}_4(V) = 4 \quad (6.44)$$

by (6.42).

For every Casimir eigenvalue c , define

$$P_c(B) := \|\Pi_c(B \odot B)\|^2. \quad (6.45)$$

Since Π_c is $SU(3)$ -equivariant and the norm is invariant, $P_c \in \mathcal{I}_4(V)$. We claim that

$$\{P_{48}, P_{36}, P_{30}, P_{24}\} \text{ is a basis of } \mathcal{I}_4(V). \quad (6.46)$$

To prove linear independence, let $B_1, \dots, B_4 \in V$ be the curvature tensors corresponding under $B \leftrightarrow A_B$ to the following four Hermitian operators. The first two are

$$A(1, 2, 4), \quad A(1, -1, 3).$$

Recall from Definition 6.3 that

$$I_6 = \text{Id}_{\text{Sym}^2 \mathbb{C}^3}, \quad v = \frac{e_1^2 + e_2^2 + e_3^2}{\sqrt{3}}, \quad P_v(u) = \langle u, v \rangle v.$$

Thus P_v is the rank-one orthogonal projection onto $\mathbb{C}v$. The third operator is the unnormalized quadric operator

$$I_6 - 6P_v = \sqrt{15} A_*. \quad (6.47)$$

By Definition 6.3, the operator in (6.47) corresponds to the curvature tensor $\sqrt{15} B_* \in V$.

The fourth is

$$A(1, 0, 2) + U \cdot A(0, 3, -1), \quad U = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where the dot denotes conjugation through the symmetric-square representation,

$$U \cdot A := (S^2 U) A (S^2 U)^{-1}.$$

Direct rational contraction gives the evaluation matrix below. Its columns are ordered as $P_{48}, P_{36}, P_{30}, P_{24}$.

$$\mathcal{M} = \begin{pmatrix} \frac{251543}{432} & \frac{41209}{540} & \frac{9487}{27} & \frac{5285}{216} \\ 119 & \frac{676}{135} & \frac{656}{21} & \frac{370}{27} \\ 280 & \frac{896}{3} & 0 & 0 \\ \frac{4656890927}{10546875} & \frac{2670069604}{52734375} & \frac{3479510416}{73828125} & \frac{233338802}{2109375} \end{pmatrix}. \quad (6.48)$$

Row reduction gives

$$\det \mathcal{M} = \frac{11610131492864}{30375} \neq 0. \quad (6.49)$$

If a linear combination of the four polynomials in (6.46) vanished identically, evaluating it at $B_1, \dots, B_4$ would give a vector in the kernel of the matrix in (6.49). The nonzero determinant forces every coefficient to vanish. Thus the four polynomials are linearly independent; since (6.44) gives $\dim_{\mathbb{R}} \mathcal{I}_4(V) = 4$, they form a basis of precisely the space $\mathcal{I}_4(V)$.

Both sides of (6.34) are invariant quartics. On the same four tensors, exact contraction gives their common values

$$\frac{41041}{160}, \quad \frac{521}{10}, \quad 0, \quad \frac{799036009}{3906250},$$

respectively. Equations (6.42) and (6.49) therefore prove the polynomial identity (6.34). Notice that this argument uses only rational arithmetic; the sum-of-squares coefficients are not numerical fits.

All four coefficients on the right of (6.34) are positive, which proves (6.35). Euler's identity and Cauchy–Schwarz now give

$$3|q_3| = |\langle B, \nabla q_3 \rangle| \leq \|B\| \|\nabla q_3\| \leq \sqrt{\frac{12}{5}} q_2^{3/2}.$$

This is (6.36). Proposition 6.4 supplies equality, proving (6.37). $\square$

The Euclidean estimate above controls the cubic density in terms of the quadratic density. To convert it into the sharp sectional estimate defining $K_{3,3}$, we next compare that quadratic density with the supremum of the holomorphic sectional polynomial.

Lemma 6.7. *For every $B \in \mathcal{B}_3$,*

$$\|h_B\|_\infty^2 \geq \frac{q_{2,3}(B)}{15}. \quad (6.50)$$

Proof. Write $h = h_B$ as a real-valued homogeneous polynomial on $\mathbb{R}^6 \cong \mathbb{C}^3$:

$$h(z) = B_{i\bar{j}k\bar{l}} z_i \bar{z}_j z_k \bar{z}_l.$$

The Euclidean Laplacian is $\Delta_{\mathbb{R}^6} = 4 \sum_i \partial_{z_i} \partial_{\bar{z}_i}$. Differentiating twice and using the Kähler symmetries gives the following explicit calculation. For each a one has, using $B_{i\bar{j}k\bar{l}} = B_{k\bar{j}i\bar{l}}$ and $B_{i\bar{j}k\bar{l}} = B_{i\bar{l}k\bar{j}}$,

$$\partial_{z_a} \partial_{\bar{z}_a} h = 4 B_{a\bar{a}k\bar{l}} z_k \bar{z}_l.$$

Consequently

$$\Delta_{\mathbb{R}^6} h = 4 \sum_a \partial_{z_a} \partial_{\bar{z}_a} h = 16 B_{a\bar{a}k\bar{l}} z_k \bar{z}_l = 0,$$

because B is Ricci-free. Thus h is a harmonic homogeneous polynomial of degree four. The polar-coordinate formula for the Euclidean Laplacian, applied to $h(r\xi) = r^4 h(\xi)$, gives

$$\Delta_{S^5} h = -4(4 + 6 - 2)h = -32h.$$

Fix $x \in S^5$ and a unit tangent vector $v \perp x$. Along the great circle $\gamma(t) = \cos t x + \sin t v$, write $\gamma(t) = ae^{it} + be^{-it}$ with $a = (x - \mathbf{i}v)/2$ and $b = (x + \mathbf{i}v)/2$. Every factor in $h(\gamma(t))$ is therefore a linear combination of e^{it} and e^{-it} , so the quartic polynomial $T(t) = h(\gamma(t))$ has the explicit Fourier form $T(t) = \sum_{r=-4}^4 c_r e^{irt}$ (and is real-valued). We now use the following pointwise form of the Bernstein–Duffin–Schaeffer inequality. If $T : \mathbb{R} \rightarrow \mathbb{R}$ is a real trigonometric polynomial of degree at most m , if T' denotes its derivative with respect to the circle parameter, and if

$$\|T\|_\infty := \max_{s \in \mathbb{R}} |T(s)|,$$

then, at every t ,

$$|T'(t)|^2 + m^2 |T(t)|^2 \leq m^2 \|T\|_\infty^2. \quad (6.51)$$

Equivalently, after dividing by $m^2 \|T\|_\infty^2$ (when the norm is nonzero), the point

$$\left(\frac{T(t)}{\|T\|_\infty}, \frac{T'(t)}{m \|T\|_\infty} \right)$$

lies in the closed unit disk. Thus (6.51) controls the value and the slope simultaneously; the usual Bernstein derivative estimate is obtained by discarding the nonnegative $m^2 T(t)^2$ term.

For completeness, put $M = \|T\|_\infty$. If $M = 0$ there is nothing to prove; otherwise, for a fixed t_0 define $\tau(s) = T(t_0 + s)/M$. Then $|\tau| \leq 1$ and τ has degree at most m . The Duffin–Schaeffer comparison principle says that, among such trigonometric polynomials with prescribed value and derivative at $s = 0$, the extremal one is a cosine polynomial. Thus the extremal boundary has the form

$$\tau(s) = \cos(ms - \phi), \quad \tau(0) = \cos \phi, \quad \tau'(0) = m \sin \phi,$$

and consequently

$$\tau(0)^2 + \frac{\tau'(0)^2}{m^2} = 1.$$

If the left-hand side were larger than 1, the cosine comparison polynomial with the same value and derivative would have amplitude strictly larger than 1, contradicting $|\tau| \leq 1$. Rescaling by M gives (6.51) at every t_0 . In our application, $T(t) = h(\gamma(t))$ is quartic, so $m = 4$. Moreover,

$$T'(0) = \langle \nabla_{S^5} h(x), v \rangle, \quad \|T\|_\infty \leq \|h\|_\infty,$$

because $\gamma(0) = x$ and $\gamma'(0) = v$. Substituting $m = 4$ into (6.51) and maximizing over all unit tangent vectors $v \perp x$ therefore yields

$$|\nabla_{S^5} h(x)|^2 + 16h(x)^2 \leq 16\|h\|_\infty^2. \quad (6.52)$$

Integrating with respect to the normalized probability measure $d\sigma$ and using integration by parts together with $-\Delta_{S^5} h = 32h$, we obtain

$$32 \int_{S^5} h^2 d\sigma + 16 \int_{S^5} h^2 d\sigma \leq 16\|h\|_\infty^2,$$

so $\int h^2 d\sigma \leq \|h\|_\infty^2 / 3$. Finally, the exact spherical moment identity gives

$$\int_{S^5} h^2 d\sigma = \frac{q_{2,3}(B)}{45},$$

and hence $q_{2,3}(B)/45 \leq \|h\|_\infty^2 / 3$, which is (6.50). $\square$

Proof of Theorem 6.5. By Theorem 6.6,

$$|q_{3,3}(B)| \leq \frac{2}{\sqrt{15}} q_{2,3}(B)^{3/2}.$$

Lemma 6.7 gives $q_{2,3}(B)^{1/2} \leq \sqrt{15} \|h_B\|_\infty$. Therefore

$$|q_{3,3}(B)| \leq \frac{2}{\sqrt{15}} q_{2,3}(B) q_{2,3}(B)^{1/2} \leq 2 \|h_B\|_\infty q_{2,3}(B).$$

Taking the supremum in the definition of $K_{3,3}$ proves $K_{3,3} \leq 2$. For B_* , Proposition 6.4 gives

$$\frac{|q_{3,3}(B_*)|}{\|h_{B_*}\|_\infty q_{2,3}(B_*)} = 2,$$

so $K_{3,3} \geq 2$. Hence $K_{3,3} = 2$, and the sign and $U(3)$ -orbit of B_* also attain equality by unitary invariance. $\square$

Proof of Corollary 1.2. The local estimate is the proof of Theorem 6.5. The global estimate follows from (1.9) and $K_{3,3} = 2$. For the pinched form, specialize (1.10) to $n = k = 3$:

$$2 \cdot \frac{2(1-\delta)}{(3+1)\delta} = \frac{1-\delta}{\delta}.$$

$\square$

7. COMPLEX QUADRICS IN ARBITRARY DIMENSION

The quadric calculation extends uniformly. Let $Q^n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface and let H denote the restricted hyperplane class. The normal sequence gives

$$c(TQ^n) = \frac{(1+H)^{n+2}}{1+2H}. \quad (7.1)$$

Hence

$$c_1 = nH, \quad (7.2)$$

$$c_2 = \frac{n^2 - n + 2}{2} H^2, \quad (7.3)$$

$$c_3 = \frac{n^3 - 3n^2 + 8n - 12}{6} H^3. \quad (7.4)$$

Substitution into (1.5) and (1.6) gives

$$\widehat{c}_2 = \frac{n+2}{2(n+1)} H^2, \quad (7.5)$$

$$\widehat{c}_3 = -\frac{(n+2)(n+3)}{3(n+1)^2} H^3. \quad (7.6)$$

Since $\int_{Q^n} H^n = 2$,

$$\frac{|\langle \widehat{c}_3 c_1^{n-3}, [Q^n] \rangle|}{\langle \widehat{c}_2 c_1^{n-2}, [Q^n] \rangle} = \frac{2(n+3)}{3n(n+1)}. \quad (7.7)$$

We next derive the curvature formula used for the standard metric.

Proposition 7.1. *Let g_Q be the standard homogeneous Kähler–Einstein metric on Q^n , normalized by $\text{Ric}(g_Q) = \mu g_Q$, and let B_Q denote its Bochner tensor. At every point there is a unitary tangent frame in which*

$$(R_Q)_{i\bar{j}k\bar{l}} = \frac{\mu}{n} (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} - \delta_{ik}\delta_{jl}), \quad (7.8)$$

$$(B_Q)_{i\bar{j}k\bar{l}} = \frac{\mu}{n(n+1)} (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} - (n+1)\delta_{ik}\delta_{jl}). \quad (7.9)$$

Consequently, for every unit vector $z \in \mathbb{C}^n$,

$$H_{Q^n}(z) = \frac{\mu}{n} \left(2 - |z_1^2 + \cdots + z_n^2|^2 \right). \quad (7.10)$$

If $m = n(n+1)/2$, $v = n^{-1/2} \sum_j e_j^2$, and P_v is orthogonal projection onto $\mathbb{C}v$, then the Hermitian operator corresponding to B_Q is

$$A_{B_Q} = \frac{2\mu}{n(n+1)} (I_m - mP_v). \quad (7.11)$$

Proof. Use the standard projective embedding $Q^n = \{[Z] \in \mathbb{P}^{n+1} : \sum Z_\alpha^2 = 0\}$. In an adapted unitary frame, differentiating the defining quadratic equation shows that the complex second fundamental form is $h_{ik} = \sqrt{c} \delta_{ik}$, where c is the curvature coefficient of the ambient Fubini–Study metric. The Kähler Gauss equation therefore gives

$$(R_Q)_{i\bar{j}k\bar{l}} = c (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} - \delta_{ik}\delta_{jl}).$$

Contracting k, l gives $\text{Ric}(g_Q) = nc, g_Q$, so $c = \mu/n$ and proves (7.8). Subtracting the scalar-curvature part in (2.2) gives (7.9), and evaluation on $(z, \bar{z}, z, \bar{z})$ gives (7.10).

Under the tensor–operator correspondence (4.4), the identity on $\text{Sym}^2 \mathbb{C}^n$ and the rank-one projection P_v correspond, respectively, to

$$\frac{1}{2}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}), \quad \frac{1}{n}\delta_{ik}\delta_{jl}.$$

Since $m/n = (n+1)/2$, equation (7.9) is equivalent to (7.11). $\square$

Since $|z_1^2 + \cdots + z_n^2|^2$ ranges over $[0, 1]$, combining (7.10) with (2.1) gives the centered sectional norm

$$\frac{\|h_{B_Q}\|_\infty}{|\mu|} = \frac{\max\{2, n-1\}}{n(n+1)}. \quad (7.12)$$

For $n \geq 3$, equations (7.7) and (7.12) yield the rigorous lower bound

$$K_{n,3} \geq \frac{2(n+3)}{3(n-1)}. \quad (7.13)$$

The same operator model gives a completely local calculation in every dimension.

Lemma 7.2. *Let $n \geq 3$, put $m = n(n+1)/2$ and $v = n^{-1/2} \sum_{j=1}^n e_j^2$, and let $P_v(u) = \langle u, v \rangle v$. For $a \in \mathbb{R}$, let $B_Q^{(a)} \in \mathcal{B}_n$ be the tensor whose corresponding operator is*

$$A_{B_Q^{(a)}} = a(I_m - mP_v). \quad (7.14)$$

Set

$$P_n = n(n-1)(n+1)(n+2).$$

Then

$$q_{2,n}(B_Q^{(a)}) = \frac{a^2 P_n}{8}, \quad q_{3,n}(B_Q^{(a)}) = -\frac{a^3 P_n (n+3)}{24}. \quad (7.15)$$

Proof. The tensor corresponding to (7.14) is

$$(B_Q^{(a)})_{i\bar{j}k\bar{l}} = \frac{a}{2}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} - (n+1)\delta_{ik}\delta_{jl}). \quad (7.16)$$

Its Ricci contraction is $a(n+1)\delta_{ij}/2 - a(n+1)\delta_{ij}/2 = 0$, so it lies in $\mathcal{B}_n$. The eigenvalues of $A_{B_Q^{(a)}}$ are $a(1-m)$ on $\mathbb{C}v$ and a on $v^\perp$. Hence

$$q_{2,n}(B_Q^{(a)}) = \frac{1}{2} \text{tr}(A_{B_Q^{(a)}}^2) = \frac{a^2 m(m-1)}{2} = \frac{a^2 P_n}{8}.$$

For the reshuffled operator $\mathcal{R}_{B_Q^{(a)}}$ in (4.29), decompose the space of $n \times n$ matrices as

$$\mathbb{C}I \oplus \text{Sym}_0^2 \mathbb{C}^n \oplus \Lambda^2 \mathbb{C}^n.$$

Equation (7.16) shows that its eigenvalues on these three summands are, respectively,

$$0, \quad -\frac{an}{2}, \quad \frac{a(n+2)}{2},$$

with dimensions 1, $m-1$, and $n(n-1)/2$. It follows that

$$\begin{aligned} \text{tr}(A_{B_Q^{(a)}}^3) &= -a^3 m(m-1)(m-2), \\ \text{tr}(\mathcal{R}_{B_Q^{(a)}}^3) &= \frac{a^3}{8} \left[-n^3(m-1) + (n+2)^3 \frac{n(n-1)}{2} \right]. \end{aligned}$$

Substitution into (4.30) and simplification give the second identity in (7.15). $\square$

Thus the quadric gives the following lower bound for the Euclidean cubic constant:

$$A_{n,3} := \sup_{B \neq 0} \frac{|q_{3,n}(B)|}{q_{2,n}(B)^{3/2}} \geq \frac{2\sqrt{2}(n+3)}{3\sqrt{n(n-1)(n+1)(n+2)}}. \quad (7.17)$$

For $n = 3$, this reduces to $2/\sqrt{15}$. We do not claim that quadrics maximize $A_{n,3}$ or $K_{n,3}$ for every n .

8. FURTHER DIRECTIONS

Theorem 1.1 separates the global geometry from a family of finite-dimensional algebraic problems. Several questions remain.

- (1) Determine the complete equality locus in Theorem 6.5. The proof above establishes sharpness and the quadric equality orbit; a full classification of all equality tensors would give the corresponding rigidity statement.
- (2) Determine $K_{n,3}$ in higher dimension and decide whether the quadric lower bound (7.13) is sharp in any further dimensions.
- (3) Determine the sharp values of $K_{n,k}$ for $k \geq 4$ and improve the effective constants $U_{n,k}$ from Theorem 4.7.

Data Availability This manuscript has no associated data.

Declarations

Conflict of interest The author states that there is no conflict of interest.

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