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A classification of finite simple amenable $\mathcal{Z}$-stable C*-algebras, II: C*-algebras with rational generalized tracial rank one

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Abstract. A classification theorem is obtained for a class of unital simple separable amenable $\mathcal{Z}$-stable C^*-algebras which exhausts all possible values of the Elliott invariant for unital stably finite simple separable amenable $\mathcal{Z}$-stable C^*-algebras. Moreover, it contains all unital simple separable amenable C^*-algebras which satisfy the UCT and have finite rational tracial rank.

Résumé. Dans cet article et le précédent on donne une classification complète, au moyen de l'invariant d'Elliott, d'une sous-classe de la classe des C*-algèbres simples, moyennables, séparables, à élément unité, absorbant l'algèbre de Jiang-Su, et satisfaisant au UCT, qui épuise l'ensemble des valeurs possibles de l'invariant pour cette class. La partie I réalise une grande partie de ce projet, et la partie II l'achève.

This is the second part of the paper entitled "A classification of finite simple amenable $\mathcal{Z}$-stable C*-algebras" (see [21]).

The main theorem of this part is the following isomorphism theorem:

Theorem (see Theorem 29.8). Let A and B be two unital separable simple amenable $\mathcal{Z}$-stable C^*-algebras which satisfy the UCT. Suppose that $gTR(A \otimes Q) \leq 1$ and $gTR(B \otimes Q) \leq 1$. Then $A \cong B$ if and only if

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

See Section 29 for a brief explanation. We also refer to the first part [21], in particular, Section 2 of [21], for the notations and definitions.

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22. Construction of Maps In this section, we will introduce some technical results on the existence of certain maps.

Recall that $\mathcal{C}$ is the class of C^*-algebras which are 1-dimensional NCCWs (see 3.1 of [21]). Let A be a unital simple C^*-algebra. We say $A \in \mathcal{B}_1$ if the following property holds: Let $\varepsilon > 0$, let $a \in A_+ \setminus \{0\}$, and let $\mathcal{F} \subset A$ be a finite subset. There exist a non-zero projection $p \in A$ and a C^*-subalgebra $C \in \mathcal{C}$ with $1_C = p$ such that

$$\|xp - px\| < \varepsilon \text{ for all } x \in \mathcal{F},$$

$$\operatorname{dist}(pxp, C) < \varepsilon \text{ for all } x \in \mathcal{F}, \text{ and} \tag{e 22.1}$$

$$1 - p \lesssim a. \tag{e 22.2}$$

If C as above can always be chosen in $\mathcal{C}_0$, that is, with $K_1(C) = \{0\}$, then we say that $A \in \mathcal{B}_0$.

Recall that we refer to the first part [21], in particular, Section 2 of [21], for the notations and definitions.

LEMMA 22.1. *Let X be a finite CW complex, let $C = PM_k(C(X))P$, and let $A_1 \in \mathcal{B}_0$ be a unital simple C^*-algebra. Assume that $A = A_1 \otimes U$ for a UHF-algebra U of infinite type. Let $\alpha \in KK_e(C, A)^{++}$ (see Definition 2.10 of [21]). Then there exists a unital monomorphism $\varphi : C \to A$ such that $[\varphi] = \alpha$. Moreover we may write $\varphi = \varphi_n' \oplus \varphi_n''$, where $\varphi_n' : C \to (1 - p_n)A(1 - p_n)$ is a unital monomorphism, $\varphi_n'' : C \to p_nAp_n$ is a unital homomorphism with $[\varphi_n''] = [\Phi]$ in $KK(C, p_nAp_n)$ for some homomorphism Φ with finite dimensional range, and*

$$\lim_{n\to\infty} \max\{\tau(1 - p_n) : \tau \in T(A)\} = 0 \text{ for all } \tau \in T(A),$$

where $p_n \in A$ is a sequence of projections.

PROOF. To simplify the matter, we may assume that X is connected. Suppose that the lemma holds for the case $C = M_k(C(X))$ for some integer $k \geq 1$. Consider the case $C = PM_k(C(X))P$. Note $C \otimes \mathcal{K} \cong C(X) \otimes \mathcal{K}$. Let $q \in M_m(A)$ be a projection (for some integer $m \geq 1$) such that $[q] = \alpha([1_{M_k(C(X))}])$. Put

$A_2 = qM_m(A)q$. Then $\alpha \in KK_e(M_k(C(X)), A_2)^{++}$. Let $\psi : M_k(C(X)) \to qM_m(A)q$ be the map given by the lemma for the case that $C = M_k(C(X))$. Note now $C = PM_k(C(X))P$. Let $\psi' = \psi|_C$. Since $P \le 1_{M_k(C(X))}$, $\psi'(1_C) = \psi'(P) \le q$. Moreover, $[\psi'(1_C)] = [1_A]$. Since $A = A_1 \otimes U$, there is a unitary $v \in A_2$ such that $v^*\psi'(1_C)v = 1_A$. Define $\varphi = \mathrm{Ad}v \circ \psi'$. We see that the general case reduces to the case $C = M_k(C(X))$. This case then reduces to the case $C = C(X)$.

Since quasitraces of C and A are traces (see 9.10 of [21]) by Corollary 3.4 of [3], $\alpha(\ker\rho_C) \subset \ker\rho_A$.

Since $K_i(C)$ is finitely generated, $i = 0, 1$, $KK(C, A) = KL(C, A)$. Let $\alpha \in KL_e(C, A)^{++}$. We may identify α with an element in $\mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(A))$ by a result in ([6]).

Write $A = \lim_{n\to\infty}(A_1 \otimes M_{r_n}, \imath_{n,n+1})$, where $r_n | r_{n+1}$, $r_{n+1} = m_n r_n$ and $\imath_{n,n+1}(a) = a \otimes 1_{M_{m_n}}$, $n = 1, 2, ...$ Since $K_*(C)$ is finitely generated and consequently, $\underline{K}(C)$ is finitely generated modulo Bockstein Λ operations, there is an element $\alpha_1 \in KK(C, A_1 \otimes M_{r_n})$ such that $\alpha = \alpha_1 \times [\imath_n]$, where $[\imath_n] \in KK(A_1 \otimes M_{r_n}, A)$ is induced by the inclusion $\imath_n : A_1 \otimes M_{r_n} \to A$. Increasing n, we may assume that $\alpha_1(\ker\rho_C) \subset \ker\rho_{A_1\otimes M_{r_n}}$ and further that $\alpha_1 \in KK_e(C, A_1 \otimes M_{r_n})^{++}$. Replacing A_1 by $A_1 \otimes M_{r_n}$, we may assume that $\alpha = \alpha_1 \times [\imath]$, where $\alpha_1 \in KK_e(C, A_1)^{++}$ and $\imath : A_1 \to A$ is the inclusion.

It induces an element $\tilde{\alpha}_1 \in KL(C \otimes U, A \otimes U)$. Let $K_0(U) = \mathbb{D}$, a dense subgroup of $\mathbb{Q}$. Note that $K_i(C \otimes U) = K_i(C) \otimes \mathbb{D}$, $i = 0, 1$, by the Künneth formula.

We verify that $\tilde{\alpha}_1(K_0(C \otimes U)_+ \setminus \{0\}) \subset K_0(A \otimes U)_+ \setminus \{0\}$. Consider $x = \sum_{i=1}^m x_i \otimes d_i \in K_0(C \otimes U)_+ \setminus \{0\}$ with $x_i \in K_0(C)$ and $d_i \in \mathbb{D}$, $i = 1, 2, ..., m$. There is a projection $p \in M_r(C)$ for some $r \ge 1$ such that $[p] = x$. Let $t \in T(C)$; then

$$\sum_{i=1}^m t(x_i) d_i > 0. \tag{e 22.3}$$

It should be noted that, since $C = C(X)$ and X is connected, $t(x_i) \in \mathbb{Z}$ and $t(x_i) = t'(x_i)$ for all $t, t' \in T(C)$. Since $\alpha_1([1_C]) = [1_{A_1}]$, $\tau \circ \alpha_1(x_i) = t(x_i)$ for any $\tau \in T(A_1)$ and $t \in T(C)$. By (e 22.3),

$$\tau(\tilde{\alpha}_1(x)) = \sum_{i=1}^m \tau \circ \alpha_1(x_i) d_i = \sum_{i=1}^m t(x_i) d_i > 0 \tag{e 22.4}$$

for all $\tau \in T(A_1)$. This shows that $\tilde{\alpha}_1$ is strictly positive. For any C^*-algebra A', in this proof, we will use $j_{A'} : A' \to A' \otimes U$ for the homomorphism $j_{A'}(a) = a \otimes 1_U$ for all $a \in A'$. Evidently,

$$\alpha = \tilde{\alpha}_1 \circ j_C = j_{A_1} \circ \alpha_1. \tag{e 22.5}$$

Let $\mathfrak{d} := p_1^{n_1} p_2^{n_2} \cdots$ be the supernatural number associated with U (and $\mathbb{D}$), where each p_i is a distinct prime number. If there are infinitely many of them, we may also write $\mathfrak{d} := \prod_{i=1}^{\infty} l_i$, where each l_i is an integer. We define $m_1 := l_1 p_{k_1}$ so that the prime number is not a factor of l_1, and $m_i := l_i p_{k_i}$ so that p_{k_i} is not a factor of l_i and $m_1 m_2 \cdots m_{i-1}$. Since $p_{k_i} \to \infty$ as $i \to \infty$, $\lim_{i\to\infty} \frac{m_i}{l_i} = \infty$. Moreover $\prod_{i=1}^{\infty} m_i = \mathfrak{d}$. If there are only finitely many distinct p_i's, write $\mathfrak{d} = p_1^{n_1} p_2^{n_2} \cdots p_f^{n_f} p_{k_1}^{\infty} \cdots p_{k_I}^{\infty}$, where $n_i < \infty$ $(1 \le i \le f)$. Let $l_1 := p_1^{n_1} p_2^{n_2} \cdots p_f^{n_f}$, $m_1 := l_1 p_{k_1} \cdots p_{k_I}$, $l_i := p_{k_1} p_{k_2} \cdots p_{k_I}$ and $m_i := p_{k_1}^i p_{k_2}^i \cdots p_{k_l}^i$ for $i \ge 2$. Then $\mathfrak{d} = \prod_{i=1}^{\infty} l_i = \prod_{i=1}^{\infty} m_i$ and $\lim_{i\to\infty} \frac{m_i}{l_i} = \infty$.

Write $U = \lim_{n\to\infty}(M_{r_n}, \imath_n)$, where $r_1 = 1$, and $r_n = \prod_{i=1}^{n-1} m_i$ $(n > 1)$, $r_{n+1} = m_n r_n$ and $\imath_n(a) = a \otimes 1_{M_{m_n}}$ for $a \in M_{r_n}$, $n = 1, 2, ...$ We may assume that $r_1 = 1$. Let $r_1' = 1$, $r_n' = \prod_{i=1}^{n-1} l_i$ $(n > 1)$. Let $U_1 := \lim_{n\to\infty}(M_{r_n'}, \imath_n')$, where $\imath_n'(b) = b \otimes M_{l_n}$ for $b \in M_{r_n'}$.

Let $\eta_n : M_{r_i'} \to M_{r_i}$ by $\eta_i(a) = a \otimes 1_{m_i/l_i}$ for $a \in M_{r_i'}$, respectively. Then $\{\eta_n\}$ induces a unital homomorphism $\eta : U_1 \to U$ which induces an isomorphism from $K_0(U_1)$ onto $K_0(U)$.

Recall that we assume that X is connected. Fix a base point $x_0 \in X$. Let $C_0 := C_0(X \setminus \{x_0\})$. Then C is KK-equivalent to $\mathbb{C} \oplus C_0$ and $\underline{K}(C) = \underline{K}(\mathbb{C}) \oplus \underline{K}(C_0)$. Let $\{x_n\}$ be a sequence of points in $X \setminus \{x_0\}$ such that $\{x_k, x_{k+1}, ..., x_n, ...\}$ is dense in X for each k and each point in $\{x_n\}$ repeated infinitely many times. Let $B = \lim_{n\to\infty}(C_n := M_{r_n}(C), \psi_n)$, where

$$\psi_n(f) = \mathrm{diag}(\underbrace{f, f ..., f}_{l_n}, f(x_1), f(x_2), ..., f(x_{m_n - l_n})) \text{ for all } f \in M_{r_n}(C),$$

$n = 1, 2, ...$ Note that ψ_n is injective. Set $e_n = \mathrm{diag}(1_{M_{r_n \cdot l_n}}, 0, ..., 0) \in M_{r_{n+1}}(C)$, $n = 1, 2, ...$

It is standard that B has tracial rank zero (see [23] and also, 3.77 and 3.79 of [31]). Moreover, $\underline{K}(B) = \underline{K}(U) \oplus \underline{K}(C_0 \otimes U_1)$ and $K_0(B)_+ = \{(d, z) : d \in \mathbb{D}_+, z \in K_0(C_0 \otimes U_1)\} \cup \{(0, 0)\}$. Note that B is a unital simple AH-algebra with no dimension growth, with real rank zero, and with a unique tracial state. Also $h := \psi_{1,\infty} : C \to B$ gives $[h]|_{\underline{K}(\mathbb{C})}(z) = z \otimes 1_{\mathbb{D}}$ for $z \in \underline{K}(\mathbb{C})$ and $[h]|_{\underline{K}(C_0)}(x) = x \otimes [1_{U_1}]$ for $x \in \underline{K}(C_0)$. Let $\eta_0 : C_0 \otimes U_1 \to C_0 \otimes U$ be defined by $\eta_0 := \mathrm{id}_{C_0} \otimes \eta$. Define $\kappa \in KL(B, B)$ by $\kappa|_{\underline{K}(\mathbb{C}\otimes U)} = \mathrm{id}_{\underline{K}(\mathbb{C}\otimes U)}$ and $\kappa|_{\underline{K}(C_0 \otimes U_1)} = \eta_0$. Note that $\kappa \in KL(B, B)^{++}$. Recall $\underline{K}(C \otimes U) = \underline{K}(\mathbb{C} \otimes U) \oplus \underline{K}(C_0 \otimes U)$. One may also view κ as an element in $\mathrm{Hom}_\Lambda(\underline{K}(\mathbb{C} \otimes U) \oplus \underline{K}(C_0 \otimes U_1), \underline{K}(C \otimes U)) = KL(B, C \otimes U)$. It has an inverse $\kappa^{-1} \in KL(C \otimes U, B)$. We have $\kappa^{-1} \circ [j_C] = [h]$.

Note that $1 - \psi_{n,\infty}(e_n)$ commutes with the image of h for all $n \ge N$. Moreover, $(1 - \psi_{n,\infty}(e_n))h(c)(1 - \psi_{n,\infty}(e_n)) = \psi_{n,\infty}((1 - e_n)\psi_{N,n} \circ \varphi_{1,N}(c)(1 - e_n))$ for all $c \in C$. Therefore the map $(1 - \psi_{n,\infty}(e_n))h(C)(1 - \psi_{n,\infty}(e_n))$ has finite dimensional range.

We also have $\tilde{\alpha}_1 \circ \kappa \in KL_e(B, A)^{++}$, where (recall) $A = A_1 \otimes U$. We also note that B has a unique tracial state. Let $\gamma : T(A) \to T(B)$ be defined by $\gamma(\tau) = t_0$ where $t_0 \in T(B)$ is the unique tracial state. It follows that $\tilde{\alpha}_1 \circ \kappa$ and γ are compatible. By Corollary 21.11 of [21], there is a unital homomorphism $H : B \to A$ such that $[H] = \tilde{\alpha}_1 \circ \kappa$. Define $\varphi : C \to A$ by $\varphi = H \circ h$. Then, φ is injective, and, by (e 22.5) and $[h] = \kappa^{-1} \circ [j_C]$, we have $[\varphi] = \alpha$.

To show the last part, define $q_n = \psi_{n+1,\infty}(e_n) \in B$, $n = N+1, N+2, ...$ Define $p_n = 1 - H(q_n)$, $n = N+1, N+2, ...$ One checks that

$$\lim_{n\to\infty} \max\{\tau(1-p_n) : \tau \in T(A)\} = \lim_{n\to\infty} \frac{l_n}{m_n} = 0. \tag{e 22.6}$$

Note that for $n > N$, q_n commutes with the image of h and the homomorphism $(1-q_n)h(1-q_n) : C \to (1-q_n)B(1-q_n)$ has finite dimensional range. Define $\varphi'_n : C \to (1-p_n)A(1-p_n)$ by $\varphi'_n(f) = H(q_n)H \circ h(f)H(q_n)$ for $f \in C$. Define $\varphi''_n(f) = (1-p_n)H \circ h(f)(1-p_n)$, which is a point-evaluation map. The lemma follows. □

We also have the following:

LEMMA 22.2. *Let $C = M_k(C(\mathbb{T}))$ and let A be a unital infinite dimensional simple C^*-algebra with stable rank one and with the property (SP). Then the conclusion of* 22.1 *also holds for a given $\alpha \in KK_e(C, A)^{++}$.*

PROOF. Let $p_1 \in C$ be a minimal rank one projection. Since $k\alpha([p_1]) = \alpha([1_C]) = [1_A]$, A contains mutually equivalent and mutually orthogonal projections $e_1, e_2, ..., e_k$ such that $\sum_{i=1}^k e_i = 1_A$. Thus $A = M_k(A')$, where $A' \cong e_1Ae_1$. Since e_iAe_i are unital infinite dimensional simple C^*-algebras with stable rank one and with (SP), the general case can be reduced to the case that $k = 1$. Fix $1 > \delta > 0$. Choose a non-zero projection $p \in A$ such that $\tau(p) < \delta$ for all $\tau \in T(A)$. Note $K_1(pAp) = K_1(A)$, since A is simple. Let $\alpha_1 : K_1(C(\mathbb{T})) \to K_1(pAp)$ be the homomorphism given by α. Let $z \in C(\mathbb{T})$ be the standard unitary generator. Let $x = \alpha_1([z]) \in K_1(pAp)$. Since pAp has stable rank one, there is a unitary $u \in pAp$ such that $[u] = x$ in $K_1(pAp) = K_1(A)$. Define $\varphi' : C(\mathbb{T}) \to pAp$ by $\varphi'(f) = f(u)$ for all $f \in C(\mathbb{T})$. Define $\varphi'' : C(\mathbb{T}) \to (1-p)A(1-p)$ by $\varphi''(f) = f(1)(1-p)$ for all $f \in C(\mathbb{T})$ (where $f(1)$ is the point evaluation at 1 on the unit circle). Define $\varphi = \varphi' \oplus \varphi'' : C(\mathbb{T}) \to A$. The map φ verifies the conclusion of lemma follows. □

COROLLARY 22.3. *Let X be a connected finite CW complex, let $C = PM_m(C(X))P$, where $P \in M_m(C(X))$ is a projection, let $A_1 \in \mathcal{B}_0$ be a unital separable simple C^*-algebra which satisfies the UCT, and let $A = A_1 \otimes U$, where U is a UHF-algebra of infinite type. Suppose that $\alpha \in KK_e(C, A)^{++}$ and $\gamma : T(A) \to T_f(C(X))$ is a continuous affine map. Then there exists a sequence of contractive completely positive linear maps $h_n : C \to A$ such that*

(1) $\lim_{n\to\infty} \|h_n(ab) - h_n(a)h_n(b)\| = 0$, *for any $a, b \in C$,*
(2) *for each h_n, the map $[h_n]$ is well defined and $[h_n] = \alpha$, and*
(3) $\lim_{n\to\infty} \max\{|\tau \circ h_n(f) - \gamma(\tau)(f)| : \tau \in T(A)\} = 0$ *for any $f \in C$.*

PROOF. By Theorem 21.10 of [21], one may assume that A is a unital C^*-algebra as described in Theorem 14.10 of [21]. It follows from Lemma 22.1 that there is a unital homomorphism $h_n : C \to A$ such that $[h_n] = \alpha$. Moreover,

$$h_n = h'_n \oplus h''_n,$$

where $h_n'' : C \to p_nAp_n$ is a homomorphism with $[h_n''] = [\Phi^{'}]$ in $KK(C, p_nAp_n)$ for some point evaluation map $\Phi^{'}$, where p_n is a projection in A with $\tau(1-p_n)$ converging to 0 uniformly as $n \to \infty$. We will modify the map $h_n = h_n' \oplus h_n''$ to get the homomorphism.

We assert that for any finite subset $\mathcal{H} \subset C_{s.a}$, and $\epsilon > 0$, and any sufficiently large n, there is a unital homomorphism $\tilde{h}_n : C \to p_nAp_n$ such that $[\tilde{h}_n] = [\Phi]$ in $KK(C, p_nAp_n)$ for a homomorphism Φ with finite dimensional range, and

$$|\tau \circ \tilde{h}_n(f) - \gamma(\tau)(f)| < \epsilon \text{ for all } \tau \in T(A)$$

for all $f \in \mathcal{H}$. The corollary then follows by replacing the map h_n'' by the map $\tilde{h}_n$—of course, we use the fact that $\lim_{n\to\infty} \tau(1-p_n) = 0$.

Let $\mathcal{H}_{1,1}$ (in place of $\mathcal{H}_{1,1}$) be the finite subset of Lemma 17.1 of [21] with respect to $\mathcal{H}$ (in place of $\mathcal{H}$), $\epsilon/8$ (in place of σ), and C (in place of C). Since $\gamma(T(A)) \subset T_f(C(X))$, there is $\sigma_{1,1} > 0$ such that

$$\gamma(\tau)(h) > \sigma_{1,1} \text{ for all } h \in \mathcal{H}_{1,1} \text{ for all } \tau \in T(A).$$

Let $\mathcal{H}_{1,2} \subset C^+$ (in place of $\mathcal{H}_{1,2}$) be the finite subset of Lemma 17.1 of [21] with respect to $\sigma_{1,1}$. Since $\gamma(T(A)) \subset T_f(C(X))$, there is $\sigma_{1,2} > 0$ such that

$$\gamma(\tau)(h) > \sigma_{1,2} \text{ for all } h \in \mathcal{H}_{1,2} \text{ for all } \tau \in T(A).$$

Let M be the constant of Lemma 17.1 of [21] with respect to $\sigma_{1,2}$. By Lemma 16.12 of [21] (also see the proof of Lemma 16.12 of [21]) for sufficiently large n, there are a C*-subalgebra $D \subset p_nAp_n \subset A$ such that $D \in \mathcal{C}_0$, and a continuous affine map $\gamma' : T(D) \to T(C)$ such that

$$|\gamma'(\frac{1}{\tau(p)}\tau|_D)(f) - \gamma(\tau)(f)| < \epsilon/4 \text{ for all } \tau \in T(A) \text{ for all } f \in \mathcal{H},$$

where $p = 1_D$, $\tau(1-p) < \epsilon/(4+\epsilon)$, and further (see part (2) of Lemma 16.12 of [21])

(e 22.7) $$\gamma'(\tau)(h) > \sigma_{1,1} \text{ for all } \tau \in T(D) \text{ for all } h \in \mathcal{H}_{1,1}, \text{ and}$$

(e 22.8) $$\gamma'(\tau)(h) > \sigma_{1,2} \text{ for all } \tau \in T(D) \text{ for all } h \in \mathcal{H}_{1,2}.$$

Since A is simple and not elementary, one may assume that the dimensions of the irreducible representations of D are at least M. Thus, by Lemma 17.1 of [21], there is a homomorphism $\varphi : C \to D$ such that $[\varphi] = [\Phi]$ in $KK(C, D)$ for a point evaluation map Φ, and that

$$|\tau \circ \varphi(f) - \gamma'(\tau)(f)| < \epsilon/4 \text{ for all } f \in \mathcal{H} \text{ for all } \tau \in T(D).$$

Pick a point $x \in X$, and define $\tilde{h} : C \to p_nAp_n$ by

$$f \mapsto f(x)(p_n - p) \oplus \varphi(f) \text{ for all } f \in C.$$

Then a calculation as in the proof of Theorem 17.3 of [21] shows that the homomorphism $h_n' \oplus \tilde{h}$ verifies the assertion. □

COROLLARY 22.4. *Let $C \in \mathbf{H}$ (see Definition 14.5 of [21]) and let $A_1 \in \mathcal{B}_0$ be a unital separable simple C^*-algebra which satisfies the UCT and let $A = A_1 \otimes U$ for some UHF-algebra U of infinite type. Suppose that $\alpha \in KK_e(C, A)^{++}$, $\lambda : U(C)/CU(C) \to U(A)/CU(A)$ is a continuous homomorphism, and $\gamma : T(A) \to T_f(C)$ is a continuous affine map such that α, λ, and γ are compatible. Then there exists a sequence of unital completely positive linear maps $h_n : C \to A$ such that*

(1) $\lim_{n\to\infty} \|h_n(ab) - h_n(a)h_n(b)\| = 0$ *for any* $a, b \in C$,
(2) *for each* h_n, *the map* $[h_n]$ *is well defined and* $[h_n] = \alpha$,
(3) $\lim_{n\to\infty} \max\{|\tau \circ h_n(f) - \gamma(\tau)(f)| : \tau \in T(A)\} = 0$ *for all* $f \in C$, *and*
(4) $\lim_{n\to\infty} \mathrm{dist}(h_n^{\ddagger}(\bar{u}), \lambda(\bar{u})) = 0$ *for any* $u \in U(C)$.

PROOF. Let $\epsilon > 0$. Let $\mathcal{U}$ be a finite subset of $U(C)$ such that $\overline{U}$ generates $J_c(K_1(C))$, where $J_c(K_1(C))$ is as in Definition 2.16 of [21]. Let $\sigma > 0$, $\delta > 0$ and $\mathcal{G}$ be the constant and finite subset of Lemma 21.5 of [21] with respect to $\mathcal{U}$, ϵ, and λ (in the place of α). Without loss of generality, one may assume that $\delta < \epsilon$.

Let $\mathcal{F}$ be a finite subset such that $\mathcal{F} \supset \mathcal{G}$. Let $\mathcal{H} \subset C$ be a finite subset of self-adjoint elements with norm at most one. By Corollary 22.3, there is a completely positive linear map $h' : C \to A$ such that h' is $\mathcal{F}$-δ-multiplicative, $[h']$ is well defined and $[h'] = \alpha$, and

$$|\tau(h'(f)) - \gamma(\tau)(f)| < \epsilon, \quad \tau \in \mathrm{T}(A),\ f \in \mathcal{H}. \tag{e 22.9}$$

By Theorem 21.9 of [21], the C*-algebra A is isomorphic to one of the model algebras constructed in Theorem 14.10 of [21], and therefore there is an inductive limit decomposition $A = \varinjlim(A_i, \varphi_i)$, where A_i and φ_i are as described in Theorem 14.10 of [21]. Without loss of generality, one may assume that $h'(C) \subset A_i$ for some i. Therefore, by Theorem 14.10 of [21], the map $\varphi_{i,\infty} \circ h'$ has a decomposition

$$\varphi_{i,\infty} \circ h' = \psi_0 \oplus \psi_1$$

such that ψ_0, ψ_1 satisfy (1)–(4) of Lemma 21.5 of [21] with the σ and δ above.

It then follows from Lemma 21.5 of [21] that there is a homomorphism $\Phi : C \to e_0Ae_0$, where $e_0 = \psi_0(1_C)$, such that

(i) Φ is homotopic to a homomorphism with finite dimensional range and

$$[\Phi]_{*0} = [\psi_0], \text{ and} \tag{e 22.10}$$

(ii) for each $w \in \mathcal{U}$, there is $g_w \in \mathrm{U}_0(B)$ with $\mathrm{cel}(g_w) < \epsilon$ such that

$$\lambda(\bar{w})^{-1}(\Phi \oplus \psi_1)^{\ddagger}(\bar{w}) = \bar{g}_w. \tag{e 22.11}$$

Consider the map $h := \Phi \oplus \psi_1$. Then h is $\mathcal{F}$-ϵ-multiplicative. By (e 22.10), one has

$$[h] = [\psi_0] \oplus [\psi_1] = [h'] = \alpha.$$

By (e 22.9) and Condition (4) of Lemma 21.5 of [21], one has, for all $f \in \mathcal{H}$,

$$|\tau(h(f)) - \gamma(\tau)(f)| \le |\tau(h'(f)) - \gamma(\tau)(f)| + \delta < \epsilon + \delta < 2\epsilon.$$

It follows from (e 22.11) that, for all $u \in \mathcal{U}$,

$$\mathrm{dist}(\overline{h(u)}, \lambda(\overline{u})) < \epsilon.$$

Since $\mathcal{F}$, $\mathcal{H}$, and ϵ are arbitrary, this proves the corollary. □

COROLLARY 22.5. *Let $C \in \mathbf{H}$ and let $A_1 \in \mathcal{B}_0$ be a unital separable simple C^*-algebra which satisfies the UCT, and let $A = A_1 \otimes U$ for some UHF-algebra U of infinite type. Suppose that $\alpha \in KL_e(C,A)^{++}$ and $\lambda : U(C)/CU(C) \to U(A)/CU(A)$ is a continuous homomorphism, and $\gamma : T(A) \to T_f(C)$ is a continuous affine map such that α, λ, and γ are compatible. Then there exists a unital homomorphism $h : C \to A$ such that*

(1) $[h] = \alpha$,
(2) $\tau \circ h(f) = \gamma(\tau)(f)$ *for any* $f \in C$, *and*
(3) $h_n^{\ddagger} = \lambda$.

PROOF. Let us construct a sequence of unital completely positive linear maps $h_n : C \to A$ which satisfies (1)–(4) of Corollary 22.4, and moreover, is such that the sequence $\{h_n(f)\}$ is Cauchy for any $f \in C$. Then the limit map $h = \lim_{n\to\infty} h_n$ is the desired homomorphism.

Let $\{\mathcal{F}_n\}$ be an increasing sequence in the unit ball of C with its union dense in the unit ball of C. Define $\Delta(a) = \min\{\gamma(\tau)(a) : \tau \in \mathrm{T}(A)\}$. Since γ is continuous and $\mathrm{T}(A)$ is compact, the map Δ is an order preserving map from $C_+^{1,q} \setminus \{0\}$ to $(0,1)$. Let $\mathcal{G}(n), \mathcal{H}_1(n), \mathcal{H}_2(n) \subset C$, $\mathcal{U}(n) \subset U_\infty(C)$, $\mathcal{P}(n) \subset \underline{K}(C)$, $\gamma_1(n)$, $\gamma_2(n)$, and $\delta(n)$ be the finite subsets and constants of Theorem 12.7 of [21] with respect to $\mathcal{F}_n$, $1/2^{n+1}$, and $\Delta/2$. We may assume that $\delta(n)$ decreases to 0 if $n \to \infty$, $\mathcal{P}(n) \subset \mathcal{P}(n+1)$, $n = 1, 2, ...,$ and $\bigcup_{n=1}^{\infty} \mathcal{P}(n) = \underline{K}(C)$.

Let $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots$ be an increasing sequence of finite subsets of C such that $\bigcup \mathcal{G}_n$ is dense in C, and let $\mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots$ be an increasing sequence of finite subsets of $U(C)$ such that $\bigcup \mathcal{U}_n$ is dense in $U(C)$. One may assume that $\mathcal{G}_n \supset \mathcal{G}(n) \cup \mathcal{G}(n-1)$, $\mathcal{G}_n \supset \mathcal{H}_1(n) \cup \mathcal{H}_1(n+1) \cup \mathcal{H}_2(n) \cup \mathcal{H}_2(n-1)$, and $\mathcal{U}_n \supset \mathcal{U}(n) \cup \mathcal{U}(n-1)$.

By Corollary 22.4, there is a $\mathcal{G}_1$-$\delta(1)$-multiplicative map $h_1' : C \to A$ such that

(4) the map $[h_1']$ is well defined and $[h_1] = \alpha$,
(5) $|\tau \circ h_n(f) - \gamma(\tau)(f)| < \min\{\gamma_1(1), \frac{1}{2}\Delta(f) : f \in \mathcal{H}_1\}$ for any $f \in \mathcal{G}_1$, and
(6) $\mathrm{dist}(h_n^{\ddagger}(\bar{u}), \lambda(\bar{u})) < \gamma_2(1)$ for any $u \in \mathcal{U}_n$.

Define $h_1 = h_1'$. Assume that $h_1, h_2, ..., h_n : C \to A$ are constructed such that

(7) h_i is $\mathcal{G}_i$-$\delta(i)$-multiplicative, $i = 1, ..., n$,
(8) the map $[h_i]$ is well defined and $[h_i] = \alpha$, $i = 1, ..., n$,

(9) $|\tau\circ h_i(f)-\gamma(\tau)(f)|<\min\{\frac{1}{2}\gamma_1(i),\frac{1}{2}\Delta(f):f\in\mathcal{H}_1(i)\}$ for any $f\in\mathcal{G}_i$, $i=1,...,n$,
(10) $\mathrm{dist}(h_i^{\ddagger}(\bar{u}),\lambda(\bar{u}))<\frac{1}{2}\gamma_2(i)$ for any $u\in\mathcal{U}_i$, $i=1,...,n$, and
(11) $\|h_{i-1}(g)-h_i(g)\|<\frac{1}{2^{i-1}}$ for all $g\in\mathcal{G}_{i-1}$, $i=2,3,...,n$.

Let us construct $h_{n+1}:C\to A$ such that

(12) h_{n+1} is $\mathcal{G}_{n+1}$-$\delta(n+1)$-multiplicative,
(13) the map $[h_{n+1}]$ is well defined and $[h_{n+1}]=\alpha$,
(14) $|\tau\circ h_{n+1}(f)-\gamma(\tau)(f)|<\min\{\frac{1}{2}\gamma_1(n+1),\frac{1}{2}\Delta(f):f\in\mathcal{H}_1(n+1)\}$ for any $f\in\mathcal{G}_{n+1}$,
(15) $\mathrm{dist}(h_{n+1}^{\ddagger}(\bar{u}),\lambda(\bar{u}))<\frac{1}{2}\gamma_2(n+1)$ for any $u\in\mathcal{U}$, $i=1,...,n$, and
(16) $\|h_n(g)-h_{n+1}(g)\|<\frac{1}{2^n}$ for all $g\in\mathcal{F}_n$.

Then the statement follows.

By Corollary 22.4, there is $\mathcal{G}(n+1)$-$\delta(n+1)$-multiplicative map $h'_{n+1}:C\to A$ such that h'_{n+1} is $\mathcal{G}_{n+1}$-$\delta(n+1)$-multiplicative, the map $[h'_{n+1}]$ is well defined and $[h'_{n+1}]=\alpha$,

$$|\tau\circ h'_{n+1}(f)-\gamma(\tau)(f)|<\min\{\frac{1}{2}\gamma_1(n+1),\frac{1}{2}\Delta(f):f\in\mathcal{H}_2(n+1)\}\tag{e 22.12}$$

for any $f\in\mathcal{G}_{n+1}$, and

$$\mathrm{dist}((h'_{n+1})^{\ddagger}(\bar{u}),\lambda(\bar{u}))<\frac{1}{2}\gamma_2(n+1)$$

for any $u\in\mathcal{U}$, $i=1,...,n$. In particular, this implies that

$$[h'_{n+1}]|_{\mathcal{P}_n}=[h_n]|_{\mathcal{P}_n},$$

and for any $f\in\mathcal{H}_2(n)$ (note that $\mathcal{H}_2(n)\subset\mathcal{G}_n$),

$$\begin{aligned}|\tau\circ h_n(f)-\tau\circ h'_{n+1}(f)| &< \gamma_1(n)/2+|\gamma(\tau)(f)-\tau\circ h'_{n+1}(f)|\\ &< \gamma_1(n)/2+\gamma_1(n+1)/2<\gamma_1(n).\end{aligned}$$

Also by (e 22.12), for any $f\in\mathcal{H}_1(n)$, one has

$$\tau(h'_{n+1}(f))\geq\gamma(\tau)(f)-\frac{1}{2}\Delta(f)>\frac{1}{2}\Delta(f).$$

By the inductive hypothesis, one also has

$$\tau(h_n(f))\geq\gamma(\tau)(f)-\frac{1}{2}\Delta(f)>\frac{1}{2}\Delta(f)\ \text{ for all }\ f\in\mathcal{H}_1(n).$$

For any $u \in \mathcal{U}(n)$, one has

$$\begin{aligned}\operatorname{dist}(\overline{h'_{n+1}(u)}, \overline{h_n(u)}) &< \frac{1}{2}\gamma_2(n+1) + \operatorname{dist}(\gamma(\overline{u}), \overline{h_n(u)}) \\ &< \frac{1}{2}\gamma_2(n+1) + \frac{1}{2}\gamma_2(n) < \gamma_1(n).\end{aligned}$$

Note that both h'_{n+1} and h_n are $\mathcal{G}(n)$-$\delta(n)$-multiplicative, and so, by Theorem 12.7 of [21], there is a unitary $W \in A$ such that

$$\|W^* h'_{n+1}(g) W - h_n(g)\| < 1/2^n \text{ for all } g \in \mathcal{F}_n.$$

Then the map $h_{n+1} := AdW \circ h'_{n+1}$ satisfies the desired conditions, and the statement is proved. □

LEMMA 22.6. *Let $C \in \mathcal{C}_0$. Let $\varepsilon > 0$, $\mathcal{F} \subset C$ be any finite subset. Suppose that B is a unital separable simple C^*-algebra in $\mathcal{B}_0$, $A = B \otimes U$ for some UHF-algebra of infinite type, and $\alpha \in KK_e(C \otimes C(\mathbb{T}), A)^{++}$. Then there is a unital $\mathcal{F}$-ε-multiplicative completely positive linear map $\varphi : C \otimes C(\mathbb{T}) \to A$ such that*

$$[\varphi] = \alpha. \tag{e 22.13}$$

PROOF. Denote by α_0 and α_1 the induced maps induced by α on K_0-groups and K_1-groups.

By Theorem 18.2 of [21], there exist an $\mathcal{F}$-ϵ-multiplicative map $\varphi_1 : C \otimes C(\mathbb{T}) \to A \otimes \mathcal{K}$ and a homomorphism $\varphi_2 : C \otimes C(\mathbb{T}) \to A \otimes \mathcal{K}$ with finite dimensional range such that

$$[\varphi_1] = \alpha + [\varphi_2] \text{ in } KK(C, A).$$

In particular, one has $(\varphi_1)_{*1} = \alpha_1$. Without loss of generality, one may assume that both φ_1 and φ_2 map C into $M_r(A)$ for some integer r.

Since $M_r(A) \in \mathcal{B}_0$, for any finite subset $\mathcal{G} \subset M_r(A)$ and any $\epsilon' > 0$, there are $\mathcal{G}$-ϵ'-multiplicative maps $L_1 : M_r(A) \to (1-p)M_r(A)(1-p)$ and $L_2 : M_r(A) \to S_0 \subset pM_r(A)p$ for a C*-subalgebra $S_0 \in \mathcal{C}_0$ with $1_{S_0} = p$ such that

(1) $||a - L_1(a) \oplus L_2(a)|| < \epsilon'$ for any $a \in \mathcal{G}$ and
(2) $\tau((1-p)) < \epsilon'$ for any $\tau \in T(M_r(A))$.

Since $K_1(S_0) = \{0\}$, choosing $\mathcal{G}$ sufficiently large and ϵ' sufficiently small, one may assume that $L_1 \circ \varphi_1$ is $\mathcal{F}$-ε-multiplicative, and

$$[L_1 \circ \varphi_1]|_{K_1(C \otimes C(\mathbb{T}))} = (\varphi_1)_{*1} = \alpha_1.$$

Moreover, since the positive cone of $K_0(C \otimes C(\mathbb{T}))$ is finitely generated, choosing ϵ' even smaller, one may assume that the map

$$\kappa := \alpha_0 - [L_1 \circ \varphi_1]|_{K_0(C \otimes C(\mathbb{T})} : K_0(C \otimes C(\mathbb{T})) \to K_0(A)$$

is positive. Pick a point $x_0 \in \mathbb{T}$, and consider the evaluation map

$$\pi : C \otimes C(\mathbb{T}) \in f \otimes g \mapsto f \cdot g(x_0) \in C.$$

Then $\pi_{*0} : K_0(C \otimes C(\mathbb{T})) \to K_0(C)$ is an order isomorphism, since $K_1(C) = 0$.

Choose a projection $q \in A$ with $[q] = \kappa([1])$. Since $qAq \in \mathcal{B}_0$, by Corollary 18.9 of [21], there is a unital homomorphism $h : C \to qAq$ such that

$$[h]_0 = \kappa \circ \pi_{*0}^{-1} \quad \text{on } K_0(C),$$

and hence one has

$$(h \circ \pi)_{*0} = \kappa, \quad \text{on } K_0(C \otimes C(\mathbb{T})).$$

Put $\varphi = (L_1 \circ \varphi_1) \oplus (h \circ \pi) : C \otimes C(\mathbb{T}) \to A$. Then it is clear that

$$\varphi_{*0} = [L_1 \circ \varphi_1]|_{K_0(C \otimes C(\mathbb{T}))} + \kappa = [L_1 \circ \varphi_1]|_{K_0(C \otimes C(\mathbb{T}))} + \alpha_0 - [L_1 \circ \varphi_1]|_{K_0(C \otimes C(\mathbb{T}))} = \alpha_0$$

$$\text{and } [\varphi]_1 = [L_1 \circ \varphi_1]|_{K_1(C \otimes C(\mathbb{T}))} = \alpha_1.$$

Since $K_*(C \otimes C(\mathbb{T}))$ is finitely generated and torsion free, one has that $[\varphi] = \alpha$ in $KK(C \otimes C(\mathbb{T}), A)$. □

Lemma 22.7. *Let $C \in \mathcal{C}_0$. Let $\varepsilon > 0$, $\mathcal{F} \subset C \otimes \mathrm{C}(\mathbb{T})$ be a finite subset, $\sigma > 0$, and $\mathcal{H} \subset (C \otimes C(\mathbb{T}))_{s.a.}$ be a finite subset. Suppose that A is a unital C^*-algebra in $\mathcal{B}_0$, $B = A \otimes U$ for some UHF-algebra U of infinite type, $\alpha \in KK_e(C \otimes C(\mathbb{T}), B)^{++}$, and $\gamma : T(B) \to T_{\mathrm{f}}(C \otimes C(\mathbb{T}))$ is a continuous affine map such that α and γ are compatible. Then there is a unital $\mathcal{F}$-ε-multiplicative completely positive linear map $\varphi : C \otimes C(\mathbb{T}) \to B$ such that*

(1) $[\varphi] = \alpha$ *and*
(2) $|\tau \circ \varphi(h) - \gamma(\tau)(h)| < \sigma$ *for any* $h \in \mathcal{H}$.

Moreover, if $A \in \mathcal{B}_1$, $\beta \in KK_e(C, A)^{++}$, $\gamma' : T(A) \to T_f(C)$ is a continuous affine map which is compatible with β, and $\mathcal{H}' \subset C_{s.a.}$ is a finite subset, then there is also a unital homomorphism $\psi : C \to A$ such that

$$\text{(e 22.14)} \quad [\psi] = \beta \ \textit{and} \ |\tau \circ \psi(h) - \gamma'(\tau)(h)| < \sigma \ \text{for all } f \in \mathcal{H}'.$$

Proof. Since $K_*(C \otimes C(\mathbb{T}))$ is finitely generated and torsion free, by the UCT, the element $\alpha \in KK(C \otimes C(\mathbb{T}), A)$ is determined by the induced maps $\alpha_0 \in \mathrm{Hom}(K_0(C \otimes C(\mathbb{T})), K_0(A))$ and $\alpha_1 \in \mathrm{Hom}(K_1(C \otimes C(\mathbb{T})), K_1(A))$. We may assume that projections in $M_r(C \otimes C(\mathbb{T}))$ (for some fixed integer $r > 0$) generate $K_0(C \otimes C(\mathbb{T}))$.

We may also assume that $\|h\| \leq 1$ for all $h \in \mathcal{H}$. Fix a finite generating set $\mathcal{G}$ of $K_0(C \otimes C(\mathbb{T}))$. Since $\gamma(\tau) \in T_{\mathrm{f}}(C \otimes \mathrm{C}(\mathbb{T}))$ for all $\tau \in T(B)$ and $\tau(B)$ is compact, one is able to define $\Delta : (C \otimes C(\mathbb{T}))_+^{q,1} \setminus \{0\} \to (0, 1)$ by

$$\Delta(\hat{h}) = \frac{1}{2} \inf\{\gamma(\tau)(h) : \ \tau \in T(B)\}.$$

Fix a finite generating set $\mathcal{G}$ of $K_0(C \otimes C(\mathbb{T}))$. Let $\mathcal{H}_1 \subset C \otimes C(\mathbb{T})$, $\delta > 0$, and $K \in \mathbb{N}$ be the finite subset and the constants of Lemma 16.10 of [21] with respect to $\mathcal{F}$, $\mathcal{H}$, ϵ, $\sigma/4$ (in place of σ), and Δ.

Since $A \in \mathcal{B}_0$ and U is of infinite type, for any finite subset $\mathcal{G}' \subset B$ and any $\epsilon' > 0$, there are unital $\mathcal{G}'$-ϵ'-multiplicative completely positive linear maps $L_1 : B \to (1-p)B(1-p)$ and $L_2 : B \to D \otimes 1_{M_K} \subset D \otimes M_K \subset pBp$ for a C*-subalgebra $D \in \mathcal{C}_0$ with $1_{D \otimes M_K} = p$ such that

(3) $||a - L_1(a) \oplus L_2(a)|| < \epsilon'$ for any $a \in \mathcal{G}'$, and
(4) $\tau((1-p)) < \min\{\epsilon', \sigma/4\}$ for any $\tau \in T(B)$.

Put $S = D \otimes M_K$. Since $K_i(C \otimes C(\mathbb{T}))$ is finite generated, $\mathrm{Hom}_\Lambda(\underline{K}(C \otimes C(\mathbb{T})), \underline{K}(C \otimes C(\mathbb{T})))$ is determined on a finitely generated subgroup G^K of $\underline{K}(C \otimes C(\mathbb{T}))$ (see Corollary 2.12 of [6]). Choosing $\mathcal{G}'$ large enough and ϵ' small enough, one may assume $[L_1]$ and $[L_2]$ are well defined on $\alpha(G^K)$, and

$$\alpha = [L_1] \circ \alpha + [j] \circ [L_2] \circ \alpha, \qquad \text{(e 22.15)}$$

where $j : S \to A$ is the embedding. Note that since $K_1(S) = \{0\}$, one has

$$\alpha_1 = [L_1] \circ \alpha|_{K_1(C \otimes C(\mathbb{T}))}.$$

Define $\kappa' = [L_2] \circ \alpha|_{K_0(C \otimes C(\mathbb{T}))}$, which is a homomorphism from $K_0(C \otimes C(\mathbb{T}))$ to $K_0(D \otimes 1_{M_K}) = K_0(D)$ (here we identify $D \otimes 1_{M_K}$ with D) which maps $[1_{C \otimes C(\mathbb{T})}]$ to $[1_{D \otimes 1_{M_K}}]$. Let $\{e_{i,j} : 1 \le i, j \le K\}$ be a system of matrix units for M_K. View $e_{i,j} \in D \otimes M_K$. Then $e_{i,j}$ commutes with the image of L_2. Define $L_2' : B \to D \otimes e_{1,1}$ by $L_2'(a) = e_{11} L_2(a) e_{1,1}$ for all $a \in B$.

Put $\kappa = [L_2'] \circ \alpha|_{K_0(C \otimes C(\mathbb{T}))}$. Put $D' = D \otimes e_{1,1}$.

Choosing $\mathcal{G}'$ larger and ϵ' smaller, if necessary, one has a continuous affine map $\gamma' : T(D') \to T(C \otimes C(\mathbb{T}))$ such that, for all $\tau \in T(A)$,

(5) $|\gamma'(\frac{1}{\tau(e_{1,1})}\tau|_{D'})(f) - \gamma(\tau)(f)| < \sigma/4$ for any $f \in \mathcal{H}$,
(6) $\gamma'(\tau)(h) > \Delta(\hat{h})$ for any $h \in \mathcal{H}_1$, and
(7) $|\gamma'(\frac{1}{\tau(e_{1,1})}\tau|_{D'})(p) - \tau(\kappa([p]))| < \delta$ for all projections $p \in M_r(C \otimes C(\mathbb{T}))$.

Then it follows from Theorem 16.10 of [21] that there is an $\mathcal{F}$-ϵ-multiplicative contractive completely positive linear map $\varphi_2 : C \otimes C(\mathbb{T}) \to M_K(D) = S$ such that

$$(\varphi_2)_{*0} = K\kappa = \kappa'$$

and

$$|(1/K)t \circ \varphi_2(h) - \gamma'(t)(h)| < \sigma/4, \quad h \in \mathcal{H},\ t \in T(D').$$

On the other hand, since $(1-p)A(1-p) \in \mathcal{B}_0$, by Lemma 22.6, there is a unital $\mathcal{F}$-ϵ-multiplicative completely positive linear map $\varphi_1 : C \otimes C(\mathbb{T}) \to (1-p)A(1-p)$ such that

$$[\varphi_1] = [L_1] \circ \alpha \quad \text{in } KK(C \otimes C(\mathbb{T}), A).$$

Define $\varphi = \varphi_1 \oplus j \circ \varphi_2 : C \otimes C(\mathbb{T}) \to (1-p)A(1-p) \oplus S \subset A$. Then, by (e 22.15), one has

$$\varphi_{*0} = (\varphi_1)_{*0} + (j \circ \varphi_2)_{*0} = ([L_1] \circ \alpha)|_{K_0(C \otimes C(\mathbb{T}))} + ([j \circ L_2] \circ \alpha)|_{K_0(C \otimes C(\mathbb{T}))} = \alpha_0$$

and

$$\varphi_{*1} = (\varphi_1)_{*1} + (j \circ \varphi_2)_{*1} = ([L_1] \circ \alpha|_{K_1(C \otimes C(\mathbb{T}))} = \alpha_1.$$

Hence $[\varphi] = \alpha$ in $KK(C \otimes C(\mathbb{T}))$.

For any $h \in \mathcal{H}$ and any $\tau \in T(A)$, one has (note that $\|h\| \le 1$ for all $h \in \mathcal{H}$, and $\tau(1-p) < \delta/4$),

$$\begin{aligned}
& |\tau \circ \varphi(h) - \gamma(\tau)(h)| \\
< \ & |\tau \circ \varphi(h) - \tau \circ j \circ \varphi_2(h)| + |\tau \circ j \circ \varphi_2(h) - \gamma(\tau)(h)| \\
< \ & \sigma/4 + |\tau \circ j \circ \varphi_2(h) - \gamma'(\frac{1}{\tau(e_{1,1})}\tau|_{D'})(h)| + |\gamma'(\frac{1}{\tau(e_{1,1})}\tau|_{D'})(h) - \gamma(\tau)(h)| \\
< \ & \sigma/4 + |\tau \circ j \circ \varphi_2(h) - \gamma'(\frac{1}{\tau(p)}\tau|_S)(h)| + |\gamma'(\frac{1}{\tau(p)}\tau|_S)(h) - \gamma(\tau)(h)| \\
< \ & \sigma/4 + \sigma/4 + \sigma/4 < \sigma,
\end{aligned}$$

where we identify $T(D')$ with $T(S)$ in a standard way for $S = D' \otimes M_K$. Hence the map φ satisfies the requirements of the lemma.

To see the last part of the lemma holds, we note that, when $C \otimes C(\mathbb{T})$ is replaced by C and A is assumed to be in $\mathcal{B}_1$, the only difference is that we cannot use 22.6. But then we can appeal to Theorem 18.7 of [21] to obtain φ_1. The semiprojectivity of C allows us actually to obtain a unital homomorphism (see Corollary 18.9 of [21]). □

Corollary 22.8. *Let $C \in \mathcal{C}_0$. Suppose that A is a unital separable simple C^*-algebra in $\mathcal{B}_0$, $B = A \otimes U$ for some UHF-algebra of infinite type, $\alpha \in KK_e(C,B)^{++}$, and $\gamma : T(B) \to T_{\mathrm{f}}(C))$ is a continuous affine map. Suppose that $(\alpha, \lambda, \gamma)$ is a compatible triple. Then there is a unital homomorphism $\varphi : C \to B$ such that*

$$[\varphi] = \alpha \ \textit{ and } \ \varphi_T = \gamma.$$

In particular, φ is a monomorphism.

Proof. The proof is exactly the same as the argument employed in 22.5 but using the second part of Lemma 22.7 instead of 22.4. The reason φ is a monomorphism is because $\gamma(\tau)$ is faithful for each $\tau \in T(A)$. □

Lemma 22.9. *Let C be a unital C^*-algebra. Let $p \in C$ be a full projection. Then, for any $u \in U_0(C)$, there is a unitary $v \in pCp$ such that*

$$\overline{u} = \overline{v \oplus (1-p)} \quad \textit{in } U_0(C)/CU(C).$$

If, furthermore, C is separable and has stable rank one, then, for any $u \in U(C)$, there is a unitary $v \in pCp$ such that

$$\overline{u} = \overline{v \oplus (1-p)} \quad \text{in } U(C)/CU(C).$$

Proof. It suffices to prove the first part of the statement. This is essentially contained in the proof of 4.5 and 4.6 of [22]. As in the proof of 4.5 of [22], for any $b \in C_{s.a.}$, there is $c \in pCp$ such that $b - c \in C_0$, where C_0 is the closed subspace of $A_{s.a.}$ consisting of elements of the form $x - y$, where $x = \sum_{n=1}^{\infty} c_n^* c_n$ and $y = \sum_{n=1}^{\infty} c_n c_n^*$ (convergence in norm) for some sequence $\{c_n\}$ in C.

Now let $u = \prod_{k=1}^{n} \exp(ib_k)$ for some $b_k \in C_{s.a.}$, $k = 1, 2, ..., n$. Then there are $c_k \in pCp$ such that $b_k - c_k \in C_0$, $k = 1, 2, ..., n$. Put $v = p(\prod_{k=1}^{n} \exp(ic_k))p$. Then $v \in U_0(pCp)$ and $v + (1-p) = \prod_{k=1}^{n} \exp(ic_k)$. By 3.1 of [52], $u^*(v + (1-p)) \in CU(C)$. □

Lemma 22.10. *Let $\mathcal{D}$ be the family of unital separable residually finite dimensional C^*-algebras and let A be a unital simple separable C^*-algebra which has the property ($L_{\mathcal{D}}$) (see 9.4 of [21]) and the property (SP). Then A satisfies the Popa condition: Let $\varepsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There exists a finite dimensional C^*-subalgebra $F \subset A$ with $P = 1_F$ such that*

$$\|[P, x]\| < \varepsilon, \ \ PxP \in_{\varepsilon} F \ \text{ and } \ \|PxP\| \geq \|x\| - \varepsilon \tag{e 22.16}$$

for all $x \in \mathcal{F}$. In particular, if $A \in \mathcal{B}_1$ and A has the property (SP), then A satisfies the Popa condition.

Proof. We may assume that $\mathcal{F} \subset A^{\mathbf{1}}$ and $0 < \varepsilon < 1/2$. Without loss of generality, we may assume that

$$d = \min\{\|x\| : x \in \mathcal{F}\} > 0.$$

Since A has property ($L_{\mathcal{D}}$), there are a projection $p \in A$ and a C^*-subalgebra $D \subset A$ with $D \in \mathcal{D}$ and $p = 1_D$ such that

$$\|px - xp\| < d\epsilon/16, \ \ pxp \in_{d\epsilon/16} D, \ \text{ and } \ \|pxp\| \geq (1 - \varepsilon/16)\|x\| \tag{e 22.17}$$

for all $x \in \mathcal{F}$ (see 9.5 of [21]).

Let $\mathcal{F}' \subset D$ be a finite subset such that, for each $x \in \mathcal{F}$, there exists $x' \in \mathcal{F}'$ such that $\|pxp - x'\| < d\varepsilon/16$. Since $D \in \mathcal{D}$, there is a unital surjective homomorphism $\pi : D \to D/\mathrm{ker}\pi$ such that $F_1 := D/\mathrm{ker}\pi$ is a finite dimensional C^*-algebra and

$$\|\pi(x')\| \geq (1 - \varepsilon/16)\|x'\| \ \text{ for all } \ x' \in \mathcal{F}'. \tag{e 22.18}$$

Let $B = \overline{(\mathrm{ker}\pi)A(\mathrm{ker}\pi)}$. B is a hereditary C^*-subalgebra of A. Let C be the closure of $D + B$. Note that $1_C = 1_D = p$. As in the proof of 5.2 of [29], B is an ideal of C and $C/B \cong D/\ker\pi = F_1$. The lemma then follows from Lemma 2.1 of [46]. In fact, since pAp has property (SP), by Lemma 2.1 of [46], there are a projection $P \in pAp$ and a monomorphism $h : F_1 \to PAP$ such that

$$h(1_F) = P, \ \|Px' - x'P\| < \varepsilon/16 \ \text{and} \tag{e 22.19}$$

$$\|h \circ \pi(x') - Px'P\| < \varepsilon \cdot d/16 \tag{e 22.20}$$

for all $x' \in \mathcal{F}'$. Put $F = h(F_1)$. Then, one estimates that, for all $x \in \mathcal{F}$,

$$\begin{aligned}
&\|Px - xP\| \le \|Ppx - Px'\| + \|Px' + x'P\| + \|x'P - xpP\| \\
&\quad < \varepsilon/16 + \varepsilon/16 + \varepsilon/16 < \varepsilon, \\
&PxP \approx_{\varepsilon/16} Px'P \in_{\varepsilon/16} F_1, \ \text{and} \\
&\|PxP\| = \|PpxpP\| \ge \|Px'P\| - d\varepsilon/16 \ge \|h \circ \pi(x')\| - d\varepsilon/8 \\
&= \|\pi(x')\| - d\varepsilon/8 \ge \|x'\| - d\varepsilon/16 - d\varepsilon/8 \ge \|pxp\| - d\varepsilon/4 \\
&\quad \ge (1 - \varepsilon/16)\|x\| - d\varepsilon/4 \ge \|x\| - \varepsilon.
\end{aligned}$$

□

LEMMA 22.11. *Let $C \in \mathcal{C}_0$. Let $\varepsilon > 0$, $\mathcal{F} \subset C$ be a finite subset, $1 > \sigma_1 > 0$, $1 > \sigma_2 > 0$, $\overline{\mathcal{U}} \subset J_c(K_1(C \otimes C(\mathbb{T}))) \subset U(C \otimes C(\mathbb{T}))/CU(C \otimes C(\mathbb{T}))$ be a finite subset (see Definition 2.16 of [21]) and $\mathcal{H} \subset (C \otimes C(\mathbb{T}))_{s.a.}$ be a finite subset. Suppose that A is a unital separable simple C^*-algebra in $\mathcal{B}_0$, $B = A \otimes U$ for some UHF-algebra U of infinite type, $\alpha \in KK_e(C \otimes C(\mathbb{T}), B)^{++}$, $\lambda : J_c(K_1(C \otimes C(\mathbb{T}))) \to U(B)/CU(B)$ is a homomorphism, and $\gamma : T(B) \to T_{\mathrm{f}}(C \otimes C(\mathbb{T}))$ is a continuous affine map. Suppose that $(\alpha, \lambda, \gamma)$ is a compatible triple. Then there is a unital $\mathcal{F}$-ε-multiplicative completely positive linear map $\varphi : C \otimes C(\mathbb{T}) \to B$ such that*

(1) $[\varphi] = \alpha$,
(2) $\mathrm{dist}(\varphi^{\ddagger}(x), \lambda(x)) < \sigma_1$, *for any $x \in \overline{\mathcal{U}}$, and*
(3) $|\tau \circ \varphi(h) - \gamma(\tau)(h)| < \sigma_2$, *for any $h \in \mathcal{H}$.*

PROOF. Note that $\underline{K}(C \otimes C(\mathbb{T}))$ is finitely generated modulo Bockstein operations and $K_0(C \otimes C(\mathbb{T}))_+$ is a finitely generated semigroup. Using the inductive limit $B = \lim_{n\to\infty}(A \otimes M_{r_n}, \imath_{n,n+1})$, one can find, for n large enough, $\alpha_n \in KK_e(C \otimes C(\mathbb{T}), A \otimes M_{r_n})^{++}$ such that $\alpha = \alpha_n \times [\imath_n]$ where $[\imath_n] \in KK(A \otimes M_{r_n}, B)$ is induced by the inclusion $\imath_n : A \otimes M_{r_n} \to B$. Replacing A by $A \otimes M_{r_n}$, we may assume that $\alpha = \alpha_1 \times [\imath]$, where $\alpha_1 \in KK_e(C \otimes C(\mathbb{T}), A)^{++}$ and $\imath : A \to A \otimes U = B$ is the inclusion. Note that $\lambda : J_c(K_1(C \otimes C(\mathbb{T}))) \to U(A \otimes U)/CU(A \otimes U)$. By the same argument as above, we know that if the integer n above is large enough, then there is a map $\lambda_n : J_c(K_1(C \otimes C(\mathbb{T}))) \to U(A \otimes M_{r_n})/CU(A \otimes M_{r_n})$ such that $|\imath_n^{\ddagger} \circ \lambda_n(u) - \lambda(u)|$ is arbitrarily small (e.g smaller than $\sigma_1/4$) for all $u \in \overline{\mathcal{U}}$. Replacing A by $A \otimes M_{r_n}$, we may assume $\lambda = \imath^{\ddagger} \circ \lambda_1$ with $\lambda_1 : J_c(K_1(C \otimes C(\mathbb{T}))) \to$

$U(A)/CU(A)$ and $\imath^{\ddagger} : U(A)/CU(A) \to U(B)/CU(B)$ induced by the inclusion map. Furthermore, we may assume that λ_1 is compatible with α_1.

Without loss of generality, we may assume that $\|h\| \le 1$ for all $h \in \mathcal{H}$. Let $p_i, q_i \in M_k(C)$ be projections such that $\{[p_1]-[q_1], ..., [p_d]-[q_d]\}$ forms a set of independent generators of $K_0(C)$ (as an abelian group) for some integer $k \ge 1$. Choosing a specific J_c, one may assume that

$$\overline{\mathcal{U}} = \{((\mathbf{1}_k - p_i) + p_i \otimes z)((\mathbf{1}_k - q_i) + q_i \otimes z^*) : 1 \le i \le d\},$$

where $z \in C(\mathbb{T})$ is the identity function on the unit circle. Put $u_i' = (\mathbf{1}_k - p_i) + p_i \otimes z)((\mathbf{1}_k - q_i) + q_i \otimes z^*)$. Hence, $\{[u_1'], ..., [u_d']\}$ is a set of standard generators of $K_1(C \otimes C(\mathbb{T})) \cong K_0(C) \cong \mathbb{Z}^d$. Then λ is a homomorphism from $\mathbb{Z}^d$ to $U(B)/CU(B)$.

Let $\pi_e : C \to F_1 = \bigoplus_{i=1}^l M_{n_i}$ be the standard evaluation map defined in Definition 3.1 of [21]. By Proposition 3.5 of [21], the map $(\pi_e)_{*0}$ induces an embedding of $K_0(C)$ in $\mathbb{Z}^l$, and the map $(\pi_e \otimes \mathrm{id})_{*1}$ induces an embedding of $K_1(C \otimes C(\mathbb{T})) \cong \mathbb{Z}^d$ in $K_1(\bigoplus_{i=1}^l M_{n_i} \otimes C(\mathbb{T})) \cong \mathbb{Z}^l$. Define $J_c(K_1(\bigoplus_{i=1}^l M_{n_i} \otimes C(\mathbb{T})))$ to be the subgroup generated by $\{\overline{e_i \otimes z_i \oplus (1-e_i)}; i = 1, ..., l\}$, where e_i is a rank one projection of M_{n_i} and z_i is the standard unitary generator of the i-th copy of $C(\mathbb{T})$. Note that the image of $J_c(K_1(C \otimes C(\mathbb{T})))$ under π_e is contained in $J_c(K_1(\bigoplus_{i=1}^l M_{n_i} \otimes C(\mathbb{T})))$. Write $w_j = e_j \otimes z_j \oplus (1-e_j)$, $1 \le j \le l$.

Let U be as in the lemma. We write $B = B_0 \otimes U_2$, and $B_0 = A \otimes U_1$, with $U = U_1 \otimes U_2$, and both U_1 and U_2 UHF algebras of infinite type. Denote by $\imath_1 : A \to B_0$, $\imath_2 : B_0 \to B$, and $\imath = \imath_2 \circ \imath_1 : A \to B$ the inclusion maps. Recall $\alpha = \alpha_1 \times [\imath] \in KK(C, B)$.

Applying Lemma 22.7, one obtains a unital $\mathcal{F}'$-ε'-multiplicative completely positive linear map $\psi : C \otimes \mathrm{C}(\mathbb{T}) \to B_0$ such that

$$[\psi] = \alpha_1 \times [\imath_1] \text{ and} \tag{e 22.21}$$

$$|\tau \circ \psi(h) - \gamma(\tau)(h)| < \min\{\sigma_1, \sigma_2\}/3 \tag{e 22.22}$$

for all $h \in \mathcal{H}$, and for all $\tau \in \mathrm{T}(B_0)$, where $\varepsilon/2 > \varepsilon' > 0$ and $\mathcal{F}_1 \supset \mathcal{F}$. (Note that $T(B_0) = T(A) = T(B)$, and the map $\gamma : T(B) \to T_{\mathrm{f}}(C \otimes C(\mathbb{T}))$ can be regarded as a map with domain $T(B_0)$). We may assume that ε' is sufficiently small and $\mathcal{F}_1$ is sufficiently large that not only (e 22.21) and (e 22.22) make sense but also that $\psi^{\ddagger}$ can be defined on $\bar{\mathcal{U}}$, and induces a homomorphism from $J_c(K_1(C \otimes C(\mathbb{T})))$ to $U(B_0)/CU(B_0)$ (see 2.17 of [21]).

Let M be the integer of Corollary 15.3 of [21] for $K_0(C) \subset \mathbb{Z}^l$ (in place of $G \subset \mathbb{Z}^l$).

For any $\varepsilon'' > 0$ and any finite subset $\mathcal{F}'' \subset B_0$, since B_0 has the Popa condition and has the property (SP) (see 22.10), there exist a non-zero projection $e \in B_0$ and a unital $\mathcal{F}''$-ε''-multiplicative completely positive linear map $L_0 : B_0 \to$

$F \subset eB_0e$, where F is a finite dimensional and $1_F = e$, and a unital $\mathcal{F}''$-ε''-multiplicative completely positive linear map $L_1 : B_0 \to (1-e)B_0(1-e)$ such that

$$\|b - \imath \circ L_0(b) \oplus L_1(b)\| < \varepsilon'' \text{ for all } b \in \mathcal{F}'', \tag{e 22.23}$$

$$\|L_0(b)\| \geq \|b\|/2 \text{ for all } b \in \mathcal{F}'', \text{ and} \tag{e 22.24}$$

$$\tau(e) < \min\{\sigma_1/2, \sigma_2/2\} \text{ for all } \tau \in T(B_0), \tag{e 22.25}$$

where $\imath : F \to eB_0e$ is the embedding and $L_1(b) = (1-p)b(1-p)$ for all $b \in B_0$.

Since the positive cone of $K_0(C \otimes C(\mathbb{T}))$ is finitely generated, with sufficiently small ε'' and sufficiently large $\mathcal{F}''$, one may assume that $[L_0 \circ \psi]|_{K_0(C\otimes C(\mathbb{T}))}$ is positive. Moreover, one may assume that $(L_0 \circ \psi)^{\ddagger}$ and $(L_1 \circ \psi)^{\ddagger}$ are well defined and induce homomorphisms from $J_c(K_1(C \otimes C(\mathbb{T})))$ to $U(B_0)/CU(B_0)$. One may also assume that $[L_1 \circ \psi]$ is well defined. Moreover, we may assume that $L_i \circ \psi$ is $\mathcal{F}$-ε-multiplicative for $i = 0, 1$.

There is a projection $E_c \in U_2$ such that E_c is a direct sum of M copies of some non-zero projections $E_{c,0} \in U_2$. Put $E = 1_{U_2} - E_c$.

Define $\varphi_0 : C \otimes C(\mathbb{T}) \to F \otimes EU_2E \to eB_0e \otimes EU_2E$ by $\varphi_0(c) = L_0 \circ \psi(c) \otimes E (\in B)$ for all $c \in C \otimes C(\mathbb{T})$ and define $\varphi_1' : C \to F \otimes E_cU_2E_c$ by $\varphi_1'(c) = L_0 \circ \psi(c) \otimes E_c$ for all $c \in C$. Note that φ_0 is also $\mathcal{F}$-ε-multiplicative and $\varphi_0^{\ddagger}$ is also well defined as $(L_0 \circ \psi)^{\ddagger}$ is. Moreover $[\varphi_1']$ is well defined. Define

$$\begin{aligned} L_2 &= \imath_2 \circ L_1 \circ \psi + \varphi_0 : C \otimes C(\mathbb{T}) \to \\ &\quad \big((1-e)B_0(1-e) \otimes 1_{U_2}\big) \oplus \big(eB_0e \otimes EU_2E\big) \ \ (\subset B). \end{aligned}$$

Denote by

$$\lambda_0 = \lambda - L_2^{\ddagger} = \lambda - \varphi_0^{\ddagger} - (\imath_2 \circ L_1 \circ \psi)^{\ddagger} : J_c(K_1(C \otimes C(\mathbb{T}))) \to U(B)/CU(B).$$

Note that L_0 factors through the finite dimensional algebra F and therefore $[L_0] = 0$ on $K_1(B_0)$. Consequently $[\varphi_0]|_{K_1(C\otimes C(\mathbb{T}))} = 0$ and $[L_1 \circ \psi] = [\imath_1] \circ [\alpha_1]$ on $K_1(C \otimes C(\mathbb{T}))$. Hence, $[\imath_2 \circ L_1 \circ \psi] = \alpha$ on K_1. Furthermore, α is compatible with λ. We know that the image of λ_0 is in $U_0(B)/CU(B)$.

Note that, by Lemma 11.5 of [21], the group $U_0(B)/CU(B)$ is divisible. It is an injective abelian group. Therefore there is a homomorphism $\tilde{\lambda} : J_c(\bigoplus_{i=1}^l M_{n_i} \otimes C(\mathbb{T})) \to U_0(B)/CU(B)$ such that

$$\tilde{\lambda} \circ (\pi_e)^{\ddagger} = \lambda_0 - L_2^{\ddagger}. \tag{e 22.26}$$

Let $\beta = [L_0 \circ \psi]|_{K_0(C)}: K_0(C) \to K_0(F) = \mathbb{Z}^n$. Let $R_0 \geq 1$ be the integer given by Corollary 15.3 of [21] for $\beta : K_0(C) \to \mathbb{Z}^n$ (in place of $\kappa : G \to \mathbb{Z}^n$; note that that $[E_c]$ is divisible by M implies that every element in $\beta(K_0(C))$ is divisible by M). There is a unital C^*-subalgebra $M_{MK} \subset E_cU_2E_c$ such that $K \geq R_0$ and such that $E_cU_2E_c$ can be written as $M_{MK} \otimes U_3$. It follows from Corollary 15.3

of [21] that there is a positive homomorphism $\beta_1 : K_0(F_1) \to K_0(F)$ such that $\beta_1 \circ (\pi_e)_{*0} = MK\beta$. Let $h : F_1 \to F \otimes M_{MK}$ be the unital homomorphism such that $h_{*0} = \beta_1$. Put $\varphi_1'' = h \circ \pi_e$: $C \to F \otimes M_{MK}$, and then one has $(\varphi_1'')_{*0} = MK\beta$. Let $J : M_{MK} \to E_cU_2E_c$ be the embedding. One verifies that

$$(\text{e}\,22.27)\quad (\imath_F \otimes J)_{*0} \circ (\varphi_1'')_{*0} = (\imath_F \otimes J)_{*0} \circ MK\beta = \tilde{\imath}_{*0} \circ (\varphi_1')_{*0},$$

where $\imath_F : F \to eB_0e$ and $\tilde{\imath} : F \otimes E_cU_2E_c \to eB_0e \otimes E_cU_2E_c$ is the unital embedding.

Choose a unitary $y_i \in (\imath_F \otimes J \circ h)(e_j)B(\imath_F \otimes J \circ h)(e_j)$ such that

$$\bar{y}_j = \tilde{\lambda}(w_j),\ \ j = 1, 2, ..., l,$$

where we recall that $w_j = e_j \otimes z_j \oplus (1 - e_j) \in F_1 \otimes C(\mathbb{T}) = \oplus_j M_{n_j} \otimes C(\mathbb{T})$ is one of the chosen generator of $K_1(M_{n_j} \otimes C(\mathbb{T}))$. Let 1_j be the unit of $M_{n_j} \subset F_1$; then $1_j = \underbrace{e_j \oplus e_j \oplus \cdots \oplus e_j}_{n_j}$.

Define $\tilde{y}_j = \text{diag}(\overbrace{y_j, y_j, ..., y_j}^{n_j}) \in (\imath_F \otimes J \circ h)(1_j)B(\imath_F \otimes J \circ h)(1_j), j = 1, 2, ..., l.$ Then $\tilde{y}_j$ commutes with $(\imath_F \otimes J)(F_1)$.

Define $\tilde{\varphi}_1 : F_1 \otimes C(\mathbb{T}) \to \big((\imath_F \otimes J) \circ \varphi_1''\big)(1_C)B\big((\imath_F \otimes J) \circ \varphi_1''\big)(1_C)$ by $\tilde{\varphi}_1(c_j \otimes f) = \big((\imath_F \otimes J) \circ \varphi_1''\big)(c_j)f(\tilde{y}_j)$ for all $c_j \in M_{n_j}$ and $f \in C(\mathbb{T})$. Define $\varphi_1 = \tilde{\varphi}_1 \circ (\pi_e \otimes \text{id}_{C(\mathbb{T})})$. Then, by identifying $K_0(C \otimes C(\mathbb{T}))$ with $K_0(C)$, one has

$$(\text{e}\,22.28)\qquad (\varphi_1)_{*0} = \tilde{\imath}_{*0} \circ (\varphi_1')_{*0} \ \text{ and } \ (\varphi_1)^{\ddagger} = \tilde{\lambda}.$$

Define $\varphi = \varphi_0 \oplus \varphi_1 \oplus \imath_2 \circ L_1 \circ \psi$. By (e 22.22) and (e 22.25),

$$|\tau \circ \varphi(h) - \gamma(\tau)(h)| < \sigma_2/3 + \sigma_2/3 = 2\sigma_2/3 \ \text{ for all } \ h \in \mathcal{H}.$$

It is ready to verify that $\varphi_{*0} = \alpha|_{K_0(C \otimes C(\mathbb{T}))}$ and $\varphi^{\ddagger} = \lambda$. Thus, since λ is compatible with α,

$$(\text{e}\,22.29)\qquad \varphi_{*1} = \alpha|_{K_1(C \otimes C(T))}.$$

Since $K_{*i}(C \otimes C(\mathbb{T})) \cong K_0(C)$ is free and finitely generated, one concludes that

$$[\varphi] = \alpha.$$

□

COROLLARY 22.12. *Let $C \in \mathcal{C}_0$ and $C_1 = C \otimes C(\mathbb{T})$. Suppose that A is a unital separable simple C^*-algebra in $\mathcal{B}_0$, $B = A \otimes U$ for some UHF-algebra U of infinite type, $\alpha \in KK_e(C_1, B)^{++}$, $\lambda : J_c(K_1(C)) \to U(B)/CU(B)$ is a homomorphism, and $\gamma : T(B) \to T_f(C_1))$ is a continuous affine map. Suppose that $(\alpha, \lambda, \gamma)$ is a compatible triple. Then there is a unital homomorphism $\varphi : C_1 \to B$ such that*

$$[\varphi] = \alpha,\ \varphi^{\ddagger}|_{J_c(K_1(C))} = \lambda \ \text{ and } \ \varphi_T = \gamma.$$

In particular, φ is a monomorphism.

PROOF. The proof is exactly the same as the argument employed in 22.5 using 22.11. □

COROLLARY 22.13. *Let $C \in \mathcal{C}_0$ and let $C_1 = C$ or $C_1 = C \otimes C(\mathbb{T})$. Suppose that A is a unital separable simple C^*-algebra in $\mathcal{B}_0$, $B = A \otimes U$ for some UHF-algebra of infinite type, $\alpha \in KK_e(C_1, B)^{++}$, and $\gamma : T(B) \to T_f(C_1)$ is a continuous affine map. Suppose that (α, γ) is compatible. Then there is a unital homomorphism $\varphi : C_1 \to B$ such that*

$$[\varphi] = \alpha \text{ and } \varphi_T = \gamma.$$

In particular, φ is a monomorphism.

PROOF. To apply 22.12, one needs a map λ. Note that $J_c(K_1(C_1))$ is isomorphic to $K_1(C_1)$ which is finitely generated. Let $J_c^{(1)} : K_1(B) \to U(B)/CU(B)$ be the splitting map defined in Definition 2.16 of [21], Define $\lambda = J_c^{(1)} \circ \alpha|_{K_1(C_1)} \circ \pi|_{J_c(K_1(C_1))}$, where $\pi : U(M_2(C_1))/CU(M_2(C_1)) \to K_1(C_1)$ is the quotient map (note that C has stable rank one and $C_1 = C \otimes C(\mathbb{T})$ has stable rank two). Then $(\alpha, \lambda, \gamma)$ is compatible. The corollary then follows from the previous one. □

LEMMA 22.14. *Let $B \in \mathcal{B}_1$ be an amenable C^*-algebra which satisfies the UCT, let $A_1 \in \mathcal{B}_0$, let $C = B \otimes U_1$, and let $A = A_1 \otimes U_2$, where U_1 and U_2 are UHF-algebras of infinite type. Suppose that $\kappa \in KL_e(C, A)^{++}$, $\gamma : T(A) \to T(C)$ is a continuous affine map and $\alpha : U(C)/CU(C) \to U(A)/CU(A)$ is a continuous homomorphism for which γ, α, and κ are compatible. Then there exists a unital monomorphism $\varphi : C \to A$ such that*

(1) $[\varphi] = \kappa$ *in* $KL_e(C, A)^{++}$,
(2) $\varphi_T = \gamma$ *and* $\varphi^{\ddagger} = \alpha$.

PROOF. The proof follows the same lines as that of Lemma 8.5 of [40]. By Theorem 9.11 of [21], every C^*-algebra $B \in \mathcal{B}_1$ has weakly unperforated $K_0(A)$. Then, by Corollary 19.3 of [21], $B \otimes U_1 \in \mathcal{B}_0$. By the classification theorem (Theorem 21.9 and Theorem 14.10 of [21]), one can write

$$C = \varinjlim (C_n, \varphi_{n,n+1})$$

where C_n is a direct sum of C^*-algebras in $\mathcal{C}_0$ or in $\mathbf{H}$. Let $\kappa_n = \kappa \circ [\varphi_{n,\infty}]$, $\alpha_n = \alpha \circ \varphi^{\ddagger}_{n,\infty}$, and $\gamma_n = (\varphi_{n,\infty})_T \circ \gamma$. Write $C_n = C_n^1 \oplus C_n^2$ with $C_n^1 \in \mathbf{H}$ and $C_n^2 \in \mathcal{C}_0$. By Corollary 22.5 applying to C_n^1 and Corollary 22.12 applying to to C_n^2, there are unital monomorphisms $\psi_n : C_n \to A$ such that

$$[\psi_n] = \kappa_n \quad \psi_n^{\ddagger} = \alpha_n, \quad \text{and} \quad (\psi_n)_T = \gamma_n.$$

(Note that $K_1(C_n^2) = 0$, Consequently, $(\psi_n|_{C_n^2})_T = (\imath_{C_n^2, C_n})_T \circ \gamma_n$ implies $(\psi_n|_{C_n^2})^{\ddagger} = \alpha_n|_{U(C_n^2)/CU(C_n^2)}$.) In particular, the sequence of monomorphisms ψ_n satisfies

$$[\psi_{n+1} \circ \varphi_{n,n+1}] = [\psi_n], \quad \psi^{\ddagger}_{n+1} \circ \varphi_{n,n+1} = \psi_n^{\ddagger}, \quad \text{and} \quad (\psi_{n+1} \circ \varphi_{n,n+1})_T = (\psi_n)_T.$$

Let $\mathcal{F}_n \subset C_n$ be a finite subset such that $\varphi_{n,n+1}(\mathcal{F}_n) \subset \mathcal{F}_{n+1}$ and $\bigcup \varphi_{n,\infty}(\mathcal{F}_n)$ is dense in C. Applying Theorem 12.7 of [21] with $\Delta(h) = \inf\{\gamma(\tau)(\varphi_{n,\infty}(h)) : \tau \in T(A)\}$, $h \in C_n^+ \setminus \{0\}$, we have a sequence of unitaries $u_n \in A$ such that

$$\mathrm{Ad}u_{n+1} \circ \psi_{n+1} \circ \varphi_{n,n+1} \approx_{1/2^n} \mathrm{Ad}u_n \circ \psi_n \quad \text{on } \mathcal{F}_n.$$

The maps $\{\mathrm{Ad}u_n \circ \psi_n : n = 1, 2, ...\}$ then converge to a unital homomorphism $\varphi : C \to A$ which satisfies the lemma. $\square$

REMARK 22.15. In the first few lines of the proof of Lemma 22.14, we recall that, if $B \in \mathcal{B}_1$ is a unital separable simple C^*-algebra with the UCT. Then $C := B \otimes U_1$ (for any infinite dimensional UHF-algebra U_1) is an inductive limit of C^*-algebras C_n, where C_n is a finite direct sum of C^*-algebras in $\mathcal{C}_0$ or in $\mathbf{H}$. This important fact which proved in the first part of this research ([21]) will be used frequently in the rest of the paper.

THEOREM 22.16. *Let X be a finite CW complex and let $C = PM_n(C(X))P$, where $n \geq 1$ is an integer and $P \in M_n(C(X))$ is a projection. Let $A_1 \in \mathcal{B}_0$ and let $A = A_1 \otimes U$ for a UHF-algebra U of infinite type. Suppose $\alpha \in KL_e(C, A)^{++}$, $\lambda : U_\infty(C)/CU_\infty(C) \to U(A)/CU(A)$ is a continuous homomorphism, and $\gamma : T(A) \to T_f(C)$ is a continuous affine map such that $(\alpha, \lambda, \gamma)$ is compatible. Then there exists a unital homomorphism $h : C \to A$ such that*

$$[h] = \alpha, \ \ h^{\ddagger} = \lambda \ \text{ and } \ h_T = \gamma. \tag{e 22.30}$$

PROOF. The proof is similar to that of 6.6 of [40]. To simplify the notation, without loss of generality, let us assume that X is connected. Furthermore, a standard argument shows that the general case can be reduced to the case $C = C(X)$. We may assume that $U(M_N(C))/U_0(M_N(C)) = K_1(C)$ for some integer N (see [48]). Therefore, in this case,

$$U(M_N(C))/CU(M_N(C)) = U_\infty(C)/CU_\infty(C).$$

Write $K_1(C) = G_1 \oplus \mathrm{Tor}(K_1(C))$, where G_1 is the torsion free part of $K_1(C)$. Fix a point $\xi \in X$ and let $C_0 = C_0(X \setminus \{\xi\})$. Note that C_0 is an ideal of C and $C/C_0 \cong \mathbb{C}$. Write

$$K_0(C) = \mathbb{Z} \cdot [1_C] \oplus K_0(C_0). \tag{e 22.31}$$

Let $B \in \mathcal{B}_0$ be a unital separable simple C^*-algebra as constructed in Corollary 14.14 of [21] such that

$$\begin{aligned}(K_0(B), K_0(B)_+, [1_B], T(B), r_B) \\ = (K_0(A), K_0(A)_+, [1_A], T(A), r_A)\end{aligned} \tag{e 22.32}$$

and $K_1(B) = G_1 \oplus \mathrm{Tor}(K_1(A))$. Put

$$\Delta(\hat{g}) = \inf\{\gamma(\tau)(g) : \tau \in T(A)\}. \tag{e 22.33}$$

For each $g \in C_+ \setminus \{0\}$, since $\gamma(\tau) \in T_f(C)$, γ is continuous and $T(A)$ is compact, $\Delta(\hat{g}) > 0$.

Let $\varepsilon > 0$, $\mathcal{F} \subset C$ be a finite subset, $1 > \sigma_1, \sigma_2 > 0$, $\mathcal{H} \subset C_{s.a.}$ be a finite subset, and $\mathcal{U} \subset U(M_N(C))/CU(M_N(C))$ be a finite subset. Without loss of generality, we may assume that $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1$, where $\mathcal{U}_0 \subset U_0(M_N(C))/CU(M_N(C))$ and $\mathcal{U}_1 \subset J_c(K_1(C)) \subset U(M_N(C))/CU(M_N(C))$.

For each $u \in \mathcal{U}_0$, write $u = \prod_{j=1}^{n(u)} \exp(\sqrt{-1}a_i(u))$, where $a_i(u) \in M_N(C)_{s.a.}$. Write

$$a_i(u) = (a_i^{(k,j)}(u))_{N\times N}, \;\; i = 1, 2, ..., n(u). \tag{e 22.34}$$

Write

$$c_{i,k,j}(u) = \frac{a_i^{(k,j)}(u) + (a_i^{(k,j)})^*}{2} \;\text{ and }\; d_{i,k,j} = \frac{a_i^{(k,j)}(u) - (a_i^{(k,j)})^*}{2i}.$$

Put

$$M = \max\{\|c\|, \|c_{i,k,j}(u)\|, \|d_{i,k,j}(u)\| : c \in \mathcal{H}, u \in \mathcal{U}_0\}. \tag{e 22.35}$$

Choose a non-zero projection $e \in B$ such that

$$\tau(e) < \frac{\min\{\sigma_1, \sigma_2\}}{16N^2(M+1)\max\{n(u) : u \in \mathcal{U}_0\}} \;\text{ for all }\; \tau \in T(B).$$

Let $B_2 = (1-e)B(1-e)$.

In what follows we will use the identification (e 22.32). Define $\kappa_0 \in \mathrm{Hom}(K_0(C), K_0(B_2))$ as follows. Define $\kappa_0(m[1_C]) = m[1-e]$ for $m \in \mathbb{Z}$ and $\kappa_0|_{K_0(C_0)} = \alpha|_{K_0(C_0)}$. Note that $K_1(B) = G_1 \oplus \mathrm{Tor}(K_1(A))$ and that α induces a map $\alpha|_{\mathrm{Tor}(K_1(C))} : \mathrm{Tor}(K_1(C)) \to \mathrm{Tor}(K_1(A))$. Using the given decomposition $K_1(C) = G_1 \oplus \mathrm{Tor}(K_1(C)$, we can define $\kappa_1 : K_1(C) \to K_1(B)$ by $\kappa_1|_{G_1} = \mathrm{id}$ and $\kappa_1|_{\mathrm{Tor}(K_1(C))} = [\alpha]|_{\mathrm{Tor}(K_1(C))}$.

By the Universal Coefficient Theorem, there is $\kappa \in KL(C, B_2)$ which gives rise to the two homomorphisms κ_0, κ_1 above. Note that $\kappa \in KL_e(C, B_2)^{++}$, since $K_0(C_0) = \mathrm{ker}\rho_C(K_0(C))$. Choose

$$\mathcal{H}_1 = \mathcal{H} \cup \{c_{i,k,j}(u), d_{i,k,j}(u) : u \in \mathcal{U}_0\}.$$

Every tracial state τ' of B_2 has the form $\tau'(b) = \tau(b)/\tau(1-e)$ for all $b \in B_2$ for some $\tau \in T(B)$. Let $\gamma' : T(B_2) \to T(C)$ be defined as follows. For $\tau' \in T(B_2)$ as above, define $\gamma'(\tau')(f) = \gamma(\tau)(f)$ for $f \in C$.

It follows from 22.3 that there exists a sequence of unital completely positive linear maps $h_n : C \to B_2$ such that

$$\lim_{n\to\infty} \|h_n(ab) - h_n(a)h_n(b)\| = 0 \text{ for all } a, b \in C,$$
$$[h_n] = \kappa \quad (K_*(C) \text{ is finitely generated}), \text{ and}$$
$$\lim_{n\to\infty} \max\{|\tau \circ h_n(c) - \gamma'(\tau)(c)| : \tau \in T(B_2)\} = 0 \text{ for all } c \in C.$$

Here we may assume that $[h_n]$ is well defined for all n and

$$|\tau \circ h_n(c) - \gamma(\tau)(c)| < \frac{\min\{\sigma_1, \sigma_2\}}{8N^2}, \ \ n = 1, 2, ... \tag{e 22.36}$$

for all $c \in \mathcal{H}_1$ and for all $\tau \in T(B_2)$. Choose $\theta \in KL(B, A)$ such that it gives the identification of (e 22.32), and, $\theta|_{G_1} = \alpha|_{G_1}$ and $\theta|_{\mathrm{Tor}(K_1(A))} = \mathrm{id}_{\mathrm{Tor}(K_1(A))}$. Let $e' \in A$ be a projection such that $[e'] \in K_0(A)$ corresponds to $[e] \in K_0(B)$ under the identification (e 22.32). Let $\beta = \alpha - \theta \circ \kappa$. Then

$$\beta([1_C]) = [e'], \ \ \beta_{K_0(C_0)} = 0, \ \text{ and } \ \beta_{K_1(C)} = 0. \tag{e 22.37}$$

Then $\beta \in KL_e(C, e'Ae')$. It follows from 22.3 that there exists a sequence of unital completely positive linear maps $\varphi_{0,n} : C \to e'Ae'$ such that

$$\lim_{n\to\infty} \|\varphi_{0,n}(ab) - \varphi_{0,n}(a)\varphi_{0,n}(b)\| = 0 \text{ and } [\varphi_{0,n}] = \beta. \tag{e 22.38}$$

Note that, for each $u \in U(M_N(C))$ with $\bar{u} \in \mathcal{U}_0$,

$$D_C(u) = \overline{\sum_{i=1}^{n(u)} \widehat{a_j(u)}}, \tag{e 22.39}$$

where $\widehat{c}(\tau) = \tau(c)$ for all $c \in C_{s.a.}$ and $\tau \in T(C)$. Since κ and λ are compatible, we compute, for $\bar{u} \in \mathcal{U}_0$,

$$\mathrm{dist}((h_n)^{\ddagger}(\bar{u}), \lambda(\bar{u})) < \sigma_2/8. \tag{e 22.40}$$

Fix a pair of large integers n, m, and define $\chi_{n,m} : J_c(G_1)(\subset U(C)/CU(C)) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ to be

$$\lambda|_{J_c(G_1)} - (h_n)^{\ddagger}|_{J_c(G_1)} - \varphi_{0,m}^{\ddagger}|_{J_c(G_1)}. \tag{e 22.41}$$

We may may also view $J_c(G_1)$ as subgroup of $J_c(K_1(B))= J_c(K_1(B_2))$. Write $J_c(K_1(B)) = J_c(G_1) \oplus J_c(\mathrm{Tor}(K_1(B_2)))$ and define $\chi_{n,m}$ to be zero on $\mathrm{Tor}(K_1(B_2))$, we obtain a homomorphism $\chi_{n,m} : J_c(K_1(B_2)) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$. It follows from Lemma 22.14 that there is a unital homomorphism $\psi : B_2 \to (1-e')A(1-e')$ such that

$$[\psi] = \theta, \ \psi_T = \mathrm{id}_{T(A)} \ \text{ and} \tag{e 22.42}$$

$$\psi^\ddagger|_{J_c(K_1(B_2))} = \chi_{n,m}|_{J_c(K_1(B_2))} + J_c \circ \theta|_{K_1(B_2)}, \tag{e 22.43}$$

where we identify $K_1(B_2)$ with $K_1(B)$. By (e 22.42),

$$\psi^\ddagger|_{\mathrm{Aff}(T(B_2))/\overline{\rho_{B_2}(K_0(B_2))}} = \mathrm{id}. \tag{e 22.44}$$

Define $L(c) = \varphi_{0,m}(c) \oplus \psi \circ h_n(c)$ for all $c \in C$. It follows, on choosing sufficiently large m and n, that L is ε-$\mathcal{F}$-multiplicative,

$$[L] = \alpha, \tag{e 22.45}$$

$$\max\{|\tau \circ \psi(f) - \gamma(\tau)(f)| : \tau \in T(A)\} < \sigma_1 \ \text{ for all } \ f \in \mathcal{H}, \ \text{ and} \tag{e 22.46}$$

$$\mathrm{dist}(L^\ddagger(\bar{u}), \lambda(\bar{u})) < \sigma_2. \tag{e 22.47}$$

This implies that that there is a sequence of contractive completely positive linear maps $\psi_n : C \to A$ such that

$$\lim_{n\to\infty} \|\psi_n(ab) - \psi_n(a)\psi_n(b)\| = 0 \ \text{ for all } \ a, b \in C, \tag{e 22.48}$$

$$[\psi_n] = \alpha, \tag{e 22.49}$$

$$\lim_{n\to\infty} \max\{|\tau \circ \psi_n(c) - \gamma(\tau)(c)| : \tau \in T(A_1)\} = 0 \quad \text{for all } \ c \in C_{s.a.}, \ \text{ and} \tag{e 22.50}$$

$$\lim_{n\to\infty} \mathrm{dist}(\psi_n^\ddagger(\bar{u}), \lambda(\bar{u})) = 0 \ \text{ for all } \ u \in U(M_N(C))/CU(M_N(C)). \tag{e 22.51}$$

Finally, applying Theorem 12.7 of [21], as in the proof of 22.5, using $\Delta/2$ above, we obtain a unital homomorphism $h : C \to A$ such that

$$[h] = \alpha, \ h_T = \gamma, \ \text{ and } \ h^\ddagger = \lambda, \tag{e 22.52}$$

as desired. □

THEOREM 22.17. *Let $C \in \mathcal{C}_0$ and let $G = K_0(C)$. Write $G = \mathbb{Z}^k$ with $\mathbb{Z}^k$ generated by*

$$\{x_1 = [p_1] - [q_1], x_2 = [p_2] - [q_2], ..., x_k = [p_k] - [q_k]\},$$

where $p_i, q_i \in M_n(C)$ (for some integer $n \geq 1$) are projections, $i = 1, ..., k$.

Let A be a simple C-algebra in $\mathcal{B}_0$, and let $B = A \otimes U$ for a UHF algebra U of infinite type. Suppose that $\varphi : C \to B$ is a monomorphism. Then, for any finite subsets $\mathcal{F} \subset C$ and $\mathcal{P} \subset \underline{K}(C)$, any $\varepsilon > 0$ and $\sigma > 0$, and any homomorphism*

$$\Gamma : \mathbb{Z}^k \to U_0(B)/CU(B),$$

there is a unitary $w \in B$ such that

(1) $\|[\varphi(f), w]\| < \varepsilon$, *for any* $f \in \mathcal{F}$,
(2) $\mathrm{Bott}(\varphi, w)|_{\mathcal{P}} = 0$, *and*
(3) $\mathrm{dist}(\overline{\langle((\mathbf{1}_n - \varphi(p_i)) + \varphi(p_i)\tilde{w})((\mathbf{1}_n - \varphi(q_i)) + \varphi(q_i)\tilde{w}^*)\rangle}, \Gamma(x_i))) < \sigma$, *for any* $1 \leq i \leq k$, *where* $\tilde{w} = \mathrm{diag}(\overbrace{w, ..., w}^{n})$.

PROOF. Write $B = \lim_{n\to\infty}(A \otimes M_{r_n}, \imath_{n,n+1})$. Using the fact that C is semiprojective (see [9]), one can construct a sequence of homomorphisms $\varphi_n : C \to A \otimes M_{r_n}$ such that $\imath_n \circ \varphi_n(c) \to \varphi(c)$ for all $c \in C$. Without loss of generality, we may assume $\varphi = \imath \circ \varphi_1$ for a homomorphism $\varphi_1 : C \to A$ (replacing A by $A \otimes M_{r_n}$), where $\imath : A \to A \otimes U = B$ is the standard inclusion.

We may assume that $||f|| \leq 1$ for any $f \in \mathcal{F}$.

For any non-zero positive element $h \in C$ with norm at most 1, define

$$\Delta(h) = \inf\{\tau(\varphi(h));\ \tau \in T(B)\}.$$

Since B is simple, one has that $\Delta(h) \in (0, 1)$.

Let $\mathcal{H}_1 \subset C_+^{\mathbf{1}} \setminus \{0\}$, $\mathcal{G} \subset C$, $\delta > 0$, $\mathcal{P} \subset \underline{K}(C)$, $\mathcal{H}_2 \subset C_{s.a.}$, and $\gamma_1 > 0$ be the finite subsets and constants of Theorem 12.7 of [21] with respect to C, $\mathcal{F}$, $\epsilon/2$, and $\Delta/2$ (since $K_1(C) = \{0\}$, one does not need $\mathcal{U}$ and γ_2).

Note that $B = A \otimes U$. Pick a unitary $z \in U$ with $\mathrm{sp}(u) = \mathbb{T}$ and consider the homomorphism $\varphi' : C \otimes C(\mathbb{T}) \to B = A \otimes U$ defined by

$$a \otimes f \mapsto \varphi_1(a) \otimes f(z).$$

(Recall that $\varphi(a) = \varphi_1(a) \otimes 1_U$.) Set

$$\gamma = (\varphi')_T : T(B) \to T_{\mathrm{f}}(C \otimes C(\mathbb{T})).$$

Also define

$$\alpha := [\varphi'] \in KK(C \otimes C(\mathbb{T}), B).$$

Note that $K_1(C \otimes C(\mathbb{T})) = K_0(C) = \mathbb{Z}^k$. Identifying $J_c(K_1(C \otimes C(\mathbb{T})))$ with $\mathbb{Z}^k$, define a map $\lambda : J_c(K_1(U(C \otimes C(\mathbb{T})))) \to U_0(B)/CU(B)$ by $\lambda(a) = \Gamma(a)$ for any $a \in \mathbb{Z}^k$.

Set

$$\mathcal{U} = \{(1_n - p_i + p_i\tilde{z'})(1_n - q_i + q_i\tilde{z'}^*) :\ i = 1, ..., k\} \subset J_c(U(C \otimes C(\mathbb{T}))),$$

where z' is the standard generator of $C(\mathbb{T})$, and set

$$\delta = \min\{\Delta(h)/4 :\ h \in \mathcal{H}_1\}.$$

Applying Lemma 22.11, one obtains a $\mathcal{F}$-$\epsilon/4$-multiplicative map $\Phi : C \otimes C(\mathbb{T}) \to B$ such that

(e 22.53) $[\Phi] = \alpha$, $\mathrm{dist}(\Phi^\ddagger(x), \lambda(x)) < \sigma$ for all $x \in \overline{\mathcal{U}}$, and

(e 22.54) $|\tau \circ \Phi(h \otimes 1) - \gamma(\tau)(h \otimes 1)| < \min\{\gamma_1, \delta\}$ for all $h \in \mathcal{H}_1 \cup \mathcal{H}_2$.

Let ψ denote the restriction of Φ to $C \otimes 1$. Then one has

$$[\psi]|_{\mathcal{P}} = [\varphi]|_{\mathcal{P}}.$$

By (e 22.54), one has that, for any $h \in \mathcal{H}_1$,

$$\tau(\psi(h)) > \gamma(\tau)(h) - \delta = \tau(\varphi'(h \otimes 1)) - \delta = \tau(\varphi(h)) - \delta > \Delta(h)/2,$$

and it is also clear that

$$\tau(\varphi(h)) > \Delta(h)/2 \text{ for all } h \in \mathcal{H}_1.$$

Moreover, for any $h \in \mathcal{H}_2$, one has

$$\begin{aligned} |\tau \circ \psi(h) - \tau \circ \varphi(h)| &= |\tau \circ \Phi(h \otimes 1) - \tau \circ \varphi'(h \otimes 1)| \\ &= |\tau \circ \Phi(h \otimes 1) - \gamma(\tau)(h \otimes 1)| < \gamma_1. \end{aligned}$$

Therefore, by Theorem 12.7 of [21], there is a unitary $W \in B$ such that

$$||W^*\psi(f)W - \varphi(f)|| < \epsilon/2 \text{ for all } f \in \mathcal{F}.$$

Then the element

$$w = W^*\Phi(1 \otimes z')W$$

is the desired unitary. □

Theorem 22.18. *Let C be a unital C^*-algebra which is a finite direct sum of C^*-algebras in $\mathcal{C}_0$ and C^*-algebras of the form $PM_n(C(X))P$, where X is a finite CW complex and P is a projection, and let $G = K_0(C)$. Write $G = \mathbb{Z}^k \bigoplus \mathrm{Tor}(G)$ with a basis for $\mathbb{Z}^k$ the set*

$$\{x_1 = [p_1] - [q_1], x_2 = [p_2] - [q_2], ..., x_k = [p_k] - [q_k]\},$$

where $p_i, q_i \in M_n(C)$ (for some integer $n \geq 1$) are projections, $i = 1, ..., k$.

Let A be a simple C-algebra in the class $\mathcal{B}_0$, and let $B = A \otimes U$ for a UHF algebra U of infinite type. Suppose that $\varphi : C \to B$ is a monomorphism. Then, for any finite subsets $\mathcal{F} \subset C$ and $\mathcal{P} \subset \underline{K}(C)$, any $\varepsilon > 0$ and $\sigma > 0$, and any homomorphism*

$$\Gamma : \mathbb{Z}^k \to U_0(M_n(B))/CU(M_n(B)),$$

there is a unitary $w \in B$ such that

(1) $\|[\varphi(f), w]\| < \varepsilon$, *for any $f \in \mathcal{F}$,*
(2) $\mathrm{Bott}(\varphi, w)|_{\mathcal{P}} = 0$, *and*
(3) $\mathrm{dist}(\overline{\langle((\mathbf{1}_n - \varphi(p_i)) + \varphi(p_i)\tilde{w})((\mathbf{1}_n - \varphi(q_i)) + \varphi(q_i)\tilde{w}^*)\rangle}, \Gamma(x_i))) < \sigma$, *for any $1 \leq i \leq k$, where $\tilde{w} = \mathrm{diag}(\overbrace{w, ..., w}^{n})$.*

Proof. By Theorem 22.17, it suffices to prove the case that $C = PM_n(C(X))P$, where X is a finite CW complex, $n \geq 1$ is an integer, and $P \in M_n(C(X))$ is a projection. The proof follows the same lines as that of Theorem 22.17 but using Lemma 22.16 instead of Lemma 22.11. □

23. A Pair of Almost Commuting Unitaries

LEMMA 23.1. *Let $C \in \mathcal{C}$. There exists a constant $M_C > 0$ satisfying the following condition: For any $\varepsilon > 0$, any $x \in K_0(C)$, and any $n \geq M_C/\varepsilon$, if*

$$|\rho_C(x)(\tau)| < \varepsilon \text{ for all } \tau \in T(C \otimes M_n), \tag{e 23.1}$$

then there are mutually inequivalent and mutually orthogonal minimal projections $p_1, p_2, ..., p_{k_1}$ and $q_1, q_2, ..., q_{k_2}$ in $C \otimes M_n$ and positive integers $l_1, l_2, ..., l_{k_1}$, $m_1, m_2, ..., m_{k_2}$ such that

$$x = [\sum_{i=1}^{k_1} l_i p_i] - [\sum_{j=1}^{k_2} m_j q_j] \text{ and} \tag{e 23.2}$$

$$\tau(\sum_{i=1}^{k_1} l_i p_i) < 4\varepsilon \text{ and } \tau(\sum_{j=1}^{k_2} m_j q_j) < 4\varepsilon \tag{e 23.3}$$

for all $\tau \in T(C \otimes M_n)$.

PROOF. Let $C = C(F_1, F_2, \varphi_1, \varphi_2)$ and $F_1 = \bigoplus_{i=1}^{l} M_{r(i)}$. By Theorem 3.15 of [21], there are only finitely many mutually inequivalent minimal projections in $C \otimes \mathcal{K}$. We can choose $N(C) > 0$ such that this set of mutually inequivalent projections is sitting in $M_{N(C)}(C)$, orthogonally. Then every projection in $C \otimes \mathcal{K}$ is equivalent to a finite direct sum of projections from this set of finitely many mutually inequivalent minimal projections (some of them may repeat in the direct sum). We also assume that, as in Definition 3.1 of [21], C is minimal. Let

$$M_C = N(C) + 2(r(1) \cdot r(2) \cdots r(l))$$

Suppose that $n \geq M_C/\varepsilon$. With the canonical embedding of $K_0(C)$ into $K_0(F_1) \cong \mathbb{Z}^l$, write

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_l \end{pmatrix} \in \mathbb{Z}^l. \tag{e 23.4}$$

By (e 23.1), for any irreducible representation π of C and any tracial state t on $M_n(\pi(C))$,

$$|t \circ \pi(x)| < \varepsilon. \tag{e 23.5}$$

It follows that

$$|x_s|/r(s)n < \varepsilon, \quad s = 1, 2, ..., l. \tag{e 23.6}$$

Let

$$T = \max\{|x_s|/r(s) : 1 \le s \le l\}. \tag{e 23.7}$$

Define

$$y = x + T\begin{pmatrix} r(1)\\ r(2)\\ \vdots\\ r(l)\end{pmatrix} \text{ and } z = T\begin{pmatrix} r(1)\\ r(2)\\ \vdots\\ r(l)\end{pmatrix}. \tag{e 23.8}$$

It is clear that $z \in K_0(C)_+$ (see Proposition 3.5 of [21]). It follows that $y \in K_0(C)$. One also computes that $y \in K_0(C)_+$. It follows that there are projections $p,\, q \in M_L(C)$ for some integer $L \ge 1$ such that $[p] = y$ and $[q] = z$. Moreover, $x = [p] - [q]$. One also computes that

$$\tau(q) < T/n < \varepsilon \text{ for all } \tau \in T(C \otimes M_n). \tag{e 23.9}$$

One also has

$$\tau(p) < 2\varepsilon \text{ for all } \tau \in T(C \otimes M_n). \tag{e 23.10}$$

There are two sets of mutually inequivalent and mutually orthogonal minimal projections $\{p_1, p_2, ..., p_{k_1}\}$ and $\{q_1, q_2, ..., q_{k_2}\}$ in $C \otimes M_n$ (since $n > N(C)$) such that

$$[p] = \sum_{i=1}^{k_1} l_i[p_i] \text{ and } [q] = \sum_{j=1}^{k_2} m_j[q_j]. \tag{e 23.11}$$

Therefore

$$x = \sum_{i=1}^{k_1} l_i[p_i] - \sum_{j=1}^{k_2} m_j[q_j]. \tag{e 23.12}$$

□

LEMMA 23.2. *Let $C \in \mathcal{C}$. There is an integer $M_C > 0$ satisfying the following condition: For any $\varepsilon > 0$ and for any $x \in K_0(C)$ with*

$$|\tau(\rho_C(x))| < \varepsilon/24\pi$$

for all $\tau \in T(C \otimes M_n)$, where $n \ge 2M_C\pi/\varepsilon$, there exists a pair of unitaries u and $v \in C \otimes M_n$ such that

$$\|uv - vu\| < \varepsilon \text{ and } \tau(\mathrm{bott}_1(u,\, v)) = \tau(x). \tag{e 23.13}$$

PROOF. We may assume that C is minimal. Applying Lemma 23.1, we obtain mutually orthogonal and mutually inequivalent minimal projections $p_1, p_2, ..., p_{k_1}, q_1, q_2, ..., q_{k_2} \in C \otimes M_n$ such that

$$\sum_{i=1}^{k_1} l_i[p_i] - \sum_{j=1}^{k_2} m_j[q_j] = x,$$

where $l_1, l_2, ..., l_{k_1}, m_1, m_2, ..., m_{k_2}$ are positive integers. Moreover,

$$\sum_{i=1}^{k_1} l_i\tau(p_i) < \varepsilon/6\pi \text{ and } \sum_{j=1}^{k_2} m_j\tau(q_j) < \varepsilon/6\pi \tag{e 23.14}$$

for all $\tau \in T(C \otimes M_n)$. Choose $N \le n$ such that $N = [2\pi/\varepsilon] + 1$. By (e 23.14),

$$\sum_{i=1}^{k_1} Nl_i\tau(p_i) + \sum_{j=1}^{k_2} Nm_j\tau(q_j) < 1/2 \text{ for all } \tau \in T(C \otimes M_n). \tag{e 23.15}$$

It follows that there are mutually orthogonal projections $d_{i,k}, d'_{j,k} \in C \otimes M_n$, $k = 1, 2, ..., N$, $i = 1, 2, ..., k_1$, and $j = 1, 2, ..., k_2$ such that

$$[d_{i,k}] = l_i[p_i] \text{ and } [d'_{j,k}] = m_j[q_j], \tag{e 23.16}$$

$i = 1, 2, ..., k_1$, $j = 1, 2, ..., k_2$ and $k = 1, 2, ..., N$. Let $D_i = \sum_{k=1}^N d_{i,k}$ and $D'_j = \sum_{k=1}^N d'_{j,k}$, $i = 1, 2, ..., k_1$ and $j = 1, 2, ..., k_2$. There are partial isometries $s_{i,k}$, $s'_{j,k} \in C \otimes M_n$ such that

$$s_{i,k}^* d_{i,k} s_{i,k} = d_{i,k+1}, \ (s'_{j,k})^* d'_{j,k} s'_{j,k} = d'_{j,k+1}, \ k = 1, 2, ..., N-1, \tag{e 23.17}$$

$$s_{i,N}^* d_{i,N} s_{i,N} = d_{i,1}, \text{ and } (s'_{j,N})^* d'_{j,N} s'_{j,N} = d'_{j,1}, \tag{e 23.18}$$

$i = 1, 2, ..., k_1$ and $j = 1, 2, ..., k_2$. Thus, we obtain unitaries $u_i \in D_i(C \otimes M_n)D_i$ and $u'_j = D'_j(C \otimes M_n)D'_j$ such that

$$\begin{aligned} &u_i^* d_{i,k} u_i = d_{i,k+1}, \ u_i^* d_{i,N} u_i = d_{i,1}, \\ &(u'_j)^* d'_{j,k} u'_j = d'_{j,k+1}, \text{ and } (u'_j)^* d'_{j,N} u'_j = d'_{j,1}, \end{aligned} \tag{e 23.19}$$

$i = 1, 2, ..., k_1$, $j = 1, 2, ..., k_2$. Define

$$v_i = \sum_{k=1}^N e^{\sqrt{-1}(2k\pi/N)} d_{i,k} \text{ and } v'_j = \sum_{k=1}^N e^{\sqrt{-1}(2k\pi/N)} d'_{j,k}.$$

We compute that

$$\|u_iv_i - v_iu_i\| < \varepsilon \text{ and } \|u_j'v_j' - v_j'u_j'\| < \varepsilon, \tag{e 23.20}$$

$$\frac{1}{2\pi\sqrt{-1}}\tau(\log v_iu_iv_i^*u_i^*) = l_i\tau(p_i), \text{ and} \tag{e 23.21}$$

$$\frac{1}{2\pi\sqrt{-1}}\tau(\log v_j'u_j'(v_j')^*(u_j')^*) = m_j\tau(q_j), \tag{e 23.22}$$

for $\tau \in T(C \otimes M_n)$, $i = 1, 2, ..., k_1$ and $j = 1, 2, ..., k_2$. Now define

$$u = \sum_{i=1}^{k_1} u_i + \sum_{j=1}^{k_2} u_j' + (1_{C\otimes M_n} - \sum_{i=1}^{k_1} D_i - \sum_{j=1}^{k_2} D_j') \text{ and} \tag{e 23.23}$$

$$v = \sum_{i=1}^{k_1} v_i + \sum_{j=1}^{k_2} (v_j')^* + (1_{C\otimes M_n} - \sum_{i=1}^{k_1} D_i - \sum_{j=1}^{k_2} D_j'). \tag{e 23.24}$$

We then compute that

$$\begin{aligned} \tau(\mathrm{bott}_1(u, v)) &= \sum_{i=1}^{k_1} \frac{1}{2\pi\sqrt{-1}}\tau(\log(v_iu_iv_i^*u_i^*)) \\ &\quad -\sum_{j=1}^{k_2} \frac{1}{2\pi\sqrt{-1}}\tau(\log v_j'u_j'(v_j')^*(u_j')^*) \\ &= \sum_{i=1}^{k_1} l_i\tau(p_i) - \sum_{j=1}^{k_2} m_j\tau(q_j) = \tau(x) \end{aligned}$$

for all $\tau \in T(C \otimes M_n)$. □

LEMMA 23.3. *Let $\varepsilon > 0$. There exists $\sigma > 0$ satisfying the following condition: Let $A = A_1 \otimes U$, where U is a UHF-algebra of infinite type and $A_1 \in \mathcal{B}_0$, let $u \in U(A)$ be a unitary with $sp(u) = \mathbb{T}$, and let $x \in K_0(A)$ with $|\tau(\rho_A(x))| < \sigma$ for all $\tau \in T(A)$ and $y \in K_1(A)$. Then there exists a unitary $v \in U(A)$ such that*

$$\|uv - vu\| < \varepsilon, \ \mathrm{bott}_1(u, v) = x, \ \textit{and } [v] = y. \tag{e 23.25}$$

PROOF. Let $\varphi_0 : C(\mathbb{T}) \to A$ be the unital monomorphism defined by $\varphi_0(f) = f(u)$ for all $f \in C(\mathbb{T})$. Let $\Delta_0 : C(\mathbb{T})_+^{q,\mathbf{1}} \setminus \{0\} \to (0, 1)$ be defined by $\Delta_0(\hat{f}) = \inf\{\tau(\varphi_0(f)) : \tau \in T(A)\}$. Let $\varepsilon > 0$ be given. Choose $0 < \varepsilon_1 < \varepsilon$ such that

$$\mathrm{bott}_1(z_1, z_2) = \mathrm{bott}_1(z_1', z_2')$$

for any two pairs of unitaries z_1, z_2 and z_1', z_2' satisfying the conditions $\|z_1 - z_1'\| < \varepsilon_1$, $\|z_2 - z_2'\| < \varepsilon_1$, $\|z_1z_2 - z_2z_1\| < \varepsilon_1$ and $\|z_1'z_2' - z_2'z_1'\| < \varepsilon_1$.

Let $\mathcal{H}_1 \subset C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\}$ be a finite subset, $\gamma_1 > 0$, $\gamma_2 > 0$, and $\mathcal{H}_2 \subset C(\mathbb{T})_{s.a.}$ be a finite subset as provided by Corollary 12.9 of [21] (for $\varepsilon_1/4$ and $\Delta_0/2$). We may assume that $\mathcal{H}_2 \subset C(\mathbb{T})^{\mathbf{1}}$.

Let

$$\delta_1 = \min\{\gamma_1/16, \gamma_2/16, \min\{\Delta_0(\hat{f}) : f \in \mathcal{H}_1\}/4\}.$$

Let $\sigma = \min\{\delta_1/16, (\delta_1/16)(\varepsilon_1/32\pi)\}$.

Let $e \in 1 \otimes U \subset A$ be a non-zero projection such that $\tau(e) < \sigma$ for all $\tau \in T(A)$. Let $B = eAe$ (then $B \cong A \otimes U'$ for some UHF-algebra U'). It follows from Corollary 18.10 of [21] that there is a unital simple C^*-algebra $C' = \lim_{n\to\infty}(C_n, \psi_n)$, where $C_n \in \mathcal{C}_0$ and $C = C' \otimes U$ such that

$$(K_0(C), K_0(C)_+, [1_C], T(C), r_C) = (\rho_A(K_0(A)), (\rho_A(K_0(A)))_+, \rho_A([e]), T(eAe), r_A).$$

Moreover, we may assume that all ψ_n are unital.

Now suppose that $x \in K_0(A)$ with $|\tau(\rho_A(x))| < \sigma$ for all $\tau \in T(A)$ and suppose that $y \in K_1(A)$. Let $z = \rho_A(x) \in K_0(C)$. We identify z with the element in $K_0(C)$ in the identification above. We claim that, there is $n_0 \geq 1$ such that there is $x' \in K_0(C_{n_0} \otimes U)$ such that $z = (\psi_{n_0,\infty})_{*0}(x') \in K_0(C)$ and $|t(\rho_{C_{n_0}\otimes U})(x')| < \sigma$ for all $t \in T(C_{n_0} \otimes U)$.

Otherwise, there is an increasing sequence n_k, $x_k \in K_0(C_{n_k} \otimes U)$ such that

$$(\psi_{n_k,\infty})_{*0}(x_k) = z \in K_0(C) \text{ and } |t_k(\rho_{C_{n_k}\otimes U})(x_k)| \geq \sigma \qquad \text{(e 23.26)}$$

for some $t_k \in T(C_{n_k} \otimes U)$, $k = 1, 2, ...$ Let $L_k : C \to C_{n_k} \otimes U$ be such that

$$\lim_{n\to\infty} \|\psi_{n,\infty} \circ L_n(c) - c\| = 0$$

for all $c \in \psi_{k,\infty}(C_{n_k} \otimes U)$, $k = 1, 2, ...$ It follows that any limit point of $t_k \circ L_k$ is a tracial state of C. Let t_0 be one such limit. Then, by (e 23.26),

$$t_0(\rho_C(z)) \geq \sigma.$$

This proves the claim.

Write $U = \lim_{n\to\infty}(M_{r(m)}, \imath_m)$, where $\imath_m : M_{r(m)} \to M_{r(m+1)}$ is a unital embedding. Repeating the argument above, we obtain $m_0 \geq 1$ and $y' \in K_0(C_{n_0} \otimes M_{r(m_0)}) = K_0(C_{n_0})$ such that $(\imath_{m_0,\infty})_{*0}(y') = x'$ and $|t(\rho_{C_{n_0}}(y'))| < \sigma$ for all $t \in T(C_{n_0} \otimes M_{r(m_0)})$. Let $M_{C_{N_0}}$ be the constant given by Lemma 23.2. Choose $r(m_1) \geq \max\{48M_{C_{n_0}}/\sigma, r(m_0)\}$ and let $y'' = (\imath_{m_0,m_1})_{*0}(y')$. Then, we compute that

$$|t(\rho_{C_{n_0}}(y''))| < \sigma \text{ for all } t \in T(C_{n_0} \otimes M_{r(m_1)}).$$

It follows from 23.2 that there exists a pair of unitaries $u_1', v_1' \in C_{n_0} \otimes M_{r(m_1)}$ such that

$$\|u_1'v_1' - v_1'u_1'\| < \varepsilon_1/4 \text{ and } \mathrm{bott}_1(u_1', v_1') = y''. \qquad \text{(e 23.27)}$$

Put $u_1 = \imath_{m_1,\infty}(u_1')$ and $v_1 = \imath_{m_1,\infty}(v_1')$. Then (e 23.27) implies that

$$\|u_1v_1 - v_1u_1\| < \varepsilon_1/4 \text{ and } \mathrm{bott}_1(u_1, v_1) = x'. \tag{e 23.28}$$

Let $h_0 : C_{n_0} \otimes U \to M_2(eAe)$ be a homomorphism as given by Corollary 18.10 of [21] such that

$$\rho_A \circ (h_0)_{*0} = (\psi_{n_0,\infty})_{*0}. \tag{e 23.29}$$

Then the projection $e' = h_0(1_{C_{n_0}\otimes U})$ satisfies $\rho_A(e') = \rho_A(e)$. Replacing e by e', we can assume h_0 is a unital homomorphism from $C_{n_0} \otimes U$ to eAe. It follows that

$$\rho_A((h_0)_{*0}(x') - x) = 0. \tag{e 23.30}$$

Let $u_2 = h_0(u_1)$ and $v_2 = h_0(v_1)$. We have

$$\rho_A(\mathrm{bott}_1(u_2, v_2) - x) = 0. \tag{e 23.31}$$

Choose another non-zero projection $e_1 \in A$ such that $e_1e = ee_1 = 0$ and $\tau(e_1) < \delta_1/16$ for all $\tau \in T(A)$. It follows from 22.1 that there is a unital homomorphism $H : C(\mathbb{T}^2) \to e_1Ae_1$ such that

$$H_{*0}(b) = x - \mathrm{bott}_1(u_2, v_2), \tag{e 23.32}$$

where b is the Bott element in $K_0(C(\mathbb{T}^2))$. (In fact, we can also apply 22.16 here.) Thus we obtain a pair of unitaries $u_3, v_3 \in e_1Ae_1$ such that

$$u_3v_3 = v_3u_3 \text{ and } \mathrm{bott}_1(u_3, v_3) = x - \mathrm{bott}_1(u_2, v_2). \tag{e 23.33}$$

Let $e_2, e_3 \in (1 - e - e_1)A(1 - e - e_1)$ be a pair of non-zero mutually orthogonal projections such that $\tau(e_2) < \delta_1/32$ and $\tau(e_3) < \delta_1/32$ for all $\tau \in T(A)$. Thus $\tau(e + e_1 + e_2 + e_3) < 3\delta_1/16$ for all $\tau \in T(A)$. Then, together with Theorem 17.3 of [21], (applied to $X = \mathbb{T}$), we obtain a unitary $u_4 \in (1 - e - e_1 - e_2 - e_3)A(1 - e - e_1 - e_2 - e_3)$ such that

$$|\tau(f(u_4)) - \tau(f(u))\| < \delta_1/4 \tag{e 23.34}$$

for all $f \in \mathcal{H}_2 \cup \mathcal{H}_1$ and for all $\tau \in T(A)$. Let $w = u_2 + u_3 + u_4 + (1 - e - e_1 - e_2 - e_3)$. It follows from Theorem 3.10 of [22] that there exists $u_5 \in U(e_2Ae_2)$ such that

$$\overline{u_5} = \bar{u}\bar{w}^* \in U(A)/CU(A). \tag{e 23.35}$$

Since A is simple and has stable rank one, there exists a unitary $v_4 \in e_3Ae_3$ such that $[v_4] = y - [v_2 + v_3 + (e_2 + e_3)] \in K_1(A)$. Now define

$$u_6 = u_2 + u_3 + u_4 + u_5 + e_3 \text{ and } v_6 = v_2 + v_3 + (1 - e - e_1 - e_2 - e_3) + e_2 + v_4.$$

Then

$$\|u_6v_6 - v_6u_6\| < \varepsilon_1/2, \ \ \mathrm{bott}_1(u_6, v_6) = x, \ \text{ and } \ [v_6] = y. \tag{e 23.36}$$

Moreover,

$$\tau(f(u_6)) \geq \Delta(\hat{f})/2 \ \text{ for all } \ f \in \mathcal{H}_1, \tag{e 23.37}$$

$$|\tau(f(u)) - \tau(f(u_6))| < \gamma_1 \ \text{ and } \ \bar{u_6} = \bar{u}. \tag{e 23.38}$$

By Corollary 12.7 of [21] and by (e 23.35) (e 23.37) and (e 23.38) that there exists a unitary $W \in A$ such that

$$\|W^* u_6 W - u\| < \varepsilon_1/2. \tag{e 23.39}$$

Now let $v = W^* v_6 W$. We compute that

$$\|uv - vu\| < \varepsilon, \ \ \mathrm{bott}_1(u, v) = \mathrm{bott}_1(u_6, v_6) = x, \ \text{ and } \ [v] = y. \tag{e 23.40}$$

□

COROLLARY 23.4. *Let $\varepsilon > 0$, $C = \bigoplus_{i=1}^k C^i = \bigoplus_{i=1}^k M_{m(i)}(C(\mathbb{T}))$. Let $\mathcal{P}_0 \subset K_0(C)$ and $\mathcal{P}_1 \subset K_1(C)$ be finite sets generating $K_0(C)$ and $K_1(C)$. There exists $\sigma > 0$ satisfying the following condition: Let $A = A_1 \otimes U$ be as in Lemma 23.3, let $\iota : C \to A$ be an embedding, and let $\alpha \in KL(A \otimes C(\mathbb{T}), A)$ be such that*

$$|\tau(\rho_A(\alpha(\boldsymbol{\beta}(w))))| < \sigma \min\{\tau'(\iota(1_{C^i}))/m(i), \ \ 1 \leq i \leq k, \ \tau' \in T(A)\},$$

for all $w \in \mathcal{P}_1$ and $\tau \in T(A)$. Then there exists a unitary $v \in \iota(1_C)A\iota(1_C)$ such that

$$\mathrm{Bott}(\iota, v)|_{\mathcal{P}_0 \cup \mathcal{P}_1} = \alpha \circ \boldsymbol{\beta}|_{\mathcal{P}_0 \cup \mathcal{P}_1}.$$

PROOF. Let $e_{11}^i \in C^i = M_{m(i)}(C(\mathbb{T}))$ be the rank one projection of the upper left corner of C^i and $u^i \in K_1(C^i)$ be the standard generator given by $ze_{11}^i + (1_{C_i} - e_{11}^i)$, where $z \in C(\mathbb{T})$ is the identity function from $\mathbb{T}$ to $\mathbb{T} \subset \mathbb{C}$. Without loss of generality, we may assume that $\mathcal{P}_0 = \{[e_{11}^i], 1 \leq i \leq k\}$ and $\mathcal{P}_1 = \{u^i, 1 \leq i \leq k\}$. Let σ be as in Lemma 23.3. For each $i \in \{1, 2, \cdots, k\}$, applying Lemma 23.3 to $\iota(e_{11}^i)A\iota(e_{11}^i)$ (in place of A), $\iota(ze_{11}^i) \in \iota(e_{11}^i)A\iota(e_{11}^i)$ (in place of u) with $x = \alpha(\boldsymbol{\beta}(u^i))$, $y = \alpha(\boldsymbol{\beta}([e_{11}^i]))$, one obtains a unitary $v_{11}^i \in \iota(e_{11}^i)A\iota(e_{11}^i)$ in place of v. Identifying $\iota(1_{C^i})A\iota(1_{C^i}) \cong \big(\iota(e_{11}^i)A\iota(e_{11}^i)\big) \otimes M_{m(i)}(\mathbb{C})$, we define $v^i = v_{11}^i \otimes 1_{m(i)}$. Finally, choose $v = v^1 \oplus v^2 \oplus \cdots \oplus v^k \in \iota(1_C)A\iota(1_C)$ to finish the proof. □

24. More Existence Theorems for Bott Elements Using Lemma 23.3, 22.1, Corollary 21.11 of [21], Lemma 18.11 of [21], and Theorem 12.11 of [21], we can show the following result:

LEMMA 24.1. *Let $A = A_1 \otimes U_1$, where A_1 is as in Theorem 14.10 of [21] and $B = B_1 \otimes U_2$, where $B_1 \in \mathcal{B}_0$ and U_1, U_2 are two UHF-algebras of infinite type. For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, and any finite subset $\mathcal{P} \subset \underline{K}(A)$, there exist $\delta > 0$ and a finite subset $\mathcal{Q} \subset K_1(A)$ satisfying the following condition: Let a unital homomorphism $\varphi : A \to B$ and $\alpha \in KL(A \otimes C(\mathbb{T}), B)$ be such that*

$$|\tau \circ \rho_B(\alpha(\boldsymbol{\beta}(x)))| < \delta \ \textit{for all} \ x \in \mathcal{Q} \ \textit{and} \ \ \textit{for all} \ \tau \in T(B). \tag{e 24.1}$$

Then there exists a unitary $u \in B$ such that

$$\|[\varphi(x), u]\| < \varepsilon \ \textit{for all} \ x \in \mathcal{F} \ \text{and} \tag{e 24.2}$$

$$\mathrm{Bott}(\varphi, u)|_{\mathcal{P}} = \alpha(\boldsymbol{\beta})|_{\mathcal{P}}. \tag{e 24.3}$$

PROOF. Let $\varepsilon_1 > 0$ and let $\mathcal{F}_1 \subset A$ be a finite subset satisfying the following condition: If

$$L, L' : A \otimes C(\mathbb{T}) \to B$$

are two unital $\mathcal{F}_1'$-ε_1-multiplicative completely positive linear maps such that

$$\|L(f) - L'(f)\| < \varepsilon_1 \ \text{for all} \ f \in \mathcal{F}_1', \tag{e 24.4}$$

where

$$\mathcal{F}_1' = \{a \otimes g : a \in \mathcal{F}_1 \ \text{and} \ g \in \{z, z^*, 1_{C(\mathbb{T})}\}\},$$

then

$$[L]|_{\boldsymbol{\beta}(\mathcal{P})} = [L']|_{\boldsymbol{\beta}(\mathcal{P})}. \tag{e 24.5}$$

Let $B_{1,n} = M_{m(1,n)}(C(\mathbb{T})) \oplus M_{m(2,n)}(C(\mathbb{T})) \oplus \cdots \oplus M_{m(k_1(1),n)}(C(\mathbb{T}))$, $B_{2,n} = PM_{r_1(n)}(C(X_n))P$, where X_n is a finite disjoint union of copies of $S^2, T_{2,k}$, and $T_{3,k}$ (for various $k \geq 1$). Let $B_{3,n}$ be a finite direct sum of C^*-algebras in $\mathcal{C}_0$ (with trivial K_1 and $\ker\rho_{B_{3,n}} = \{0\}$—see Proposition 3.5 of [21]), $n = 1, 2, ...$ Put $C_n = B_{1,n} \oplus B_{2,n} \oplus B_{3,n}$, $n = 1, 2, ...$ We may write that $A = \lim_{n\to\infty}(C_n, \imath_n)$ as in Theorem 14.10 of [21]. with the maps $\imath_n$ injective (applying Theorem 14.10 of [21] to A_1),

$$\ker\rho_A \subset (\imath_{n,\infty})_{*0}(\ker\rho_{C_n}), \ \text{and} \tag{e 24.6}$$

$$\lim_{n\to\infty} \sup\{\tau(1_{B_{1,n}} \oplus 1_{B_{2,n}}) : \tau \in T(B)\} = 0. \tag{e 24.7}$$

Let $\varepsilon_2 = \min\{\varepsilon_1/4, \varepsilon/4\}$ and let $\mathcal{F}_2 = \mathcal{F}_1 \cup \mathcal{F}$.

Let $\mathcal{P}_{1,1} \subset \underline{K}(B_{1,n_1})$, $\mathcal{P}_{2,1} \subset \underline{K}(B_{2,n_1})$ and $\mathcal{P}_{3,1} \subset \underline{K}(B_{3,n_1})$ be finite subsets such that

$$\mathcal{P} \subset [\imath_{n_1,\infty}](\mathcal{P}_{1,1}) \oplus [\imath_{n_1,\infty}](\mathcal{P}_{2,1}) \oplus [\imath_{n_1,\infty}])(\mathcal{P}_{3,1})$$

for some $n_1 \geq 1$. To simplify the notation, without lose of generality, we may assume $\mathcal{P} \subset [\imath_{n_1,\infty}](\mathcal{P}_{1,1}) \cup [\imath_{n_1,\infty}](\mathcal{P}_{2,1}) \cup [\imath_{n_1,\infty}])(\mathcal{P}_{3,1})$. Let $\mathcal{Q}'$ be a finite set of generators of $K_1(C_{n_1})$ and let $\mathcal{Q} = [\imath_{n_1,\infty}](\mathcal{Q}')$. Since $K_i(B_{1,n_1}),\ i = 0, 1$, are finitely generated free abelian groups, without loss of generality, we may assume that $\mathcal{P}_{1,1} \subset K_0(B_{1,n_1}) \cup K_1(B_{1,n_1})$ and generates $K_0(B_{1,n_1}) \oplus K_1(B_{1,n_1})$.

Without loss of generality, we may assume that $\mathcal{F}_1 \cup \mathcal{F} \subset \imath_{n_1,\infty}(C_{n_1})$. Let $\mathcal{F}_{1,1} \subset B_{1,n_1}$, $\mathcal{F}_{2,1} \subset B_{2,n_2}$, and $\mathcal{F}_{3,1} \subset B_{3,n_1}$ be finite subsets such that

$$\mathcal{F}_1 \cup \mathcal{F} \subset \imath_{n_1,\infty}(\mathcal{F}_{1,1} \cup \mathcal{F}_{2,1} \cup \mathcal{F}_{3,1}). \tag{e 24.8}$$

Let $e_1 = \imath_{n_1,\infty}(1_{B_{1,n_1}})$, $e_2 = \imath_{n_1,\infty}(1_{B_{2,n_1}})$, and $e_3 = 1 - e_1 - e_2$. Note that $B_{1,n_1} = \oplus_{i=1}^{s(n_1)} B^i_{1,n_1}$, where $s(n_1)$ is an integer depending on n_1 and $B^i_{1,n_1} = M_{m(i,n_1)}(C(\mathbb{T}))$. We may write $e_1 = \oplus_{i=1}^{s(n_1)} e_1^i$ with $e_1^i = \imath_{n_1,\infty}(1^i_{B_{1,n_1}})$. Let $\Delta_1 : (B_{2,n_1})_+^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be defined by

$$\Delta_1(\hat{h}) = (1/2)\inf\{\tau(\varphi(\imath_{n_1,\infty}(h)) : \tau \in T(B)\} \text{ for all } h \in (B_{2,n_1})_+^{\mathbf{1}} \setminus \{0\}.$$

Let $\Delta_2 : B_{3,n_1}^{q,\mathbf{1}} \setminus \{0\} \to (0,1)$ be defined by

$$\Delta_2(\hat{h}) = (1/2)\inf\{\tau(\varphi(\imath_{n_1,\infty}(h)) : \tau \in T(B)\} \text{ for all } h \in (B_{3,n_1})_+^{\mathbf{1}} \setminus \{0\}.$$

Note that B_{2,n_1} has the form C of Theorem 12.7 of [21]. So we will apply Theorem 12.7 of [21]. Let $\mathcal{H}_{2,1} \subset (B^{\mathbf{1}}_{2,n_1})_+ \setminus \{0\}$ (in place of $\mathcal{H}_1$), $\gamma_{2,1} > 0$ (in place of γ_1), $\delta_{2,1} > 0$ (in place of δ), $\mathcal{G}_{2,1} \subset B_{2,n_1}$ (in place of $\mathcal{G}$), $\mathcal{P}_{2,2} \subset \underline{K}(B_{2,n_1})$ (in place of $\mathcal{P}$), and $\mathcal{H}_{2,2} \subset (B_{2,n_1})_{s.a.}$ (in place of $\mathcal{H}_2$) be the constants and finite subsets provided by Theorem 12.7 of [21] for $\varepsilon_2/16$, $\mathcal{F}_{2,1}$, and Δ_1 (we do not need the set $\mathcal{U}$ in Theorem 12.7 of [21] since $K_1(B_{2,n_1})$ is torsion or zero; see Corollary 12.8 of [21]).

Recall that $B_{1,n_1} = \bigoplus_{i=1}^{s(n_1)} B^i_{1,n_1}$ with $B^i_{1,n_1} = M_{m(i,n_1)}(C(\mathbb{T}))$, and $e_1 = \bigoplus_{i=1}^{s(n_1)} e_1^i$ with $e_1^i = \imath_{n_1,\infty}(1^i_{B_{1,n_1}})$. Now let $\sigma > 0$ be as provided by Corollary 23.4 (see 23.3 also) for $\mathcal{P}_{1,1}$ and $\varepsilon_2/4$ (in place of ε). Let $\delta = \sigma \cdot \inf\{\tau(e_1^i)/m(i,n_1) : 1 \leq i \leq s(n_1),\ \tau \in T(A)\}$. It follows from 23.4 that if $|\tau \circ \rho_B(\alpha(\beta(x)))| < \delta$ for all $x \in [\imath_{n_1,\infty}])(\mathcal{P}_{1,1})$ then there is a unitary $v_1 \in e_1Be_1$ such that

$$\mathrm{Bott}(\varphi \circ \imath_{n_1,\infty}, v_1)|_{\mathcal{P}_{1,1}} = \alpha \circ \beta \circ [\imath_{n_1,\infty}]|_{(\mathcal{P}_{1,1})}. \tag{e 24.9}$$

Note that $K_1(B_{2,n_1})$ is a finite group. Therefore,

$$\alpha(\boldsymbol{\beta}([\imath_{n_1,\infty}])(K_1(B_{2,n_1})) \subset \ker\rho_B. \tag{e 24.10}$$

Define $\kappa_1 \in KK(B_{2,n_1} \otimes C(\mathbb{T}), A)$ by $\kappa_1|_{\underline{K}(B_{2,n_1})} = [\varphi \circ \imath_{n_1,\infty}|_{B_{2,n_1}}]$ and $\kappa_1|_{\boldsymbol{\beta}(\underline{K}(B_{2,n_1}))} = \alpha|_{\boldsymbol{\beta}(\underline{K}(B_{2,n_1}))}$. Since $\imath_{n_1,\infty}$ is injective, by (e 24.10), $\kappa_1 \in KK_e(B_{2,n_1} \otimes C(\mathbb{T}), e_2Be_2)^{++}$.

Let

$$\sigma_0 = \min\{\gamma_{2,1}/2, \min\{\Delta_1(\hat{h}) : h \in \mathcal{H}_{2,1}\} \cdot \inf\{\tau(e_2) : \tau \in T(A)\}.$$

Define $\gamma_0 : T(e_2Ae_2) \to T_f(B_{2,n_1} \otimes C(\mathbb{T}))$ by $\gamma_0(\tau)(f \otimes 1_{C(\mathbb{T})}) = \tau \circ \varphi \circ \imath_{n_1,\infty}(f)$ for all $f \in B_{2,n_1}$ and $\gamma_0(1 \otimes g) = \int_{\mathbb{T}} g(t)dt$ for all $g \in C(\mathbb{T})$. It follows from 22.16, applied to the space $X_{n_1} \times \mathbb{T}$, that there is a unital monomorphism $\Phi : B_{2,n_1} \otimes C(\mathbb{T}) \to e_2Ae_2$ such that $[\Phi] = \kappa_1$ and $\Phi_T = \gamma_0$. Put $L_2 = \Phi|_{B_{2,n_1}}$ (identifying B_{2,n_1} with $B_{2,n_1} \otimes 1_{C(\mathbb{T})}$) and $v_2' = \Phi(1 \otimes z)$, where $z \in C(\mathbb{T})$ is the identity function on the unit circle. Then L_2 is a unital monomorphism from B_{2,n_1} to e_2Ae_2. We also have the following facts:

(e 24.11) $$[L_2] = [\varphi \circ \imath_{n_1,\infty}], \ \ \|[L_2(f), v_2']\| = 0,$$

(e 24.12) $$\mathrm{Bott}(L_2, v_2')|_{\mathcal{P}_{2,2}} = \alpha(\boldsymbol{\beta}([\imath_{n_1,\infty}]))|_{\mathcal{P}_{2,2}}, \text{ and}$$

(e 24.13) $$|\tau \circ L_2(f) - \tau \circ \varphi \circ \imath_{n_1,\infty}(f)| = 0 \text{ for all } f \in \mathcal{H}_{2,1} \cup \mathcal{H}_{2,2}$$

and for all $\tau \in T(e_2Ae_2)$. It follows from (e 24.13) that

(e 24.14) $$\tau(L_2(f)) \geq \Delta_1(\hat{f}) \cdot \tau(e_2) \text{ for all } f \in \mathcal{H}_{2,1} \text{ and } \tau \in T(A).$$

By Theorem 12.7 of [21] (see also Corollary 12.8 of [21]), there exists a unitary $w \in e_2Ae_2$ such that

(e 24.15) $$\|\mathrm{Ad}\, w \circ L_2(f) - \varphi \circ \imath_{n_1,\infty}(f)\| < \varepsilon_2/16 \text{ for all } f \in \mathcal{F}_{2,1}.$$

Define $v_2 = w^* v_2' w$. Then, for all $f \in \mathcal{F}_{2,1}$,

(e 24.16) $$\|[\varphi \circ \imath_{n_1,\infty}(f), v_2]\| < \varepsilon_2/8 \text{ and}$$
$$\mathrm{Bott}(\varphi \circ \imath_{n_1,\infty}, v_2)|_{\mathcal{P}_{2,1}} = \alpha(\boldsymbol{\beta}([\imath_{n_2,\infty}]))|_{\mathcal{P}_{2,1}}.$$

Note that B_{3,n_1} has the form C of Theorem 12.7 of [21]. Let $\mathcal{H}_{3,1} \subset (B_{3,n_1})_+^{\mathbf{1}} \setminus \{0\}$ (in place of $\mathcal{H}_1$), $\gamma_{3,1} > 0$ (in place of γ_1), $\delta_{3,1} > 0$ (in place of δ), $\mathcal{G}_{3,1} \subset B_{3,N_1}$ (in place of $\mathcal{G}$), $\mathcal{P}_{3,2} \subset \underline{K}(B_{3,n_1})$ (in place of $\mathcal{P}$), and $\mathcal{H}_{3,2} \subset (B_{3,n_1})_{s.a.}$ (note that $K_1(B_{3,n_1}) = \{0\}$) be constants and finite subsets as provided by Theorem 12.7 of [21] for $\varepsilon_2/16$, $\mathcal{F}_{3,1}$, and Δ_2 (see also Corollary 12.8 of [21]).

Let

$$\sigma_1 = (\gamma_{3,1}/2) \min\{\tau(e_3) : \tau \in T(A)\} \cdot \min\{\Delta_1(\hat{f}) : f \in \mathcal{H}_{3,1}\}.$$

Note that $\ker\rho_{B_{3,n_1}} = \{0\}$ and $K_1(B_{3,n_1}) = \{0\}$ (see Proposition 3.5 of [21]) Therefore, $\ker\rho_{B_{3,n_1} \otimes C(\mathbb{T})} = \ker\rho_{B_{3,n_1}} = \{0\}$. Define $\kappa_2 \in KK(B_{3,n_1} \otimes C(\mathbb{T}), A)$ as follows:

$$\kappa_2|_{\underline{K}(B_{3,n_1})} = [\varphi \circ \imath_{n_1,\infty}]|_{B_{3,n_1}} \text{ and } \kappa_2|_{\boldsymbol{\beta}(\underline{K}(B_{3,n_1}))} = \alpha(\boldsymbol{\beta}(\imath_{n_1,\infty})|_{\underline{K}(B_{3,n_1})}.$$

Thus $\kappa_2 \in KK_e(B_{3,n_1} \otimes C(\mathbb{T}), e_3Ae_3)^{++}$. It follows from 22.7 that there is a unital $\mathcal{G}_{3,1}$-$\min\{\varepsilon_2/16, \delta_{3,1}/2\}$-multiplicative completely positive linear map $L_3 : B_{3,n_1} \to e_3Ae_3$ and a unitary $v_3' \in e_3Ae_3$ such that

$$[L_3] = [\varphi], \ \ \|[L_3(f), v_3']\| < \varepsilon_2/16 \text{ for all } f \in \mathcal{G}_{3,1}, \tag{e 24.17}$$

$$\mathrm{Bott}(L_3, v_3')|_{\mathcal{P}_{3,1}} = \kappa_2|_{\boldsymbol{\beta}(\mathcal{P}_{3,2})}, \text{ and} \tag{e 24.18}$$

$$|\tau \circ L_3(f) - \tau \circ \varphi \circ \imath_{n_1,\infty}(f)| < \sigma_1 \text{ for all } f \in \mathcal{H}_{3,1} \cup \mathcal{H}_{3,2} \tag{e 24.19}$$

and for all $\tau \in T(e_3Ae_3)$. It follows from (e 24.19) that

$$\tau(L_3(f)) \geq \Delta_1(\hat{f})\tau(e_3) \tag{e 24.20}$$

for all $f \in \mathcal{H}_{3,1}$ and for all $\tau \in T(A)$. It follows from Theorem 12.7 of [21], and its corollary (see part (2) of Corollary 12.8 of [21]), that there exists a unitary $w_1 \in e_3Ae_3$ such that

$$\|\mathrm{Ad}\, w_1 \circ L_2(f) - \varphi \circ \imath_{n_1,\infty}(f)\| < \varepsilon_2/16 \text{ for all } f \in \mathcal{F}_{3,1}. \tag{e 24.21}$$

Define $v_3 = w_1^* v_3' w_1$. Then

$$\begin{aligned} &\|[\varphi \circ \imath_{n_1,\infty}(f), v_3]\| < \varepsilon_2/8 \text{ and} \\ &\mathrm{Bott}(\varphi \circ \imath_{n_1,\infty}, v_3)|_{\mathcal{P}_{3,1}} = \mathrm{Bott}(L_3, v_3')|_{\mathcal{P}_{3,1}}. \end{aligned} \tag{e 24.22}$$

Let $v = v_1 + v_2 + v_3$. Then

$$\|[\varphi(f), v]\| < \varepsilon \text{ for all } f \in \mathcal{F}. \tag{e 24.23}$$

Moreover, we compute that

$$\mathrm{Bott}(\varphi, v)|_{\mathcal{P}} = \alpha|_{\boldsymbol{\beta}(\mathcal{P})}. \tag{e 24.24}$$

□

We have actually proved the following result:

LEMMA 24.2. *Let $A = A_1 \otimes U_1$, where A_1 is as in Theorem 14.10 of [21] and $B = B_1 \otimes U_2$, where $B_1 \in \mathcal{B}_0$ is a unital simple C^*-algebra and where U_1, U_2 are two UHF-algebras of infinite type. Write $A = \lim_{n\to\infty}(C_n, \imath_n)$ as described in Theorem 14.10 of [21]. For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, and any finite subset $\mathcal{P} \subset \underline{K}(A)$, there exists an integer $n \geq 1$ such that $\mathcal{P} \subset [\imath_{n,\infty}](\underline{K}(C_n))$ and there is a finite subset $\mathcal{Q} \subset K_1(C_n)$ which generates $K_1(C_n)$ and there exists $\delta > 0$ satisfying the following condition: Let $\varphi : A \to B$ be a unital homomorphism and let $\alpha \in KK(C_n \otimes C(\mathbb{T}), B)$ such that*

$$|\tau \circ \rho_B(\alpha(\boldsymbol{\beta}(x)))| < \delta \textit{ for all } x \in \mathcal{Q} \textit{ and for all } \tau \in T(B).$$

Then there exists a unitary $u \in B$ such that

$$\|[\varphi(x), u]\| < \varepsilon \textit{ for all } x \in \mathcal{F} \text{ and } \mathrm{Bott}(\varphi \circ [\imath_{n,\infty}], u) = \alpha(\boldsymbol{\beta}).$$

PROOF. Note that, in the proof of Lemma 24.1, $K_i(B_{j,1})$ is finitely generated, $i = 0, 1$, and $j = 1, 2, 3$. Then $KK(B_{j,1}, A) = KL(B_{j,1}, B)$ (for any unital C^*-algebra B), $j = 1, 2$. Moreover (see [6]), there exists an integer $N_0 > 1$ such that elements in $\mathrm{Hom}_\Lambda(\underline{K}(B_{j,1}), \underline{K}(B))$ are determined by their restrictions to $K_i(B_{j,1})$ and $K_i(B_{j,1}, \mathbb{Z}/m\mathbb{Z})$, $m = 2, 3, ..., N_0$. In particular, we may assume, in the proof of 24.1, that $\mathcal{P}_{j,1}$ generates $K_i(B_{j,1}) \oplus \bigoplus_{m=2}^{N_0} K_i(B_{j,1}, \mathbb{Z}/m\mathbb{Z})$, $j = 1, 2, 3$. □

REMARK 24.3. Note that, in the statement above, if an integer n works, any integer $m \geq n$ also works. In the terminology of Definition 3.6 of [44], the statement above also implies that B has properties (B1) and (B2) associated with C.

COROLLARY 24.4. *Let $B \in \mathcal{B}_0$, let $A_1 \in \mathcal{B}_0$, let $C = B \otimes U_1$, and let $A = A_1 \otimes U_2$, where U_1 and U_2 are UHF-algebras of infinite type. Suppose that B satisfies the UCT and suppose that $\kappa \in KK_e(C, A)^{++}$, $\gamma : T(A) \to T(C)$ is a continuous affine map, and $\alpha : U(C)/CU(C) \to U(A)/CU(A)$ is a continuous homomorphism for which γ, α, and κ are compatible. Then, there exists a unital monomorphism $h : C \to A$ such that*

(1) *$[h] = \kappa$ in $KK_e(C, A)^{++}$,*
(2) *$h_T = \gamma$ and $h^\ddagger = \alpha$.*

PROOF. The proof follows the same lines as that of Theorem 8.6 of [40], following the proof of Theorem 3.17 of [44]. Denote by $\overline{\kappa} \in KL(C, A)$ the image of κ. It follows from Lemma 22.14 that there is a unital monomorphism $\varphi : C \to A$ such that

$$[\varphi] = \overline{\kappa}, \quad \varphi^\ddagger = \alpha, \quad \text{and} \quad (\varphi)_T = \gamma.$$

Note that it follows from the UCT that (as an element of $KK(C, A)$)

$$\kappa - [\varphi] \in \mathrm{Pext}(K_*(C), K_{*+1}(A)).$$

By Lemmas 24.2 (see also Remark 22.15) and 23.3, the C^*-algebra A has Property (B1) and Property (B2) associated with C in the sense of [44]. Since A contains a unital copy of U_2, it is infinite dimensional, simple and antiliminal. It follows from a result in [1] that A contains an element b with $sp(b) = [0, 1]$. Moreover, A is approximately divisible. It follows from Theorem 3.17 of [44] that there is a unital monomorphism $\psi_0 : A \to A$ which is approximately inner and such that

$$[\psi_0 \circ \varphi] - [\varphi] = \kappa - [\varphi] \quad \text{in } KK(C, A).$$

Then the map

$$h := \psi_0 \circ \varphi$$

satisfies the requirements of the corollary. □

LEMMA 24.5. *Let $A = A_1 \otimes U_1$, where A_1 is as in Theorem 14.10 of [21] and $B = B_1 \otimes U_2$, where $B_1 \in \mathcal{B}_0$ is a unital simple C^*-algebra and where U_1, U_2 are two UHF-algebras of infinite type. Let $A = \lim_{n\to\infty}(C_n, \imath_n)$ be as described in Theorem 14.10 of [21], For any $\varepsilon > 0$, any $\sigma > 0$, any finite subset $\mathcal{F} \subset A$, any finite subset $\mathcal{P} \subset \underline{K}(A)$, and any projections $p_1, p_2, ..., p_k, q_1, q_2, ..., q_k \in A$ such that $\{x_1, x_2, ..., x_k\}$ generates a free abelian subgroup G of $K_0(A)$, where $x_i = [p_i] - [q_i]$, $i = 1, 2, ..., k$, there exists an integer $n \geq 1$ such that $x_j \in \mathcal{P} \subset [\imath_{n,\infty}](\underline{K}(C_n))$ and there is a finite subset $\Omega \subset K_1(C_n)$ which generates $K_1(C_n)$ and there exists $\delta > 0$ satisfying the following condition: Let $\varphi : A \to B$ be a unital homomorphism, let $\Gamma : G \to U(B)/CU(B)$ be a homomorphism and let $\alpha \in KK(C_n \otimes C(\mathbb{T}), B)$ such that*

$$\alpha(\boldsymbol{\beta}(g))) = \kappa_1^B(\Gamma(g)) \text{ for all } g \in {\imath_{n,\infty}}_{*0}^{-1}(G) \text{ and}$$
$$|\tau \circ \rho_B(\alpha(\boldsymbol{\beta}(x)))| < \delta \text{ for all } x \in \Omega \text{ and for all } \tau \in T(B).$$

Then there exists a unitary $u \in B$ such that

$$\|[\varphi(x), u]\| < \varepsilon \text{ for all } x \in \mathcal{F}, \ \mathrm{Bott}(\varphi \circ [\imath_{n,\infty}], u) = \alpha(\boldsymbol{\beta}),$$

and

$$\mathrm{dist}(\overline{\langle((1-\varphi(p_i)) + \varphi(p_i)u)((1-\varphi(q_i)) + \varphi(q_i)u^*)\rangle}, \Gamma(x_i)) < \sigma, \ i = 1, 2, ..., k.$$

PROOF. This follows from Lemma 24.2 and Theorem 22.18. In fact, for any $0 < \varepsilon_1 < \varepsilon/2$ and finite subset $\mathcal{F}_1 \supset \mathcal{F}$, by 24.2, there exists an integer $n \geq 1$, a finite subset $\Omega \subset K_1(C_n)$, and $\delta > 0$ as described above, and a unitary $u_1 \in U_0(B)$, such that

$$\|[\varphi(x), u_1]\| < \varepsilon_1 \text{ for all } x \in \mathcal{F}_1$$

and

$$\mathrm{Bott}(\varphi \circ \imath_{n,\infty}, u_1) = \alpha(\boldsymbol{\beta})|_{\mathcal{P}}.$$

Choosing a smaller ε_1 and a larger $\mathcal{F}_1$, if necessary, we may assume that the class

$$\overline{\langle((1-\varphi(p_i)) + \varphi(p_i)u_1)((1-\varphi(q_i)) + \varphi(q_i)u_1^*)\rangle} \in U(B)/CU(B)$$

is well defined for all $1 \leq i \leq k$. Define a map $\Gamma_1 : G \to U(B)/CU(B)$ by

$$\Gamma_1(x_i) = \overline{\langle((1-\varphi(p_i)) + \varphi(p_i)u_1)(1-\varphi(q_i)) + \varphi(q_i)u_1^*)\rangle}, \quad i = 1, 2, ..., k.$$

Choosing a large enough n, without loss of generality, we may assume that there are projections $p_1', p_2', ..., p_k', q_1', q_2,' ..., q_k' \in C_n$ such that $\imath_{n,\infty}(p_i') = p_i$ and $\imath_{n,\infty}(q_i') = q_i$, $i = 1, 2, ..., k$. Moreover, we may assume that $\mathcal{F}_1 \subset \imath_{n,\infty}(C_n)$.

Let $\Gamma_2 : G \to U_0(B)/CU(B)$ be defined by $\Gamma_2(x_i) = \Gamma_1(x_i)^*\Gamma(x_i)$, $i = 1, 2, ..., k$. It follows by Theorem 22.18 that is a unitary $v \in U_0(B)$ such that

$$\|[\varphi(x),\, v]\| < \varepsilon/2 \;\text{ for all }\; x \in \mathcal{F}, \tag{e 24.25}$$

$$\mathrm{Bott}(\varphi \circ \imath_{n,\infty}, v) = 0, \;\text{ and} \tag{e 24.26}$$

$$\mathrm{dist}(\overline{\langle((1-\varphi(p_i)) + \varphi(p_i)v)((1-\varphi(q_i)) + \varphi(q_i)v^*)\rangle}, \Gamma_2(x_i)) \;<\; \sigma,$$

$i = 1, 2, ..., k$. Define $u = u_1 v$,

$$X_i \;=\; \overline{\langle((1-\varphi(p_i)) + \varphi(p_i)u_1)((1-\varphi(q_i)) + \varphi(q_i)u_1^*)\rangle}, \;\text{ and} \tag{e 24.27}$$

$$Y_i \;=\; \overline{\langle((1-\varphi(p_i)) + \varphi(p_i)v)((1-\varphi(q_i)) + \varphi(q_i)v^*)\rangle}, \tag{e 24.28}$$

$i = 1, 2, ..., k$. We then compute that

$$\begin{aligned} &\|[\varphi(x),\, u]\| < \varepsilon_1 + \varepsilon/2 < \varepsilon \;\text{ for all }\; x \in \mathcal{F}, \\ &\mathrm{Bott}(\varphi \circ \imath_{n,\infty}, u) = \mathrm{Bott}(\varphi \circ \imath_{n,\infty}, u_1) = \alpha(\boldsymbol{\beta}), \quad \text{and} \\ &\mathrm{dist}(\overline{\langle((1-\varphi(p_i)) + \varphi(p_i)u)((1-\varphi(q_i)) + \varphi(q_i)u^*)\rangle}, \Gamma(x_i)) \\ &\le \mathrm{dist}(X_iY_i, \Gamma_1(x_i)Y_i) + \mathrm{dist}(\Gamma_1(x_i)Y_i, \Gamma(x_i)) \\ &= \mathrm{dist}(X_i, \Gamma_1(x_i)) + \mathrm{dist}(Y_i, \Gamma_2(x_i)) < \sigma, \end{aligned} \tag{e 24.29}$$

for $i = 1, 2, ..., k$. □

25. Another Basic Homotopy Lemma

LEMMA 25.1. *Let A be a unital C*-algebra and let U be an infinite dimensional UHF-algebra. Then there is a unitary $w \in U$ such that for any unitary $u \in A$, one has*

$$\tau(f(u \otimes w)) = \tau(f(1_A \otimes w)) = \int_{\mathbb{T}} f\,dm, \quad f \in \mathrm{C}(\mathbb{T}),\; \tau \in T(A \otimes U), \tag{e 25.1}$$

where m is normalized Lebesgue measure on $\mathbb{T}$. Furthermore, for any $a \in A$ and $\tau \in T(A \otimes U)$, $\tau(a \otimes w^j) = 0$ if $j \neq 0$.

PROOF. Denote by τ_U the unique trace of U. Then any trace $\tau \in T(A \otimes U)$ is a product trace, i.e.,

$$\tau(a \otimes b) = \tau(a \otimes 1) \otimes \tau_U(b), \quad a \in A, b \in U.$$

Pick a unitary $w \in U$ such that the spectral measure of w is Lebesgue measure (a Haar unitary). Such a unitary always exists. (It can be constructed directly; or, one can consider a strictly ergodic Cantor system (Ω, σ) such that $K_0(\mathrm{C}(\Omega) \rtimes_\sigma \mathbb{Z}) \cong K_0(U)$. One then notes that the canonical unitary in $\mathrm{C}(\Omega) \rtimes_\sigma \mathbb{Z}$ is a Haar unitary. Embedding $C(\Omega) \rtimes_\sigma \mathbb{Z}$ into U, one obtains a Haar unitary in U.) Then one has, for each $n \in \mathbb{Z}$,

$$\tau_U(w^n) = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, for any $\tau \in T(A \otimes U)$, one has, for each $n \in \mathbb{Z}$,

$$\tau((u \otimes w)^n) = \tau(u^n \otimes w^n) = \tau(u^n \otimes 1)\tau_U(w^n) = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise;} \end{cases}$$

and therefore,

$$\tau(P(u \otimes w)) = \tau(P(1 \otimes w)) = \int_{\mathbb{T}} P(z)dm$$

for any polynomial P. Similarly, $\tau(P(u \otimes w)^*) = \int_T P(\bar{z})dm$ for any polynomial P. Since polynomials in z and z^{-1} are dense in $C(\mathbb{T})$, one has

$$\tau(f(u \otimes w)) = \tau(f(1 \otimes w)) = \int_{\mathbb{T}} f dm, \quad f \in C(\mathbb{T}),$$

as desired. □

LEMMA 25.2. *Let A be a unital separable amenable C^*-algebra and let $L : A \otimes C(\mathbb{T}) \to B$ be a unital completely positive linear map, where B is another unital amenable C^*-algebra. Suppose that C is a unital C^*-algebra and $u \in C$ is a unitary. Then, there is a unique pair of unital completely positive linear maps $\Phi_1, \Phi_2 : A \otimes C(\mathbb{T}) \to B \otimes C$ such that*

$$\Phi_i|_{A\otimes 1_{C(\mathbb{T})}} = \imath \circ L|_{A\otimes 1_{C(\mathbb{T})}} \ (i = 0, 1) \ \textit{and} \quad \Phi_1(a \otimes z^j) = L(a \otimes z^j) \otimes u^j \ \textit{and}$$

$$\Phi_2(a \otimes z^j) = L(a \otimes 1_{C(\mathbb{T})}) \otimes u^j$$

for any $a \in A$ and any integer j, where $\imath : B \to B \otimes C$ is the standard inclusion.

Furthermore, if $\delta > 0$ and $\mathcal{G} \subset A \otimes C(\mathbb{T})$ is a finite subset, there are $\delta_1 > 0$ and finite set $\mathcal{G}_1 \subset A \otimes C(\mathbb{T})$ (which do not depend on L) such that if L is $\mathcal{G}_1$-δ_1-multiplicative, then Φ_i is $\mathcal{G}$-δ-multiplicative.

PROOF. Considering the map $L' : A \otimes C(\mathbb{T}) \to B$ by $L'(a \otimes f) = L(a \otimes f(1))$ for all $a \in A$ and $f \in C(\mathbb{T})$, where $f(1)$ is the evaluation of f at 1 (a point on the unit circle), we see that it suffices to prove the statement for Φ_1 only.

Denote by C_0 the unital C*-subalgebra of C generated by u. Then the tensor product map

$$L \otimes \mathrm{id}_{C_0} : A \otimes C(\mathbb{T}) \otimes C_0 \to B \otimes C_0$$

is unital and completely positive (see, for example, Theorem 3.5.3 of [4]). Define the homomorphism $\psi : C(\mathbb{T}) \to C(\mathbb{T}) \otimes C_0$ by

$$\psi(z) = z \otimes u.$$

By Theorem 3.5.3 of [4] again, the tensor product map

$$\mathrm{id}_A \otimes \psi : A \otimes C(\mathbb{T}) \to A \otimes C(\mathbb{T}) \otimes C_0$$

is unital and completely positive. Then, the map

$$\Phi := (L \otimes \mathrm{id}_{C_0}) \circ (\mathrm{id}_A \otimes \psi)$$

satisfies the requirement of the first part of the lemma.

Let us consider the second part of the lemma. Let $\delta > 0$ and $\mathcal{G} \subset A \otimes C(\mathbb{T})$ be a finite subset. Without loss of generality, we may assume that elements in $\mathcal{G}$ have the form $\sum_{-n\le i\le n} a_i \otimes z^i$. Let $N = \max\{n : \sum_{-n\le i\le n} a_i \otimes z^i \in \mathcal{G}\}$, let $\delta_1 = \delta/2N^2$, and let $\mathcal{G}_1 \supset \{a_i \otimes z^i : -n \le i \le n : \sum_{-n\le i\le n} a_i \otimes z^i \in \mathcal{G}\}$.

Then

$$\begin{aligned}
&\Phi((\sum_{-n\le i\le n} a_i \otimes z^i)(\sum_{-n\le i\le n} b_i \otimes z^i)) \\
&= \sum_{i,j} \Phi(a_i b_j \otimes z^{i+j}) \\
&= \sum_{i,j} L(a_i b_j \otimes z^{i+j}) \otimes u^{i+j} \\
&\approx_\delta (\sum_{-n\le i\le n} L(a_i \otimes z^i) \otimes u^i)(\sum_{-n\le i\le n} L(b_i \otimes z^i) \otimes u^i) \\
&= \Phi(\sum_{-n\le i\le n} a_i \otimes z^i)\Phi(\sum_{-n\le i\le n} b_i \otimes z^i),
\end{aligned}$$

if $\sum_{-n\le i\le n} a_i \otimes z^i, \sum_{-n\le i\le n} b_i \otimes z^i \in \mathcal{G}$. It follows that Φ is $\mathcal{G}$-δ-multiplicative.

Let $P(\mathbb{T}) = \{\sum_{i=-n}^{n} c_i z^i, c_i \in \mathbb{C}\}$ denote the algebra of Laurent polynomials. Uniqueness of Φ follows from the fact that $A \otimes C(\mathbb{T})$ is the closure of the algebraic tensor product $A \otimes_{alg} P(\mathbb{T})$. □

The following corollary follows immediately from 25.2 and 25.1.

Corollary 25.3. *Let C be a unital C^*-algebra and let U be an infinite dimensional UHF-algebra. For any $\delta > 0$ and any finite subset $\mathcal{G} \subset C \otimes C(\mathbb{T})$, there exist $\delta_1 > 0$ and a finite subset $\mathcal{G}_1 \subset C \otimes C(\mathbb{T})$ satisfying the following condition: For any $1 > \sigma_1, \sigma_2 > 0$, any finite subset $\mathcal{H}_1 \subset C(\mathbb{T})_+ \setminus \{0\}$, any finite subset $\mathcal{H}_2 \subset (C \otimes C(\mathbb{T}))_{s.a.}$, and any unital $\mathcal{G}_1$-δ_1-multiplicative completely positive linear map $L : C \otimes C(\mathbb{T}) \to A$, where A is another unital C^*-algebra, there exists a unitary $w \in U$ satisfying the following conditions:*

(e 25.2) $|\tau(L_1(f)) - \tau(L_2(f))| < \sigma_1$ *for all* $f \in \mathcal{H}_2,\ \tau \in T(B)$, *and*

(e 25.3) $$\tau(g(1_A \otimes w)) \ge \sigma_2(\int g dm) \text{ for all } g \in \mathcal{H}_1,\ \tau \in T(B),$$

where $B = A \otimes U$ and m is normalized Lebesgue measure on $\mathbb{T}$, and $L_1, L_2 : C \otimes C(\mathbb{T}) \to A \otimes U$ are $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps as given by Lemma 25.2 (as Φ_1, Φ_2) such that $L_i(c \otimes 1_{C(\mathbb{T})}) = L(c \otimes 1_{C(\mathbb{T})}) \otimes 1_U$ ($i = 1, 2$), $L_1(c \otimes z^j) = L(c \otimes z^j) \otimes w^j$, and $L_2(c \otimes z^j) = L(c \otimes 1_{C(\mathbb{T})}) \otimes w^j$ for all $c \in C$ and all integers j.

PROOF. Fix a $\delta > 0$ and a finite subset $\mathcal{G}$. Let $\delta_1 > 0$ and $\mathcal{G}_1 \subset C \otimes C(\mathbb{T})$ be as given by Lemma 25.2 for A (in place of B).

Let $0 < \sigma_1, \sigma_2 < 1$, $\mathcal{H}_1$, and $\mathcal{H}_2$ be as given in the statement. There are a finite subset $\mathcal{F}_C \subset C$ and an integer $N > 0$ such that, for any $h \in \mathcal{H}_2$,

$$\|h - \sum_{j=-N}^{N} a_{h,i} \otimes z^i\| < \sigma_1/2, \qquad \text{(e 25.4)}$$

where $a_{h,i} \in \mathcal{F}_C \cup \{0\}$. Set $\mathcal{H}_2' = \{\sum_{i=-N}^{N} a_{h,i} \otimes z^j : h \in \mathcal{H}_2\}$.

Now assume that L is as stated for $\mathcal{G}_1$ and δ_1 mentioned above.

Choose $w \in U$ as in Lemma 25.1. Let $L_1, L_2 : C \otimes C(\mathbb{T}) \to A \otimes U$ be as described in the corollary. In other words, let $L_1 : C \otimes C(\mathbb{T}) \to A \otimes U$ be the map as Φ_1 given by Lemma 25.2 (with C in place of A, A in place of B, U in place of C, and $w \in U$ in place $u \in U$), and let $L_2 : C \otimes C(\mathbb{T}) \to A \otimes U$ be defined by $L_2(c \otimes f) = L(c \otimes 1_{C(\mathbb{T})}) \otimes f(w)$ for all $c \in C$ and $f \in C(\mathbb{T})$ (as Φ_2 in Lemma 25.2). By the choice of $\mathcal{G}_1$ and δ_1, L_1 and L_2 are $\mathcal{G}$-δ-multiplicative (as in 25.2).

By Lemma 25.1 (and by Lemma 25.2), for $h \in \mathcal{H}_2'$,

$$\begin{aligned}
\tau(L_1(h)) &= \tau(\sum_{i=-N}^{N} L_1(a_{h,i} \otimes z^i)) = \sum_{i=-N}^{N} \tau(L(a_{h,i} \otimes z^i) \otimes w^i) \\
&= \tau(L(a_{h,0} \otimes 1_{C(\mathbb{T})})) = \tau(L_2(a_{h,0} \otimes 1_{C(\mathbb{T})})) \\
&= \tau(\sum_{i=-N}^{N} L(a_{h,i} \otimes 1_{C(\mathbb{T})}) \otimes w^i) = \tau(L_2(h)).
\end{aligned}$$

Thus, combining (e 25.4), inequality (e 25.2) holds. By (e 25.1), (e 25.3) also holds.

□

LEMMA 25.4. *Let $A = A_1 \otimes U_1$, where $A_1 \in \mathcal{B}_0$ and satisfies the UCT and U_1 is a UHF-algebra of infinite type. For any $1 > \varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, $\sigma > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\{p_1, p_2, ..., p_k, q_1, q_2, ..., q_k\}$ of projections of A such that $\{[p_1] - [q_1], [p_2] - [q_2], ..., [p_k] - [q_k]\}$ generates a free abelian subgroup G_u of $K_0(A)$, and a finite subset $\mathcal{P} \subset \underline{K}(A)$, satisfying the following condition:*

Let $B = B_1 \otimes U_2$, where B_1 is in $\mathcal{B}_0$ and satisfies the UCT and U_2 is a UHF-algebra of infinite type. Suppose that $\varphi : A \to B$ is a unital homomorphism.

If $u \in U(B)$ is a unitary such that

$$\|[\varphi(x),\, u]\| < \delta \ \textit{for all} \ x \in \mathcal{G}, \tag{e 25.5}$$

$$\mathrm{Bott}(\varphi,\, u)|_{\mathcal{P}} = 0, \tag{e 25.6}$$

$$\mathrm{dist}(\overline{\langle((1-\varphi(p_i)) + \varphi(p_i)u)(1-\varphi(q_i)) + \varphi(q_i)u^*)\rangle}, \bar{1}) < \sigma, \ \textit{and} \tag{e 25.7}$$

$$\mathrm{dist}(\bar{u}, \bar{1}) < \sigma, \tag{e 25.8}$$

then there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\} \subset U(B)$ such that

$$u(0) = u, \ \ u(1) = 1_B, \tag{e 25.9}$$

$$\mathrm{dist}(u(t), CU(B)) < \varepsilon \ \textit{for all} \ t \in [0,1], \tag{e 25.10}$$

$$\|[\varphi(a),\, u(t)]\| < \varepsilon \ \textit{for all} \ a \in \mathcal{F} \ \textit{and for all} \ t \in [0,1], \ \textit{and} \tag{e 25.11}$$

$$\mathrm{length}(\{u(t)\}) \le 2\pi + \varepsilon. \tag{e 25.12}$$

PROOF. It is enough to prove the statement under the assumption that $u \in CU(B)$.

Recall that every C*-algebra in $\mathcal{B}_0$ has stable rank one (see Theorem 9.7 of [21]). Define

$$\Delta(f) = (1/2)\int f dm \ \text{ for all } \ f \in C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\},$$

where m is normalized Lebesgue measure on the unit circle $\mathbb{T}$. Let $A_2 = A \otimes C(\mathbb{T})$. Let $\mathcal{F}_1 = \{x \otimes f : x \in \mathcal{F}, f = 1, z, z^*\}$. We may assume that $\mathcal{F}$ is a subset of the unit ball of A. Let $1 > \delta_1 > 0$ (in place of δ), $\mathcal{G}_1 \subset A_2$ (in place of $\mathcal{G}$), $1/4 > \sigma_1 > 0$, $1/4 > \sigma_2 > 0$, $\widetilde{\mathcal{P}} \subset \underline{K}(A_2)$, $\mathcal{H}_1 \subset C(\mathbb{T})_+^{\mathbf{1}} \setminus \{0\}$, $\mathcal{H}_2 \subset (A_2)_{s.a.}$, and $\mathcal{U} \subset U(M_2(A_2))/CU(M_2(A))$ be the constants and finite subsets provided by Theorem 12.11 part (b) of [21] for $\varepsilon/4$ (in place of ε), $\mathcal{F}_1$ (in place of $\mathcal{F}$), Δ, and A_2 (in place of A).

We may assume $\mathcal{H}_2 \subset A_{s.a.}^{\mathbf{1}}$. that

$$\mathcal{G}_1 = \{a \otimes f : a \in \mathcal{G}_2 \ \text{ and } \ f = 1, z, z^*\},$$

where $\mathcal{G}_2 \subset A$ is a finite subset, and $\widetilde{\mathcal{P}} = \mathcal{P}_1 \cup \boldsymbol{\beta}(\mathcal{P}_2)$, where $\mathcal{P}_1, \mathcal{P}_2 \subset \underline{K}(A)$ are finite subsets. Define $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$.

We may assume that $(2\delta_1, \tilde{\mathcal{P}}, \mathcal{G}_1)$ is a KL-triple for A_2, $(2\delta_1, \mathcal{P}_1, \mathcal{G}_2)$ is a KL-triple for A, and $1_{A_1} \otimes \mathcal{H}_1 \subset \mathcal{H}_2$.

We may choose σ_1 and σ_2 such that

$$\max\{\sigma_1, \sigma_2\} < (1/4)\inf\{\Delta(f) : f \in \mathcal{H}_1\}. \tag{e 25.13}$$

Let δ_2 (in place of δ_1) and a finite subset $\mathcal{G}_3$ (in place of $\mathcal{G}_1$) be as provided by 25.3 for A (in place of C), $\delta_1/4$ (in place of δ), and $\mathcal{G}_1$ (in place of $\mathcal{G}$). Choosing smaller δ_2, without loss of generality, we may assume that $\mathcal{G}_3 = \{a \otimes f : g \in \mathcal{G}_2' \text{ and } f = 1, z, z^*\}$ for a large finite subset $\mathcal{G}_2' \supset \mathcal{G}_2$. We may assume that $\delta_2 < \delta_1$.

We may further assume that

$$\mathcal{U} = \mathcal{U}_1 \cup \{\overline{1 \otimes z}\} \cup \mathcal{U}_2, \tag{e 25.14}$$

where $\mathcal{U}_1 = \{\overline{a \otimes 1} : a \in \mathcal{U}_1' \subset U(A)\}$, $\mathcal{U}_1'$ is a finite subset, and $\mathcal{U}_2 \subset U(A_2)/CU(A_2)$ is a finite subset whose elements represent a finite subset of $\boldsymbol{\beta}(K_0(A))$. So we may assume that $\mathcal{U}_2 \in J_c(\boldsymbol{\beta}(K_0(A)))$. As in Remark 12.12 of [21], we may assume that the subgroup of $J_c(\boldsymbol{\beta}(K_0(A)))$ generated by $\mathcal{U}_2$ is free abelian. Let $\mathcal{U}_2'$ be a finite subset of unitaries such that $\{\bar{x} : x \in \mathcal{U}_2'\} = \mathcal{U}_2$. We may also assume that the unitaries in $\mathcal{U}_2'$ have the form

$$((1-p_i) + p_i \otimes z)((1-q_i) + q_i \otimes z^*), \ \ i = 1, 2, ..., k. \tag{e 25.15}$$

We may further assume that $p_i \otimes z, q_i \otimes z \in \mathcal{G}_1$, $i = 1, 2, ..., k$. Choose $\delta_3 > 0$ and a finite subset $\mathcal{G}_4' \subset A$ (and write $\mathcal{G}_4 := \{g \otimes f : g \in \mathcal{G}_4', f = 1, z, z^*\}$) such that, for any two unital $\mathcal{G}_4$-δ_3-multiplicative completely positive linear maps $\Psi_1, \Psi_2 : A \otimes C(\mathbb{T}) \to C$ (any unital C^*-algebra C), any $\mathcal{G}_4'$-δ_3-multiplicative contractive completely positive linear map $\Psi_0 : A \to C$ and unitary $V \in C$ $(1 \le i \le k)$, if

$$\|\Psi_0(g) - \Psi_1(g \otimes 1)\| < \delta_3 \ \text{ for all } \ g \in \mathcal{G}_4', \tag{e 25.16}$$

$$\|\Psi_1(z) - V\| < \delta_3, \ \text{ and } \|\Psi_1(g) - \Psi_2(g)\| < \delta_3 \ \text{ for all } \ g \in \mathcal{G}_4, \tag{e 25.17}$$

then

$$\langle (1 - \Psi_0(p_i) + \Psi_0(p_i)V)(1 - \Psi_0(q_i) + \Psi_0(q_i)V^*) \rangle \tag{e 25.18}$$

$$\approx_{\sigma_2/2^{10}} \langle \Psi_1(((1-p_i) + p_i \otimes z)((1-q_i) + q_i \otimes z^*)) \rangle, \tag{e 25.19}$$

$$\|\langle \Psi_1(x) \rangle - \langle \Psi_2(x) \rangle\| < \sigma_2/2^{10} \ \text{ for all } \ x \in \mathcal{U}_2', \ \text{ and} \tag{e 25.20}$$

$$\Psi_1(((1-p_i) + p_i \otimes z)((1-q_i) + q_i \otimes z^*)) \tag{e 25.21}$$

$$\approx_{\sigma_2/2^{10}} \Psi_1((1-p_i) + p_i \otimes z)\Psi_1((1-q_i) + q_i \otimes z^*), \tag{e 25.22}$$

and, furthermore, for $d_i^{(1)} = p_i$, $d_i^{(2)} = q_i$, there are projections $\bar{d}_i^{(j)} \in C$ and unitaries $\bar{z}_i^{(j)} \in \bar{d}_i^{(j)} C \bar{d}_i^{(j)}$ such that

$$\Psi_1(((1 - d_i^{(j)}) + d_i^{(j)} \otimes z)) \approx_{\frac{\sigma_2}{2^{12}}} (1 - \bar{d}_i^{(j)}) + \bar{z}_i^{(j)}, \tag{e 25.23}$$

$$\bar{d}_i^{(j)} \approx_{\frac{\sigma_2}{2^{12}}} \Psi_1(d_i^{(j)}), \ \bar{z}_i^{(1)} \approx_{\frac{\sigma_2}{2^{12}}} \Psi_1(p_i \otimes z), \ \text{ and } \ \bar{z}_i^{(2)} \approx_{\frac{\sigma_2}{2^{12}}} \Psi_1(q_i \otimes z^*), \tag{e 25.24}$$

where $1 \le i \le k$, $j = 1, 2$. Choose $\sigma > 0$ so it is smaller than $\min\{\sigma_1/16, \varepsilon/16, \sigma_2/16, \delta_2/16, \delta_3/16\}$.

Choose $\delta_5 > 0$ and a finite subset $\mathcal{G}_5 \subset A$ satisfying the following condition: there is a unital $\mathcal{G}_4$-$\sigma/8$-multiplicative completely positive linear map $L : A \otimes C(\mathbb{T}) \to B'$ such that

$$\|L(a \otimes 1) - \varphi'(a)\| < \sigma/8 \text{ for all } a \in \mathcal{G}_4' \text{ and } \|L(1 \otimes z) - u'\| < \sigma/8 \tag{e 25.25}$$

for any unital homomorphism $\varphi' : A \to B'$ and any unitary $u' \in B'$ such that

$$\|\varphi'(g)u' - u'\varphi'(g)\| < \delta_5 \text{ for all } g \in \mathcal{G}_5.$$

Let $\delta = \min\{\delta_5/4, \sigma\}$ and $\mathcal{G} = \mathcal{G}_5 \cup \mathcal{G}_4' \cup \mathcal{G}_2'$.

Now suppose that $\varphi : A \to B$ is a unital homomorphism and $u \in CU(B)$ satisfies the assumptions (e 25.5) to (e 25.7) for the above mentioned δ, σ, $\mathcal{G}$, $\mathcal{P}$, p_i, and q_i. Choose an isomorphism $s : U_2 \otimes U_2 \to U_2$. Note that $s \circ \imath$ (since it is unital) is approximately unitarily equivalent to the identity map on U_2, where $\imath : U_2 \to U_2 \otimes U_2$ is defined by $\imath(a) = a \otimes 1$ (for all $a \in U_2$). To simplify notation, let us assume that $\varphi(A) \subset B \otimes 1 \subset B \otimes U_2$. Suppose that $u \in U(B) \otimes 1_{U_2}$ is a unitary which satisfies the assumption of the lemma. As mentioned at the beginning, we may assume that $u \in CU(B) \otimes 1_{U_2}$. Without loss of generality, we may further assume that $u = \prod_{j=1}^{m_1} c_j d_j c_j^* d_j^*$, where $c_j, d_j \in U(B) \otimes 1_{U_2}$, $1 \le j \le m_1$. Let $\mathcal{F}_2 = \{c_j, d_j : 1 \le j \le m_1\}$.

Let $L : A \otimes C(\mathbb{T}) \to B$ be a unital $\mathcal{G}_4$-$\delta_2/8$-multiplicative completely positive linear map such that

$$\|L(1 \otimes z) - u\| < \sigma/8 \text{ and } \|L(a \otimes 1) - \varphi(a)\| < \sigma/8 \tag{e 25.26}$$

for all $a \in \mathcal{G}_4'$. Since $\mathrm{Bott}(\varphi, u)|_{\mathcal{P}} = 0$, we may also assume that

$$[L]|_{\mathcal{P}_1} = [\varphi]|_{\mathcal{P}_1} \text{ and } [L]|_{\boldsymbol{\beta}(\mathcal{P}_2)} = 0. \tag{e 25.27}$$

Since B is in $\mathcal{B}_0$, there is a projection $p \in B$ and a unital C^*-subalgebra $C \in \mathcal{C}_0$ with $1_C = p$ satisfying the following condition:

$$\|L(g) - [(1-p)L(g)(1-p) + L_1(g)]\| < \sigma^2/32(m_1+1) \text{ for all } g \in \mathcal{G}_4 \tag{e 25.28}$$

$$\text{and } \|(1-p)x - x(1-p)\| < \sigma^2/32(m_1+1) \text{ for all } x \in \mathcal{F}_2, \tag{e 25.29}$$

where $L_1 : A \otimes C(\mathbb{T}) \to C$ is a unital $\mathcal{G}_4$-$\min\{\delta_2/8, \varepsilon/8\}$-multiplicative completely positive linear map,

$$\tau(1-p) < \min\{\sigma_1/16, \sigma_2/16\} \text{ for all } \tau \in T(B), \tag{e 25.30}$$

and, using (e 25.7), (e 25.8), (e 25.26), and (e 25.18) to (e 25.22) we have that

$$\mathrm{dist}(L_2^{\ddagger}(x), \bar{1}) < \sigma_2/4 \text{ for all } x \in \{1 \otimes \bar{z}\} \cup \mathcal{U}_2 \text{ and} \tag{e 25.31}$$

$$\mathrm{dist}(L_2^{\ddagger}(x), \overline{\varphi(x') \otimes 1_{C(\mathbb{T})}}) < \sigma_2/4 \text{ for all } x \in \mathcal{U}_1, \tag{e 25.32}$$

where $\overline{x'\otimes 1_{C(\mathbb{T})}} = x$ and $L_2(a) = (1-p)L(a)(1-p)+L_1(a)$ for all $a \in A\otimes C(\mathbb{T})$. Note that we also have

$$[L_2|_A]|_{\mathcal{P}_1} = [\varphi]|_{\mathcal{P}_1} \text{ and } \|\varphi(g) - L_2(g\otimes 1)\| < \sigma/2 \tag{e 25.33}$$

for all $g \in \mathcal{G}_4'$. By (e 25.29) and the choice of $\mathcal{F}_2$, there are a unitary $v_0 \in CU(C)$ and a unitary
$v_{00} \in CU((1-p)B(1-p))$ such that

$$\|L_1(1\otimes z) - v_0\| < \min\{\delta_2/2, \varepsilon/8\} \text{ and} \tag{e 25.34}$$

$$\|(1-p)L(1\otimes z)(1-p) - v_{00}\| < \min\{\delta_2/2, \varepsilon/8\}. \tag{e 25.35}$$

By the choice of δ_2 and $\mathcal{G}_3$, applying Corollary 25.3, we obtain a unitary $w \in U(U_2)$ for example) such that

$$|t(L_3'(g))) - t(L_1'(g))| < \sigma_1/128, \quad g \in \mathcal{H}_2, \text{ and} \tag{e 25.36}$$

$$t(g(p\otimes w)) \geq \frac{1}{2(1-\sigma_1/2)}\int_{\mathbb{T}} gdm \text{ for all } g \in \mathcal{H}_1 \tag{e 25.37}$$

for all $t \in T(pBp\otimes U_2)$, where $L_1', L_3' : A\otimes C(\mathbb{T}) \to pBp\otimes U_2$ is the unital $\mathcal{G}_1$-$\delta_1/4$-multiplicative completely positive linear map defined by

$$L_1'(a\otimes 1) = L_1(a\otimes 1)\otimes 1_{U_2},\ L_1'(a\otimes z^j) = L_1(a\otimes 1)\otimes w^j, \tag{e 25.38}$$

$$L_3'(a\otimes 1) = L_1(a\otimes 1)\otimes 1_{U_2} \text{ and } L_3'(a\otimes z^j) = L_1(a\otimes z^j)\otimes (w)^j \tag{e 25.39}$$

for all $a \in A$ and all integers j as given by Lemma 25.2.

Let $L_3'' : A\otimes C(\mathbb{T}) \to B\otimes U_2$ be defined by $L_3''(a\otimes z^j) = (1-p)L(a\otimes z^j)(1-p)\otimes w^j$ for all $a \in A$ and for all $j \in \mathbb{Z}$ as described in Lemma 25.2 which is $\mathcal{G}_1$-$\delta_1/4$-multiplicative completely positive linear map.

Define $L_3 : A\otimes C(\mathbb{T}) \to B\otimes U_2$ by $L_3(c) = L_3'(c)+L_3''(c)$ for all $c \in A\otimes C(\mathbb{T})$. Thus $L_3(a\otimes z^j) = L_2(a\otimes z^j)\otimes w^j$ for all $a \in A$ and integer $j \in \mathbb{Z}$. Define $\Phi : A\otimes C(\mathbb{T}) \to B\otimes U_2$ by $\Phi(a\otimes 1) = \varphi(a) \subset B\otimes 1_{U_2}$ for all $a \in A$ and $\Phi(1\otimes f) = 1_A\otimes f(w)$ for all $f \in C(\mathbb{T})$ and for some $\lambda \in \mathbb{T}$.

One then checks that, for all $t \in T(B\otimes U_2)$,

$$|t(L_3(g))) - t(\Phi(g))| < \sigma_1, \quad g \in \mathcal{H}_2, \text{ and} \tag{e 25.40}$$

$$t(g(1\otimes w)) \geq \frac{1}{2}\int_{\mathbb{T}} gdm \text{ for all } g \in \mathcal{H}_1. \tag{e 25.41}$$

Note that $CU(U_2) = U(U_2)$ (see Theorem 4.1 of [22]). It is also known (by working in matrices, for example) that there is a continuous path of unitaries in $CU(U_2)$ connecting 1_{U_2} to w with length no more than $\pi + \varepsilon/256$. Therefore one obtains a continuous path of unitaries $\{v(t) : t \in [1/4, 1/2]\} \subset CU(U_2)$ such that

$$v(1/4) = 1_{U_2}, \ \ v(1/2) = w, \ \text{ and}$$
$$\text{length}(\{v(t) : t \in [1/4, 1/2]\}) \le \pi + \varepsilon/256. \tag{e 25.42}$$

Note that $\varphi(a)\Phi(1 \otimes z) = \Phi(1 \otimes z)\varphi(a)$ for all $a \in A$. So, in particular, Φ is a unital homomorphism and

$$[\Phi]|_{\boldsymbol{\beta}(\underline{K}(A))} = 0. \tag{e 25.43}$$

Define a unital completely positive linear map $L_t : A_2 \to C([2,3], B \otimes U_2)$ by

$$L_t(f \otimes 1) = L_2(f \otimes 1) \ \text{ and } \ L_t(a \otimes z^j) = L_2(a \otimes z^j) \otimes (v((t-2)/4 + 1/4))^j$$

for all $a \in A$ and integers j and $t \in [2,3]$. Note that $L_t(1 \otimes z) \approx_{\min(\delta_2/2, \varepsilon/8)} (v_0 \oplus v_{00}) \otimes v((t-2)/4+1/4))$, and, since $v(s) \in CU(U_2)$, $L_t(1 \otimes z) \in_{\min(\delta_2/2, \varepsilon/8)} CU(B \otimes U_2)$ for all $t \in [2,3]$. Note also that, L_t is $\mathcal{G}_1$-$\delta_1/4$-multiplicative. Note that at $t = 2$, $L_t = L_2$ and at $t = 3$, $L_t = L_3$. It follows that

$$[L_3]|_{\mathcal{P}_1} = [L_2]|_{\mathcal{P}_1} = [\varphi]|_{\mathcal{P}_1}, \ \ [L_3]|_{\boldsymbol{\beta}(\mathcal{P}_2)} = 0, \ \text{ and} \tag{e 25.44}$$
$$L_3^{\ddagger}(x) = L_2^{\ddagger}(x) \ \text{ for all } \ x \in \mathcal{U}_1. \tag{e 25.45}$$

If $v = (e \otimes z) + (1 - e)$ for some projection $e \in A$, then

$$L_3(v) = L_2(e \otimes z) \otimes w + L_2((1-e)). \tag{e 25.46}$$

Since $w \in CU(U_2)$, one computes from (e 25.22) that that, with $x = ((1-p_i) + p_i \otimes z)((1-q_i) + q_i \otimes z^*)$,

$$\begin{aligned}\overline{\langle L_3(x)\rangle} &\approx_{\sigma_2/2^{10}} \overline{(\bar{z}_i^{(1)} \otimes w + (1-\bar{p}_i))(\bar{z}_i^{(2)} \otimes w + (1-\bar{q}_i))}\\ &= \overline{(\bar{z}_i^{(1)} + (1-\bar{p}_i))(\bar{p}_i \otimes w + (1-\bar{p}_i) \otimes 1_{U_2})(\bar{z}_i^{(2)} + (1-\bar{q}_i))(\bar{q}_i \otimes w + (1-\bar{q}_i))}\\ &= \overline{(\bar{z}_i^{(1)} + (1-\bar{p}_i))(\bar{z}_i^{(2)} + (1-\bar{q}_i))} = \overline{\langle L_2(x)\rangle},\end{aligned}$$

where $\bar{p}_i, \bar{q}_i, \bar{z}_i^{(1)}, \bar{z}_i^{(2)}$ are as above (see the lines following (e 25.22)), with Ψ_1 replaced by L_2. It follows that

$$\text{dist}(L_3^{\ddagger}(x), \bar{1}) < \sigma_2/2 \ \text{ for all } \ x \in \{\overline{1 \otimes z}\} \cup \mathcal{U}_2. \tag{e 25.47}$$

Note that, since $w \in CU(U_2)$ and $\varphi(q) \in B \otimes 1_{U_2}$,

$$\Phi(q \otimes z + (1-q) \otimes 1) = \varphi(q) \otimes w + \varphi(1-q) \in CU(B \otimes U_2)$$

for any projection $q \in A$. It follows that

$$\Phi^{\ddagger}(x) \in CU(B \otimes U_2) \text{ for all } x \in \{\overline{1 \otimes z}\} \cup \mathcal{U}_2. \tag{e 25.48}$$

Therefore (see also (e 25.43))

$$[L_3]|_{\mathcal{P}} = [\Phi]|_{\mathcal{P}} \text{ and } \mathrm{dist}(\Phi^{\ddagger}(x), L_3^{\ddagger}(x)) < \sigma_2 \text{ for all } x \in \mathcal{U}. \tag{e 25.49}$$

It follows from (e 25.41) that

$$\tau(\Phi(f)) \geq \Delta(f), \quad f \in \mathcal{H}_1, \ \tau \in T(B \otimes U_2), \tag{e 25.50}$$

and it follows from (e 25.40) that

$$|\tau(\Phi(f)) - \tau(L_3(f))| < \sigma_1, \quad f \in \mathcal{H}_2, \ \tau \in T(B \otimes U_2). \tag{e 25.51}$$

Applying Theorem 12.11 of [21], we obtain a unitary $w_1 \in B \otimes U_2$ such that

$$\|w_1^* \Phi(f) w_1 - L_3(f)\| < \varepsilon/4 \text{ for all } f \in \mathcal{F}_1. \tag{e 25.52}$$

Since $w \in U_2$, there is a continuous path of unitaries $\{w(t) : t \in [3/4, 1]\} \subset CU(U_2)$ (recall that $CU(U_2) = U_0(U_2)$) such that

$$w(3/4) = w, \ w(1) = 1_{U_2} \text{ and}$$
$$\mathrm{length}(\{w(t) : t \in [3/4, 1]\}) \leq \pi + \varepsilon/256. \tag{e 25.53}$$

Note that

$$\Phi(a \otimes 1)w(t) = w(t)\Phi(a \otimes 1) \text{ for all } a \in A \text{ and } t \in [3/4, 1]. \tag{e 25.54}$$

It follows from (e 25.52) that there exists a continuous path of unitaries $\{u(t) : t \in [1/2, 3/4]\} \subset B \otimes U_2$ such that (see (e 25.34), (e 25.35), and (e 25.38))

$$u(1/2) = (v_{00} + (v_0)) \otimes w, \ u(3/4) = w_1^* \Phi(1 \otimes z) w_1, \text{ and} \tag{e 25.55}$$

$$\|u(t) - u(1/2)\| < \varepsilon/4 \text{ for all } t \in [1/2, 3/4]. \tag{e 25.56}$$

It follows from (e 25.26), (e 25.34), and (e 25.35) that there exists a continuous path of unitaries $\{u(t) : t \in [0, 1/4]\} \subset B$ such that

$$u(0) = u, \ u(1/4) = v_{00} + v_0, \text{ and} \tag{e 25.57}$$

$$\|u(t) - u\| < \varepsilon/4 \text{ for all } t \in [0, 1/4]. \tag{e 25.58}$$

Also, define $u(t) = (v_{00} \oplus v_0) \otimes v(t)$ for all $t \in [1/4, 1/2]$. It follows that

$$\|\varphi(g)u(t) - u(t)\varphi(g)\| < \varepsilon/4 + \delta < 5\varepsilon/16 \text{ for all } g \in \mathcal{G}. \tag{e 25.59}$$

Then define

$$u(t) = w_1^*(p \otimes w(t) + (1-p) \otimes 1_{U_2})w_1 \text{ for all } t \in [3/4, 1]. \tag{e 25.60}$$

Then $\{u(t) : t \in [0,1]\} \subset B \otimes U_2$ is a continuous path of unitaries such that $u(0) = u$ and $u(1) = 1$. Moreover, by (e 25.58), (e 25.59), (e 25.55), (e 25.56), (e 25.52), (e 25.60), (e 25.26), (e 25.42), and (e 25.53),

$$\|\varphi(f)u(t) - u(t)\varphi(f)\| < \varepsilon \text{ for all } f \in \mathcal{F} \text{ and } \mathrm{length}(\{u(t)\}) \leq 2\pi + \varepsilon.$$

□

REMARK 25.5. Note, in the statement of Theorem 25.4, if $[1_A] \in \mathcal{P}$ (as an element of $K_0(A)$), by 2.14 of [21], condition (e 25.6) implies $[u] = 0$ in $K_0(B)$. In other words, by making $[1_A] \in \mathcal{P}$, (e 25.6) implies $[u] = 0$.

One also notices that if, for some i, $p_i = 1_A$ and $q_i = 0$, then (e 25.7) implies (e 25.8). In fact, (e 25.8) is redundant. To see this, let A be a unital simple separable amenable C^*-algebra with stable rank one. Let $G_0 \subset K_0(A)$ be a finitely generated subgroup containing $[1_A]$. Let $G_r = \rho_A(G_0)$. Then $\rho_A([1_A]) \neq 0$ and G_r is a finitely generated free abelian group. Then we may write $G_0 = G_0 \cap \ker\rho_A \oplus G_r'$, where $\rho_A(G_r') = G_r$ and $G_r' \cong G_r$. Note that $G_0 \cap \ker\rho_A$ is a finitely generated group. We may therefore write $G_0 \cap \ker\rho_A = G_{00} \oplus G_{01}$, where G_{00} is a torsion group and G_{01} is free abelian. Note that $G_{01} \oplus G_r'$ is free abelian. Therefore $G_0 = \mathrm{Tor}(G_0) \oplus F$, where F is a finitely generated free abelian subgroup. Note that there is an integer $m \geq 1$ such that $m[1_A] \in F$. Let $z \in C(\mathbb{T})$ be the standard unitary generator. Consider $A \otimes C(\mathbb{T})$. Then $\boldsymbol{\beta}(G_0) \subset \boldsymbol{\beta}(K_0(A))$ is a subgroup of $K_1(A \otimes C(\mathbb{T}))$. Moreover, $\boldsymbol{\beta}([1_A])$ may be identified with $[1 \otimes z]$.

If we choose $\mathcal{U}_2$ in the proof of 25.4 to generate $\boldsymbol{\beta}(F)$, then $m[1_A]$ is in the subgroup generated by $\{[p_i] - [q_i] : 1 \leq i \leq k\}$ (see the last paragraph). Thus, for any $\sigma_1 > 0$, we may assume that

$$\mathrm{dist}(\overline{u^m}, \bar{1}) < \sigma_1 \tag{e 25.61}$$

provided that (e 25.7) holds for a sufficiently small σ. Recall that B has stable rank one (see Theorem 9.7 of [21]) and $u \in U_0(B)$ (see the beginning of this remark). We may write $u = \exp(ih)v$ for some $h \in B_{s.a.}$ and $v \in CU(B)$. Recall, in this case, $U_0(B)/CU(B) = \mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))}$, where $\overline{\rho_B(K_0(B))}$ is a closed vector subspace of $\mathrm{Aff}(T(B))$ (see the proof of Lemma 11.5 of [21]). The image of $\overline{u^m}$ in $\mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))}$ is the same as m times the image of $\overline{u}$ in $\mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))}$. It follows from (e 25.61) that

$$\mathrm{dist}(\bar{u}, \bar{1}) < \sigma_1. \tag{e 25.62}$$

This implies that (with sufficiently small σ) the condition (e 25.8) is redundant and therefore can be omitted.

26. Stable Results

LEMMA 26.1. *Let C be a unital amenable separable C^*-algebra which is residually finite dimensional and satisfies the UCT. For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset C$, any finite subset $\mathcal{P} \subset \underline{K}(C)$, any unital homomorphism $h : C \to A$, where A is any unital C^*-algebra, and any $\kappa \in \mathrm{Hom}_\Lambda(\underline{K}(SC), \underline{K}(A))$, there exists an integer $N \geq 1$, a unital homomorphism $h_0 : C \to M_N(\mathbb{C}) \subset M_N(A)$, and a unitary $u \in U(M_{N+1}(A))$ such that*

$$\|[H(c),\, u]\| < \varepsilon \text{ for all } c \in \mathcal{F} \quad \text{and} \quad \mathrm{Bott}(H,\, u)|_{\mathcal{P}} = \kappa \circ \beta|_{\mathcal{P}}, \tag{e 26.1}$$

where $H(c) = \mathrm{diag}(h(c), h_0(c))$ for all $c \in C$.

PROOF. Define $S = \{z, 1_{C(\mathbb{T})}\}$, where z is the identity function on the unit circle. Define $x \in \mathrm{Hom}_\Lambda(\underline{K}(C \otimes C(\mathbb{T})), \underline{K}(A))$ as follows:

$$x|_{\underline{K}(C)} = [h] \text{ and } x|_{\boldsymbol{\beta}(\underline{K}(C))} = \kappa. \tag{e 26.2}$$

Fix a finite subset $\mathcal{P}_1 \subset \boldsymbol{\beta}(\underline{K}(C))$. Choose $\varepsilon_1 > 0$ and a finite subset $\mathcal{F}_1 \subset C$ satisfying the following condition:

$$[L']|_{\mathcal{P}_1} = [L'']|_{\mathcal{P}_1} \tag{e 26.3}$$

for any pair of $(\mathcal{F}_1 \otimes S)$-$\varepsilon_1$-multiplicative contractive completely positive linear maps $L', L'' : C \otimes C(\mathbb{T}) \to B$ (for any unital C^*-algebra B), whenever

$$L' \approx_{\varepsilon_1} L'' \text{ on } \mathcal{F}_1 \otimes S. \tag{e 26.4}$$

Let a positive number $\varepsilon > 0$, a finite subset $\mathcal{F}$ and a finite subset $\mathcal{P} \subset \underline{K}(C)$ be given. We may assume, without loss of generality, that

$$\mathrm{Bott}(H',\, u')|_{\mathcal{P}} = \mathrm{Bott}(H',\, u'')|_{\mathcal{P}} \tag{e 26.5}$$

whenever $\|u' - u''\| < \varepsilon$ for any unital homomorphism H' from C. Put $\varepsilon_2 = \min\{\varepsilon/2, \varepsilon_1/2\}$ and $\mathcal{F}_2 = \mathcal{F} \cup \mathcal{F}_1$ (choosing $\mathcal{P}_1 = \boldsymbol{\beta}(\mathcal{P})$ above).

Let $\delta > 0$, a finite subset $\mathcal{G} \subset C$, and a finite subset $\mathcal{P}_0 \subset \underline{K}(C)$ (in place of $\mathcal{P}$) be as provided by Lemma 4.17 of [21] for $\varepsilon_2/2$ (in place of ε) and $\mathcal{F}_2$ (in place of $\mathcal{F}$). We may assume that $\mathcal{F}_2$ and $\mathcal{G}$ are in the unit ball of C and $\delta < \min\{1/2, \varepsilon_2/16\}$. Fix another finite subset $\mathcal{P}_2 \subset \underline{K}(C)$ and define $\mathcal{P}_3 = \mathcal{P}_0 \cup \boldsymbol{\beta}(\mathcal{P}_2)$ (as a subset of $\underline{K}(C \otimes C(\mathbb{T}))$). We may assume that $\mathcal{P}_1 \subset \boldsymbol{\beta}(\mathcal{P}_2)$.

It follows from Theorem 18.2 of [21] that there are integers $k_1, k_2, ..., k_m$ and K_1, a homomorphism $h_1' : C \otimes C(\mathbb{T}) \to \bigoplus_{j=1}^m M_{k_j}(\mathbb{C}) \to M_K(A)$, and a unital $(\mathcal{G} \otimes S)$-$\delta/2$-multiplicative completely positive linear map $L' : C \otimes C(\mathbb{T}) \to M_{K+1}(A)$ such that

$$[L']|_{\mathcal{P}_3} = (x + [h_1'])|_{\mathcal{P}_3}. \tag{e 26.6}$$

Write $h_1' = \oplus_{j=1}^m H_j'$, where $H_j' = \psi_j \circ \pi_j$, $\pi_j : A \otimes C(\mathbb{T}) \to M_{k_j}(\mathbb{C})$ is a finite dimensional representation and $\psi_j : M_{k_j}(\mathbb{C}) \to M_K(A)$ is a homomorphism. Let e_j be a minimal projection of $M_{k_j}(\mathbb{C})$ and $q_j = \psi_j(e_j) \in M_K(A)$, and $Q_j = \psi_j(1_{M_{k_j}(\mathbb{C})}) \in M_K(A)$. Set $p_j = 1_{M_K(A)} - q_j$, $j = 1, 2, ..., m$. Then $M_{k_jK}(\mathbb{C})$ can be identified with $M_{k_j}(M_K(A)) = M_{k_j}\big((q_j \oplus p_j)M_K(A)(q_j \oplus p_j)\big)$ (since $q_j + p_j = 1_{M_K(A)}$) in such a way that $Q_j(M_K(A))Q_j$ is identified with $M_{k_j}(q_j M_K(A) q_j)$. Define $\psi_j' : M_{k_j}(\mathbb{C}) \to M_{k_j}(\mathbb{C} \cdot 1_{M_K(A)}) \subset M_{k_jK}(A)$ by sending e_j to p_j. Define $H_j'' : C \otimes C(\mathbb{T}) \to M_{k_jK}(A)$ by $H_j''(c) = \psi_j \circ \pi_j(c) \oplus \psi_j' \circ \pi_j$ (conjugating a unitary). Note we require H_j'' maps into the scalar matrices of $M_{k_jK}(A)$. Let $H' = \oplus_{j=1}^m \psi_j' : C \otimes C(\mathbb{T}) \to M_{k_j}(p_j M_K(A) p_j) \subset M_{(\sum_j k_j)K}(A)$ (conjugating a suitable unitary). Let $N_1 = (\sum_{j=1}^m k_j)K$. Define $h_1 = h_1' \oplus H'$ and $L = L' \oplus H'$. Then h_1 maps $C \otimes C(\mathbb{T})$ into $M_{N_1}(\mathbb{C} \cdot 1_A) \subset M_{N_1}(A)$.

In other words, there are an integer $N_1 \geq 1$, a unital homomorphism $h_1' : C \otimes C(\mathbb{T}) \to M_{N_1}(\mathbb{C}) \subset M_{N_1}(A)$, and a unital $(\mathcal{G} \otimes S)$-$\delta/2$-multiplicative completely positive linear map $L : C \otimes C(\mathbb{T}) \to M_{N_1+1}(A)$ such that

(e 26.7) $$[L]|_{\mathcal{P}_3} = (x + [h_1])|_{\mathcal{P}_3}.$$

We may assume that there is a unitary $v_0 \in M_{N_1+1}(A)$ such that

(e 26.8) $$\|L(1 \otimes z) - v_0\| < \varepsilon_2/2.$$

Define $H_1 : C \to M_{N_1+1}(A)$ by

(e 26.9) $$H_1(c) = h(c) \oplus h_1(c \otimes 1) \text{ for all } c \in C.$$

Define $L_1 : C \to M_{N_1+1}(A)$ by $L_1(c) = L(c \otimes 1)$ for all $c \in C$. Note that

(e 26.10) $$[L_1]|_{\mathcal{P}_0} = [H_1]|_{\mathcal{P}_0}.$$

It follows from Lemma 4.17 of [21] that there exists an integer $N_2 \geq 1$, a unital homomorphism $h_2 : C \to M_{N_2(N_1+1)}(\mathbb{C}) \subset M_{N_2(N_1+1)}(A)$, and a unitary $W \in M_{(N_2+1)(1+N_1)}(A)$ such that

(e 26.11) $$W^*(L_1(c) \oplus h_2(c))W \approx_{\varepsilon/4} H_1(c) \oplus h_2(c) \text{ for all } c \in \mathcal{F}_2.$$

Put $N = N_2(N_1+1) + N_1$. Now define $h_0 : C \to M_N(\mathbb{C})$ and $H : C \to M_{N+1}(A)$ by

(e 26.12) $$h_0(c) = h_1(c \otimes 1) \oplus h_2(c) \text{ and } H(c) = h(c) \oplus h_0(c)$$

for all $c \in C$. Define $u = W^*(v_0 \oplus 1_{M_{N_2(N_1+1)}})W$. Then, by (e 26.11), and as L_1 is $(\mathcal{G} \otimes S)$-$\delta/2$- multiplicative, we have

$$\begin{aligned}\|[H(c), u]\| &\leq \|(H(c) - \operatorname{Ad} W \circ (L_1(c) \oplus h_2(c)))u]\| \\ &\quad + \|\operatorname{Ad} W \circ (L_1(c) \oplus h_2(c)), u]\| + \|u(H(c) - \operatorname{Ad} W \circ (L_1(c) \oplus h_2(c)))\| \\ &< \varepsilon/4 + \delta/2 + \varepsilon/4 < \varepsilon \text{ for all } c \in \mathcal{F}_2.\end{aligned}$$

Define $L_2 : C \to M_{N+1}(A)$ by $L_2(c) = L_1(c) \oplus h_2(c)$ for all $c \in C$. Then, we compute that

$$\begin{aligned}&\mathrm{Bott}(H, u)|_{\mathcal{P}}\\ &= \mathrm{Bott}(\mathrm{Ad}\, W \circ L_2, u)|_{\mathcal{P}} = \mathrm{Bott}(L_2, v_0 \oplus 1_{M_{N_2(N_1+1)}})|_{\mathcal{P}}\\ &= \mathrm{Bott}(L_1, v_0)|_{\mathcal{P}} + \mathrm{Bott}(h_2, 1_{M_{N_2(N_1+1)}})|_{\mathcal{P}}\\ &= [L]|_{\boldsymbol{\beta}(\mathcal{P})} + 0 = (x + [h])|_{\boldsymbol{\beta}(\mathcal{P})} = \kappa|_{\mathcal{P}}.\end{aligned}$$

□

THEOREM 26.2. *Let C be a unital amenable separable C^*-algebra which is residually finite dimensional and satisfies the UCT. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there are $\delta > 0$, a finite subset $\mathcal{G} \subset C$, and a finite subset $\mathcal{P} \subset \underline{K}(C)$ satisfying the following condition:*

Suppose that A is a unital C^*-algebra, suppose that $h : C \to A$ is a unital homomorphism and suppose that $u \in U(A)$ is a unitary such that

$$\|[h(a), u]\| < \delta \text{ for all } a \in \mathcal{G} \text{ and } \mathrm{Bott}(h, u)|_{\mathcal{P}} = 0. \tag{e 26.13}$$

Then there exist an integer $N \geq 1$, a unital homomorphism $H_0 : C \otimes C(\mathbb{T}) \to M_N(\mathbb{C})$ ($\subset M_N(A)$) (with finite dimensional range), and a continuous path of unitaries $\{U(t) : t \in [0, 1]\}$ in $M_{N+1}(A)$ such that

$$U(0) = u', \ U(1) = 1_{M_{N+1}(A)}, \text{ and } \|[h'(a), U(t)]\| < \varepsilon \text{ for all } a \in \mathcal{F},$$

where

$$u' = \mathrm{diag}(u, H_0(1 \otimes z))$$

and $h'(f) = h(f) \oplus H_0(f \otimes 1)$ for $f \in C$, and $z \in C(\mathbb{T})$ is the identity function on the unit circle.

Moreover,

$$\mathrm{Length}(\{U(t)\}) \leq \pi + \varepsilon. \tag{e 26.14}$$

PROOF. Let $\varepsilon > 0$ and $\mathcal{F} \subset C$ be given. We may assume that $\mathcal{F}$ is in the unit ball of C.

Let $\delta_1 > 0$, $\mathcal{G}_1 \subset C \otimes C(\mathbb{T})$, and $\mathcal{P}_1 \subset \underline{K}(C \otimes C(\mathbb{T}))$ be as provided by Lemma 4.17 of [21] for $\varepsilon/4$ and $\mathcal{F} \otimes S$. We may assume that $\mathcal{G}_1 = \mathcal{G}_1' \otimes S$, where $\mathcal{G}_1'$ is in the unit ball of C and $S = \{1_{C(\mathbb{T})}, z\} \subset C(\mathbb{T})$. Moreover, we may assume that $\mathcal{P}_1 = \mathcal{P}_2 \cup \mathcal{P}_3$, where $\mathcal{P}_2 \subset \underline{K}(C)$ and $\mathcal{P}_3 \subset \boldsymbol{\beta}(\underline{K}(C))$. Let $\mathcal{P} = \mathcal{P}_2 \cup \beta^{-1}(\mathcal{P}_3) \subset \underline{K}(C)$. Furthermore, we may assume that any δ_1-$\mathcal{G}_1$-multiplicative contractive completely positive linear map L' from $C \otimes C(\mathbb{T})$ to a unital C^*-algebra gives rise to a well defined map $[L']|_{\mathcal{P}_1}$.

Let $\delta_2 > 0$ and a finite subset $\mathcal{G}_2 \subset C$ be as provided by 2.8 of [38] for $\delta_1/2$ and $\mathcal{G}_1'$ above.

Let $\delta = \min\{\delta_2/2, \delta_1/2, \varepsilon/2\}$ and $\mathcal{G} = \mathcal{F} \cup \mathcal{G}_2$.

Suppose that h and u satisfy the assumption with δ, $\mathcal{G}$ and $\mathcal{P}$ as above. Thus, by 2.8 of [38], there is a $\delta_1/2$-$\mathcal{G}_1$-multiplicative contractive completely positive linear map $L : C \otimes C(\mathbb{T}) \to A$ such that

$$\|L(f \otimes 1) - h(f)\| < \delta_1/2 \text{ for all } f \in \mathcal{G}_1' \text{ and} \tag{e 26.15}$$

$$\|L(1 \otimes z) - u\| < \delta_1/2. \tag{e 26.16}$$

Define $y \in \mathrm{Hom}_\Lambda(\underline{K}(C \otimes C(\mathbb{T})), \underline{K}(A))$ as follows:

$$y|_{\underline{K}(C)} = [h]|_{\underline{K}(C)} \text{ and } y|_{\boldsymbol{\beta}(\underline{K}(C))} = 0.$$

It follows from $\mathrm{Bott}(h, u)|_{\mathcal{P}} = 0$ that $[L]|_{\beta(\mathcal{P})} = 0$.

Then

$$[L]|_{\mathcal{P}_1} = y|_{\mathcal{P}_1}. \tag{e 26.17}$$

Define $H : C \otimes C(\mathbb{T}) \to A$ by

$$H(c \otimes g) = h(c) \cdot g(1) \cdot 1_A$$

for all $c \in C$ and $g \in C(\mathbb{T})$, where $\mathbb{T}$ refers to the unit circle (and $1 \in \mathbb{T}$).

It follows that

$$[H]|_{\mathcal{P}_1} = y|_{\mathcal{P}_1} = [L]|_{\mathcal{P}_1}. \tag{e 26.18}$$

It follows from Lemma 4.17 of [21] that there are an integer $N \geq 1$, a unital homomorphism $H_0 : C \otimes C(\mathbb{T}) \to M_N(\mathbb{C})$ $(\subset M_N(A))$ with finite dimensional range, and a unitary $W \in U(M_{1+N}(A))$ such that

$$W^*(H(c) \oplus H_0(c))W \approx_{\varepsilon/4} L(c) \oplus H_0(c) \text{ for all } c \in \mathcal{F} \otimes S. \tag{e 26.19}$$

Since H_0 has finite dimensional range and since $H_0(1 \otimes z)$ is in the center of $\mathrm{range}(H_0) \subset M_N(\mathbb{C})$, it is easy to construct a continuous path $\{V'(t) : t \in [0, 1]\}$ in a finite dimensional C^*-subalgebra of $M_N(\mathbb{C})$ such that

$$V'(0) = H_0(1 \otimes z), \;\; V'(1) = 1_{M_N(A)} \text{ and} \tag{e 26.20}$$

$$H_0(c \otimes 1)V'(t) = V'(t)H_0(c \otimes 1) \tag{e 26.21}$$

for all $c \in C$ and $t \in [0, 1]$. Moreover, we may ensure that

$$\mathrm{Length}(\{V'(t)\}) \leq \pi. \tag{e 26.22}$$

Now define $U(1/4 + 3t/4) = W^*\mathrm{diag}(1, V'(t))W$ for $t \in [0, 1]$ and

$$u' = u \oplus H_0(1_A \otimes z) \text{ and } h'(c) = h(c) \oplus H_0(c \otimes 1)$$

for $c \in C$ for $t \in [0,1]$. Then, by (e 26.19),

$$\|u' - U(1/4)\| < \varepsilon/4 \text{ and } \|[U(t), h'(a)]\| < \varepsilon/4 \tag{e 26.23}$$

for all $a \in \mathcal{F}$ and $t \in [1/4, 1]$. The desired conclusion follows by connecting $U(1/4)$ with u' with a short path as follows: There is a self-adjoint element $a \in M_{1+N}(A)$ with $\|a\| \le \varepsilon\pi/8$ such that

$$\exp(ia) = u'U(1/4)^* \tag{e 26.24}$$

Then the path of unitaries $U(t) = \exp(i(1-4t)a)U(1/4)$ for $t \in [0, 1/4)$ satisfies the requirements. □

LEMMA 26.3. *Let C be a unital separable C^*-algebra whose irreducible representations have bounded dimension and let B be a unital C^*-algebra with $T(B) \neq \varnothing$. Suppose that $\varphi_1, \varphi_2 : C \to B$ are two unital monomorphisms such that*

$$[\varphi_1] = [\varphi_2] \text{ in } KK(C, B),$$

Let $\theta : \underline{K}(C) \to \underline{K}(M_{\varphi_1,\varphi_2})$ be the splitting map defined in Equation (e 2.46) in Definition 2.21 of [21].

For any $1/2 > \varepsilon > 0$, any finite subset $\mathcal{F} \subset C$ and any finite subset $\mathcal{P} \subset \underline{K}(C)$, there are integers $N_1 \ge 1$, a unital $\varepsilon/2$-$\mathcal{F}$-multiplicative completely positive linear map $L : C \to M_{1+N_1}(M_{\varphi_1,\varphi_2})$, a unital homomorphism $h_0 : C \to M_{N_1}(\mathbb{C})$ (later, $M_{N_1}(\mathbb{C})$ can be regarded as unital subalgebra of $M_{N_1}(B)$ and also of $M_{N_1}(M_{\varphi_1,\varphi_2})$), and a continuous path of unitaries $\{V(t) : t \in [0, 1-d]\}$ in $M_{1+N_1}(B)$ for some $1/2 > d > 0$, such that $[L]|_{\mathcal{P}}$ is well defined, $V(0) = 1_{M_{1+N_1}(B)}$,

$$[L]|_{\mathcal{P}} = (\theta + [h_0])|_{\mathcal{P}}, \tag{e 26.25}$$

$$\pi_t \circ L \approx_\varepsilon \operatorname{Ad} V(t) \circ (\varphi_1 \oplus h_0) \text{ on } \mathcal{F} \tag{e 26.26}$$

for all $t \in (0, 1-d]$,

$$\pi_t \circ L \approx_\varepsilon \operatorname{Ad} V(1-d) \circ (\varphi_1 \oplus h_0) \text{ on } \mathcal{F} \tag{e 26.27}$$

for all $t \in (1-d, 1)$, and

$$\pi_1 \circ L \approx_\varepsilon \varphi_2 \oplus h_0 \text{ on } \mathcal{F}, \tag{e 26.28}$$

where $\pi_t : M_{\varphi_1,\varphi_2} \to B$ is the point evaluation at $t \in (0,1)$.

Proof. Let $\varepsilon > 0$ and let $\mathcal{F} \subset C$ be a finite subset. Let $\delta_1 > 0$, a finite subset $\mathcal{G}_1 \subset C$, and a finite subset $\mathcal{P} \subset \underline{K}(C)$ be as provided by 26.2 for $\varepsilon/4$ and $\mathcal{F}$ above. In particular, we assume that $\delta_1 < \delta_{\mathcal{P}}$ (see Definition 2.14 of [21]). By Lemma 2.15 of [40], we may further assume that δ_1 is sufficiently small that

$$\text{(e 26.29)} \qquad \mathrm{Bott}(\Phi, U_1U_2U_3)|_{\mathcal{P}} = \sum_{i=1}^{3} \mathrm{Bott}(\Phi, U_i)|_{\mathcal{P}}$$

whenever $\|[\Phi(a), U_i]\| < \delta_1$ for all $a \in \mathcal{G}_1$, $i = 1, 2, 3$.

Let $\varepsilon_1 = \min\{\delta_1/2, \varepsilon/16\}$ and $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{G}_1$. We may assume that $\mathcal{F}_1$ is in the unit ball of C. We may also assume that $[L']|_{\mathcal{P}}$ is well defined for any ε_1-$\mathcal{F}_1$-multiplicative contractive completely positive linear map L' from C to any unital C*-algebra.

Let $\delta_2 > 0$, $\mathcal{G} \subset C$, and $\mathcal{P}_1 \subset \underline{K}(C)$ be a constant and finite subsets as provided by Lemma 4.17 of [21] for $\varepsilon_1/2$ and $\mathcal{F}_1$. We may assume that $\delta_2 < \varepsilon_1/2$, $\mathcal{G} \supset \mathcal{F}_1$, and $\mathcal{P}_1 \supset \mathcal{P}$. We also assume that $\mathcal{G}$ is in the unit ball of C.

It follows from Theorem 18.2 of [21] that there exist an integer $K_1 \geq 1$, a unital homomorphism $h_0' : C \to M_{K_1}(\mathbb{C})$ (see also lines around (e 26.7)), and a $\delta_2/2$-$\mathcal{G}$-multiplicative contractive completely positive linear map $L_1 : C \to M_{K_1+1}(M_{\varphi_1,\varphi_2})$ such that

$$\text{(e 26.30)} \qquad [L_1]|_{\mathcal{P}_1} = (\theta + [h_0'])|_{\mathcal{P}_1}.$$

Note that $[\pi_0] \circ \theta = [\varphi_1]$ and $[\pi_1] \circ \theta = [\varphi_2]$ and, for each $t \in (0, 1)$,

$$\text{(e 26.31)} \qquad [\pi_t] \circ \theta = [\varphi_1] = [\varphi_2].$$

By Lemma 4.17 of [21], we obtain an integer K_0, a unitary $V \in U(M_{1+K_1+K_0}((C)))$, and a unital homomorphism $h : C \to M_{K_0}(\mathbb{C})$ such that

$$\text{(e 26.32)} \qquad \mathrm{Ad}\, V \circ (\pi_e \circ L_1 \oplus h) \approx_{\varepsilon_1/2} (\mathrm{id} \oplus h_0' \oplus h) \text{ on } \mathcal{F}_1,$$

where $\pi_e : M_{\varphi_1,\varphi_2} \to C$ is the canonical projection.

(Here and below, we will identify a homomorphism mapping to $M_k(\mathbb{C})$ with a homomorphism to $M_k(A)$ for any unital C^* algebra A, without introducing new notation.)

Write $V_{00} = \varphi_1(V)$ and $V_{00}' = \varphi_2(V)$. The assumption that $[\varphi_1] = [\varphi_2]$ implies that $[V_{00}] = [V_{00}']$ in $K_1(B)$. By adding another homomorphism to h in (e 26.32), replacing K_0 by $2K_0$, and replacing V by $V \oplus 1_{M_{K_0}}$, if necessary, we may assume that V_{00} and V_{00}' are in the same connected component of $U(M_{1+K_1+K_0}(B))$. (Note that $[V_{00}] = [V_{00}']$.)

One obtains a continuous path of unitaries $\{Z(t) : t \in [0, 1]\}$ in $M_{1+K_1+K_0}(B)$ such that

$$\text{(e 26.33)} \qquad Z(0) = V_{00} \text{ and } Z(1) = V_{00}'.$$

It follows that $Z \in M_{1+K_1+K_0}(M_{\varphi_1,\varphi_2})$. By replacing L_1 by $\mathrm{ad}\, Z \circ (L_1 \oplus h)$ and using a new h_0', we may assume that

$$\pi_0 \circ L_1 \approx_{\varepsilon_1/2} \varphi_1 \oplus h_0' \ \text{ and } \ \pi_1 \circ L_1 \approx_{\varepsilon_1/2} \varphi_2 \oplus h_0' \ \text{ on } \ \mathcal{F}_1. \tag{e 26.34}$$

Define $\lambda : C \to M_{1+K_1+K_0}(C)$ by $\lambda(c) = \mathrm{diag}(c, h_0'(c))$, where we also identify $M_{K_0+K_1}(\mathbb{C})$ with the scalar matrices in $M_{K_0+K_1}(C)$. In particular, since φ_i is unital, $\varphi_i \otimes \mathrm{id}_{M_{K_1+K_0}}$ is the identity on $M_{K_0+K_1}(\mathbb{C})$, $i = 1, 2$. Consequently, $(\varphi_i \otimes \mathrm{id}_{M_{K_0+K_1}}) \circ h_0' = h_0'$. Therefore, one may write

$$\varphi_i(c) \oplus h_0'(c) = (\varphi_i \otimes \mathrm{id}_{M_{K_0+K_1+1}}) \circ \lambda(c) \ \text{ for all } \ c \in C.$$

There is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that

$$\pi_{t_i} \circ L_1 \approx_{\delta_2/8} \pi_t \circ L_1 \ \text{ on } \ \mathcal{G} \tag{e 26.35}$$

for all $t_i \le t \le t_{i+1}$, $i = 1, 2, ..., n-1$. Applying Lemma 4.17 of [21] again, we obtain an integer $K_2 \ge 1$, a unital homomorphism $h_{00} : C \to M_{K_2}(\mathbb{C})$, and a unitary $V_{t_i} \in M_{1+K_0+K_1+K_2}(B)$ such that

$$\mathrm{Ad}\, V_{t_i} \circ (\varphi_1 \oplus h_0' \oplus h_{00}) \approx_{\varepsilon_1/2} (\pi_{t_i} \circ L_1 \oplus h_{00}) \ \text{ on } \ \mathcal{F}_1. \tag{e 26.36}$$

Note that, by (e 26.35), (e 26.36), and (e 26.34),

$$\|[\varphi_1 \oplus h_0' \oplus h_{00}(a), V_{t_i} V_{t_{i+1}}^*]\| < \delta_2/4 + \varepsilon_1 \ \text{ for all } \ a \in \mathcal{F}_1.$$

Define $\eta_{-1} = 0$ and

$$\eta_k = \sum_{i=0}^{k} \mathrm{Bott}(\varphi_1 \oplus h_0' \oplus h_{00}, V_{t_i} V_{t_{i+1}}^*)|_{\mathcal{P}}, \ k = 0, 1, ..., n-1.$$

Now we will construct, for each i, a unital homomorphism $F_i : C \to M_{J_i}(\mathbb{C}) \subset M_{J_i}(B)$ and a unitary $W_i \in M_{1+K_0+K_1+K_2+\sum_{k=1}^{i} J_i}(B)$ such that

$$\|[H_i(a), W_i]\| < \delta_2/4 \ \text{ for all } \ a \in \mathcal{F}_1 \ \text{ and } \ \mathrm{Bott}(H_i, W_i) = \eta_{i-1}, \tag{e 26.37}$$

where $H_i = \varphi_1 \oplus h_0' \oplus h_{00} \oplus \bigoplus_{k=1}^{i} F_i$, $i = 1, 2, ..., n-1$.

Let $W_0 = 1_{M_{1+K_0+K_1+K_2}}$. It follows from Lemma 26.1 that there are an integer $J_1 \ge 1$, a unital homomorphism $F_1 : C \to M_{J_1}(\mathbb{C})$, and a unitary $W_0 \in U(M_{1+K_0+K_1+K_2+J_1}(B))$ such that

$$\|[H_1(a), W_1]\| < \delta_2/4 \ \text{ for all } \ a \in \mathcal{F}_1 \ \text{ and } \ \mathrm{Bott}(H_1, W_1) = \eta_0, \tag{e 26.38}$$

where $H_1 = \varphi_1 \oplus h_0' \oplus h_{00} \oplus F_1$.

Assume that we have constructed the required F_i and W_i for $i = 0, 1, ..., k < n-1$. It follows from Lemma 26.1 that there are an integer $J_{k+1} \geq 1$, a unital homomorphism $F_{k+1} : C \to M_{J_{k+1}}(\mathbb{C})$, and a unitary

$$W_{k+1} \in U(M_{1+K_0+K_1+K_2+\sum_{i=1}^{k+1} J_i}(B))$$

such that

$$\|[H_{k+1}(a),\, W_{k+1}]\| < \delta_2/4 \text{ for all } a \in \mathcal{F}_1 \qquad (e\,26.39)$$
$$\text{and } \mathrm{Bott}(H_{k+1}, W_{k+1}) = \eta_k$$

where $H_{k+1} = \varphi_1 \oplus h_0' \oplus h_{00} \oplus \bigoplus_{i=1}^{k+1} F_i$. This finished the construction of F_i, W_i and H_i for $i = 0, 1, ..., n-1$.

Now define $F_{00} = h_{00} \oplus \bigoplus_{i=0}^{n-1} F_i$ and define $K_3 = 1 + K_0 + K_1 + K_2 + \sum_{i=1}^{n-1} J_i$. Define

$$v_{t_k} = \mathrm{diag}(W_k \mathrm{diag}(V_{t_k}, \mathrm{id}_{1_{M_{\sum_{i=1}^{k} J_i}}}), 1_{M_{\sum_{i=k+1}^{n-1} J_i}}),$$

$k = 1, 2, ..., n-1$ and $v_{t_0} = 1_{M_{1+K_0+K_1+K_2+\sum_{i=1}^{n-1} J_i}}$. Then

$$\mathrm{Ad}\, v_{t_i} \circ (\varphi_1 \oplus h_0' \oplus F_{00}) \approx_{\delta_2+\varepsilon_1} \pi_{t_i} \circ (L_1 \oplus F_{00}) \text{ on } \mathcal{F}_1,$$
$$\|[\varphi_1 \oplus h_0' \oplus F_{00}(a),\, v_{t_i} v_{t_{i+1}}^*]\| < \delta_2/2 + 2\varepsilon_1 \text{ for all } a \in \mathcal{F}_1, \text{ and}$$
$$\mathrm{Bott}(\varphi_1 \oplus h_0' \oplus F_{00}, v_{t_i} v_{t_{i+1}}^*)$$
$$= \mathrm{Bott}(\varphi_1', W_i') + \mathrm{Bott}(\varphi_1', V_{t_i}'(V_{t_{i+1}}')^*) + \mathrm{Bott}(\varphi_1', (W_{i+1}')^*)$$
$$= \eta_{i-1} + \mathrm{Bott}(\varphi_1', V_{t_i} V_{t_{i+1}}^*) - \eta_i = 0,$$

where $\varphi_1' = \varphi_1 \oplus h_0' \oplus F_{00}$, $W_i' = \mathrm{diag}(W_i, 1_{M_{\sum_{j=i+1}^{n-1} J_i}})$ and $V_{t_i}' = \mathrm{diag}(V_{t_i}, 1_{M_{\sum_{i=1}^{n-1} J_i}})$, $i = 0, 1, 2, ..., n-2$.

It follows by Lemma 26.2 that there are an integer $N_1 \geq 1$, a unital homomorphism $F_0' : C \to M_{N_1}(\mathbb{C})$, and a continuous path of unitaries $\{w_i(t) : t \in [t_{i-1}, t_i]\}$ in $M_{K_3}(B)$ such that

$$w_i(t_{i-1}) = v_{i-1}'(v_i')^*, w_i(t_i) = 1, \text{ and} \qquad (e\,26.40)$$

$$\|[\varphi_1 \oplus h_0' \oplus F_{00} \oplus F_0'(a),\, w_i(t)]\| < \varepsilon/4 \text{ for all } a \in \mathcal{F}, \qquad (e\,26.41)$$

where $v_i' = \mathrm{diag}(v_i, 1_{M_{N_1}}(B))$, $i = 1, 2, ..., n-1$. Define $V(t) = w_i(t) v_i'$ for $t \in [t_{i-1}, t_i]$, $i = 1, 2, ..., n-1$. Then $V(t) \in C([0, t_{n-1}], M_{K_3+N_1}(B))$. Moreover,

$$\mathrm{Ad}\, V(t) \circ (\varphi_1 \oplus h_0' \oplus F_{00} \oplus F_0') \qquad (e\,26.42)$$
$$\approx_\varepsilon \pi_t \circ L_1 \oplus F_{00} \oplus F_0' \text{ on } \mathcal{F}.$$

Define $h_0 = h_0' \oplus F_{00} \oplus F_0'$, $L = L_1 \oplus F_{00} + F_0'$, and $d = 1 - t_{n-1}$. Then, by (e 26.42), (e 26.26) and (e 26.27) hold. From (e 26.34), it follows that (e 26.28) also holds.

□

27. Asymptotic Unitary Equivalence

LEMMA 27.1. *Let C_1 and A_1 be two unital separable simple C^*-algebras in $\mathcal{B}_1$, let U_1 and U_2 be two UHF-algebras of infinite type and consider the C^*-algebras $C = C_1 \otimes U_1$ and $A = A_1 \otimes U_2$. Suppose that $\varphi_1, \varphi_2 : C \to A$ are two unital monomorphisms. Suppose also that*

$$[\varphi_1] = [\varphi_2] \text{ in } KL(C, A), \tag{e 27.1}$$

$$(\varphi_1)_T = (\varphi_2)_T \text{ and } \varphi_1^{\ddagger} = \varphi_2^{\ddagger}. \tag{e 27.2}$$

Then φ_1 and φ_2 are approximately unitarily equivalent.

PROOF. This follows immediately from Theorem 12.11 part (a) of [21]. Note that both A and C are in $\mathcal{B}_1$. □

LEMMA 27.2. *Let B be a unital C^*-algebra and let $u_1, u_2, ..., u_n \subset U(B)$ be unitaries. Suppose that $v_1, v_2, ..., v_m \subset U(B)$ are also unitaries such that $[v_j] \subset G$, $j = 1, ..., m$, where G is the subgroup of $K_1(B)$ generated by $[u_1], [u_2], ..., [u_n]$. There exist $\delta > 0$ and a finite subset $\mathcal{F} \subset B$ satisfying the following condition: For any unital C^*-algebra A and any unital monomorphisms $\varphi_1, \varphi_2 : B \to A$, if $\tau \circ \varphi_1 = \tau \circ \varphi_2$ for all $\tau \in T(A)$ and if there is a unitary $w \in U(B)$ such that*

$$\|w^*\varphi_1(b)w - \varphi_2(b)\| < \delta \text{ for all } b \in \mathcal{F}, \tag{e 27.3}$$

then there exists a group homomorphism $\alpha : G \to \mathrm{Aff}(T(A))$ such that

$$\frac{1}{2\pi i}\tau(\log(\varphi_2(u_k)w^*\varphi_1(u_k^*)w) = \alpha([u_k])(\tau) \text{ and} \tag{e 27.4}$$

$$\frac{1}{2\pi i}\tau(\log(\varphi_2(v_j)w^*\varphi_1(v_j^*)w) = \alpha([v_j])(\tau), \tag{e 27.5}$$

for any $\tau \in T(A)$, $k = 1, 2, ..., n$ and $j = 1, 2, ..., m$.

PROOF. The proof is essentially contained in the proofs of 6.1, 6.2, and 6.3 of [36]. Note that there is a typo in Lemma 6.2 and Lemma 6.3 in [36]: "$\tau(\alpha(a)) = a$" should be "$\tau(\alpha(a)) = \tau(a)$". Here the condition $\tau \circ \varphi_1 = \tau \circ \varphi_2$ plays the role of condition $\tau(\alpha(a)) = \tau(a)$ there.

□

LEMMA 27.3. *Let C_1 be a unital simple C^*-algebra as in Theorem 14 .10 of [21], let A_1 be a unital separable simple C^*-algebra in $\mathcal{B}_0$, and let U_1 and U_2 be UHF-algebras of infinite type. Let $C = C_1 \otimes U_1$ and $A = A_1 \otimes U_2$. Suppose that $\varphi_1, \varphi_2 : C \to A$ are unital monomorphisms. Suppose also that*

$$[\varphi_1] = [\varphi_2] \text{ in } KL(C, A), \tag{e 27.6}$$

$$\varphi_1^{\ddagger} = \varphi_2^{\ddagger}, \ (\varphi_1)_T = (\varphi_2)_T, \text{ and} \tag{e 27.7}$$

$$R_{\varphi,\psi}(K_1(M_{\varphi_1,\varphi_2})) \subset \rho_A(K_0(A)). \tag{e 27.8}$$

Then, for any increasing sequence of finite subsets $\{\mathcal{F}_n\}$ of C whose union is dense in C, any increasing sequence of finite subsets $\mathcal{P}_n$ of $K_1(C)$ with $\bigcup_{n=1}^{\infty}\mathcal{P}_n = K_1(C)$, and any decreasing sequence of positive numbers $\{\delta_n\}$ with $\sum_{n=1}^{\infty}\delta_n < \infty$, there exists a sequence of unitaries $\{u_n\}$ in $U(A)$ such that

(e 27.9) $$\mathrm{Ad}u_n\circ\varphi_1 \approx_{\delta_n} \varphi_2 \ \text{ on } \mathcal{F}_n \ \text{ and}$$

(e 27.10) $$\rho_A(\mathrm{bott}_1(\varphi_2, u_n^*u_{n+1})(x)) = 0 \text{ for all } x\in\mathcal{P}_n(\subset K_1(C))$$

and for all sufficiently large n.

PROOF. Note that $A\cong A\otimes U_2$. Therefore, as U_2 is of infinite type, there is a unital homomorphism $s: A\otimes U_2\to A$ such that $s\circ\imath$ is approximately unitarily equivalent to the identity map on A, where $\imath: A\to A\otimes U_2$ is defined by $a\to a\otimes 1_{U_2}$ for all $a\in A$. Therefore, we may assume that $\varphi_1(C)$, $\varphi_2(C)\subset A\otimes 1_{U_2}$. By Lemma 27.1, there exists a sequence of unitaries $\{v_n\}\subset A$ such that

(e 27.11) $$\lim_{n\to\infty}\mathrm{Ad}\, v_n\circ\varphi_1(c) = \varphi_2(c) \ \text{ for all } \ c\in C.$$

We may assume that the set $\mathcal{F}_n$ are in the unit ball of C, with dense union. For the next four paragraphs of the proof, fix $n = 1, 2, ...$

Put $\varepsilon_n' = \min\{1/2^{n+1}, \delta_n/2\}$. Let $C_n\subset C$ be a unital C^*-subalgebra (in place of C_n) such that $K_i(C_n)$ is finitely generated ($i = 0, 1$), and let $\mathcal{Q}_n$ be a finite set of generators of $K_1(C_n)$, let $\delta_n' > 0$ (in place of δ) be as in Lemma 24.2 for C (in place of A), ε_n' (in place of ε), $\mathcal{F}_n$ (in place of $\mathcal{F}$), and $[\imath_n](\mathcal{Q}_{n-1})$ (in place of $\mathcal{P}$), where $\imath_n: C_n\to C$ is the embedding. Note that we assume that

(e 27.12) $$[\imath_{n+1}](\mathcal{Q}_{n+1})\supset\mathcal{P}_{n+1}\cup[\imath_n](\mathcal{Q}_n).$$

Write $K_1(C_n) = G_{n,f}\oplus\mathrm{Tor}(K_1(C_n))$, where $G_{n,f}$ is a finitely generated free abelian group. Let $z_{1,n}, z_{2,n}, ..., z_{f(n),n}$ be independent generators of $G_{n,f}$ and $z_{1,n}', z_{2,n}', ..., z_{t(n),n}'$ be generators of $\mathrm{Tor}(K_1(C_n))$. We may assume that

$$\mathcal{Q}_n = \{z_{1,n}, z_{2,n}, ..., z_{f(n),n}, z_{1,n}', z_{2,n}', ..., z_{t(n),n}'\}.$$

Choose $1/2 > \varepsilon_n'' > 0$ so that $\mathrm{bott}_1(h', u')|_{K_1(C_n)}$ is a well defined group homomorphism, $\mathrm{bott}_1(h', u')|_{\mathcal{Q}_n}$ is well defined, and $(\mathrm{bott}_1(h', u')|_{K_1(C_n)})|_{\mathcal{Q}_n} = \mathrm{bott}_1(h', u')|_{\mathcal{Q}_n}$ for any unital homomorphism $h': C\to A$ and any unitary $u'\in A$ for which

(e 27.13) $$\|[h'(c), u']\| < \varepsilon_n'' \ \text{ for all } \ c\in\mathcal{G}_n'$$

for some finite subset $\mathcal{G}_n'\subset C$ which contains $\mathcal{F}_n$.

Let $w_{1,n}, w_{2,n}, ..., w_{f(n),n}, w_{1,n}', w_{2,n}', ..., w_{t(n),n}'\in C$ be unitaries (note that, by Theorem 9.7 of [21], C has stable rank one) such that $[w_{i,n}] = (\imath_n)_{*1}(z_{i,n})$ and $[w_{j,n}'] = (\imath_n)_{*1}(z_{j,n}')$, $i = 1, 2, ..., f(n)$, $j = 1, 2, ..., t(n)$, and $n = 1, 2, ...$

Since we may choose larger $\mathcal{G}'_n$, without loss of generality, we may assume that $w_{i,n} \in \mathcal{G}'_n$.

Let $\delta''_1 = 1/2$ and, for $n \geq 2$, let $\delta''_n > 0$ (in place of δ) and $\mathcal{G}''_n$ (in place of $\mathcal{F}$) be as in Lemma 27.2 associated with $w_{1,n}, w_{2,n}, ..., w_{f(n),n}, w'_{1,n}, w'_{2,n}, ..., w'_{t(n),n}$ (in place of $u_1, u_2, ..., u_n$) and

$$\{w_{1,n-1}, w_{2,n-1}, ..., w_{f(n-1),n-1}, w'_{1,n-1}, w'_{2,n-1}, ..., w'_{t(n-1),n-1}\}$$

(in place of $v_1, v_2, ..., v_m$).

Now consider all $n = 1, 2, ...$ Put $\varepsilon_n = \min\{\varepsilon''_n/2, \varepsilon'_n/2, \delta'_n, \delta''_n/2\}$ and $\mathcal{G}_n = \mathcal{G}'_n \cup \mathcal{G}''_n$. By (e 27.11), we may assume that

$$\mathrm{Ad}\, v_n \circ \varphi_1 \approx_{\varepsilon_n} \varphi_2 \ \text{ on } \ \mathcal{G}_n, \ \ n = 1, 2, ... \tag{e 27.14}$$

Thus, $\mathrm{bott}_1(\varphi_2 \circ \imath_n, v_n^* v_{n+1})$ is well defined. Since $\mathrm{Aff}(T(A))$ is torsion free,

$$\tau\big(\mathrm{bott}_1(\varphi_2 \circ \imath_n, v_n^* v_{n+1})|_{\mathrm{Tor}(K_1(C_n))}\big) = 0. \tag{e 27.15}$$

From (e 27.14), we have

$$\|\varphi_2(w_{j,n})\mathrm{Ad}\, v_n(\varphi_1(w_{j,n})^*) - 1\| < \varepsilon_n, \tag{e 27.16}$$

for all $n = 1, 2, ...$ Define

$$h_{j,n} = \frac{1}{2\pi i}\log(\varphi_2(w_{j,n})\mathrm{Ad}\, v_n(\varphi_1(w_{j,n})^*)), \tag{e 27.17}$$

for $j = 1, 2, ..., f(n),\ n = 1, 2, ...$ Then, for any $\tau \in T(A)$, $|\tau(h_{j,n})| < \varepsilon_n < \delta'_n,\ j = 1, 2, ..., f(n), n = 1, 2, ...$ Since $\mathrm{Aff}(T(A))$ is torsion free, and the classes $[w'_{j,n}]$ are torsion, it follows from Lemma 27.2 that

$$\tau(\frac{1}{2\pi i}\log(\varphi_2(w'_{j,n})\mathrm{Ad}\, v_n(\varphi_1(w'^*_{j,n})))) = 0, \tag{e 27.18}$$

$j = 1, 2, ..., t(n)$ and $n = 1, 2, ...$ By the assumption that $R_{\varphi_1,\varphi_2}(K_1(M_{\varphi_1,\varphi_2})) \subset \rho_A(K_0(A))$, by Exel's formula (see [24]), and by Lemma 3.5 of [37], we conclude that

$$\widehat{h_{j,n}}(\tau) = \tau(h_{j,n}) \in R_{\varphi_1,\varphi_2}(K_1(M_{\varphi_1,\varphi_2})) \subset \rho_A(K_0(A)).$$

Now define $\alpha'_n : K_1(C_n) \to \rho_A(K_0(A))$ by

$$\alpha'_n(z_{j,n})(\tau) = \widehat{h_{j,n}}(\tau) = \tau(h_{j,n}), \ \ j = 1, 2, ..., f(n)$$

and

$$\alpha'_n(z'_{j,n}) = 0, \ \ j = 1, 2, ..., t(n), \tag{e 27.19}$$

$n = 1, 2, ...$ Since $\alpha'_n(K_1(C_n))$ is free abelian, it follows that there is a homomorphism $\alpha_n^{(1)} : K_1(C_n) \to K_0(A)$ such that

$$(\rho_A \circ \alpha_n^{(1)}(z_{j,n}))(\tau) = \tau(h_{j,n}),\ \ j = 1, 2, ..., f(n),\ \ \tau \in T(A), \tag{e 27.20}$$

and

$$\alpha_n^{(1)}(z'_{j,n}) = 0,\ \ j = 1, 2, ..., t(n). \tag{e 27.21}$$

Define $\alpha_n^{(0)} : K_0(C_n) \to K_1(A)$ by $\alpha_n^{(0)} = 0$. By the UCT, there is $\kappa_n \in KL(SC_n, A)$ such that $\kappa_n|_{K_i(C_n)} = \alpha_n^{(i)}$, $i = 0, 1$, where SC_n is the suspension of C_n (here, we identify $K_i(C_n)$ with $K_{i+1}(SC_n)$).

By the UCT again, there is $\alpha_n \in KL(C_n \otimes C(\mathbb{T}), A)$ such that $\alpha_n \circ \boldsymbol{\beta}|_{\underline{K}(C_n)} = \kappa_n$. In particular, $\alpha_n \circ \boldsymbol{\beta}|_{K_1(C_n)} = \alpha_n^{(1)}$. It follows from Lemma 24.2 that there exists a unitary $U_n \in U_0(A)$ such that

$$\|[\varphi_2(c), U_n]\| < \varepsilon''_n \text{ for all } c \in \mathcal{F}_n \text{ and} \tag{e 27.22}$$

$$\rho_A(\mathrm{bott}_1(\varphi_2, U_n)(z_{j,n})) = -\rho_A \circ \alpha_n^{(1)}(z_{j,n}), \tag{e 27.23}$$

$j = 1, 2, ..., f(n)$. We also have

$$\rho_A(\mathrm{bott}_1(\varphi_2, U_n)(z'_{j,n})) = 0,\ \ j = 1, 2, ..., t(n), \tag{e 27.24}$$

as the elements $z_{j,n}$ are torsion. By the Exel trace formula (see [24]), (e 27.20), and (e 27.23), we have

$$\begin{aligned} \tau(h_{j,n}) &= -\rho_A(\mathrm{bott}_1(\varphi_2, U_n)(z_{j,n})(\tau) \\ &= -\tau(\frac{1}{2\pi i}\log(U_n\varphi_2(w_{j,n})U_n^*\varphi_2(w_{j,n}^*))) \end{aligned} \tag{e 27.25}$$

for all $\tau \in T(A)$, $j = 1, 2, ..., f(n)$. Define $u_n = v_nU_n$, $n = 1, 2, ...$ By 6.1 of [36], (e 27.25), and (e 27.23), we compute that

$$\begin{aligned} &\tau(\frac{1}{2\pi i}\log(\varphi_2(w_{j,n})\mathrm{Ad}u_n(\varphi_1(w_{j,n}^*))))) \\ &= \tau(\frac{1}{2\pi i}\log(U_n\varphi_2(w_{j,n})U_n^*v_n^*\varphi_1(w_{j,n}^*)v_n))) \\ &= \tau(\frac{1}{2\pi i}\log(U_n\varphi_2(w_{j,n})U_n^*\varphi_2(w_{j,n}^*)\varphi_2(w_{j,n})v_n^*\varphi_1(w_{j,n}^*)v_n))) \\ &= \tau(\frac{1}{2\pi i}\log(U_n\varphi_2(w_{j,n})U_n^*\varphi_2(w_{j,n}^*)))) \\ &\quad +\tau(\frac{1}{2\pi i}\log(\varphi_2(w_{j,n})v_n^*\varphi_1(w_{j,n}^*)v_n))) \\ &= \rho_A(\mathrm{bott}_1(\varphi_2, U_n)(z_{j,n}))(\tau) + \tau(h_{j,n}) = 0 \end{aligned} \tag{e 27.26}$$

for all $\tau \in T(A)$, $j = 1, 2, ..., f(n)$ and $n = 1, 2, ...$ By (e 27.18) and (e 27.24),

$$\tau(\frac{1}{2\pi i}\log(\varphi_2(w'_{j,n})\mathrm{Ad}u_n(\varphi_1((w'_{j,n})^*)))) = 0, \tag{e 27.27}$$

$j = 1, 2, ..., t(n)$ and $n = 1, 2, ...$ Let

$$b_{j,n} = \frac{1}{2\pi i}\log(u_n\varphi_2(w_{j,n})u_n^*\varphi_1(w_{j,n}^*)), \tag{e 27.28}$$

$$b'_{j,n} = \frac{1}{2\pi i}\log(\varphi_2(w_{j,n})u_n^*u_{n+1}\varphi_2(w_{j,n}^*)u_{n+1}^*u_n), \text{ and} \tag{e 27.29}$$

$$b''_{j,n+1} = \frac{1}{2\pi i}\log(u_{n+1}\varphi_2(w_{j,n})u_{n+1}^*\varphi_1(w_{j,n}^*)), \tag{e 27.30}$$

$j = 1, 2, ..., f(n)$ and $n = 1, 2, ...$ We have, by (e 27.26),

$$\begin{aligned} \tau(b_{j,n}) &= \tau(\frac{1}{2\pi i}\log(u_n\varphi_2(w_{j,n})u_n^*\varphi_1(w_{j,n}^*))) \\ &= \tau(\frac{1}{2\pi i}\log(\varphi_2(w_{j,n})u_n^*\varphi_1(w_{j,n}^*)u_n)) = 0 \end{aligned} \tag{e 27.31}$$

for all $\tau \in T(A)$, $j = 1, 2, ..., f(n)$, and $n = 1, 2, ...$ Note that $\tau(b_{j,n+1}) = 0$ for all $\tau \in T(A)$, $j = 1, 2, ..., f(n+1)$. It follows from Lemma 27.2 and (e 27.12) that

$$\tau(b''_{j,n+1}) = 0 \text{ for all } \tau \in T(A), \ \ j = 1, 2, ..., f(n), \ \ n = 1, 2, ...$$

Note that

$$u_n e^{2\pi i b'_{j,n}} u_n^* = e^{2\pi i b_{j,n}} \cdot e^{-2\pi i b''_{j,n+1}}, \ \ j = 1, 2, ..., f(n).$$

Hence, using 6.1 of [36], we compute that

$$\tau(b'_{j,n}) = \tau(b_{j,n}) - \tau(b''_{j,n+1}) = 0 \text{ for all } \tau \in T(A). \tag{e 27.32}$$

By the Exel formula (see [24]) and (e 27.32),

$$\begin{aligned} &\rho_A(\mathrm{bott}_1(\varphi_2, u_n^*u_{n+1}))(w_{j,n}^*)(\tau) \\ &= \tau(\frac{1}{2\pi i}\log(u_n^*u_{n+1}\varphi_2(w_{j,n})u_{n+1}^*u_n\varphi_2(w_{j,n}^*))) \\ &= \tau(\frac{1}{2\pi i}\log(\varphi_2(w_{j,n})u_n^*u_{n+1}\varphi_2(w_{j,n}^*)u_{n+1}^*u_n)) = 0 \end{aligned}$$

for all $\tau \in T(A)$ and $j = 1, 2, ..., f(n)$. Thus,

$$\rho_A(\mathrm{bott}_1(\varphi_2, u_n^*u_{n+1})(w_{j,n}))(\tau) = 0 \text{ for all } \tau \in T(A), \tag{e 27.33}$$

$j = 1, 2, ..., f(n)$, and $n = 1, 2, ...$ We also have

$$\rho_A(\mathrm{bott}_1(\varphi_2, u_n^* u_{n+1})(w'_{j,n}))(\tau) = 0 \text{ for all } \tau \in T(A), \tag{e 27.34}$$

$j = 1, 2, ..., f(n)$, and $n = 1, 2, ...$ By 27.2, we have that

$$\rho_A(\mathrm{bott}_1(\varphi_2, u_n^* u_{n+1})(z)) = 0 \text{ for all } z \in \mathcal{P}_n, \tag{e 27.35}$$

$n = 1, 2, ...$ □

REMARK 27.4. Let C be a unital separable amenable C*-algebra satisfying the UCT with finitely generated $K_i(C)$ $(i = 0, 1)$, let A be a unital separable C*-algebra and let $\varphi_1, \varphi_2 : C \to A$ be two unital homomorphisms. In what follows, we will continue to use φ_1 and φ_2 for the induced homomorphisms from $M_k(C)$ to $M_k(A)$. Suppose that $v \in U(A)$ and

$$\|v^* \varphi_1(a) v - \varphi_2(a)\| < \varepsilon < 1/8, \quad a \in \{z_1, z_2, ..., z_n\} \cup \mathcal{F}$$

for a finite subset $\mathcal{F} \subseteq M_k(C)$ and some $z_1, z_2, ..., z_n \in U(M_k(C))$ such that $[z_1], [z_2], ..., [z_n]$ generate $K_1(C)$. Define $W_j(t) \in U(M_2(C([0,1], M_k(A))))$ as follows

$$W_j(t) = (T_t V T_t^{-1})^* \mathrm{diag}(\varphi_1(z_j), 1_{M_k}) T_t V T_t^{-1},$$

where

$$V = \mathrm{diag}(v, 1_{M_k}) \quad \text{and} \quad T_t = \begin{pmatrix} \cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}.$$

Note that $W_j(0) = \mathrm{diag}(v^* \varphi_1(z_j) v, 1)$ and $W_j(1) = \mathrm{diag}(\varphi_1(z_j), 1)$. Connecting $W_j(0)$ with $\mathrm{diag}(\varphi_2(z), 1)$ by a continuous path, we obtain a continuous path of unitaries $Z_j(t)$ such that $Z_j(0) = \mathrm{diag}(\varphi_2(z_j), 1)$, $Z(1/4) = W(0)$ and $Z_j(1) = \mathrm{diag}(\varphi_1(z_j), 1)$ and $\|Z_j(t) - Z_j(1/4)\| < 1/8$ for $t \in [0, 1/4)$. Thus $Z_j \in M_{2k}(M_{\varphi_1, \varphi_2})$. With sufficiently small $\varepsilon > 0$, since $K_1(C)$ is finitely generated, the map

$$K_1(C) \ni [z_j] \mapsto [Z_j] \in K_1(M_{\varphi_1, \varphi_2}), \quad j = 1, 2, ..., n,$$

induces a homomorphism.

Set

$$h_j = \frac{1}{i} \mathrm{diag}(\log(\varphi_2(z_j)^* V^* \varphi_1(z_j) V), 1), \quad j = 1, 2, ..., n. \tag{e 27.36}$$

We may specifically use

$$Z_j(t) = \mathrm{diag}(\varphi_2(z_j), 1) \exp(i4th_j), \quad t \in [0, 1/4].$$

Still use φ_1 and φ_2 for the induced homomorphisms from $M_k(C \otimes C')$ to $M_k(A \otimes C')$, where C' is a commutative C*-algebra C' with finitely generated $K_i(C')$ $(i = 0, 1)$. Fix a finite set of unitaries $z_1, ..., z_n \in M_\infty(C \otimes C')$

which generates $K_1(C \otimes C')$. We also obtain a homomorphism $K_1(C \otimes C') \to K_1(M_{\varphi_1,\varphi_2} \otimes C')$ provided that ε is small.

Let $\mathcal{F} \subset C$ be a finite subset and $\varepsilon > 0$. Suppose that there is a unitary $v \in U(A)$ such that

$$\mathrm{ad} v \circ \varphi_1 \approx_\varepsilon \varphi_2 \quad \text{on } \mathcal{F}.$$

Let $U'(t) = T_t V T_t^{-1}$. Define

$$L(c)(t) = (U'(\frac{4t-1}{3}))^* \mathrm{diag}(\varphi_1(c), 1) U'(\frac{4t-1}{3}), \quad t \in [1/4, 1] \tag{e 27.37}$$

and

$$L(c)(t) = 4tL(c)(1/4) + (1-4t)\mathrm{diag}(\varphi_2(c), 1), \quad t \in [0, 1/4].$$

Note that L maps C into $M_2(M_{\varphi_1,\varphi_2})$. Thus, since $K_i(C)$ $(i = 0, 1)$ is finitely generated, by Corollary 2.11 of [6], there is $N_1 > 0$ such that any element of $\mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(A))$ is determined by its restriction to $K_i(A, \mathbb{Z}/n\mathbb{Z})$, $i = 0, 1$, $n = 0, 1, ..., N_1$. Hence, if ε is sufficiently small and $\mathcal{F}$ is sufficiently large, there is

$$\gamma_{\varphi_1,\varphi_2,v} \in \mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(M_{\varphi_1,\varphi_2})) \tag{e 27.38}$$

such that

$$[L]|_{\mathcal{P}} = \gamma_{\varphi_1,\varphi_2,v}|_{\mathcal{P}} \tag{e 27.39}$$

for any given finite subset $\mathcal{P} \subset \underline{K}(C)$.

One computes that

$$\int_0^1 \tau(\frac{dZ_j(t)}{dt} Z_j(t))dt = \tau(h_j), \quad \tau \in T(A).$$

Therefore, if $R_{\varphi_1,\varphi_2} \circ \gamma_{\varphi_1,\varphi_2,v}(K_1(C)) = 0$, then

$$\tau(h_j) = 0, \quad \tau \in T(A). \tag{e 27.40}$$

On the other hand, for any given $\eta > 0$ and a finite set $\{z_1, z_2, ..., z_n\}$ of generators of $K_1(C)$, by (e 27.36),

$$|\tau(h_i)| < \eta, \quad \tau \in T(A), \tag{e 27.41}$$

provided that ε is sufficiently small and $\mathcal{F}$ is sufficiently large.

Now, assume that $\varphi_1 = \varphi_2$. Then, with sufficiently large $\mathcal{F}$ and sufficiently small ε, the element $\mathrm{Bott}(\varphi_1, v) : \underline{K}(C) \to \underline{K}(SA)$ is well defined.

We have the following splitting short exact sequence:

$$0 \longrightarrow \mathrm{S}A \xrightarrow{\imath} M_{\varphi_1,\varphi_1} \xrightarrow{\pi_e} C \longrightarrow 0 \ .$$

Define $\theta : C \to M_{\varphi_1,\varphi_1}$ by $\theta(b) = \varphi_1(b)$ as a constant element in M_{φ_1,φ_1}. Then θ may be identified with a splitting map and $\underline{K}(M_{\varphi_1,\varphi_1})$ may be written as $\underline{K}(SA) \oplus \underline{K}(C)$.

Let $P : \underline{K}(M_{\varphi_1,\varphi_1}) \cong \underline{K}(C) \oplus \underline{K}(SA) \to \underline{K}(SA)$ be the standard projection map. One can verify that for any two elements $x, y \in \underline{K}(C)$ if $\mathrm{Bott}(\varphi_1, v)(x) = \mathrm{Bott}(\varphi_1, v)(y)$, then $P \circ \gamma_{\varphi_1,\varphi_1,v}(x) = P \circ \gamma_{\varphi_1,\varphi_1,v}(y)$. So we will also use $\Gamma(\mathrm{Bott}(\varphi_1, v))$ to denote the map $P \circ \gamma_{\varphi_1,\varphi_1,v} \in \mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(SA))$.

By shifting the index, we see $\Gamma(\mathrm{Bott}(\varphi_1, v))|_{\mathcal{P}}$ maps $\mathcal{P}$ to $\underline{K}(A)$. One may identify P with $\mathrm{id}_{M_{\varphi_1,\varphi_1}} - [\theta] \circ [\pi_e]$. Note that

$$\pi_e \circ \theta = \mathrm{id}_C \ \text{ and } \ \pi_e \circ L = \mathrm{id}_C.$$

So

$$\Gamma(\mathrm{Bott}(\varphi_1, v))|_{\mathcal{P}} = \gamma_{\varphi_1,\varphi_1,v}|_{\mathcal{P}} - \theta|_{\mathcal{P}}. \tag{e 27.42}$$

Furthermore, it is shown in 10.6 of [37] that $\Gamma(\mathrm{Bott}(\varphi_1, v)) = 0$ if and only if $\mathrm{Bott}(\varphi_1, v) = 0$.

Note that since the K-theory of C is finitely generated, by Corollary 2.11 of [6], one has that any element of $\mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(A))$ is determined by its restriction to $K_i(A, \mathbb{Z}/n\mathbb{Z})$, $i = 0, 1$, $n = 0, 1, ..., N_1$. Fix separable commutative C*-algebras $C_0 = \mathbb{C}$, C_1, ..., C_{N_1}, C_{N_1+1}, ..., C_{2N_1+1} with

$$K_0(C_n) = \mathbb{Z}/n\mathbb{Z} \quad \text{and} \quad K_1(C_n) = \{0\}, \quad n = 0, 1, ..., N_1,$$

and $C_{N_1+i} = SC_{i-1}$, $i = 1, 2, ..., N_1 + 1$. For each $C \otimes C'$, where C' is one of the $C_0, C_1, ..., C_{2N_1+1}$, fix a finite set of unitaries $z_1^{(n)}, z_2^{(n)}, ..., z_{k(n)}^{(n)}$ of $M_{N_2}(\widetilde{C \otimes C'}) \subset M_{N_2}(C \otimes \widetilde{C}')$ (for some $N_2 \geq 1$) which generates $K_1(C \otimes C_n)$, $n = 0, 1, ..., 2N_1+1$. Let $C_i' = M_{N_2}(C_i)$, $i = 0, 1, ..., 2N_1 + 1$.

Let $1/4 > \varepsilon > 0$ and $1/4 > \eta > 0$. Choose $0 < \delta < \varepsilon/2$ sufficiently small and a finite set $\mathcal{F} \subseteq A$ sufficiently large such that if

$$u^* \varphi_1 u \approx_\delta \varphi_2 \quad \text{on } \mathcal{F}$$

for some unitary $u \in A$, then, for each C_n,

$$u^* \tilde{\varphi}_1 u \approx_{\varepsilon/2} \tilde{\varphi}_2 \text{ on } \mathcal{F}_{0,n}, \tag{e 27.43}$$

where $\mathcal{F}_{0,n}$ is a finite subset which contains $\{z_1^{(n)}, z_2^{(n)}, ..., z_{k(n)}^{(n)}\}$, $n = 0, 1, ..., 2N_1+1$. We also assume that δ is sufficiently small and $\mathcal{F}$ is sufficiently large so that (e 27.38), (e 27.39), (e 27.41), (e 27.42) hold.

Suppose that there are unitaries $u_1, u_2 \in A$ such that

$$u_i^*\varphi_1 u_i \approx_{\delta/2} \varphi_2 \ \text{ on } \ \mathcal{F},$$

$i = 1, 2$. Then, as in (e 27.43), for each C_n,

$$u_i^*\tilde{\varphi}_1 u_i \approx_{\varepsilon/2} \tilde{\varphi}_2 \text{ on } \mathcal{F}_{0,n}, \tag{e 27.44}$$

where $\mathcal{F}_{0,n}$ is a finite subset which contains $\{z_1^{(n)}, z_2^{(n)}, ..., z_{k(n)}^{(n)}\}$, $n = 0, 1, ..., 2N_1 + 1$. Let $L_i : C \to M_{\varphi_1,\varphi_2}$ and $\gamma_{\varphi_1,\varphi_2,u_i} \in \mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(M_{\varphi_1,\varphi_2}))$ be the element defined by the pair (φ_1, u_i) $(i = 1, 2)$ as above.

On the other hand, one also has that

$$u_2 u_1^* \varphi_1 u_1 u_2^* \approx_\delta \varphi_1 \ \text{ on } \ \mathcal{F}.$$

Note that $\pi_e \circ (L_1 - L_2) = 0$.

Fix $n \in \{0, 1, ..., 2N_1 + 1\}$. Consider $z \in \{z_1^{(n)}, z_2^{(n)}, ..., z_{k(n)}^{(n)}\}$ and $\tilde{u}_i = u_i \otimes 1_{\widetilde{C'_n}}$, $i = 1, 2$. We also write $\tilde{\varphi}_i$ for $\varphi_i \otimes \mathrm{id}_{\widetilde{C'_n}}$.

Define $\tilde{T}(t) = T_{2(t-1/4)}$ for $t \in [1/4, 3/4]$ and $\tilde{T}_t = T_1$ for $t \in [3/4, 1]$. Let $W_i(t) = \tilde{T}_t \begin{pmatrix} \tilde{u}_i^* & 0 \\ 0 & 1 \end{pmatrix} \tilde{T}_t^*$, $t \in [1/4, 1]$, and $W_i(t) = \mathrm{diag}(1, 1)$ for $t \in [0, 1/4]$, $i = 0, 1$. Note that

$$\|\mathrm{diag}(\tilde{u}_1^*\tilde{\varphi}(z)\tilde{u}_1\tilde{u}_2^*\tilde{\varphi}(z)^*\tilde{u}_2, 1) - \mathrm{diag}(1, 1)\| < \varepsilon < 1/8. \tag{e 27.45}$$

There is a continuous path $d(z)(t)$ (for $t \in [0, 1/4]$) such that $d(z)(0) = \mathrm{diag}(1, 1)$ and $d(z)(1/4) = \mathrm{diag}(\tilde{u}_1^*\tilde{\varphi}(z)\tilde{u}_1\tilde{u}_2^*\tilde{\varphi}(z)^*\tilde{u}, 1)$ and

$$\|d(z)(t) - \mathrm{diag}(1, 1)\| < \varepsilon \ \text{ for all } \ t \in [0, 1/4]. \tag{e 27.46}$$

Define, for $t \in [1/4, 1]$,

$$\begin{aligned} d(z)(t) &= \big(W_1(t)\mathrm{diag}(\tilde{\varphi}(z), 1)W_1(t)^*\big)\big(W_2(t)\mathrm{diag}(\tilde{\varphi}(z)^*, 1)W_2(t)^*\big) \\ &= \tilde{T}_t \begin{pmatrix} \tilde{u}_1^* & 0 \\ 0 & 1 \end{pmatrix} \tilde{T}_t^* \begin{pmatrix} \tilde{\varphi}_1(z) & 0 \\ 0 & 1 \end{pmatrix} \tilde{T}_t \begin{pmatrix} \tilde{u}_1\tilde{u}_2^* & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad \tilde{T}_t^* \begin{pmatrix} \tilde{\varphi}_1(z)^* & 0 \\ 0 & 1 \end{pmatrix} \tilde{T}_t \begin{pmatrix} \tilde{u}_2 & 0 \\ 0 & 1 \end{pmatrix} \tilde{T}_t^*. \end{aligned}$$

On $[3/4, 1]$, define $d(z)(t) = \mathrm{diag}(1, 1)$. Define $U(t) = \mathrm{diag}(W_2(t)^*, W_2(t))$ for $t \in (1/4, 3/4)$, On $[0, 1/4]$, there is a continuous path $U(t)$ of unitaries in $M_4(A \otimes C'_n)$ with $U(0) = \mathrm{diag}(1, 1, 1, 1)$ and $U(1/4) = \mathrm{diag}(W_2(0)^*, W_2(0))$. On $[3/4, 1]$, there is a continuous path $U(t)$ of unitaries in $M_4(A \otimes C'_n)$ with $U(3/4) = \mathrm{diag}(W_2(3/4)^*, W_2(3/4))$ and $U(1) = \mathrm{diag}(1, 1, 1, 1)$.

For $n=0$ (so z is represented by unitaries in $M_{N_2}(C)$), we may also assume that (see (e 27.41)),

$$|\tau(h_z)|<\eta \ \text{ for all } \ \tau\in T(A), \tag{e 27.47}$$

where

$$h_z=\mathrm{diag}(\log(\tilde\varphi_2(z)^*\tilde u_1^*\tilde\varphi_1(z)\tilde u_1,1).$$

Note that $d(z)(0)=d(z)(1)=1$ and

$$[d(z)]=\gamma_{\varphi_1,\varphi_2,u_1}(z)-\gamma_{\varphi_1,\varphi_2,u_2}(z), \tag{e 27.48}$$

where $\gamma_{\varphi_1,\varphi_2,u_i}$, $i=1,2$, are the maps defined above (see (e 27.37)).

One has, on $[1/4,3/4]$,

$$\begin{aligned} &U(t)^*\mathrm{diag}(d(z)(t),1,1)U(t) \\ &= \mathrm{diag}\Bigg(\Bigg(\tilde T_t\begin{pmatrix}\tilde u_2 & 0\\ 0 & 1\end{pmatrix}\tilde T_t^*\Bigg) \\ &\quad\Bigg(\tilde T_t\begin{pmatrix}\tilde u_1^* & 0\\ 0 & 1\end{pmatrix}\tilde T_t^*\begin{pmatrix}\tilde\varphi_1(z) & 0\\ 0 & 1\end{pmatrix}\tilde T_t\begin{pmatrix}\tilde u_1\tilde u_2^* & 0\\ 0 & 1\end{pmatrix} \\ &\quad\tilde T_t^*\begin{pmatrix}\tilde\varphi_1(z)^* & 0\\ 0 & 1\end{pmatrix}\Bigg),\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Bigg) \\ &= \mathrm{diag}\Bigg(\tilde T_t\begin{pmatrix}\tilde u_2\tilde u_1^* & 0\\ 0 & 1\end{pmatrix}\tilde T_t^*\begin{pmatrix}\tilde\varphi_1(z) & 0\\ 0 & 1\end{pmatrix}\tilde T_t\begin{pmatrix}\tilde u_1\tilde u_2^* & 0\\ 0 & 1\end{pmatrix} \\ &\quad\tilde T_t^*\begin{pmatrix}\tilde\varphi_1(z)^* & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Bigg), \end{aligned} \tag{e 27.49}$$

on $[0,1/4]$, and on $[3/4,1]$,

$$\|U(t)^*\mathrm{diag}(d(z)(t),1,1)U(t)-\mathrm{diag}(1,1,1,1)\|<\varepsilon. \tag{e 27.50}$$

Moreover, $U(0)^*\mathrm{diag}(d(z)(0),1,1)U(0)=U^*(1)\mathrm{diag}(d(z)(1),1,1)U(1)=\mathrm{diag}(1,1,1,1)$. Therefore

$$[d(z)]=[U^*d(z)U] \text{ in } K_1(SA\otimes\widetilde{C_n}). \tag{e 27.51}$$

Since the short exact sequence $0\to SA\otimes C_n\to SA\otimes\widetilde{C_n}\to SA\to 0$ splits, we conclude that

$$[d(z)]=[U^*d(z)U] \text{ in } K_1(SA\otimes C_n). \tag{e 27.52}$$

On the other hand, the class $\Gamma(\mathrm{Bott}(\varphi_1, u_1u_2^*))(z)$ is represented by the path

$$r(t) = \tilde{T}_t \begin{pmatrix} \tilde{u}_2\tilde{u}_1^* & 0 \\ 0 & 1 \end{pmatrix} \tilde{T}_t^* \begin{pmatrix} \tilde{\varphi}_1(z) & 0 \\ 0 & 1 \end{pmatrix} \tilde{T}_t \begin{pmatrix} \tilde{u}_1\tilde{u}_2^* & 0 \\ 0 & 1 \end{pmatrix} \tilde{T}_t^* \begin{pmatrix} \tilde{\varphi}_1(z)^* & 0 \\ 0 & 1 \end{pmatrix}$$

for all $t \in [1/4, 3/4]$, and

$$\|r(t) - \mathrm{diag}(1,1)\| < \varepsilon \text{ for all } t \in [0,1/4] \cup [3/4,1]. \tag{e 27.53}$$

Hence (see (e 27.49)), by (e 27.48) and (e 27.52),

$$\Gamma(\mathrm{Bott}(\varphi_1, u_1u_2^*)) = \gamma_{\varphi_1,\varphi_2,u_1} - \gamma_{\varphi_1,\varphi_2,u_2}. \tag{e 27.54}$$

THEOREM 27.5. *Let C_1 be a unital simple C^*-algebra as in Theorem 14.10 of [21], let A_1 be a unital separable simple C^*-algebra in $\mathcal{B}_0$, let $C = C_1 \otimes U_1$ and let $A = A_1 \otimes U_2$, where U_1 and U_2 are UHF-algebras of infinite type. Suppose that $\varphi_1, \varphi_2 : C \to A$ are two unital monomorphisms. Then φ_1 and φ_2 are asymptotically unitarily equivalent if and only if*

$$[\varphi_1] = [\varphi_2] \text{ in } KK(C, A), \tag{e 27.55}$$

$$\varphi^\ddagger = \psi^\ddagger, \ (\varphi_1)_T = (\varphi_2)_T, \text{ and } \overline{R_{\varphi_1,\varphi_2}} = 0. \tag{e 27.56}$$

PROOF. We will prove the "if " part only. The "only if" part follows from 4.3 of [40]. Note $C = C_1 \otimes U_1$ can be also regarded as a C^*-algebra as in Theorem 14.10 of [21]. Let $C = \lim_{n\to\infty}(C_n, \imath_n)$ be as in Theorem 14.10 of [21], where each $\imath_n : C_n \to C_{n+1}$ is an injective homomorphism. Let $\mathcal{F}_n \subset C$ be an increasing sequence of finite subsets of C such that $\bigcup_{n=1}^\infty \mathcal{F}_n$ is dense in C. Put

$$M_{\varphi_1,\varphi_2} = \{(f, c) \in C([0,1], A) \oplus C : f(0) = \varphi_1(c) \text{ and } f(1) = \varphi_2(c)\}.$$

Since C satisfies the UCT, the assumption that $[\varphi_1] = [\varphi_2]$ in $KK(C, A)$ implies that the following exact sequence splits:

$$0 \to \underline{K}(SA) \to \underline{K}(M_{\varphi_1,\varphi_2}) \rightleftharpoons^{\pi_e}_{\theta} \underline{K}(C) \to 0 \tag{e 27.57}$$

for some $\theta \in \mathrm{Hom}(\underline{K}(C), \underline{K}(A))$, where $\pi_e : M_{\varphi_1,\varphi_2} \to C$ is the projection to C defined in Definition 2.20 of [21]. Furthermore, since $\tau \circ \varphi_1 = \tau \circ \varphi_2$ for all $\tau \in T(A)$, and $\overline{R_{\varphi_1,\varphi_2}} = 0$, we may also assume that

$$R_{\varphi_1,\varphi_2}(\theta(x)) = 0 \text{ for all } x \in K_1(C). \tag{e 27.58}$$

By [6], we have

$$\lim_{n\to\infty}(\underline{K}(C_n), [\imath_n]) = \underline{K}(C). \tag{e 27.59}$$

Since $K_i(C_n)$ is finitely generated, there exists $K(n) \ge 1$ such that

(e 27.60) $$\mathrm{Hom}_\Lambda(F_{K(n)}\underline{K}(C_n),\, F_{K(n)}\underline{K}(A)) = \mathrm{Hom}_\Lambda(\underline{K}(C_n), \underline{K}(A))$$

(see also [6] for the notation F_m there).

Let $\delta_n' > 0$ (in place of δ), $\sigma_n' > 0$ (in place of σ), $\mathcal{G}_n' \subset C$ (in place of $\mathcal{G}$), $\{p_{1,n}', p_{2,n}', ..., p_{I(n),n)}', q_{1,n}', q_{2,n}', ..., q_{I(n),n}'\}$ (in place of $\{p_1, p_2, ..., p_k, q_1, q_2, ..., q_k\}$), $\mathcal{P}_n' \subset \underline{K}(C)$ (in place of $\mathcal{P}$) corresponding to $1/2^{n+2}$ (in place of ε), and $\mathcal{F}_n$ (in place of $\mathcal{F}$) be as provided by Lemma 25.4 (see also Remark 25.5). Note that, by the choice as in 25.4, we may assume that $G_{u,n}'$, the subgroup generated by $\{[p_{i,n}'] - [q_{i,n}'] : 1 \le i \le I(n)\}$ is free abelian.

Without loss of generality, we may assume that $\mathcal{G}_n' \subset \imath_{n,\infty}(\mathcal{G}_n)$ and $\mathcal{P}_n' \subset [\imath_{n,\infty}](\mathcal{P}_n)$ for some finite subset $\mathcal{G}_n \subset C_n$, and for some finite subset $\mathcal{P}_n \subset \underline{K}(C_n)$, and we may assume that $p_{i,n}' = \imath_{n,\infty}(p_{i,n})$ and $q_{i,n}' = \imath_{n,\infty}(q_{i,n})$ for some projections $p_{i,n}, q_{i,n} \in C_n$, $i = 1, 2, ..., I(n)$. We may also assume that the subgroup $G_{n,u}$ generated by $\{[p_{i,n}] - [q_{i,n}] : 1 \le i \le I(n)\}$ is free abelian and $p_{i,n}, q_{i,n} \in \mathcal{G}_n$, $n = 1, 2, ..., I(n)$.

We may assume that $\mathcal{P}_n$ contains a set of generators of $F_{K(n)}\underline{K}(C_n)$, $\mathcal{F}_n \subset \mathcal{G}_n'$, and $\delta_n' < 1/2^{n+3}$. We may also assume that $\mathrm{Bott}(h', u')|_{\mathcal{P}_n}$ is well defined whenever $\|[h'(a), u']\| < \delta_n'$ for all $a \in \mathcal{G}_n'$ and for any unital homomorphism h' from C_n and unitary u' in the target algebra. Note that $\mathrm{Bott}(h', u')|_{\mathcal{P}_n}$ defines $\mathrm{Bott}(h'\, u')$. We may further assume that

(e 27.61) $$\mathrm{Bott}(h,\, u)|_{\mathcal{P}_n} = \mathrm{Bott}(h', u)|_{\mathcal{P}_n}$$

provided that $h \approx_{\delta_n'} h'$ on $\mathcal{G}_n'$. We may also assume that δ_n' is smaller than $\delta/16$ for the δ defined in 2.15 of [40] for C_n (in place of A) and $\mathcal{P}_n$ (in place of $\mathcal{P}$). Let $k(n) \ge n$ (in place of n), $\eta_n' > 0$ (in place of δ), and $\mathcal{Q}_{k(n)} \subset K_1(C_{k(n)})$ be as provided by Lemma 24.5 for $\delta_{k(n)}'/4$ (in place of ε), $\imath_{n,\infty}(\mathcal{G}_{k(n)})$ (in place of $\mathcal{F}$), $\mathcal{P}_{k(n)}$ (in place of $\mathcal{P}$), $\{p_{i,n}, q_{i,n}, : i = 1, 2, ..., k(n)\}$ (in place of $\{p_i, q_i : i = 1, 2, ..., k\}$), and $\sigma_{k(n)}'/16$ (in place of σ). We may assume that $\mathcal{Q}_{k(n)}$ generates the group $K_1(C_{k(n)})$. Since $\mathcal{P}$ generates $F_{K(n)}\underline{K}(C_{k(n+1)})$, we may assume that $\mathcal{Q}_n \subset \mathcal{P}_{k(n)}$.

Since $K_i(C_n)$ $(i = 0, 1)$ is finitely generated, by (e 27.60), we may further assume that $[\imath_{k(n),\infty}]$ is injective on $[\imath_{n,k(n)}](\underline{K}(C_n))$, $n = 1, 2, ...$ Passing to a subsequence, we may also assume that $k(n) = n+1$. Let $\delta_n = \min\{\eta_n, \sigma_n', \delta_n'/2\}$. By Lemma 27.3, there are unitaries $v_n \in U(A)$ such that

(e 27.62) $$\mathrm{Ad}\, v_n \circ \varphi_1 \approx_{\delta_{n+1}/4} \varphi_2 \;\text{ on }\; \imath_{n,\infty}(\mathcal{G}_{n+1}),$$

(e 27.63) $$\rho_A(\mathrm{bott}_1(\varphi_2, v_n^* v_{n+1}))(x) = 0$$

for all $x \in (\imath_{n,\infty})_{*1}(K_1(C_{n+1}))$, and

(e 27.64) $$\|[\varphi_2(c),\, v_n^* v_{n+1}]\| < \delta_{n+1}/2 \;\text{ for all }\; a \in \imath_{n,\infty}(\mathcal{G}_{n+1})$$

(Recall that $K_1(C_{n+1})$ is finitely generated). Note that, by (e 27.61), we may also assume that

$$\text{(e 27.65)}\quad \mathrm{Bott}(\varphi_1,\, v_{n+1}v_n^*)|_{[\imath_{n,\infty}](\mathcal{P}_n)} = \mathrm{Bott}(v_n^*\varphi_1 v_n,\, v_n^* v_{n+1})|_{[\imath_{n,\infty}](\mathcal{P}_n)} = \mathrm{Bott}(\varphi_2,\, v_n^* v_{n+1}])|_{[\imath_{n,\infty}](\mathcal{P}_n)}.$$

In particular,

$$\text{(e 27.66)}\quad \mathrm{bott}_1(v_n^*\varphi_1 v_n,\, v_n^* v_{n+1})(x) = \mathrm{bott}_1(\varphi_2,\, v_n^* v_{n+1})(x)$$

for all $x \in (\imath_{n,\infty})_{*1}(K_1(C_{n+1}))$.

Applying 10.4 and 10.5 of [37] (see also Remark 27.4), we may assume that the pair (φ_1, φ_2) and v_n define an element

$$\gamma_n := \gamma_{\varphi_1|_{C_{n+1}}, \varphi_1|_{C_{n+1}}, v_n} \in \mathrm{Hom}_\Lambda(\underline{K}(C_{n+1}), \underline{K}(M_{\varphi_1,\varphi_2}))$$

and $[\pi_e] \circ \gamma_n = [\mathrm{id}_{C_{n+1}}]$ (see Remark 27.4 for the definition of γ_n). Moreover, we may assume (see (e 27.47)) that

$$\text{(e 27.67)}\quad |\tau(\log(\varphi_2 \circ \imath_{n,\infty}(z_j^*)\tilde{v}_n\varphi_1 \circ \imath_{n,\infty}(z_j)\tilde{v}_n))| < \delta_{n+1},$$

$j = 1, 2, ..., r(n)$, where $\{z_1, z_2, ..., z_{r(n)}\} \subset U(M_k(C_{n+1}))$, and this set generates $K_1(C_{n+1})$, and where $\tilde{v}_n = \mathrm{diag}(\overbrace{v_n, v_n, ..., v_n}^{k})$. We may assume that $z_j \in \mathcal{Q}_n \subset \mathcal{P}_n$, $j = 1, 2, ..., r(n)$.

Let $H_n = [\imath_{n+1}](\underline{K}(C_{n+1})) \subset \underline{K}(C_{n+2})$. Since $\bigcup_{n=1}[\imath_{n+1,\infty}](\underline{K}(C_n)) = \underline{K}(C)$ and $[\pi_e] \circ \gamma_n = [\mathrm{id}_{C_{n+1}}]$, we conclude that

$$\text{(e 27.68)}\quad \underline{K}(M_{\varphi_1,\varphi_2}) = \underline{K}(SA) + \bigcup_{n=1}^{\infty} \gamma_{n+1}(H_n).$$

Thus, passing to a subsequence, we may further assume that

$$\text{(e 27.69)}\quad \gamma_{n+1}(H_n) \subset \underline{K}(SA) + \gamma_{n+2}(H_{n+1}),\ \ n = 1, 2, ...$$

Identifying H_n with $\gamma_{n+1}(H_n)$, let us write $j_n : \underline{K}(SA) \oplus H_n \to \underline{K}(SA) \oplus H_{n+1}$ for the inclusion in (e 27.69). By (e 27.68), the inductive limit is $\underline{K}(M_{\varphi_1,\varphi_2})$. From the definition of γ_n, we note that $\gamma_n - \gamma_{n+1} \circ [\imath_{n+1}]$ maps $\underline{K}(C_{n+1})$ into $\underline{K}(SA)$. By Remark 27.4 (see (e 27.54)), the map

$$\Gamma(\mathrm{Bott}(\varphi_1,\, v_n v_{n+1}^*))|_{H_n} = (\gamma_{n+1} - \gamma_{n+2} \circ [\imath_{n+2}])|_{H_n}$$

(see 27.4 for the definition of $\Gamma(\mathrm{Bott}(,)))$ is then a homomorphism $\xi_n : H_n \to \underline{K}(SA)$. Put $\zeta_n = \gamma_{n+1}|_{H_n}$. Then

$$\text{(e 27.70)}\quad j_n(x, y) = (x + \xi_n(y), [\imath_{n+2}](y))$$

for all $(x,y)\in \underline{K}(SA)\oplus H_n$. Thus we obtain the following diagram:

$$\begin{array}{ccccccc}
0\to & \underline{K}(SA) & \to & \underline{K}(SA)\oplus H_n & \to & H_n & \to 0\\
 & \| & & \| \quad \swarrow_{\xi_n} \downarrow_{[\imath_{n+2,\infty}]} & & \downarrow_{[\imath_{n+2,\infty}]} & \\
0\to & \underline{K}(SA) & \to & \underline{K}(SA)\oplus H_{n+1} & \to & H_{n+1} & \to 0\\
 & \| & & \| \quad \swarrow_{\xi_{n+1}} \downarrow_{[\imath_{n+3,\infty}]} & & \downarrow_{[\imath_{n+3,\infty}]} & \\
0\to & \underline{K}(SA) & \to & \underline{K}(SA)\oplus H_{n+2} & \to & H_{n+2} & \to 0.
\end{array}$$

By the assumption that $\bar{R}_{\varphi_1,\varphi_2}=0$, the map θ also gives the following decomposition:

$$\ker R_{\varphi_1,\varphi_2}=\ker\rho_A\oplus K_1(C). \tag{e 27.71}$$

Define $\theta_n=\theta\circ[\imath_{n+2,\infty}]$ and $\kappa_n=\zeta_n-\theta_n$. Note that

$$\theta_n=\theta_{n+1}\circ[\imath_{n+2}]. \tag{e 27.72}$$

We also have that

$$\zeta_n-\zeta_{n+1}\circ[\imath_{n+2}]=\xi_n. \tag{e 27.73}$$

Since $[\pi_e]\circ(\zeta_n-\theta_n)|_{H_n}=0$, κ_n maps H_n into $\underline{K}(SA)$. It follows that

$$\begin{aligned}\kappa_n-\kappa_{n+1}\circ[\imath_{n+2}] &= \zeta_n-\theta_n-\zeta_{n+1}\circ[\imath_{n+2}]+\theta_{n+1}\circ[\imath_{n+2}]\\ &= \zeta_n-\zeta_{n+1}\circ[\imath_{n+2}]=\xi_n.\end{aligned} \tag{e 27.74}$$

It follows from Lemma 26.3 that there are an integer $N_1\ge 1$, a unital $\frac{\delta_{n+1}}{4}$-$\imath_{n+1}(\mathcal{G}_{n+1})$-multiplicative completely positive linear map $L_n:\imath_{n,\infty}(C_{n+1})\to M_{1+N_1}(M_{\varphi_1,\varphi_2})$, a unital homomorphism $h_0:\imath_{n+1,\infty}(C_{n+1})\to M_{N_1}(\mathbb{C})$, and a continuous path of unitaries $\{V_n(t): t\in[0,3/4]\}$ in $M_{1+N_1}(A)$ such that $[L_n]|_{\mathcal{P}'_{n+1}}$ is well defined, $V_n(0)=1_{M_{1+N_1}(A)}$,

$$[L_n\circ\imath_{n,\infty}]|_{\mathcal{P}_n}=(\theta\circ[\imath_{n+1,\infty}]+[h_0\circ\imath_{n+1,\infty}])|_{\mathcal{P}_n},$$

$$\pi_t\circ L_n\circ\imath_{n+1,\infty}\approx_{\delta_{n+1}/4}\mathrm{Ad}\,V_n(t)\circ((\varphi_1\circ\imath_{n+1,\infty})\oplus(h_0\circ\imath_{n+1,\infty}))$$

on $\imath_{n+1,\infty}(\mathcal{G}_{n+1})$ for all $t\in(0,3/4]$,

$$\pi_t\circ L_n\circ\imath_{n+1,\infty}\approx_{\delta_{n+1}/4}\mathrm{Ad}\,V_n(3/4)\circ((\varphi_1\circ\imath_{n+1,\infty})\oplus(h_0\circ\imath_{n+1,\infty}))$$

on $\imath_{n+1,\infty}(\mathcal{G}_{n+1})$ for all $t\in(3/4,1)$, and

$$\pi_1\circ L_n\circ\imath_{n+1,\infty}\approx_{\delta_{n+1}/4}\varphi_2\circ\imath_{n+1,\infty}\oplus h_0\circ\imath_{n+1,\infty}$$

on $\imath_{n+1,\infty}(\mathcal{G}_{n+1})$, where $\pi_t: M_{\varphi_1,\varphi_2}\to A$ is the point evaluation at $t\in(0,1)$.

Note that $R_{\varphi_1,\varphi_2}(\theta(x))=0$ for all $x\in \imath_{n+1,\infty}(K_1(C_{n+1}))$. As in (e 27.40) (see also 10.4 of [37]),

$$\tau(\log((\varphi_2(x)\oplus h_0(x)^*V_n(3/4)^*(\varphi_1(x)\oplus h_0(x))V_n(3/4)))=0 \tag{e 27.75}$$

for $x=\imath_{n+1,\infty}(y)$, where y is in a set of generators of $K_1(C_{n+1})$, and for all $\tau\in T(A)$.

Define $W_n'=\mathrm{diag}(v_{n+1},1)\in M_{1+N_1}(A)$. Then

$$\tilde{\kappa}_n:=\mathrm{Bott}((\varphi_1\oplus h_0)\circ \imath_{n+1,\infty},\, W_n'(V_n(3/4)^*) \tag{e 27.76}$$

defines a homomorphism in $\mathrm{Hom}_\Lambda(\underline{K}(C_{n+1}),\underline{K}(SA))$. By (e 27.67)

$$|\tau(\log((\varphi_2\oplus h_0)\circ \imath_{n+1,\infty}(z_j)^*(W_n')^*(\varphi_1\oplus h_0)\circ \imath_{n+1,\infty}(z_j)W_n'))| < \delta_{n+1}, \tag{e 27.77}$$

$j=1,2,...,r(n)$. One computes (see (e 27.42)) that

$$\begin{aligned}\Gamma(\mathrm{Bott}(\varphi_1\circ \imath_{n+1,\infty}\oplus h_0,\, W_n'V(3/4)^*)|_{\mathcal{P}_n}\\ =(\gamma_{n+1}-\theta)[\imath_n]|_{\mathcal{P}_n}.\end{aligned} \tag{e 27.78}$$

Put $\tilde{V}_n=V_n(3/4)$. Let

$$b_{j,n} = \frac{1}{2\pi i}\log(\tilde{V}_n^*(\varphi_1\oplus h_0)\imath_{n+1,\infty}(z_j)\tilde{V}_n(\varphi_2\oplus h_0)\circ \imath_{n+1,\infty}(z_j)^*), \tag{e 27.79}$$

$$\begin{aligned} b_{j,n}' = \frac{1}{2\pi i}\log((\varphi_1\oplus h_0)\circ \imath_{n+1,\infty}(z_j)\tilde{V}_n(W_n')^*(\varphi_1\oplus h_0)\\ \circ \imath_{n+1,\infty}(z_j)^*W_n'\tilde{V}_n^*),\end{aligned} \tag{e 27.80}$$

and

$$\begin{aligned} b_{j,n}'' = \frac{1}{2\pi i}\log((\varphi_2\oplus h_0)\imath_{n+1,\infty}(z_j)(W_n')^*(\varphi_1\oplus h_0)\\ \circ \imath_{n+1,\infty}(z_j)^*W_n'),\end{aligned} \tag{e 27.81}$$

$j=1,2,...,r(n)$. By (e 27.75) and (e 27.77),

$$\tau(b_{j,n})=0 \ \text{ and } \ |\tau(b_{j,n}'')|<\delta_{n+1} \tag{e 27.82}$$

for all $\tau\in T(A)$. Note that

$$\tilde{V}_n^*e^{2\pi i b_{j,n}'}\tilde{V}_n=e^{2\pi i b_{j,n}}e^{2\pi i b_{j,n}''}. \tag{e 27.83}$$

Then, by 6.1 of [36] and by (e 27.82),

$$\tau(b_{j,n}') = \tau(b_{j,n})-\tau(b_{j,n}'')=\tau(b_{j,n}'') \ \text{ and } \ |\tau(b_{j,n}')|<\delta_{n+1} \tag{e 27.84}$$

for all $\tau \in T(A)$. It follows from this, (e 27.76), and (**??**) that

$$|\rho_A(\tilde{\kappa}_n(z_j))(\tau)| < \delta_{n+1}, \;\; j = 1, 2, ..., \tag{e 27.85}$$

for all $\tau \in T(A)$. It follows from 24.5 that there is a unitary $w_n' \in U(A)$ such that

$$\|[\varphi_1(a), w_n']\| < \delta_{n+1}'/4 \;\text{ for all }\; a \in \imath_{n+1,\infty}(\mathcal{G}_{n+1}) \;\text{ and} \tag{e 27.86}$$

$$\mathrm{Bott}(\varphi_1 \circ \imath_{n+1,\infty},\, w_n') = -\tilde{\kappa}_n \circ [\imath_{n+1}]. \tag{e 27.87}$$

By (e 27.61),

$$\mathrm{Bott}(\varphi_2 \circ \imath_{n+1,\infty},\, v_n^* w_n' v_n)|_{\mathcal{P}_n} = -\tilde{\kappa}_n \circ [\imath_{n+1}]|_{\mathcal{P}_n}. \tag{e 27.88}$$

It follows from (e 27.42) (see also 10.6 of [37]) and (e 27.78) that

$$\Gamma(\mathrm{Bott}(\varphi_1 \circ \imath_{n+1,\infty}, w_n')) \;=\; -\kappa_n \circ [\imath_{n+1}] \;\text{ and} \tag{e 27.89}$$

$$\Gamma(\mathrm{Bott}(\varphi_1 \circ \imath_{n+2,\infty}, w_{n+1}')) \;=\; -\kappa_{n+1} \circ [\imath_{n+2}]. \tag{e 27.90}$$

We also have

$$\Gamma(\mathrm{Bott}(\varphi_1 \circ \imath_{n+1,\infty}, v_n v_{n+1}^*))|_{H_n} = \zeta_n - \zeta_{n+1} \circ [\imath_{n+2}] = \xi_n. \tag{e 27.91}$$

But, by (e 27.74) and (e 27.75),

$$(-\kappa_n + \xi_n + \kappa_{n+1} \circ [\imath_{n+2}]) = 0. \tag{e 27.92}$$

By 10.6 of [37] (see also Remark 27.4), $\Gamma(\mathrm{Bott}(.,.)) = 0$ if and only if $\mathrm{Bott}(.,.) = 0$. Thus, by (e 27.88), (e 27.89), and (e 27.91),

$$\begin{aligned} &-\mathrm{Bott}(\varphi_1 \circ \imath_{n+1,\infty},\, w_n') + \mathrm{Bott}(\varphi_1 \circ \imath_{n+1,\infty},\, v_n v_{n+1}^*) \\ &\quad +\mathrm{Bott}(\varphi_1 \circ \imath_{n+1,\infty}, w_{n+1}') = 0. \end{aligned} \tag{e 27.93}$$

Put $w_n = v_n^*(w_n')v_n$ and $u_n = v_n w_n^*$, $n = 1, 2, ...$ Then, by (e 27.62) and (e 27.86),

$$\mathrm{Ad}\, u_n \circ \varphi_1 \approx_{\delta_n'/2} \varphi_2 \;\text{ for all }\; a \in \imath_{n+1,\infty}(\mathcal{G}_{n+1}). \tag{e 27.94}$$

From (e 27.65), (e 27.61), and (e 27.93), we compute that

$$\begin{aligned} &\mathrm{Bott}(\varphi_2 \circ \imath_{n+1,\infty}, u_n^* u_{n+1}) \\ &\quad= \mathrm{Bott}(\varphi_2 \circ \imath_{n+1,\infty}, w_n v_n^* v_{n+1} w_{n+1}^*) \\ &\quad= \mathrm{Bott}(\varphi_2 \circ \imath_{n+1,\infty}, w_n) + \mathrm{Bott}(\varphi_2 \circ \imath_{n+1,\infty}, v_n^* v_{n+1}) \\ &\qquad +\mathrm{Bott}(\varphi_2 \circ \imath_{n+1,\infty}, w_{n+1}^*) \\ &\quad= \mathrm{Bott}(\varphi_1 \circ \imath_{n+1,\infty}, w_n') + \mathrm{Bott}(\varphi_1 \circ \imath_{n+1,\infty}, v_{n+1} v_n^*) \\ &\qquad +\mathrm{Bott}(\varphi_1 \circ \imath_{n+1,\infty}, (w_{n+1}')^*) \\ &\quad= -[-\mathrm{Bott}(\varphi_1 \circ \imath_{n+1,\infty}, w_n') + \mathrm{Bott}(\varphi_1 \circ \imath_{n+1,\infty}, v_n v_{n+1}^*) \\ &\qquad +\mathrm{Bott}(\varphi_1 \circ \imath_{n+1,\infty}, w_{n+1}')] = 0. \end{aligned} \tag{e 27.95}$$

Let $x_{i,n} = [p_{i,n}] - [q_{i,n}]$, $1 \le i \le I(n)$. Note that we assume that $G_{u,n}$ is a free abelian group generated by $\{x_{i,n} : 1 \le i \le I(n)\}$. Without loss of generality, we may assume that these generators are independent. Put $e_{i,n} = \varphi_2 \circ \iota_{n+1,\infty}(p_{i,n})$, $e'_{i,n} = \varphi_2 \circ \iota_{n+1,\infty}(q_{i,n})$, $i = 1, 2, ..., I(n)$. Put $s_1 = 1$ and $\tilde{u}_1 = u_1 s_1^* = u_1$. Define a homomorphism $\Lambda_1 : G_{u,1} \to U_0(A)/CU(A)$ (see (e 27.95)) by

$$\Lambda_1(x_{i,1}) = \overline{\langle((1 - e_{i,1}) + e_{i,1}u_1^*u_2)((1 - e'_{i,1}) + e'_{i,1}u_2^*u_1)\rangle}. \tag{e 27.96}$$

Since Λ_1 factors through $G'_{u,1}$, applying Theorem 24.5 (or just Theorem 22.18) to $\varphi_2 \circ \iota_{2,\infty}$, one obtains a unitary $s_2 \in A$ such that

$$\|[\varphi_2 \circ \iota_{2,\infty}(f), s_2]\| < \delta'_2/4 \ \text{ for all } \ f \in \mathcal{G}_2, \tag{e 27.97}$$

$$\mathrm{Bott}(\varphi_2 \circ \iota_{2,\infty}, s_2)|_{\mathcal{P}_2} = 0, \ \text{ and} \tag{e 27.98}$$

$$\mathrm{dist}(\overline{\langle((1 - e_{i,1}) + e_{i,1}s_2^*)((1 - e'_{i,1}) + e'_{i,1}s_2)\rangle}, \Lambda_1(-x_{i,1})) < \sigma'_2/16. \tag{e 27.99}$$

Define $\tilde{u}_2 = u_2 s_2^*$. In what follows, we will construct unitaries $s_2, ..., s_n, ...$ in A such that

$$\|[\varphi_2 \circ \iota_{j+1,\infty}(f), s_{j+1}]\| < \delta'_{j+1}/4 \ \text{ for all } \ f \in \mathcal{G}_{j+1}, \tag{e 27.100}$$

$$\mathrm{Bott}(\varphi_2 \circ \iota_{+1,\infty}, s_{j+1})|_{\mathcal{P}_j} = 0, \ \text{ and} \tag{e 27.101}$$

$$\mathrm{dist}(\overline{\langle((1 - e_{i,j}) + e_{i,j}s_{n+1}^*)((1 - e'_{i,j}) + e'_{i,j}s_{j+1})\rangle}, \Lambda_n(-x_{i,j})) < \sigma'_n/16, \tag{e 27.102}$$

where $\Lambda_j : G_{u,j} \to U_0(A)/CU(A)$ is a homomorphism defined by

$$\Lambda_j(x_{i,j}) = \overline{\langle((1 - e_{i,j}) + e_{i,}\widetilde{u_j}^*u_{j+1})((1 - e'_{i,j}) + e'_{i,j}u_{j+1}^*\widetilde{u_j})\rangle} \tag{e 27.103}$$

(see (e 27.95) and (e 27.101) for j), and $\tilde{u}_j = u_j s_j^*$, $j = 1, 2, ...$

Assume that $s_2, ..., s_n$ are already constructed. Let us construct s_{n+1}. Note that by (e 27.95) the K_1 class of the unitary $u_n^* u_{n+1}$ is trivial. In particular, the K_1 class of $s_n u_n^* u_{n+1}$ is trivial. Since Λ_n factors through $G'_{u,n}$, applying Theorem 24.5 to $\varphi_2 \circ \iota_{n+1,\infty}$, one obtains a unitary $s_{n+1} \in A$ such that

$$\|[\varphi_2 \circ \iota_{n+1,\infty}(f), s_{n+1}]\| < \delta'_{n+1}/4 \ \text{ for all } \ f \in \mathcal{G}_{n+1}, \tag{e 27.104}$$

$$\mathrm{Bott}(\varphi_2 \circ \iota_{n+1,\infty}, s_{n+1})|_{\mathcal{P}_n} = 0, \ \text{ and} \tag{e 27.105}$$

$$\mathrm{dist}(\overline{\langle((1 - e_{i,n}) + e_{i,n}s_{n+1}^*)((1 - e'_{i,n}) + e'_{i,n}s_{n+1})\rangle}, \Lambda_n(-x_{i,n})) < \sigma'_n/16, \tag{e 27.106}$$

$i = 1, 2, ..., I(n+1)$. Then $s_1, s_2, ..., s_{n+1}$ satisfy (e 27.100), (e 27.101), and (e 27.102).

Put $\widetilde{u_n + 1} = u_{n+1}s_{n+1}^*$. Then by (e 27.94) and (e 27.100), one has

$$\text{(e 27.107)} \qquad \mathrm{ad}\,\widetilde{u_n} \circ \varphi_1 \approx_{\delta_n'} \varphi_2 \;\text{ for all }\; a \in \imath_{n+1,\infty}(\mathcal{G}_{n+1}).$$

By (e 27.95) and (e 27.101), one has

$$\text{(e 27.108)} \qquad \mathrm{Bott}(\varphi_2 \circ \imath_{n+1,\infty}, (\widetilde{u_n})^*\widetilde{u_{n+1}})|_{\mathcal{P}_n} = 0.$$

Note that

$$\text{(e 27.109)} \qquad \overline{\langle (1 - e_{i,n}) + e_{i,n}\widetilde{u_n}^*\widetilde{u_{n+1}}\rangle\langle (1 - e_{i,n}') + e_{i,n}'\widetilde{u_{n+1}}^*\widetilde{u_n}\rangle} = \overline{c_1c_2c_4c_3} = \overline{c_1c_3c_2c_4},$$

where

$$\text{(e 27.110)} \quad c_1 = \langle (1 - e_{i,n}) + e_{i,n}\widetilde{u_n}^*u_{n+1}\rangle,\; c_2 = \langle (1 - e_{i,n}) + e_{i,n}s_{n+1}^*\rangle.$$

$$\text{(e 27.111)} \quad c_3 = \langle (1 - e_{i,n}') + e_{i,n}'u_{n+1}^*\widetilde{u_n}\rangle,\; c_4 = \langle (1 - e_{i,n}') + e_{i,n}'s_{n+1}\rangle.$$

Therefore, by (e 27.104) and (e 27.103), one has

$$\text{(e 27.112)} \quad \mathrm{dist}(\overline{\langle((1 - e_{i,n}) + e_{i,n}\widetilde{u_n}^*\widetilde{u_{n+1}})((1 - e_{i,n}') + e_{i,n}'\widetilde{u_{n+1}}^*\widetilde{u_n}))\rangle}, \bar{1})$$

$$\text{(e 27.113)} \quad < \sigma_n'/16 + \mathrm{dist}(\Lambda_n(x_{i,n})\Lambda_n(-x_{i,n}), \bar{1}) = \sigma_n'/16,$$

$i = 1, 2, ..., I(n)$. Therefore, by Lemma 25.4 (and Remark 25.5), there exists a continuous and piecewise smooth path of unitaries $\{z_n(t) : t \in [0, 1]\}$ of A such that

$$\text{(e 27.114)} \qquad z_n(0) = 1,\;\; z_n(1) = (\widetilde{u_n})^*\widetilde{u_{n+1}} \;\text{ and}$$

$$\text{(e 27.115)} \qquad \|[\varphi_2(a),\, z_n(t)]\| < 1/2^{n+2} \;\text{ for all }\; a \in \mathcal{F}_n \;\text{ and }\; t \in [0, 1].$$

Define

$$u(t + n - 1) = \widetilde{u_n}z_{n+1}(t) \;\; t \in (0, 1].$$

Note that $u(n) = \widetilde{u_{n+1}}$ for all integers n and $\{u(t) : t \in [0, \infty)\}$ is a continuous path of unitaries in A. One estimates that, by (e 27.94) and (e 27.115),

$$\mathrm{Ad}\,u(t + n - 1) \circ \varphi_1 \approx_{\delta_n'} \mathrm{Ad}\,z_{n+1}(t) \circ \varphi_2 \approx_{1/2^{n+2}} \varphi_2 \quad \text{on } \;\mathcal{F}_n$$

for all $t \in (0, 1)$. It then follows that

$$\text{(e 27.116)} \qquad \lim_{t\to\infty} u^*(t)\varphi_1(a)u(t) = \varphi_2(a) \;\text{ for all }\; a \in C.$$

□

28. Rotation Maps and Strong Asymptotic Equivalence

LEMMA 28.1. *Let A be a unital separable simple C^*-algebra of stable rank one. Suppose that $u \in CU(A)$. Then, for any continuous and piecewise smooth path $\{u(t) : t \in [0,1]\} \subset U(A)$ with $u(0) = u$ and $u(1) = 1_A$,*

$$D_A(\{u(t)\}) \in \overline{\rho_A(K_0(A))} \qquad \text{(recall Definition 2.16 of [21] for } D_A\text{)}.$$

(e 28.1)

PROOF. It follows from Corollary 11.11 of [21] that the map $j : u \mapsto \mathrm{diag}(u, 1, ..., 1)$ from $U(A)$ to $U(M_n(A))$ induces an isomorphism from $U(A)/CU(A)$ to $U(M_n(A))/CU(M_n(A))$. Then the conclusion follows from 3.1 and 3.2 of [52]. □

LEMMA 28.2. *Let A be a unital separable simple C^*-algebra of stable rank one. Suppose that B is a unital separable C^*-algebra and suppose that $\varphi, \psi : B \to A$ are two unital monomorphisms such that*

$$[\varphi] = [\psi] \text{ in } KK(B, A), \tag{e 28.2}$$

$$\varphi_T = \psi_T \text{ and } \varphi^\ddagger = \psi^\ddagger. \tag{e 28.3}$$

Then

$$R_{\varphi,\psi} \in \mathrm{Hom}(K_1(B), \overline{\rho_A(K_0(A))}). \tag{e 28.4}$$

PROOF. Let $z \in K_1(B)$ be represented by a unitary $u \in U(M_m(B))$ for some integer m. Then, by (e 28.3),

$$(\varphi \otimes \mathrm{id}_{M_m})(u)(\psi \otimes \mathrm{id}_{M_m})(u)^* \in CU(M_m(A)).$$

Suppose that $\{u(t) : t \in [0,1]\}$ is a continuous and piecewise smooth path in $M_m(U(A))$ such that $u(0) = (\varphi \otimes \mathrm{id}_{M_m})(u)$ and $u(1) = (\psi \otimes \mathrm{id}_{M_m})(u)$. Put $w(t) = (\psi \otimes \mathrm{id}_{M_m})(u)^* u(t)$. Then $w(0) = (\psi \otimes \mathrm{id}_{M_m})(u)^*(\varphi \otimes \mathrm{id}_{M_m})(u) \in CU(A)$ and $w(1) = 1_A$. Thus,

$$\begin{aligned} R_{\varphi,\psi}(z)(\tau) &= \frac{1}{2\pi i}\int_0^1 \tau(\frac{du(t)}{dt}u^*(t))dt = \frac{1}{2\pi i}\int_0^1 \tau(\psi(u)^*\frac{du(t)}{dt}u^*(t)\psi(u))dt \\ &= \frac{1}{2\pi i}\int_0^1 \tau(\frac{dw(t)}{dt}w^*(t))dt \end{aligned}$$

for all $\tau \in T(A)$. By 28.1,

$$R_{\varphi,\psi}(z) \in \overline{\rho_A(K_0(A))}.$$

It follows that

$$R_{\varphi,\psi} \in \mathrm{Hom}(K_1(B), \overline{\rho_A(K_0(A))}).$$

□

THEOREM 28.3. *Let $C_1, C_2 \in \mathcal{B}_0$ be unital separable simple C^*-algebras, and $A = C_1 \otimes U_1$, $B = C_2 \otimes U_2$, where U_1 and U_2 are UHF-algebras of infinite type, and B satisfies the UCT. Suppose that B is a unital C^*-subalgebra of A, and denote by $\imath$ the embedding. For any $\lambda \in \mathrm{Hom}(K_1(B), \overline{\rho_A(K_0(A))})$, there exists $\varphi \in \overline{\mathrm{Inn}}(B, A)$ (see Definition 2.8 of [21]) such that there are homomorphisms $\theta_i : K_i(B) \to K_i(M_{\imath,\varphi})$ with $(\pi_0)_{*i} \circ \theta_i = \mathrm{id}_{K_i(B)}$, $i = 0, 1$, and the rotation map $R_{\imath,\varphi} : K_1(M_{\imath,\varphi}) \to \mathrm{Aff}(T(A))$ is given by*

$$R_{\imath,\varphi}(x) = \rho_A(x - \theta_1(\pi_0)_{*1}(x)) + \lambda \circ (\pi_0)_{*1}(x)) \tag{e 28.5}$$

for all $x \in K_1(M_{\imath,\varphi})$. In other words,

$$[\varphi] = [\imath] \text{ in } KK(B, A) \tag{e 28.6}$$

and the rotation map $R_{\imath,\varphi} : K_1(M_{\imath,\varphi}) \to \mathrm{Aff}(T(A))$ is given by

$$R_{\imath,\varphi}(a, b) = \rho_A(a) + \lambda(b) \tag{e 28.7}$$

for some identification of $K_1(M_{\imath,\varphi})$ with $K_0(A) \oplus K_1(B)$.

PROOF. The proof is exactly the same as that of Theorem 4.2 of [44]. By Lemma 23.3 and Lemma 24.1 (see also Lemma 24.2 and Remark 22.15), we have the properties (B1) and (B2) associated with B (defined in 3.6 of [44]) as in Theorem 4.2 of [44]. In 4.2 of [44], it is also assumed that $\rho_A(K_0(A))$ is dense in $\mathrm{Aff}(T(A))$, which is only used to get that $\psi(K_1(B)) \subset \overline{\rho_A(K_0(A))}$, which corresponds to the assumption $\lambda(K_1(B)) \subset \overline{\rho_A(K_0(A))}$ here. □

DEFINITION 28.4. Let A be a unital C^*-algebra and let C be a unital separable C^*-algebra. Denote by $\mathrm{Mon}_{asu}^e(C, A)$ the set of all asymptotic unitary equivalence classes of unital monomorphisms from C into A. Denote by $\boldsymbol{K} : \mathrm{Mon}_{asu}^e(C, A) \to KK_e(C, A)^{++}$ the map defined by

$$\varphi \mapsto [\varphi] \text{ for all } \varphi \in \mathrm{Mon}_{asu}^e(C, A).$$

Let $\kappa \in KK_e(C, A)^{++}$. Denote by $\langle \kappa \rangle$ the set of classes of all $\varphi \in \mathrm{Mon}_{asu}^e(C, A)$ such that $\boldsymbol{K}(\varphi) = \kappa$.

Denote by $KKUT_e(A, B)^{++}$ the set of triples (κ, α, γ) for which $\kappa \in KK_e(A, B)^{++}$, $\alpha : U(A)/CU(A) \to U(B)/CU(B)$ is a homomorphism, $\gamma : T(B) \to T(A)$ is a continuous affine map, and both α and γ are compatible with κ. Denote by $\mathfrak{K}$ the map from $\mathrm{Mon}_{asu}^e(C, A)$ into $KKUT(C, A)^{++}$ defined by

$$\varphi \mapsto ([\varphi], \varphi^{\ddagger}, \varphi_T) \text{ for all } \varphi \in \mathrm{Mon}_{asu}^e(C, A).$$

Denote by $\langle \kappa, \alpha, \gamma \rangle$ the subset of $\varphi \in \mathrm{Mon}_{asu}^e(C, A)$ such that $\mathfrak{K}(\varphi) = (\kappa, \alpha, \gamma)$.

THEOREM 28.5. *Let C and A be two unital separable amenable C^*-algebras. Suppose that $\varphi_1, \varphi_2, \varphi_3 : C \to A$ are three unital monomorphisms for which*

$$[\varphi_1] = [\varphi_2] = [\varphi_3] \text{ in } KK(C,A)) \text{ and } (\varphi_1)_T = (\varphi_2)_T = (\varphi_3)_T.$$

Then

$$\overline{R}_{\varphi_1,\varphi_2} + \overline{R}_{\varphi_2,\varphi_3} = \overline{R}_{\varphi_1,\varphi_3}.$$

PROOF. The proof is exactly the same as that of Theorem 9.6 of [40]. □

LEMMA 28.6. *Let A and B be two unital separable amenable C^*-algebras. Suppose that $\varphi_1, \varphi_2 : A \to B$ are two unital monomorphisms such that*

$$[\varphi_1] = [\varphi_2] \text{ in } KK(A,B) \text{ and } (\varphi_1)_T = (\varphi_2)_T.$$

Suppose that $(\varphi_2)_T : T(B) \to T(A)$ is an affine homeomorphism. Suppose also that there is $\alpha \in Aut(B)$ such that

$$[\alpha] = [\mathrm{id}_B] \text{ in } KK(B,B) \text{ and } \alpha_T = \mathrm{id}_T.$$

Then

$$\overline{R}_{\varphi_1,\alpha\circ\varphi_2} = \overline{R}_{\mathrm{id}_B,\alpha} \circ (\varphi_2)_{*1} + \overline{R}_{\varphi_1,\varphi_2} \tag{e 28.8}$$

in $\mathrm{Hom}(K_1(A), \mathrm{Aff}(T(B)))/\mathcal{R}_0$.

PROOF. Using 28.5, we compute that

$$\overline{R}_{\varphi_1,\alpha\circ\varphi_2} = \overline{R}_{\varphi_1,\varphi_2} + \overline{R}_{\varphi_2,\alpha\circ\varphi_2} = \overline{R}_{\varphi_1,\varphi_2} + \overline{R}_{\mathrm{id}_B,\alpha} \circ (\varphi_2)_{*1}.$$

□

THEOREM 28.7. *Let B be a unital separable simple amenable C^*-algebra in $\mathcal{B}_0$ satisfying the UCT, let $C = B \otimes U_1$, where U_1 is a UHF-algebra of infinite type, let A_1 be a unital separable amenable simple C^*-algebra in $\mathcal{B}_0$, and let $A = A_1 \otimes U_2$, where U_2 is another UHF-algebra of infinite type. Then the map $\boldsymbol{\mathfrak{K}} : \mathrm{Mon}^e_{asu}(C,A) \to KKUT(C,A)^{++}$ is surjective. Moreover, for each $(\kappa, \alpha, \gamma) \in KKUT(C,A)^{++}$, there exists a bijection*

$$\eta : \langle \kappa, \alpha, \gamma \rangle \to \mathrm{Hom}(K_1(C), \overline{\rho_A(K_0(A))})/\mathcal{R}_0.$$

PROOF. It follows from Lemma 24.4 (also Remark 22.15) that $\boldsymbol{\mathfrak{K}}$ is surjective.

Fix a triple $(\kappa, \alpha, \gamma) \in KKT(C,A)^{++}$ and choose a unital monomorphism $\varphi : C \to A$ such that $[\varphi] = \kappa$, $\varphi^{\ddagger} = \alpha$, and $\varphi_{\mathrm{T}} = \gamma$. If $\varphi_1 : C \to A$ is another unital monomorphism such that $\boldsymbol{\mathfrak{K}}(\varphi_1) = \boldsymbol{\mathfrak{K}}(\varphi)$, then by Lemma 28.2,

$$\overline{R}_{\varphi,\varphi_1} \in \mathrm{Hom}(K_1(C), \overline{\rho_A(K_0(A))})/\mathcal{R}_0. \tag{e 28.9}$$

Let $\lambda \in \mathrm{Hom}(K_1(C), \overline{\rho_A(K_0(A))})$ be a homomorphism. It follows from Theorem 28.3 that there is a unital monomorphism $\psi \in \overline{\mathrm{Inn}}(\varphi(C), A)$ with $[\psi\circ\varphi] = [\varphi]$ in $KK(C, A)$ such that there exists a homomorphism $\theta : K_1(C) \to K_1(M_{\varphi,\psi\circ\varphi})$ with $(\pi_0)_{*1} \circ \theta = \mathrm{id}_{K_1(C)}$ for which $R_{\varphi,\psi\circ\varphi} \circ \theta = \lambda$. Let $\beta = \psi \circ \varphi$. Then $R_{\varphi,\beta} \circ \theta = \lambda$. Note also that, since $\psi \in \overline{\mathrm{Inn}}(\varphi(C), A)$, $\beta^\ddagger = \varphi^\ddagger$ and $\beta_{\mathrm{T}} = \varphi_{\mathrm{T}}$. In particular, $\boldsymbol{\mathfrak{K}}(\beta) = \boldsymbol{\mathfrak{K}}(\varphi)$.

Thus, for each unital monomorphism φ, we obtain a well-defined and surjective map

$$\eta_\varphi : \langle [\varphi], \varphi^\ddagger, \varphi_T \rangle \to \mathrm{Hom}(K_1(A), \overline{\rho_A(K_0(A))})/\mathcal{R}_0.$$

To see that η_φ is injective, consider two monomorphisms $\varphi_1, \varphi_2 : C \to A$ in $\langle [\varphi], \varphi^\ddagger, \varphi_T \rangle$ such that

$$\overline{R}_{\varphi,\varphi_1} = \overline{R}_{\varphi,\varphi_2}.$$

Then, by Theorem 28.5,

$$\overline{R}_{\varphi_1,\varphi_2} = \overline{R}_{\varphi_1,\varphi} + \overline{R}_{\varphi,\varphi_2} = -\overline{R}_{\varphi,\varphi_1} + \overline{R}_{\varphi,\varphi_2} = 0. \qquad \text{(e 28.10)}$$

It follows from Theorem 27.5 that φ_1 and φ_2 are asymptotically unitarily equivalent. The map η_φ is the desired bijection η as $\langle [\varphi], \varphi^\ddagger, \varphi_T \rangle = \langle \kappa, \alpha, \gamma \rangle$. □

Definition 28.8. Denote by $KKUT_e^{-1}(A, A)^{++}$ the subset of those elements $(\kappa, \alpha, \gamma) \in KKUT_e(A, A)^{++}$ for which $\kappa|_{K_i(A)}$ is an isomorphism ($i = 0, 1$), α is an isomorphism, and γ is an affine homeomorphism. Recall from the proof of Theorem 28.7 that

$$\eta_{\mathrm{id}_A} : \langle [\mathrm{id}_A], \mathrm{id}_A^\ddagger, (\mathrm{id}_A)_T \rangle \to \mathrm{Hom}(K_1(A), \overline{\rho_A(K_0(A))})/\mathcal{R}_0$$

is a bijection.

Denote by $\langle \mathrm{id}_A \rangle$ the class of those automorphisms ψ which are asymptotically unitarily equivalent to id_A—this subset of $\mathrm{Aut}(A)$ gives rise to a single element in $\mathrm{Mon}^e_{asu}(A, A)$ which should not be confused with the subset $\langle [\mathrm{id}_A], \mathrm{id}_A^\ddagger, (\mathrm{id}_A)_T \rangle \subset \mathrm{Mon}^e_{asu}(A, A)$. Note that, if $\psi \in \langle \mathrm{id}_A \rangle$, then ψ is *asymptotically inner*, i.e., there exists a continuous path of unitaries $\{u(t) : t \in [0, \infty)\} \subset A$ such that

$$\psi(a) = \lim_{t\to\infty} u(t)^* a u(t) \ \text{ for all } \ a \in A.$$

Note that $\langle \mathrm{id}_A \rangle$ is a normal subgroup of $\mathrm{Aut}(A)$.

Corollary 28.9. *Let $A_1 \in \mathcal{B}_0$ be a unital simple amenable C^*-algebra satisfying the UCT and let $A = A_1 \otimes U$ for some UHF-algebra U of infinite type. Then one has the following short exact sequence:*

$$0 \to \mathrm{Hom}(K_1(A), \overline{\rho_A(K_0(A))})/\mathcal{R}_0 \stackrel{\eta^{-1}_{\mathrm{id}_A}}{\to} \mathrm{Aut}(A)/\langle \mathrm{id}_A \rangle \stackrel{\boldsymbol{\mathfrak{K}}}{\to} KKUT_e^{-1}(A, A)^{++} \to 0.$$

In particular, if $\varphi, \psi \in \mathrm{Aut}(A)$ *are such that*

$$\boldsymbol{\mathfrak{K}}(\varphi) = \boldsymbol{\mathfrak{K}}(\psi) = \boldsymbol{\mathfrak{K}}(\mathrm{id}_A),$$

then

$$\eta_{\mathrm{id}_A}(\varphi \circ \psi) = \eta_{\mathrm{id}_A}(\varphi) + \eta_{\mathrm{id}_A}(\psi).$$

PROOF. It follows from Lemma 24.4 (see also Remark 22.15) that, for any $\langle \kappa, \alpha, \gamma \rangle$, there is a unital monomorphism $h : A \to A$ such that $\boldsymbol{\mathfrak{K}}(h) = \langle \kappa, \alpha, \gamma \rangle$. The fact that $\kappa \in KK_e^{-1}(A, A)^{++}$ implies that there is $\kappa_1 \in KK_e^{-1}(A, A)^{++}$ such that

$$\kappa \times \kappa_1 = \kappa_1 \times \kappa = [\mathrm{id}_A].$$

Using Lemma 24.4, choose $h_1 : A \to A$ such that

$$\boldsymbol{\mathfrak{K}}(h) = \langle \kappa_1, \alpha^{-1}, \gamma^{-1} \rangle.$$

It follows from Lemma 27.1 that $h_1 \circ h$ and $h \circ h_1$ are approximately unitarily equivalent. Applying the standard approximate intertwining argument of G. A. Elliott (Theorem 2.1 of [12]), one obtains two isomorphisms φ and φ^{-1} such that there is a sequence of unitaries $\{u_n\}$ in A such that

$$\varphi(a) = \lim_{n\to\infty} \mathrm{Ad}\, u_{2n+1} \circ h(a) \ \text{ and } \ \varphi^{-1}(a) = \lim_{n\to\infty} \mathrm{Ad}\, u_{2n} \circ h_1(a)$$

for all $a \in A$. Thus, $[\varphi] = [h]$ in $KL(A, A)$ and $\varphi^{\ddagger} = h^{\ddagger}$ and $\varphi_T = h_T$. Then, as in the proof of 24.4, there is $\psi_0 \in \overline{\mathrm{Inn}}(A, A)$ such that $[\psi_0 \circ \varphi] = [\mathrm{id}_A]$ in $KK(A, A)$ as well as $(\psi_0 \circ \varphi)^{\ddagger} = h^{\ddagger}$ and $(\psi_0 \circ \varphi)_T = h_T$. So we have $\psi_0 \circ \varphi \in \mathrm{Aut}(A, A)$ such that $\boldsymbol{\mathfrak{K}}(\psi_0 \circ \varphi) = \langle \kappa, \alpha, \gamma \rangle$. This implies that $\mathfrak{K}$ is surjective.

Now let $\lambda \in \mathrm{Hom}(K_1(C), \overline{\mathrm{Aff}(T(A))})/\mathcal{R}_0$. The proof Theorem 28.7 says that there is $\psi_{00} \in \overline{\mathrm{Inn}}(A, A)$ (in place of ψ) such that $\boldsymbol{\mathfrak{K}}(\psi_{00} \circ \mathrm{id}_A) = \boldsymbol{\mathfrak{K}}(\mathrm{id}_A)$ and

$$\overline{R}_{\mathrm{id}_A, \psi_{00}} = \lambda.$$

Note that ψ_{00} is again an automorphism. The last part of the lemma then follows from Lemma 28.6. □

DEFINITION 28.10 (Definition 10.2 of [37] and see also [41]). Let A be a unital C^*-algebra and B be another C^*-algebra. Recall ([41]) that

$$H_1(K_0(A), K_1(B)) = \{x \in K_1(B) : \varphi([1_A]) = x \text{ for some } \varphi \in \mathrm{Hom}(K_0(A), K_1(B))\}.$$

PROPOSITION 28.11 (Proposition 12.3 of [37])). *Let A be a unital separable C^*-algebra and let B be a unital C^*-algebra. Suppose that $\varphi : A \to B$ is a unital homomorphism and $u \in U(B)$ is a unitary. Suppose that there is a continuous path of unitaries $\{u(t) : t \in [0, \infty)\} \subset B$ such that*

$$u(0) = 1_B \ \textit{ and } \ \lim_{t\to\infty} \mathrm{Ad}\, u(t) \circ \varphi(a) = \mathrm{Ad}\, u \circ \varphi(a) \tag{e 28.11}$$

for all $a \in A$. Then

$$[u] \in H_1(K_0(A), K_1(B)).$$

Lemma 28.12. *Let $C = C' \otimes U$ for some $C' = \varinjlim(C_n, \psi_n)$ and a UHF algebra U of infinite type, where each C_n is a direct sum of C*-algebras in $\mathcal{C}_0$ and $\mathbf{H}$. Assume that ψ_n is unital and injective. Let $A \in \mathcal{B}_1$. Let $\varphi_1, \varphi_2 : C \to A$ be two monomorphisms such that there is an increasing sequence of finite subsets $\mathcal{F}_n \subset C$ with dense union, an increasing sequence of finite subsets $\mathcal{P}_n \subset K_1(C)$ with union equal to $K_1(C)$, a sequence of positive numbers (δ_n) with $\sum \delta_n < 1$ and a sequence of unitaries $\{u_n\} \subset A$ such that*

$$\mathrm{Ad}u_n \circ \varphi_1 \approx_{\delta_n} \varphi_2 \quad \text{on } \mathcal{F}_n \text{ and } \rho_A(\mathrm{bott}_1(\varphi_2, u_n^* u_{n+1})) = 0 \text{ for all } x \in \mathcal{P}_n.$$

Suppose that $H_1(K_0(C), K_1(A)) = K_1(A)$. Then there exists a sequence of unitaries $v_n \in U_0(A)$ such that

$$\mathrm{Ad}v_n \circ \varphi_1 \approx_{\delta_n} \varphi_2 \quad \text{on } \mathcal{F}_n \text{ and} \tag{e 28.12}$$

$$\rho_A(\mathrm{bott}_1(\varphi_2, v_n^* v_{n+1})) = 0, \quad x \in \mathcal{P}_n. \tag{e 28.13}$$

Proof. Let $x_n = [u_n] \in K_1(A)$. Since $H_1(K_0(C), K_1(A)) = K_1(A)$, there is a homomorphism

$$\kappa_{n,0} : K_0(C) \to K_1(A)$$

such that $\kappa_{n,0}([1_C]) = -x_n$. Since C satisfies the Universal Coefficient Theorem, there is $\kappa_n \in KL(C \otimes C(\mathbb{T}), A)$ such that

$$(\kappa_n)|_{\boldsymbol{\beta}(K_0(C))} = \kappa_{n,0} \quad \text{and} \quad (\kappa_n)|_{\boldsymbol{\beta}(K_1(C))} = 0.$$

Without loss of generality, we may assume that $[1_C] \in \mathcal{P}_n$, $n = 1, 2, \ldots..$ For each δ_n, choose a positive number $\eta_n < \delta_n$, such that

$$\mathrm{Ad}u_n \circ \varphi_1 \approx_{\eta_n} \varphi_2 \quad \text{on } \mathcal{F}_n.$$

By Lemma 24.1, there is a unitary $w_n \in U(A)$ such that

$$\|[\varphi_2(a), w_n]\| < (\delta_n - \eta_n)/2 \quad \text{for all } a \in \mathcal{F}_n \text{ and } \mathrm{Bott}(\varphi_2, w_n)|_{\mathcal{P}_n} = \kappa_n|_{\boldsymbol{\beta}(\mathcal{P}_n)}.$$

Put $v_n = u_n w_n$, $n = 1, 2, \ldots.$ Then

$$\mathrm{Ad}v_n \circ \varphi_1 \approx_{\delta_n} \varphi_2 \quad \text{on } \mathcal{F}_n, \;\; \rho_A(\mathrm{bott}_1(\varphi_2, v_n^* v_{n+1}))|_{\mathcal{P}_n} = 0$$

and, since $[1_C] \in \mathcal{P}_n$,

$$[v_n] = [u_n] - x_n = 0,$$

as desired. □

Theorem 28.13. *Let $B \in \mathcal{B}_1$ be a unital separable simple C*-algebra which satisfies the UCT, let $A_1 \in \mathcal{B}_1$ be a unital separable simple C*-algebra, and let $C = B \otimes U_1$ and $A = A_1 \otimes U_2$, where U_1 and U_2 are unital infinite dimensional UHF-algebras. Suppose that $H_1(K_0(C), K_1(A)) = K_1(A)$ and suppose* *that $\varphi_1, \varphi_2 : C \to A$ are two unital monomorphisms which are asymptotically unitarily equivalent. Then φ_1 and φ_2 are strongly asymptotically unitarily equivalent, that is, there exists a continuous path of unitaries $\{u(t) : t \in [0, \infty)\} \subset A$ such that*

$$u(0) = 1 \ \textit{and} \ \lim_{t\to\infty} \mathrm{Ad}u(t) \circ \varphi_1(a) = \varphi_2(a) \ \textit{for all} \ a \in C.$$

PROOF. By 4.3 of [40], one has

$$[\varphi_1] = [\varphi_2] \quad \text{in} \ KK(C, A),$$
$$\varphi^{\ddagger} = \psi^{\ddagger}, \ (\varphi_1)_T = (\varphi_2)_T \ \text{and} \ \overline{R_{\varphi_1,\varphi_2}} = 0.$$

Then by Lemma 28.12 (see also Remark 22.15), one may assume that $v_n \in U_0(A)$ ($n = 1, 2, ...$) in the proof of Theorem 27.5. It follows that $\xi_n([1_C]) = 0$, $n = 1, 2, ...$, and therefore $\kappa_n([1_C]) = 0$. This implies that $\gamma_n \circ \beta([1_C]) = 0$. Hence $w_n \in U_0(A)$, and also $u_n \in U_0(A)$. Therefore, the continuous path of unitaries $\{u(t)\}$ constructed in Theorem 27.5 is in $U_0(A)$, and then one may require that $u(0) = 1_A$ by connecting $u(0)$ to 1_A. □

29. The General Classification Theorem

LEMMA 29.1. *Let $A_1 \in \mathcal{B}_0$ be a unital separable simple C^*-algebra, let $A = A_1 \otimes U$ for some infinite dimensional UHF-algebra, and let $\mathfrak{p}$ be a supernatural number of infinite type. Then the homomorphism $\imath : a \mapsto a \otimes 1$ induces an isomorphism from $U_0(A)/CU(A)$ to $U_0(A \otimes M_{\mathfrak{p}})/CU(A \otimes M_{\mathfrak{p}})$.*

PROOF. There are sequences of positive integers $\{m(n)\}$ and $\{k(n)\}$ such that $A \otimes M_{\mathfrak{p}} = \lim_{n\to\infty}(A \otimes M_{m(n)}, \imath_n)$, where

$$\imath_n : M_{m(n)}(A) \to M_{m(n+1)}(A)$$

is defined by $\imath(a) = \mathrm{diag}(\overbrace{a, a, ..., a}^{k(n)})$ for all $a \in M_{m(n)}(A)$, $n = 1, 2, ...$ Note, $M_{m(n)}(A) = M_{m(n)}(A_1) \otimes U$ and $M_{m(n)}(A_1) \in \mathcal{B}_0$. Let

$$j_n : U(M_{m(n)}(A))/CU(M_{m(n)}(A))) \to U(M_{m(n+1)}(A))/CU(M_{m(n+1)}(A))$$

be defined by

$$j_n(\bar{u}) = \overline{\mathrm{diag}(u, \underbrace{1, 1, ..., 1}_{k(n)-1})} \ \text{for all} \ u \in U((M_{m(n)}(A)).$$

It follows from Corollary11.11 of [21] that j_n is an isomorphism. By Corollary 11.7 of [21], the abelian group $U_0(M_{m(n)}(A))/CU(M_{m(n)}(A))$ is divisible. For each n and i, there is a unitary $U_i \in M_{m(n+1)}(A)$ such that

$$U_i^* E_{1,1} U_i = E_{i,i}, \ \ i = 2, 3, ..., k(n),$$

where $E_{i,i} = \sum_{j=(i-1)m(n)+1}^{im(n)} e_{j,j}$ and $\{e_{i,j}\}$ is a system of matrix units for $M_{m(n+1)}$. Then

$$\imath_n(u) = u'(U_2^* u' U_2)(U_3^* u' U_3)\cdots(U_{k(n)}^* u' U_{k(n)}),$$

where $u' = \mathrm{diag}(u, \overbrace{1, 1, ..., 1})$, for all $u \in M_{m(n)}(A)$. Thus,

$$\imath_n^{\ddagger}(\bar{u}) = k(n)\jmath_n(\bar{u}).$$

It follows that $\imath_n^{\ddagger}|_{U_0(M_{m(n)}(A))/CU(M_{m(n)}(A))}$ is injective, since

$$U_0(M_{m(n+1)}(A))/CU(M_{m(n+1)}(A))$$

is torsion free (see Lemma 11.5 of [21]) and $\jmath_n$ is injective. For each $z \in U_0(M_{m(n+1)}(A)/CU(M_{m(n+1)})$, there is a unitary $v \in M_{m(n+1)}(A)$ such that

$$\jmath_n(\bar{v}) = z,$$

since $\jmath_n$ is an isomorphism. By the divisibility of $U_0(M_{m(n)}(A)/CU(M_{m(n)})$, there is $u \in M_{m(n)}(A)$ such that

$$\overline{u^{k(n)}} = \overline{u}^{k(n)} = \overline{v}.$$

As above,

$$\imath_n^{\ddagger}(\bar{u}) = k(n)\jmath_n(\bar{v}) = z.$$

So $\imath_n^{\ddagger}|_{U_0(M_{m(n)}(A))/CU(M_{m(n)}(A))}$ is surjective. It follows that

$$\imath_{n,\infty}^{\ddagger}|_{U_0(M_{m(n)}(A))/CU(M_{m(n)}(A))}$$

is an isomorphism. One then concludes that $\imath^{\ddagger}|_{U_0(A)/CU(A)}$ is an isomorphism. □

LEMMA 29.2. *Let A_1 and B_1 be two unital separable simple C^*-algebras in $\mathcal{B}_0$, let $A = A_1 \otimes U_1$ and let $B = B_1 \otimes U_2$, where U_1 and U_2 are two UHF-algebras of infinite type. Let $\varphi : A \to B$ be an isomorphism and let $\beta : B \otimes M_{\mathfrak{p}} \to B \otimes M_{\mathfrak{p}}$ be an automorphism such that $\beta_{*1} = \mathrm{id}_{K_1(B \otimes M_{\mathfrak{p}})}$ for some supernatural number $\mathfrak{p}$ of infinite type. Then*

$$\psi^{\ddagger}(U(A)/CU(A)) = (\varphi_0)^{\ddagger}(U(A)/CU(A)) = U(B)/CU(B),$$

where $\varphi_0 = \imath \circ \varphi$, $\psi = \beta \circ \imath \circ \varphi$ and where $\imath : B \to B \otimes M_{\mathfrak{p}}$ is defined by $\imath(b) = b \otimes 1$ for all $b \in B$. Moreover, there is an isomorphism $\mu : U(B)/CU(B) \to U(B)/CU(B)$ with $\mu(U_0(B)/CU(B)) \subset U_0(B)/CU(B)$ such that

$$\imath^{\ddagger} \circ \mu \circ \varphi^{\ddagger} = \psi^{\ddagger} \text{ and } q_1 \circ \mu = q_1,$$

where $q_1 : U(B)/CU(B) \to K_1(B)$ is the quotient map.

PROOF. The proof is exactly the same as that of Lemma 11.3 of [40]. $\square$

LEMMA 29.3. *Let A_1 and B_1 be two unital simple amenable C^*-algebras in $\mathcal{B}_0$ satisfying the UCT, let $A = A_1 \otimes U_1$, and let $B = B_1 \otimes U_2$, where U_1 and U_2 are UHF-algebras of infinite type. Suppose that $\varphi_1, \varphi_2 : A \to B$ are two isomorphisms such that $[\varphi_1] = [\varphi_2]$ in $KK(A, B)$. Then there exists an automorphism $\beta : B \to B$ such that $[\beta] = [\mathrm{id}_B]$ in $KK(B, B)$ and $\beta \circ \varphi_2$ is asymptotically unitarily equivalent to φ_1. Moreover, if $H_1(K_0(A), K_1(B)) = K_1(B)$, then β can be chosen so that $\beta \circ \varphi_1$ and $\beta \circ \varphi_2$ are strongly asymptotically unitarily equivalent.*

PROOF. It follows from Theorem 28.7 that there is an automorphism $\beta_1 : B \to B$ satisfying the following condition:

$$[\beta_1] = [\mathrm{id}_B] \text{ in } KK(B, B), \tag{e 29.1}$$

$$\beta_1^{\ddagger} = \varphi_1^{\ddagger} \circ (\varphi_2^{-1})^{\ddagger} \text{ and } (\beta_1)_T = (\varphi_1)_T \circ (\varphi_2)_T^{-1}. \tag{e 29.2}$$

By Corollary 28.9, there is automorphism $\beta_2 \in \mathrm{Aut}(B)$ such that

$$[\beta_2] = [\mathrm{id}_B] \text{ in } KK(B, B), \tag{e 29.3}$$

$$\beta_2^{\ddagger} = \mathrm{id}_B^{\ddagger}, \ (\beta_2)_T = (\mathrm{id}_B)_T, \text{ and} \tag{e 29.4}$$

$$\overline{R}_{\mathrm{id}_B, \beta_2} = -\overline{R}_{\varphi_1, \beta_1 \circ \varphi_2} \circ (\varphi_2)_{*1}^{-1}. \tag{e 29.5}$$

Put $\beta = \beta_2 \circ \beta_1$. It follows that

$$[\beta \circ \varphi_2] = [\varphi_1] \text{ in } KK(A, B), (\beta \circ \varphi_2)^{\ddagger} = \varphi_1^{\ddagger}, \text{ and} \quad (\beta \circ \varphi_2)_T = (\varphi_1)_T. \tag{e 29.6}$$

Moreover, by 28.6,

$$\begin{aligned} \overline{R}_{\varphi_1, \beta \circ \varphi_2} &= \overline{R}_{\mathrm{id}_B, \beta_2} \circ (\varphi_2)_{*1} + \overline{R}_{\varphi_1, \beta_1 \circ \varphi_2} \\ &= (-\overline{R}_{\varphi_1, \beta_1 \circ \varphi_2} \circ (\varphi_2)_{*1}^{-1}) \circ (\varphi_2)_{*1} + \overline{R}_{\varphi_1, \beta_1 \circ \varphi_2} = 0. \end{aligned} \tag{e 29.7}$$

It follows from 28.7 that $\beta \circ \varphi_2$ and φ_1 are asymptotically unitarily equivalent.

In the case that $H_1(K_0(A), K_1(B)) = K_1(B)$, it follows from Theorem 28.13 that $\beta \circ \varphi_2$ and φ_1 are strongly asymptotically unitarily equivalent. $\square$

LEMMA 29.4. *Let A_1 and B_1 be two unital simple amenable C^*-algebras in $\mathcal{B}_0$ satisfying the UCT and let $A = A \otimes U_1$ and $B = B_1 \otimes U_2$ for UHF-algebras U_1 and U_2 of infinite type. Let $\varphi : A \to B$ be an isomorphism. Suppose that $\beta \in \mathrm{Aut}(B \otimes M_{\mathfrak{p}})$ is such that*

$$[\beta] = [\mathrm{id}_{B \otimes M_{\mathfrak{p}}}] \ \textit{in} \ KK(B \otimes M_{\mathfrak{p}}, B \otimes M_{\mathfrak{p}}) \ \textit{and} \ \beta_T = (\mathrm{id}_{B \otimes M_{\mathfrak{p}}})_T$$

for some supernatural number $\mathfrak{p}$ of infinite type.

Then there exists an automorphism $\alpha \in Aut(B)$ with $[\alpha] = [\mathrm{id}_B]$ in $KK(B, B)$ such that $\imath \circ \alpha \circ \varphi$ and $\beta \circ \imath \circ \varphi$ are asymptotically unitarily equivalent, where $\imath : B \to B \otimes M_{\mathfrak{p}}$ is defined by $\imath(b) = b \otimes 1$ for all $b \in B$.

PROOF. It follows from Lemma 29.2 that there is an isomorphism $\mu : U(B)/CU(B) \to U(B)/CU(B)$ such that

$$\imath^{\ddagger} \circ \mu \circ \varphi^{\ddagger} = (\beta \circ \imath \circ \varphi)^{\ddagger}.$$

Note that $\imath_T : T(B \otimes M_{\mathfrak{p}}) \to T(B)$ is an affine homeomorphism.

It follows from Theorem 28.7 that there is an automorphism $\alpha : B \to B$ such that

$$\begin{aligned}
&[\alpha] = [\mathrm{id}_B] \text{ in } KK(B,B),\\
&\alpha^{\ddagger} = \mu, \;\; \alpha_T = (\beta \circ \imath \circ \varphi)_T \circ ((\imath \circ \varphi)_T)^{-1} = (\mathrm{id}_{B \otimes M_{\mathfrak{p}}})_T \text{ and}\\
&\overline{R}_{\mathrm{id}_B, \alpha}(x)(\tau) = -\overline{R}_{\beta\circ\imath\circ\varphi,\, \imath\circ\varphi}(\varphi_{*1}^{-1}(x))(\imath_T(\tau)) \text{ for all } x \in K_1(A)
\end{aligned}$$

and for all $\tau \in T(B)$.

Denote by $\psi = \imath \circ \alpha \circ \varphi$. Then we have, by Lemma 28.6,

$$\begin{aligned}
&[\psi] = [\imath \circ \varphi] = [\beta \circ \imath \circ \varphi] \text{ in } KK(A, B \otimes M_{\mathfrak{p}})\\
&\psi^{\ddagger} = \imath^{\ddagger} \circ \mu \circ \varphi^{\ddagger} = (\beta \circ \imath \circ \varphi)^{\ddagger}, \text{ and}\\
&\psi_T = (\imath \circ \alpha \circ \varphi)_T = (\imath \circ \varphi)_T = (\beta \circ \imath \circ \varphi)_T.
\end{aligned}$$

Moreover, for any $x \in K_1(A)$ and $\tau \in T(B \otimes M_{\mathfrak{p}})$,

$$\begin{aligned}
&\overline{R}_{\beta\circ\imath\circ\varphi,\psi}(x)(\tau)\\
&= \overline{R}_{\beta\circ\imath\circ\varphi,\imath\circ\varphi}(x)(\tau) + \overline{R}_{\imath,\imath\circ\alpha} \circ \varphi_{*1}(x)(\tau)\\
&= \overline{R}_{\beta\circ\imath\circ\varphi,\imath\circ\varphi}(x)(\tau) + \overline{R}_{\mathrm{id}_B,\imath\circ\alpha} \circ \varphi_{*1}(x)(\imath_T^{-1}(\tau))\\
&= \overline{R}_{\beta\circ\imath\circ\varphi,\imath\circ\varphi}(x)(\tau) - \overline{R}_{\beta\circ\imath\circ\varphi,\imath\circ\varphi}(\varphi_{*1}^{-1})(\varphi_{*1}(x))(\tau) = 0.
\end{aligned}$$

It follows from Theorem 27.5 that $\imath\circ\alpha\circ\varphi$ and $\beta\circ\imath\circ\varphi$ are asymptotically unitarily equivalent. □

Let $\mathcal{N}$ be the class of separable amenable C^*-algebras which satisfy the UCT.

THEOREM 29.5. *Let A and B be two unital separable simple C^*-algebras in $\mathcal{N}$. Suppose that there is an isomorphism*

$$\Gamma : \mathrm{Ell}(A) \to \mathrm{Ell}(B).$$

Suppose also that, for some pair of relatively prime supernatural numbers $\mathfrak{p}$ and $\mathfrak{q}$ of infinite type such that $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} \cong Q$, we have $A \otimes M_{\mathfrak{p}} \in \mathcal{B}_0$, $B \otimes M_{\mathfrak{p}} \in \mathcal{B}_0$, $A \otimes M_{\mathfrak{q}} \in \mathcal{B}_0$, and $B \otimes M_{\mathfrak{q}} \in \mathcal{B}_0$. Then,

$$A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}.$$

PROOF. The proof is almost identical to that of 11.7 of [40], with a few necessary modifications. Note that Γ induces an isomorphism

$$\Gamma_{\mathfrak{p}} : \mathrm{Ell}(A \otimes M_{\mathfrak{p}}) \to \mathrm{Ell}(B \otimes M_{\mathfrak{p}}).$$

Since $A \otimes M_{\mathfrak{p}} \in \mathcal{B}_0$ and $B \otimes M_{\mathfrak{p}} \in \mathcal{B}_0$, by Theorem 21.10 of [21], there is an isomorphism $\varphi_{\mathfrak{p}} : A \otimes M_{\mathfrak{p}} \to B \otimes M_{\mathfrak{p}}$. Moreover (by the proof of Theorem 21.10 of [21]), $\varphi_{\mathfrak{p}}$ carries $\Gamma_{\mathfrak{p}}$. In the same way, Γ induces an isomorphism

$$\Gamma_{\mathfrak{q}} : \mathrm{Ell}(A \otimes M_{\mathfrak{q}}) \to \mathrm{Ell}(B \otimes M_{\mathfrak{q}})$$

and there is an isomorphism $\psi_{\mathfrak{q}} : A \otimes M_{\mathfrak{q}} \to B \otimes M_{\mathfrak{q}}$ which induces $\Gamma_{\mathfrak{q}}$.

Put $\varphi = \varphi_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}} : A \otimes Q \to B \otimes Q$ and $\psi = \psi_{\mathfrak{q}} \otimes \mathrm{id}_{M_{\mathfrak{p}}} : A \otimes Q \to B \otimes Q$. Note that

$$(\varphi)_{*i} = (\psi)_{*i} \ (i = 0, 1) \ \text{ and } \ \varphi_T = \psi_T$$

(all four of these maps are induced by Γ). Note that φ_T and ψ_T are affine homeomorphisms. Since $K_{*i}(B \otimes Q)$ is divisible, we in fact have $[\varphi] = [\psi]$ (in $KK(A\otimes Q, B\otimes Q)$). It follows from Lemma 29.3 that there is an automorphism $\beta : B \otimes Q \to B \otimes Q$ such that

$$[\beta] = [\mathrm{id}_{B\otimes Q}] \ \text{ in } KK(B \otimes Q, B \otimes Q)$$

and such that φ and $\beta\circ\psi$ are asymptotically unitarily equivalent. Since $K_1(B\otimes Q)$ is divisible, $H_1(K_0(A \otimes Q), K_1(B \otimes Q)) = K_1(B \otimes Q)$. It follows that φ and $\beta \circ \psi$ are strongly asymptotically unitarily equivalent. Note also in this case

$$\beta_T = (\mathrm{id}_{B\otimes Q})_T.$$

Let $\imath : B \otimes M_{\mathfrak{q}} \to B \otimes Q$ be defined by $\imath(b) = b \otimes 1$ for $b \in B$. We consider the pair $\beta \circ \imath \circ \psi_{\mathfrak{q}}$ and $\imath \circ \psi_{\mathfrak{q}}$. Applying Lemma 29.4, we obtain an automorphism $\alpha : B\otimes M_{\mathfrak{q}} \to B\otimes M_{\mathfrak{q}}$ such that $\imath\circ\alpha\circ\psi_{\mathfrak{q}}$ and $\beta\circ\imath\circ\psi_{\mathfrak{q}}$ are asymptotically unitarily equivalent (in $B \otimes Q$). So, by Lemma 29.3, they are strongly asymptotically unitarily equivalent in $B \otimes Q$. Moreover,

$$[\alpha] = [\mathrm{id}_{B\otimes M_{\mathfrak{q}}}] \ \text{ in } \ KK(B \otimes M_{\mathfrak{q}}, B \otimes M_{\mathfrak{q}}).$$

We will show that $\beta \circ \psi$ and $(\alpha \circ \psi_{\mathfrak{q}}) \otimes \mathrm{id}_{M_{\mathfrak{p}}}$ are strongly asymptotically unitarily equivalent. Define $\beta_1 = (\beta\circ\imath\circ\psi_{\mathfrak{q}})\otimes\mathrm{id}_{M_{\mathfrak{p}}} : B\otimes Q\otimes M_{\mathfrak{p}} \to B\otimes Q\otimes M_{\mathfrak{p}}$. Let $j : Q \to Q \otimes M_{\mathfrak{p}}$ be defined by $j(b) = b \otimes 1$. There is an isomorphism $s : M_{\mathfrak{p}} \to M_{\mathfrak{p}} \otimes M_{\mathfrak{p}}$ such that the homomorphism $\mathrm{id}_{M_{\mathfrak{q}}} \otimes s : M_{\mathfrak{q}} \otimes M_{\mathfrak{p}}(= Q) \to M_{\mathfrak{q}} \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{p}}(= Q \otimes M_{\mathfrak{p}})$ induces $(\mathrm{id}_{M_{\mathfrak{q}}} \otimes s)_{*0} = j_{*0}$. In this case, $[\mathrm{id}_{M_{\mathfrak{q}}} \otimes s] = [j]$. Since $K_1(M_{\mathfrak{p}}) = 0$, by Theorem 27.5, $\mathrm{id}_{M_{\mathfrak{q}}} \otimes s$ is strongly asymptotically unitarily equivalent to j. It follows that $(\alpha \circ \psi_{\mathfrak{q}}) \otimes \mathrm{id}_{M_{\mathfrak{p}}}$ and $(\beta \circ \imath \circ \psi_{\mathfrak{q}}) \otimes \mathrm{id}_{M_{\mathfrak{p}}}$ are strongly asymptotically unitarily equivalent (note that $\imath\circ\alpha\circ\psi_{\mathfrak{q}}$ and $\beta\circ\imath\circ\psi_{\mathfrak{q}}$ are strongly asymptotically unitarily equivalent). Consider the C^*-subalgebra $C = \beta \circ \psi(1 \otimes M_{\mathfrak{p}}) \otimes M_{\mathfrak{p}} \subset B \otimes Q \otimes M_{\mathfrak{p}}$. In C, $\beta \circ \varphi|_{1 \otimes M_{\mathfrak{p}}}$ and j_0 are strongly asymptotically unitarily equivalent, where $j_0 : M_{\mathfrak{p}} \to C$ is defined by $j_0(a) = 1 \otimes a$ for all $a \in M_{\mathfrak{p}}$. In particular, there exists a continuous path of unitaries $\{v(t) : t \in [0, \infty)\} \subset C$ such that

$$\lim_{t \to \infty} \mathrm{Ad}\, v(t) \circ \beta \circ \varphi(1 \otimes a) = 1 \otimes a \ \text{ for all } \ a \in M_{\mathfrak{p}}. \tag{e 29.8}$$

It follows that $\beta \circ \psi$ and β_1 are strongly asymptotically unitarily equivalent. Therefore $\beta \circ \psi$ and $(\alpha \circ \psi_{\mathfrak{q}}) \otimes \mathrm{id}_{M_{\mathfrak{p}}}$ are strongly asymptotically unitarily equivalent. Finally, we conclude that $(\alpha \circ \psi_{\mathfrak{q}}) \otimes \mathrm{id}_{M_{\mathfrak{p}}}$ and φ are strongly asymptotically unitarily equivalent. Note that $\alpha \circ \psi_{\mathfrak{q}}$ is an isomorphism which induces $\Gamma_{\mathfrak{q}}$.

Let $\{u(t) : t \in [0, 1)\}$ be a continuous path of unitaries in $B \otimes Q$ with $u(0) = 1_{B \otimes Q}$ such that

$$\lim_{t \to \infty} \mathrm{Ad}\, u(t) \circ \varphi(a) = \alpha \circ \psi_{\mathfrak{q}} \otimes \mathrm{id}_{M_{\mathfrak{p}}}(a) \ \text{ for all } \ a \in A \otimes Q.$$

One then obtains a unitary suspended isomorphism which lifts Γ along $Z_{p,q}$ (see [56]). It follows from Theorem 7.1 of [56] that $A \otimes \mathcal{Z}$ and $B \otimes \mathcal{Z}$ are isomorphic. □

DEFINITION 29.6. Denote by $\mathcal{N}_0$ the class of those unital simple C^*-algebras A in $\mathcal{N}$ for which $A \otimes M_{\mathfrak{p}} \in \mathcal{N} \cap \mathcal{B}_0$ for any supernatural number $\mathfrak{p}$ of infinite type.

Of course $\mathcal{N}_0$ contains all unital simple amenable C^*-algebras in $\mathcal{B}_0$ which satisfy the UCT. It contains all unital simple inductive limits of C^*-algebras in $\mathcal{C}_0$. It should be noted that, by Theorem 19.3 of [21], $\mathcal{N}_0 = \mathcal{N}_1$.

COROLLARY 29.7. *Let A and B be two C^*-algebras in $\mathcal{N}_0$. Then $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$ if and only if* $\mathrm{Ell}(A \otimes \mathcal{Z}) \cong \mathrm{Ell}(B \otimes \mathcal{Z})$.

PROOF. This follows from Theorem 29.5 immediately. □

THEOREM 29.8. *Let A and B be two unital separable simple amenable $\mathcal{Z}$-stable C^*-algebras which satisfy the UCT. Suppose that $gTR(A \otimes Q) \le 1$ and $gTR(B \otimes Q) \le 1$. Then $A \cong B$ if and only if*

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

PROOF. It follows from Corollary 19.3 of [21] that $A \otimes U, B \otimes U \in \mathcal{B}_0$ for any UHF-algebra U of infinite type. The theorem follows immediately by Corollary 29.7. □

COROLLARY 29.9. *Let A and B be two unital separable amenable simple C^*-algebras which satisfy the UCT. Suppose that $gTR(A) \le 1$ and $gTR(B) \le 1$. Then $A \cong B$ if and only if*

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

COROLLARY 29.10. *Let A and B be two unital simple C^*-algebras in $\mathcal{B}_1 \cap \mathcal{N}$. Then $A \cong B$ if and only if*

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

PROOF. It follows from Theorem 10.7 of [21] that $A \otimes \mathcal{Z} \cong A$ and $B \otimes \mathcal{Z} \cong B$. The corollary then follows from Theorem 29.8. □

REMARK 29.11. Soon after this work was completed in 2015, it was shown (see [16]) C^*-algebras A with finite decomposition rank which satisfy the UCT have $gTR(A \otimes U) \leq 1$ for all UHF-algebras of infinite type. Therefore, by Theorem 29.8, they are classified by the Elliott invariant.

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