Classifying embeddings of C*-algebras

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Indigenous affirmation

I pay respect to the Algonquin people, who are the traditional guardians of this land. I acknowledge their longstanding relationship with this territory, which remains unceded. I pay respect to all Indigenous people in this region, from all nations across Canada, who call Ottawa home. I acknowledge the traditional knowledge keepers, both young and old. And I honour their courageous leaders: past, present, and future.

Classifying embeddings

Definition

Let A, B be C*-algebras with B unital. Two *-homomorphisms $\phi, \psi : A \to B$ are *approximately unitarily (a.u.) equivalent* if there exist unitaries (u_i) such that
$$\|u_i\phi(a)u_i^* - \psi(a)\| \to 0 \quad \forall a \in A.$$

Classifying embeddings means finding a nice description of the a.u. equivalence classes of injective *-homomorphisms $A \to B$.

- "Nice description": ideally this means K-theory (plus...).
- Often we restrict the *-homomorphisms under consideration: nuclearity, fullness.

Why?

- Interesting in its own right.
- Leads to classification of C*-algebras (via intertwining).

History

Usually when C*-algebras are classified, some homomorphisms are classified. So this list is not exhaustive!

- Elliott ('76): classified *-homs. between AF algs., using K_0.
- Thomsen ('92): classified *-homs. between AI algebras, using a certain semigroup.
- Ciuperca–Elliott–Santiago, Robert ('08-'12): recast and extended Thomsen; classified *-homs. from certain ASH domains to stable rank 1 codomains.
- Kirchberg ('??): classified nuclear embeddings from sep. unital exact domains to simple unital p.i. codomains, up to *asymptotic* unitary equivalence, using KK-theory.
- Phillips ('00): similar result.

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- Kirchberg ('??): classified nuclear embeddings from sep. unital exact domains to simple unital p.i. codomains, up to *asymptotic* unitary equivalence, using KK-theory.
- Phillips ('00): similar result.
- Using Dadarlat–Loring (and ideas of Rørdam), get classification up to approximate unitary equivalence, using total K-theory, provided the domain also satisfies the UCT.
- Matui ('11): classified embeddings of unital [AH or rationally TAF] domains to unital sep. simple exact rationally TAF codomains, using K-theoretic invariant.
- Lin ('17): classified embeddings of unital AH domains into unital simple tracial rank one codomains.

Existence and uniqueness

Let $I : \{\text{C*-algebras}\} \to \mathcal{E}$ be a functor which collapses a.u. equivalence classes (e.g., K-theory and traces).

Definition

Let A, B be C*-algebras. The embeddings from A to B are *classified* by the invariant I if for every $\alpha \in \mathrm{Hom}(I(A), I(B))$:

(i) there is an embedding $\phi : A \to B$ such that $I(\phi) = \alpha$ (Existence), and

(ii) the embedding ϕ is unique up to approximate unitary equivalence (Uniqueness).

Isomorphism via intertwining

Theorem (intertwining argument)

Let A, B be separable C*-algebras. If there exist *-homomorphisms $\phi : A \to B$ and $\psi : B \to A$ such that both $\phi \circ \psi$ and $\psi \circ \phi$ are approximately unitarily equivalent to the identity, then $A \cong B$.

If A, B have isomorphic invariants and we can classify embeddings between A and B then:

- Existence provides *-homomorphisms ϕ, ψ realizing the isomorphism $I(A) \cong I(B)$.
- Uniqueness tells us that $\psi \circ \phi$ is a.u. equivalent to id_A.
- Likewise $\phi \circ \psi$ is a.u. equivalent to id_B.
- Hence by the intertwining argument, $A \cong B$.

Isomorphism via intertwining

Caveat: the invariant needed to classify *-homs. is bigger than the invariant used to classify C*-algebras.

This is okay, because the smaller invariant determines the bigger invariant on objects – but not in a natural way, and therefore it doesn't determine it on morphisms.

Localized versions

There are localized versions of classifying *-homs., which can often be expressed using many quantifiers:

Prototype approximate uniqueness

Given a finite set $\mathcal{F} \subset A$ and $\epsilon > 0$, there exists a finite set $\mathcal{G} \subset A$ and $\delta > 0$ and a finite set $\mathcal{H} \subset I(A)$, such that if $\phi, \psi : A \to B$ are $(\mathcal{G}, \delta)$-approximately multiplicative *-linear maps, and $I(\phi), I(\psi)$ (which are only partially defined) agree on $\mathcal{H}$, then there is a unitary $u \in A$ such that

$$\|\phi(a) - u\psi(a)u^*\| < \epsilon, \quad \forall a \in \mathcal{F}.$$

Equivalent approach: classify into a sequence algebra

$$A_\infty := l_\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A).$$

Localized versions: a means to an end

Another intertwining argument allows passing from approximate existence/uniqueness to (exact) existence:

Theorem (intertwining argument)

Let A, B be separable C*-algebras with B unital. If $\phi : A \to B_\infty$ is a *-homomorphism that is a.u. equivalent to any reparametrization of itself, then ϕ is a.u. equivalent to a *-homomorphism $A \to B$.

If we can classify *-homs. $A \to B_\infty$ then given $\alpha \in \mathrm{Hom}(I(A), I(B))$:

- Existence provides a *-homomorphism $\phi : A \to B_\infty$ lifting $I(\iota_{B\subseteq B_\infty}) \circ \alpha$.
- Uniqueness tells us that ϕ is a.u. equivalent to any reparametrization of itself.
- Hence we get a *-homomorphism $\psi : A \to B$.
- If $I(\iota_{B\subseteq B_\infty})$ is injective then ψ lifts α.

The classification-of-embeddings theorem

Theorem (CGSTW)

Let A be a separable exact C*-algebra which satisfies the UCT.

Let B be a separable $\mathcal{Z}$-stable C*-algebra with $T(B)$ compact and with strict comparison with respect to traces.

Then the full nuclear *-homomorphisms from A to B (or B_∞) are classified by total K-theory, traces, and hausdorffized unitary algebraic K-theory.

Corollary (Elliott–Gong–Lin–Niu, ...)

Simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebras satisfying the UCT are classified up to isomorphism, by K-theory paired with traces.

The invariant

To discuss the invariant in more detail, let us stick to the unital case. Then the invariant $\underline{K}T_u(A)$ consists of:

(i) K-theory: $K_0(A), K_1(A)$, along with $[1_A] \in K_0(A)$;

(ii) Traces: $T(A)$;

(iii) Total K-theory $\underline{K}(A)$ (a.k.a. K-theory with coefficients): $K_i(A; \mathbb{Z}/n) := K_{1-i}(A \otimes \mathbb{I}_n)$ where $\mathbb{I}_n$ is a nuclear UCT C*-algebra satisfying $K_*(\mathbb{I}_n) = 0 \oplus \mathbb{Z}/n$ (plus Bockstein maps $\mu_{i,A}^{(n)}, \nu_{i,A}^{(n)}, \kappa_{i,A}^{(m,n)}$);

(iv) Hausdorffized unitary algebraic K_1: $\overline{K}_1^{\mathrm{alg}}(A) := \bigcup_n U_n(A) / \bigcup_n \overline{DU_n(A)}$;

(v) Maps $K_0(A) \xrightarrow{\rho_A} \mathrm{Aff}(T(A)) \xrightarrow{\mathrm{Th}_A} \overline{K}_1^{\mathrm{alg}}(A) \xrightarrow{\not{a}_A} K_1(A)$;

(vi) Maps $K_0(A; \mathbb{Z}/n) \xrightarrow{\zeta_A^{(n)}} \overline{K}_1^{\mathrm{alg}}(A)$.

$\zeta_A^{(n)}$ is a natural map $K_0(A;\mathbb{Z}/n) \to \overline{K}_1^{\mathrm{alg}}(A)$, readily constructed when we take $\mathbb{I}_n$ to be a dimension drop algebra.

Properties:

- The Bockstein map $\nu_0^{(n)} : K_0(A;\mathbb{Z}/n) \to K_1(A)$ factorizes as
$$K_0(A;\mathbb{Z}/n) \xrightarrow{\zeta_A^{(n)}} \overline{K}_1^{\mathrm{alg}}(A) \xrightarrow{a_A} K_1(A).$$
- For a projection $p \in M_m(A)$,
$$\zeta_A^{(n)}([p]_{K_0(A;\mathbb{Z}/n)}) = [e^{2\pi i p/n}]_{\overline{K}_1^{\mathrm{alg}}}.$$
- $\mathrm{Tor}(\overline{K}_1^{\mathrm{alg}}(A)) = \bigcup_n \mathrm{Im}(\zeta_A^{(n)})$.

The trace-kernel extension

Our argument combines von Neumann- and lifting-techniques.

Definition

Let B be a unital C*-alg. with $T(B) \neq \emptyset$. The *trace-kernel ideal* is
$$J_B := \{(a_n)_{n=1}^{\infty} : \lim_{n\to\infty} \max_{\tau\in T(B)} \tau(a^*a) = 0\} \subseteq B_\infty.$$

The *trace-kernel extension* is

$$0 \to J_B \to B_\infty \to B^\infty := B_\infty / J_B \to 0.$$

Von Neumann algebraic techniques: B^∞ has von Neumann "fibres", whose nice behaviour can be "glued".

Lifting techniques: We use KK-theory to lift existence/uniqueness from $A \to B^\infty$ to $A \to B_\infty$.

Existence

Say we have $\alpha \in \mathrm{Hom}(\underline{K}T_u(A), \underline{K}T_u(B_\infty))$.

This means α consists of maps $\alpha_{K_i}, \alpha_T, \alpha_{K_i(\cdot,\mathbb{Z}/n)}, \alpha_{\overline{K}_1^{\mathrm{alg}}}$ between the different components. They intertwine the various maps $(\mu, \nu, \kappa, \rho, \mathrm{Th}, \not{\mathrm{a}}, \zeta)$.

Step 1

Use von Neumann algebraic structure of B^∞ to find a *-hom. $A \to B^\infty$ which lifts α_T. (Note that $T(B^\infty) = T(B_\infty)$.)

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$$\begin{array}{ccc} A & \dashrightarrow & B_\infty \\ & \searrow & \downarrow \\ & & B^\infty \end{array}$$

Using the multi-UCT, the maps $\alpha_{K_i}, \alpha_{K_i(\cdot,\mathbb{Z}/n)}$ give us a KK-lift.

Step 2

Use KK-existence to get $\phi : A \to B_\infty$ which lifts $\alpha_T, \alpha_{K_i}, \alpha_{K_i(\cdot,\mathbb{Z}/n)}$.

Existence

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A computation shows that the difference between $\alpha_{\overline{K}_1^{\mathrm{alg}}}$ and $\overline{K}_1^{\mathrm{alg}}(\phi)$ factors through a rotation map

$$K_1(A)/\mathrm{Tor}(K_1(A)) \to \ker(\not{a}_B) \subseteq \overline{K}_1^{\mathrm{alg}}(B).$$

ζ plays a role here! This can be encoded as $x \in KK(A, J_B)$.

Step 3

Use a different KK-existence to get $\psi : A \to B_\infty$ such that ϕ, ψ is a Cuntz pair and $[\psi, \phi] = x$.
Consequently, ψ lifts α.