MINICOURSE: QUANTUM SYMMETRIC PAIRS AND APPLICATIONS

WEIQIANG WANG

Outline

Since its introduction by Drinfeld and Jimbo in 1980s, the theory of quantum groups has found many developments and applications, including R-matrices, canonical bases, q-Schur duality, connections to Hall algebras, geometry of type A flag varieties and quiver varieties, modular representations, Kazhdan-Lusztig theory of (super) type A, and categorification.

Quantum symmetric pairs $(\mathbf{U}, \mathbf{U}^\imath)$ are quantizations of the symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\imath)$, where $\imath$ is an involution on a simple Lie algebra $\mathfrak{g}$. The symmetric pairs are in bijection with the real simple Lie algebras. In recent years, it has become increasingly clear (thanks to Huanchen Bao and others) that the quantum symmetric pairs provide a right setting for natural generalizations of the aforementioned constructions and applications of quantum groups.

Here is a detailed description of the mini-course:

(1) We review the quantum group associated to a simple Lie algebra $\mathfrak{g}$, with an emphasis on the type A case. We explain the q-Schur duality between $\mathbf{U}$ and Hecke algebra for the symmetric group.

(1-i) We introduce an example of quantum symmetric pair of type AIII, $(\mathbf{U}, \mathbf{U}^\imath)$, where $\mathbf{U} = \mathbf{U}_q(\mathfrak{sl}_N)$. We shall explain an $\imath$-Schur duality between $\mathbf{U}^\imath$ and Hecke algebra of type B (of equal or unequal parameter).

(2) We review the canonical bases on half a quantum group and its finite-dimensional simple modules. We introduce the quasi-R matrix of Lusztig, and use it to define canonical bases of tensor product $\mathbf{U}$-modules.

(2-i) We introduce a bar involution on $\mathbf{U}^\imath$ and a generalization of quasi-R matrix for $(\mathbf{U}, \mathbf{U}^\imath)$. We establish a new canonical basis (called $\imath$-canonical basis) for tensor product $\mathbf{U}$-modules when viewed as $\mathbf{U}^\imath$-modules.

(3) We explain how to view the BGG category $\mathcal{O}$ of Lie algebra of type A (and its super analogue) as a categorification of the Schur duality, and how the Kazhdan-Lusztig theory for $\mathcal{O}$ can be reformulated in terms of canonical basis.

(3-i) We explain how the BGG category $\mathcal{O}^\imath$ of Lie algebras of type B/C/D (and its super analogue) as a categorification of the $\imath$-Schur duality, and how the Kazhdan-Lusztig theory for $\mathcal{O}^\imath$ can be reformulated in terms of $\imath$-canonical basis.

(4) Time permitting, we review the construction of canonical basis on the modified quantum group $\dot{\mathbf{U}}$, and explain the construction of $\imath$-canonical basis

on the modified form of $\mathbf{U}^{\imath}$. Alternatively, we may discuss some geometric constructions.