

A NON-HOMOGENEOUS UNIFORM RANK-NINE BUNDLE ON $\mathbb{P}^5$

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ABSTRACT. In 1980, Drézet conjectured that every uniform vector bundle of rank $r < 2n$ on $\mathbb{P}^n$ is homogeneous. We show that this conjecture fails on $\mathbb{P}^5$ by exhibiting a uniform vector bundle of rank nine and splitting type $(0^5, -1^4)$ which is not homogeneous under $\mathrm{PGL}(6)$. The bundle twisted by $\mathcal{O}(1)$, as well as its dual, is globally generated; in particular, the example is the image of a 14×14 matrix of linear forms of constant rank nine. We also show that its isomorphism class determines its defining bivector up to scale, and compute its K -class and all its Chern classes.

Keywords. Uniform vector bundle; homogeneous vector bundle; projective space; matrix of constant rank.

1. INTRODUCTION

A vector bundle E on $\mathbb{P}^n$ is called *uniform* if the splitting type of $E|_L$, written in non-increasing order, is independent of the line $L \subset \mathbb{P}^n$. It is called *homogeneous* if $g^*E \simeq E$ for every $g \in \mathrm{Aut}(\mathbb{P}^n)$. Since projective space is covered by lines, uniformity is a natural first approximation to homogeneity. It is nevertheless a global condition of a rather different nature: it records the restriction of a bundle to each line, but not how these restrictions are glued as the line varies.

The classification problem goes back to the 1970s. Van de Ven treated rank two in 1972 [12]. Sato obtained splitting results below the dimension and classified the rank- n case in odd dimension [11]; the remaining rank- n case was completed by Elencwajg, Hirschowitz, and Schneider [7]. The work of Ellia and Ballico then settled rank $n+1$ [1, 8]. Together with the rank-three analysis on the plane [5], these results showed that the uniform bundles occurring in the low-rank range are the expected homogeneous ones.

The first counterexamples to the unrestricted assertion appeared soon afterwards. Elencwajg constructed a non-homogeneous uniform rank-four bundle on $\mathbb{P}^2$ in 1979 [6]. Further examples in rank $3n-1$ on $\mathbb{P}^n$, for $n \geq 3$, were obtained by Hirschowitz; see [10, Chapter I]. In 1980, Drézet constructed non-homogeneous uniform bundles of rank $2n$ on $\mathbb{P}^n$ and, in the same paper, proposed the following conjecture [3].

Conjecture. Every uniform vector bundle of rank $r < 2n$ on $\mathbb{P}^n$ is homogeneous.

The low-rank classification results recalled above settle the range $r \leq n+1$, so the only remaining interval is $n+1 < r < 2n$. In this interval, the conjecture was proved for $n=3$ by Ballico and Ellia in 1983 [2]. The case $n=4$ was established much more recently by the author and Zhou [4]. Thus $\mathbb{P}^5$ is the first dimension in which the remaining interval was still open.

One reason for the persistence of the problem is that the standard method becomes rapidly less rigid beyond rank $n+1$. On the incidence variety

$$F(1, 2; V) = \{(x, L) : x \in L \subset \mathbb{P}(V)\}$$

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with projections

$$p : F(1, 2; V) \longrightarrow \mathbb{P}(V), \quad q : F(1, 2; V) \longrightarrow \mathrm{Gr}(2, V),$$

the bundle p^*E has a relative Harder–Narasimhan filtration with respect to q . Uniformity fixes the degrees and multiplicities of its factors along the fibers, but the factors themselves are bundles on the Grassmannian and the extension data remain to be determined. In small rank, Chern-class identities and descent force these data into a short list. In the interval $n + 1 < r < 2n$, several HN blocks can occur simultaneously, and the resulting identities in the Chow or K -theory of the Grassmannian no longer determine the extension classes. The related problem of classifying linear spaces of matrices of constant rank encounters the same loss of rigidity [9].

The purpose of this paper is to show that Drézet’s conjecture is false. More precisely, we construct a rank-nine counterexample on $\mathbb{P}^5$. The numerical coincidence

$$\mathrm{rk} \Omega_{\mathbb{P}^5}^2(2) - 1 = \binom{5}{2} - 1 = 9 = 2 \cdot 5 - 1$$

is the starting point. Let V be a complex vector space of dimension six, write $X = \mathbb{P}(V)$ for the space of one-dimensional subspaces of V , and let

$$(1.1) \quad 0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow V \otimes \mathcal{O}_X \longrightarrow \mathcal{Q} \longrightarrow 0$$

be the universal sequence. Thus

$$\Omega_X^1(1) \simeq \mathcal{Q}^\vee, \quad \Omega_X^2(2) \simeq \bigwedge^2 \mathcal{Q}^\vee.$$

A bivector $\alpha \in \bigwedge^2 V$ maps, by the exterior square of the universal quotient, to a section of $\bigwedge^2 \mathcal{Q}$. Pairing with this section gives

$$\iota_\alpha : \bigwedge^2 \mathcal{Q}^\vee \longrightarrow \mathcal{O}_X.$$

Our main result is the following.

Theorem 1.1. *Assume that $\alpha^{\wedge 3} \neq 0$. Then the morphism ι_α is surjective and its kernel $\mathcal{I}_\alpha$ has the following properties.*

- (i) $\mathcal{I}_\alpha$ has rank nine and, for every line $L \subset X$,

$$\mathcal{I}_\alpha|_L \simeq \mathcal{O}_L(-1)^4 \oplus \mathcal{O}_L^5.$$

In particular, $\mathcal{I}_\alpha$ is uniform.

- (ii) Both $\mathcal{I}_\alpha(1)$ and $\mathcal{I}_\alpha^\vee$ are globally generated, and

$$h^0(\mathcal{I}_\alpha(1)) = h^0(\mathcal{I}_\alpha^\vee) = 14.$$

Consequently, $\mathcal{I}_\alpha$ is the image of a 14×14 matrix of linear forms having constant rank nine on X .

- (iii) *The bundle $\mathcal{I}_\alpha$ is not homogeneous under $\mathrm{PGL}(V)$.*
 (iv) *Its total Chern class is*

$$c(\mathcal{I}_\alpha) = 1 - 4H + 9H^2 - 14H^3 + 14H^4,$$

where $H = c_1(\mathcal{O}_X(1))$; in particular, $c_5(\mathcal{I}_\alpha) = \cdots = c_9(\mathcal{I}_\alpha) = 0$.

The word homogeneous in Theorem 1.1 refers to invariance under the full group $\mathrm{PGL}(V)$. The stabilizer of $[\alpha]$ is a projective symplectic group acting transitively on $\mathbb{P}(V)$, and $\mathcal{I}_\alpha$ is equivariant for this smaller group. This distinction is harmless for uniformity but essential for the formulation of Drézet’s conjecture.

The proof uses only elementary linear algebra and standard cohomology on projective space, apart from the simplicity of $\Omega_X^3(4)$. The key point is that a bivector contained in $U \wedge V$, with $\dim U = 2$, has rank at most four. A non-degenerate bivector on V has rank

six and therefore avoids all these subspaces. This observation proves uniformity without analyzing the relative HN equations.

The paper is organized as follows. Section 2 establishes local freeness of the kernel. Section 3 computes its restriction to every line. Section 4 proves the two global-generation statements and constructs the constant-rank matrix. Section 5 recovers $[\alpha]$ from the isomorphism class and deduces non-homogeneity. Section 6 records the K -class and Chern classes and discusses the parameter space and the special role of dimension five.

2. BIVECTORS AND THE CONTRACTION MORPHISM

We first record the elementary rank facts used below. For $\lambda \in V^\vee$, let ι_λ denote contraction by λ on the exterior algebra of V ; on a decomposable bivector it is characterized by

$$\iota_\lambda(u \wedge v) = \lambda(u)v - \lambda(v)u.$$

If $\gamma \in \bigwedge^2 V$, its rank is the rank of the skew-symmetric linear map

$$\gamma^\sharp : V^\vee \longrightarrow V, \quad \lambda \longmapsto \iota_\lambda \gamma.$$

Equivalently, $\text{rk}(\gamma)$ is twice the number of nonzero summands in a Darboux normal form for γ .

Lemma 2.1. *Let $0 \neq U \subset V$.*

- (a) *If $\dim U = 1$ and $\gamma \in U \wedge V$, then $\text{rk}(\gamma) \leq 2$.*
- (b) *If $\dim U = 2$ and $\gamma \in U \wedge V$, then $\text{rk}(\gamma) \leq 4$.*

Proof. For (a), write $U = \langle u \rangle$. Then every element of $U \wedge V$ has the form $u \wedge v$, so its support is contained in $\langle u, v \rangle$.

For (b), choose a basis u_1, u_2 of U . Every element of $U \wedge V$ can be written

$$\gamma = u_1 \wedge v_1 + u_2 \wedge v_2$$

for suitable $v_1, v_2 \in V$; a possible multiple of $u_1 \wedge u_2$ can be absorbed into either summand. Hence the support of γ is contained in $\langle u_1, u_2, v_1, v_2 \rangle$, which has dimension at most four. The rank of a bivector cannot exceed the dimension of its support. $\square$

For a point $x = [\ell] \in X$, the fiber of $\mathcal{Q}$ is V/ℓ . Let

$$\pi_\ell : V \longrightarrow V/\ell$$

be the quotient map and set

$$\bar{\alpha}_\ell = \left(\bigwedge^2 \pi_\ell \right) (\alpha) \in \bigwedge^2 (V/\ell).$$

The fiber of ι_α at x is the functional

$$(2.1) \quad (\iota_\alpha)_x : \bigwedge^2 (V/\ell)^\vee \longrightarrow \mathbb{C}, \quad \eta \longmapsto \eta(\bar{\alpha}_\ell).$$

Proposition 2.2. *If $\text{rk}(\alpha) = 6$, then ι_α is surjective. Consequently,*

$$(2.2) \quad 0 \longrightarrow \mathcal{I}_\alpha \longrightarrow \Omega_X^2(2) \xrightarrow{\iota_\alpha} \mathcal{O}_X \longrightarrow 0$$

is an exact sequence of vector bundles and $\text{rk}(\mathcal{I}_\alpha) = 9$.

Proof. If $(\iota_\alpha)_x$ were zero at $x = [\ell]$, then $\bar{\alpha}_\ell = 0$. The kernel of

$$\bigwedge^2 V \longrightarrow \bigwedge^2 (V/\ell)$$

is $\ell \wedge V$. Thus $\alpha \in \ell \wedge V$, and Lemma 2.1(a) would imply $\text{rk}(\alpha) \leq 2$. This contradicts $\text{rk}(\alpha) = 6$. Hence every fiber map in (2.1) is a nonzero functional and is therefore surjective. Since $\text{rk} \Omega_X^2(2) = \binom{5}{2} = 10$, its kernel is locally free of rank nine. $\square$

3. UNIFORMITY

Let $L = \mathbb{P}(U) \subset X$ be a line, where $U \subset V$ has dimension two. Choose a complement W such that $V = U \oplus W$. Restricting (1.1) to L gives

$$(3.1) \quad \mathcal{Q}|_L \simeq \mathcal{O}_L(1) \oplus W \otimes \mathcal{O}_L.$$

Taking the second exterior power of the dual yields

$$(3.2) \quad \Omega_X^2(2)|_L \simeq \mathcal{O}_L(-1)^4 \oplus \mathcal{O}_L^6.$$

The decomposition $V = U \oplus W$ gives

$$\bigwedge^2 V = \bigwedge^2 U \oplus (U \wedge W) \oplus \bigwedge^2 W.$$

Under (3.2), the restriction of ι_α has the form

$$(3.3) \quad (\ell_1, \dots, \ell_4, c_1, \dots, c_6) : \mathcal{O}_L(-1)^4 \oplus \mathcal{O}_L^6 \longrightarrow \mathcal{O}_L,$$

where the ℓ_i are linear forms on L and the c_j are constants. The constant vector $(c_1, \dots, c_6)$ is the component of α in $\bigwedge^2 W$, or, intrinsically, the image of α in $\bigwedge^2(V/U)$.

Lemma 3.1. *If $\text{rk}(\alpha) = 6$, then the constant part of (3.3) is nonzero for every line $L = \mathbb{P}(U)$.*

Proof. The constant part vanishes precisely when the image of α in $\bigwedge^2(V/U)$ is zero. The kernel of

$$\bigwedge^2 V \longrightarrow \bigwedge^2(V/U)$$

is $U \wedge V$. If the constant part vanished, then $\alpha \in U \wedge V$, and Lemma 2.1(b) would give $\text{rk}(\alpha) \leq 4$, a contradiction. $\square$

Proposition 3.2. *For every line $L \subset X$,*

$$\mathcal{I}_\alpha|_L \simeq \mathcal{O}_L(-1)^4 \oplus \mathcal{O}_L^5.$$

In particular, $\mathcal{I}_\alpha$ is uniform.

Proof. By Lemma 3.1, a constant change of basis in the $\mathcal{O}_L^6$ summand changes (3.3) into

$$(\ell_1, \dots, \ell_4, 0, \dots, 0, 1) : \mathcal{O}_L(-1)^4 \oplus \mathcal{O}_L^5 \oplus \mathcal{O}_L \longrightarrow \mathcal{O}_L.$$

The kernel is the image of the injective morphism

$$\begin{aligned} \mathcal{O}_L(-1)^4 \oplus \mathcal{O}_L^5 &\longrightarrow \mathcal{O}_L(-1)^4 \oplus \mathcal{O}_L^5 \oplus \mathcal{O}_L, \\ (a_1, \dots, a_4, b_1, \dots, b_5) &\longmapsto (a_1, \dots, a_4, b_1, \dots, b_5, -\sum_i \ell_i a_i). \end{aligned}$$

It follows that the kernel is $\mathcal{O}_L(-1)^4 \oplus \mathcal{O}_L^5$, independently of L . $\square$

Remark 3.3. The rank-six hypothesis is exactly what removes jumping lines from this construction. Suppose instead that $\text{rk}(\alpha) = 4$, set $S = \text{Im}(\alpha^\sharp)$, and let $\omega \in \bigwedge^2 S^\vee$ be the non-degenerate alternating form inverse to the bivector $\alpha \in \bigwedge^2 S$. Define the jumping-line locus by

$$(3.4) \quad J(\alpha) = \{[U] \in \text{Gr}(2, V) : \alpha \in U \wedge V\}.$$

By (3.3), this is precisely the locus on which the constant part of the restricted contraction morphism vanishes. In fact,

$$(3.5) \quad J(\alpha) = \text{LG}(2, S, \omega) = \{[U] \in \text{Gr}(2, S) : \omega|_U = 0\} \simeq Q^3.$$

Here $\text{LG}(2, S, \omega)$ denotes the Lagrangian Grassmannian of two-dimensional ω -isotropic subspaces of S . Indeed, if $\alpha \in U \wedge V$ and $A = U \cap S$, then the image of α in $\bigwedge^2(S/A)$ is zero. If $A = 0$, this would force $\alpha = 0$; if $\dim A = 1$, it would put a rank-four bivector in $A \wedge S$, contradicting Lemma 2.1(a). Hence $\dim A = 2$ and $U \subset S$.

It remains to analyze a two-plane $U \subset S$. If U is ω -isotropic, choose a Darboux basis e_1, e_2, f_1, f_2 of S with $U = \langle e_1, e_2 \rangle$ and

$$\alpha = e_1 \wedge f_1 + e_2 \wedge f_2.$$

Then $\alpha \in U \wedge S$. If $\omega|_U$ is non-degenerate, choose a Darboux basis with $U = \langle e_1, f_1 \rangle$. The image of $\alpha = e_1 \wedge f_1 + e_2 \wedge f_2$ in $\bigwedge^2(S/U)$ is $e_2 \wedge f_2 \neq 0$, so $\alpha \notin U \wedge S$. This proves the first equality in (3.5).

In the Plücker embedding, $\text{Gr}(2, S)$ is a smooth quadric $Q^4 \subset \mathbb{P}(\bigwedge^2 S) \simeq \mathbb{P}^5$, and the condition $\omega|_U = 0$ cuts it by the hyperplane defined by ω . A tangent hyperplane to this Plücker quadric corresponds to a decomposable, hence rank-two, alternating form on S . Since ω is non-degenerate of rank four, its hyperplane is not tangent. Therefore $\text{LG}(2, S, \omega)$ is a smooth quadric threefold Q^3 .

The two restriction types can also be read off from (3.3). If $[U] \notin J(\alpha)$, the proof of Proposition 3.2 gives

$$\mathcal{I}_\alpha|_{\mathbb{P}(U)} \simeq \mathcal{O}_{\mathbb{P}(U)}(-1)^4 \oplus \mathcal{O}_{\mathbb{P}(U)}^5.$$

If $[U] \in J(\alpha)$, all constant entries vanish. The fiberwise argument in Proposition 2.2, which only excludes bivectors of rank at most two, shows that ι_α is still surjective. Thus the remaining four linear entries have no common zero. After changing bases they are $(x_0, x_1, 0, 0)$, and hence

$$\mathcal{I}_\alpha|_{\mathbb{P}(U)} \simeq \mathcal{O}_{\mathbb{P}(U)}(-2) \oplus \mathcal{O}_{\mathbb{P}(U)}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}(U)}^6.$$

Thus (3.5) is the entire jumping-line locus, with its two restriction types explicitly distinguished.

4. GLOBAL GENERATION AND A CONSTANT-RANK MATRIX

Dualizing (2.2) gives

$$(4.1) \quad 0 \longrightarrow \mathcal{O}_X \xrightarrow{s_\alpha} \bigwedge^2 \mathcal{Q} \longrightarrow \mathcal{I}_\alpha^\vee \longrightarrow 0.$$

Since $\mathcal{Q}$ is globally generated, so is $\bigwedge^2 \mathcal{Q}$, and hence $\mathcal{I}_\alpha^\vee$ is globally generated. Moreover,

$$H^0(X, \bigwedge^2 \mathcal{Q}) \simeq \bigwedge^2 V.$$

Applying global sections to s_α gives the linear map

$$H^0(s_\alpha) : H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \bigwedge^2 \mathcal{Q}).$$

Under the preceding identification, it sends 1 to α ; in particular, its image is $\mathbb{C}\alpha$. One can verify this identification directly from

$$0 \longrightarrow \mathcal{O}_X(-1) \otimes \mathcal{Q} \longrightarrow \bigwedge^2 V \otimes \mathcal{O}_X \longrightarrow \bigwedge^2 \mathcal{Q} \longrightarrow 0 :$$

the universal sequence twisted by $\mathcal{O}_X(-1)$ shows that $H^0(\mathcal{O}_X(-1) \otimes \mathcal{Q}) = H^1(\mathcal{O}_X(-1) \otimes \mathcal{Q}) = 0$. Since $H^1(\mathcal{O}_X) = 0$, cohomology of (4.1) therefore gives

$$(4.2) \quad H^0(X, \mathcal{I}_\alpha^\vee) \simeq \bigwedge^2 V / \mathbb{C}\alpha, \quad h^0(\mathcal{I}_\alpha^\vee) = 14.$$

It remains to prove the less immediate global generation of $\mathcal{I}_\alpha(1)$. Twisting (2.2) gives

$$(4.3) \quad 0 \longrightarrow \mathcal{I}_\alpha(1) \longrightarrow \Omega_X^2(3) \xrightarrow{\iota_\alpha} \mathcal{O}_X(1) \longrightarrow 0.$$

The standard Koszul description gives

$$(4.4) \quad H^0(X, \Omega_X^2(3)) \simeq \bigwedge^3 V^\vee.$$

For completeness, this is the degree-three part of the Koszul complex of the Euler morphism:

$$0 \longrightarrow H^0(\Omega_X^2(3)) \longrightarrow \bigwedge^2 V^\vee \otimes V^\vee \longrightarrow V^\vee \otimes S^2 V^\vee \longrightarrow S^3 V^\vee \longrightarrow 0.$$

The alternating map

$$\bigwedge^3 V^\vee \longrightarrow \bigwedge^2 V^\vee \otimes V^\vee, \quad a \wedge b \wedge c \longmapsto a \wedge b \otimes c - a \wedge c \otimes b + b \wedge c \otimes a$$

identifies $\bigwedge^3 V^\vee$ with the first kernel in this complex, which proves (4.4) without invoking Bott's theorem. Under this identification, the map on global sections in (4.3) is contraction by the bivector α :

$$(4.5) \quad \Lambda_\alpha : \bigwedge^3 V^\vee \longrightarrow V^\vee, \quad \tau \longmapsto \iota_\alpha \tau.$$

We give the fiberwise linear-algebra argument needed to prove that the kernel of (4.5) generates $\mathcal{I}_\alpha(1)$.

Lemma 4.1. *Let $\alpha \in \bigwedge^2 V$ be non-degenerate and let $0 \neq v \in V$. Then contraction induces a surjection*

$$\Lambda_\alpha : \bigwedge^3 \text{Ann}(v) \longrightarrow \text{Ann}(v),$$

where $\text{Ann}(v) = \{\lambda \in V^\vee : \lambda(v) = 0\}$.

Proof. The inverse of $\alpha^\sharp : V^\vee \rightarrow V$ defines a non-degenerate alternating form on V . Choose f_0 pairing nontrivially with v , rescale it so that the pairing is one, and apply the symplectic Gram–Schmidt procedure to the orthogonal complement of $\langle v, f_0 \rangle$. We thereby obtain a Darboux basis

$$e_0, f_0, e_1, f_1, e_2, f_2$$

for V such that $v = e_0$ and

$$\alpha = e_0 \wedge f_0 + e_1 \wedge f_1 + e_2 \wedge f_2.$$

Write $e_i^\vee, f_i^\vee$ for the dual basis. Then

$$\text{Ann}(e_0) = \langle f_0^\vee, e_1^\vee, f_1^\vee, e_2^\vee, f_2^\vee \rangle.$$

Up to irrelevant signs, contraction by α maps

$$\begin{aligned} f_0^\vee \wedge e_1^\vee \wedge f_1^\vee &\longmapsto f_0^\vee, \\ e_1^\vee \wedge e_2^\vee \wedge f_2^\vee &\longmapsto e_1^\vee, \\ f_1^\vee \wedge e_2^\vee \wedge f_2^\vee &\longmapsto f_1^\vee, \\ e_2^\vee \wedge e_1^\vee \wedge f_1^\vee &\longmapsto e_2^\vee, \\ f_2^\vee \wedge e_1^\vee \wedge f_1^\vee &\longmapsto f_2^\vee. \end{aligned}$$

These five images span $\text{Ann}(e_0)$. □

Proposition 4.2. *The bundle $\mathcal{I}_\alpha(1)$ is globally generated and*

$$H^0(X, \mathcal{I}_\alpha(1)) \simeq \ker(\Lambda_\alpha : \bigwedge^3 V^\vee \longrightarrow V^\vee).$$

This vector space has dimension fourteen.

Proof. The map $\Lambda_\alpha : \bigwedge^3 V^\vee \rightarrow V^\vee$ is surjective. Indeed, in a Darboux basis each basis covector is obtained by contracting its exterior product with the two covectors belonging to one of the other symplectic pairs. Thus its kernel has dimension $20 - 6 = 14$. The exact sequence (4.3), together with (4.4), identifies this kernel with $H^0(\mathcal{I}_\alpha(1))$.

We now prove that these sections generate every fiber. Fix $x = [\ell] \in X$ and choose $0 \neq v \in \ell$. Evaluation of $\tau \in \bigwedge^3 V^\vee$ at x is represented by

$$\iota_v \tau \in \bigwedge^2 (V/\ell)^\vee;$$

the omitted factor in $\ell^\vee$ only records the choice of v . Consequently, the fiber of $\mathcal{I}_\alpha(1)$ consists of the $\eta \in \bigwedge^2 (V/\ell)^\vee$ satisfying

$$(4.6) \quad \langle \bar{\alpha}_\ell, \eta \rangle = 0.$$

Let η satisfy (4.6). Choose $\tau_0 \in \bigwedge^3 V^\vee$ such that $\iota_v \tau_0 = \eta$. Such a lift exists after choosing a complement of ℓ in V . Put $\lambda = \Lambda_\alpha(\tau_0)$. Compatibility of the two contractions gives

$$\lambda(v) = \langle \bar{\alpha}_\ell, \iota_v \tau_0 \rangle = \langle \bar{\alpha}_\ell, \eta \rangle = 0,$$

so $\lambda \in \text{Ann}(v)$. By Lemma 4.1, there is $\sigma \in \bigwedge^3 \text{Ann}(v)$ with $\Lambda_\alpha(\sigma) = -\lambda$. Since $\iota_v \sigma = 0$, the form

$$\tau = \tau_0 + \sigma$$

satisfies

$$\Lambda_\alpha(\tau) = 0, \quad \iota_v \tau = \eta.$$

Thus every element of the fiber is the value of a global section in $\ker \Lambda_\alpha$, proving global generation. $\square$

Corollary 4.3. *There is a morphism*

$$\Phi_\alpha : \mathcal{O}_X(-1)^{14} \longrightarrow \mathcal{O}_X^{14}$$

represented by a 14×14 matrix of linear forms, such that Φ_α has rank nine at every point and $\text{im } \Phi_\alpha \simeq \mathcal{I}_\alpha$.

Proof. The evaluation morphism of $\mathcal{I}_\alpha(1)$ and its twist by $\mathcal{O}_X(-1)$ give a surjection

$$H^0(\mathcal{I}_\alpha(1)) \otimes \mathcal{O}_X(-1) \twoheadrightarrow \mathcal{I}_\alpha.$$

The evaluation morphism of $\mathcal{I}_\alpha^\vee$, after dualizing, gives an injection

$$\mathcal{I}_\alpha \hookrightarrow H^0(\mathcal{I}_\alpha^\vee)^\vee \otimes \mathcal{O}_X.$$

Their composite is Φ_α . Both vector spaces of sections have dimension fourteen by (4.2) and Proposition 4.2. The first arrow is fiberwise surjective and the second is fiberwise injective, so the image is $\mathcal{I}_\alpha$ and the rank is nine at every point. $\square$

5. COHOMOLOGY AND NON-HOMOGENEITY

Set

$$\mathcal{F}_\alpha = \mathcal{I}_\alpha^\vee, \quad \mathcal{C} = \bigwedge^2 \mathcal{Q} \simeq \Omega_X^3(4).$$

Then (4.1) becomes

$$(5.1) \quad 0 \longrightarrow \mathcal{O}_X \xrightarrow{s_\alpha} \mathcal{C} \longrightarrow \mathcal{F}_\alpha \longrightarrow 0.$$

We will show that the isomorphism class of $\mathcal{F}_\alpha$, and hence that of $\mathcal{I}_\alpha$, determines the projective class $[\alpha]$.

Lemma 5.1. *One has*

$$H^0(X, \Omega_X^2(2)) = H^1(X, \Omega_X^2(2)) = 0.$$

Consequently,

$$H^0(X, \mathcal{I}_\alpha) = 0, \quad H^1(X, \mathcal{I}_\alpha) \simeq \mathbb{C}.$$

Proof. The exterior Euler sequence and the Euler sequence twisted by one are

$$(5.2) \quad 0 \longrightarrow \Omega_X^2(2) \longrightarrow \bigwedge^2 V^\vee \otimes \mathcal{O}_X \longrightarrow \Omega_X^1(2) \longrightarrow 0,$$

$$(5.3) \quad 0 \longrightarrow \Omega_X^1(2) \longrightarrow V^\vee \otimes \mathcal{O}_X(1) \longrightarrow \mathcal{O}_X(2) \longrightarrow 0.$$

On global sections, the last map in (5.3) is the multiplication map

$$V^\vee \otimes V^\vee \longrightarrow S^2 V^\vee.$$

Its kernel is $\bigwedge^2 V^\vee$. Hence $H^0(\Omega_X^1(2)) \simeq \bigwedge^2 V^\vee$, and $H^1(\Omega_X^1(2)) = 0$. The map on global sections in (5.2) is the natural nonzero $\mathrm{GL}(V)$ -equivariant endomorphism of $\bigwedge^2 V^\vee$, hence an isomorphism. The long exact sequence of (5.2) now gives

$$H^0(\Omega_X^2(2)) = H^1(\Omega_X^2(2)) = 0.$$

Apply cohomology to (2.2). The segment

$$0 \longrightarrow H^0(\mathcal{I}_\alpha) \longrightarrow 0 \longrightarrow \mathbb{C} \longrightarrow H^1(\mathcal{I}_\alpha) \longrightarrow 0$$

gives the remaining assertions. $\square$

Lemma 5.2. *The bundle $\mathcal{C} = \Omega_X^3(4)$ is simple:*

$$\mathrm{Hom}_X(\mathcal{C}, \mathcal{C}) = \mathbb{C}.$$

Proof. The cotangent bundle Ω_X^1 is slope-stable with respect to $\mathcal{O}_X(1)$, and each exterior power Ω_X^p is slope-stable on projective space; see [10, Chapter II, Section 1]. Twisting preserves slope stability, so $\Omega_X^3(4)$ is stable. A stable bundle over $\mathbb{C}$ is simple [10, Chapter II, Section 1], as follows. The coefficients of the characteristic polynomial of an endomorphism are global regular functions and hence constants. Choose an eigenvalue λ . If $\varphi - \lambda \mathrm{id}$ were nonzero, its kernel and image would be nonzero proper torsion-free subsheaves. Stability would make both of their slopes strictly smaller than the slope of the original bundle. This is impossible because the slope of the middle term in the exact sequence

$$0 \longrightarrow \ker(\varphi - \lambda \mathrm{id}) \longrightarrow \mathcal{C} \longrightarrow \mathrm{im}(\varphi - \lambda \mathrm{id}) \longrightarrow 0$$

is the rank-weighted average of the other two slopes. Hence every endomorphism is scalar, and therefore $\mathrm{Hom}_X(\mathcal{C}, \mathcal{C}) = \mathbb{C}$. $\square$

Proposition 5.3. *Let $\alpha, \beta \in \bigwedge^2 V$ be non-degenerate. Then*

$$\mathcal{I}_\alpha \simeq \mathcal{I}_\beta \iff [\alpha] = [\beta] \text{ in } \mathbb{P}(\bigwedge^2 V).$$

Proof. If β is a nonzero scalar multiple of α , the two contraction morphisms have the same kernel.

Conversely, suppose $\mathcal{I}_\alpha \simeq \mathcal{I}_\beta$, equivalently $\mathcal{F}_\alpha \simeq \mathcal{F}_\beta$. By Lemma 5.1,

$$\mathrm{Ext}_X^1(\mathcal{F}_\alpha, \mathcal{O}_X) \simeq H^1(X, \mathcal{F}_\alpha^\vee) = H^1(X, \mathcal{I}_\alpha) \simeq \mathbb{C},$$

and similarly for β . The extension (5.1) is non-split: otherwise projection onto its $\mathcal{O}_X$ summand would give a nontrivial idempotent endomorphism of the simple bundle $\mathcal{C}$. Thus it represents a nonzero generator of this one-dimensional extension group.

An isomorphism $f : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ induces an isomorphism of the two extension groups. Pulling back the nonzero extension for β gives a nonzero scalar multiple of the extension for α . Rescaling the left copy of $\mathcal{O}_X$, equivalence of extensions produces a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{s_\alpha} & \mathcal{C} & \longrightarrow & \mathcal{F}_\alpha \longrightarrow 0 \\ & & \downarrow \lambda & & \downarrow u & & \downarrow f \\ 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{s_\beta} & \mathcal{C} & \longrightarrow & \mathcal{F}_\beta \longrightarrow 0, \end{array}$$

where $\lambda \neq 0$ and u is an automorphism. By Lemma 5.2, u is scalar. The left square therefore says that the sections s_α and s_β span the same line in $H^0(\mathcal{C}) = \bigwedge^2 V$. Hence $[\alpha] = [\beta]$. $\square$

Theorem 5.4. *The bundle $\mathcal{I}_\alpha$ is not homogeneous under $\mathrm{PGL}(V)$.*

Proof. For $g \in \mathrm{PGL}(V)$, functoriality of the universal quotient sequence gives, up to the harmless choice of inverse in the action convention,

$$g^* \mathcal{I}_\alpha \simeq \mathcal{I}_{g\alpha}.$$

If $\mathcal{I}_\alpha$ were homogeneous, Proposition 5.3 would imply

$$[g\alpha] = [\alpha] \quad \text{for every } g \in \mathrm{PGL}(V).$$

This is impossible. For example, choose a Darboux basis in which

$$\alpha = e_0 \wedge f_0 + e_1 \wedge f_1 + e_2 \wedge f_2.$$

The projective transformation induced by the linear map that sends e_0 to $2e_0$ and fixes the other five basis vectors sends α to

$$2e_0 \wedge f_0 + e_1 \wedge f_1 + e_2 \wedge f_2,$$

which is not proportional to α . Hence $g^* \mathcal{I}_\alpha \not\simeq \mathcal{I}_\alpha$ for this g . $\square$

6. CHERN CLASSES AND FURTHER REMARKS

Let $A^*(X)$ denote the Chow ring of X , and set $H = c_1(\mathcal{O}_X(1))$. Thus $A^*(X) \simeq \mathbb{Z}[H]/(H^6)$. Let $K_0(X)$ denote the Grothendieck group of algebraic vector bundles on X . The Euler–Koszul sequences give, in $K_0(X)$,

$$[\Omega_X^1(2)] = 6[\mathcal{O}_X(1)] - [\mathcal{O}_X(2)]$$

and, from (5.2),

$$[\Omega_X^2(2)] = 15[\mathcal{O}_X] - 6[\mathcal{O}_X(1)] + [\mathcal{O}_X(2)].$$

Using (2.2),

$$(6.1) \quad [\mathcal{I}_\alpha] = 14[\mathcal{O}_X] - 6[\mathcal{O}_X(1)] + [\mathcal{O}_X(2)].$$

Therefore

$$(6.2) \quad \mathrm{ch}(\mathcal{I}_\alpha) = 14 - 6e^H + e^{2H} = 9 - 4H - H^2 + \frac{1}{3}H^3 + \frac{5}{12}H^4 + \frac{13}{60}H^5.$$

The same K -class gives the total Chern class directly:

$$(6.3) \quad c(\mathcal{I}_\alpha) = \frac{c(\mathcal{O}_X(2))}{c(\mathcal{O}_X(1))^6} = \frac{1 + 2H}{(1 + H)^6} = 1 - 4H + 9H^2 - 14H^3 + 14H^4.$$

Consequently,

$$c_1(\mathcal{I}_\alpha) = -4H, \quad c_2(\mathcal{I}_\alpha) = 9H^2, \quad c_3(\mathcal{I}_\alpha) = -14H^3, \quad c_4(\mathcal{I}_\alpha) = 14H^4, \\ c_5(\mathcal{I}_\alpha) = c_6(\mathcal{I}_\alpha) = \cdots = c_9(\mathcal{I}_\alpha) = 0.$$

Here $c_5 = 0$ is the degree-five coefficient of (6.3), while $c_i = 0$ for $i > 5$ because $A^i(X) = 0$ in those degrees.

For comparison, the standard homogeneous bundles with splitting type $(0^5, -1^4)$ have second Chern class $6H^2$ or $7H^2$. Indeed,

$$\mathcal{O}_X(-1)^4 \oplus \mathcal{O}_X^5$$

has $c_2 = 6H^2$, while

$$\mathcal{Q}^\vee \oplus \mathcal{O}_X(-1)^3 \oplus \mathcal{O}_X, \quad \mathcal{Q}(-1) \oplus \mathcal{O}_X^4$$

both have $c_2 = 7H^2$. Thus the value $c_2 = 9H^2$ provides a useful independent check that $\mathcal{I}_\alpha$ is not one of the expected homogeneous models.

Remark 6.1 (The parameter space). The non-degenerate bivectors form the open set

$$\mathcal{U} = \{[\alpha] \in \mathbb{P}(\bigwedge^2 V) : \alpha^{\wedge 3} \neq 0\},$$

the complement of the Pfaffian cubic hypersurface. By Proposition 5.3, the assignment $[\alpha] \mapsto [\mathcal{I}_\alpha]$ is injective on isomorphism classes over the fixed projective space X . The group $\mathrm{PGL}(V)$ is transitive on $\mathcal{U}$, so all members become equivalent after a projective change of coordinates, but an individual member is not fixed by the whole automorphism group.

Remark 6.2 (Why dimension five is exceptional). On $\mathbb{P}^n$, the same one-bivector construction has rank

$$\binom{n}{2} - 1.$$

It equals $2n - 1$ only for $n = 5$. More generally, quotienting $\bigwedge^2 \mathcal{Q}$ by a space $W \subset \bigwedge^2 V$ preserves the two-step restriction type only if $\mathbb{P}(W)$ avoids the Pfaffian locus of bivectors of rank at most four. That locus has codimension

$$\binom{n-3}{2}$$

in $\mathbb{P}(\bigwedge^2 V)$. The dimension required to reduce the rank to $2n - 1$ exceeds this codimension by $n - 5$, explaining the special role of $\mathbb{P}^5$ in the present construction.

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