Modular curvatures for toric noncommutative manifolds

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Noncommutative spaces

In noncommutative geometry, a geometric space is implemented by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$:

- The algebra $\mathcal{A}$ represents the "coordinate functions" on the underlying space, elements in $\mathcal{A}$ are bounded operators on $\mathcal{H}$ that do not necessary commute with each other as in quantum physics.
- D is an self-adjoint unbounded operator on $\mathcal{H}$ with the first order condition: all the commutators $[a, D]$ are bounded where $a \in \mathcal{A}$.

A typical example is the spectral triple for Dirac model:

$$(C^\infty(M), L^2(\not{S}^-) \oplus L^2(\not{S}^+), \not{D}),$$

where M is a closed Spin manifold with spinor bundle $S = S^+ \oplus S^-$, and $\not{D}$ is the associated Dirac operator.

Global geometry

General algebraic-topological and analytical tools for global treatment of the usual spaces have been successfully adapted and upgraded to the noncommutative context, such as:

- K-theory;
- cyclic cohomology;
- Morita equivalence;
- operator-theoretic index theorems;
- Hopf algebra symmetry, etc.

Local Geometry

- By contrast, the fundamental local geometric concepts, in particular, the notion of intrinsic curvature, which lies at the very core of geometry, has only recently begun to be comprehended via the study of modular geometry on noncommutative two tori.
- Proposed in A. Connes and H. Moscovici's recent work(2014) "It is the high frequency behavior of the spectrum of D coupled with the action of the algebra $\mathcal{A}$ in $\mathcal{H}$ which detects the local curvature of the geometry."

Spectral Geometry

On a closed Riemannian manifold M, let Δ be a Laplacian type operator, the Schwartz kernel of the heat operator operator has the following asymptotic expansion on the diagonal:

$$e^{-t\Delta}(x,x,t) \backsim_{t\searrow 0} \sum_{j\geq 0} V_j(x) t^{(j-d)/2}, \;\; d = \dim M.$$

The coefficients V_j are polynomial functions on in the curvature tensor and its covariant derivatives.
For example, let Δ be the scalar Laplacian, then upto a factor $(4\pi)^{-d/2}$:

$$V_2(x,\Delta) = \frac{1}{6} S_\Delta$$
$$V_4(x,\Delta) = \frac{1}{360}\left(-12\Delta S_\Delta + 5S_\Delta^2 - 2\left|\mathrm{Ric}\right|^2 + 2\left|R\right|^2\right)$$

here S_Δ is the scalar curvature function, Ric and R are the Ricci curvature tensor and the full curvature tensor respectively.

Scalar curvature functional

The diagonal of the heat kernel $e^{-t\Delta}(x, x, t)$ does not make sense for our noncomutative spaces. The operator-theoretic counterpart is the trace functional

$$f \mapsto \mathrm{Tr}(fe^{-t\Delta}), \ \forall f \in C^\infty(M).$$

As before, it has an asymptotic expansion as $t \to 0$

$$f \mapsto \mathrm{Tr}\left(fe^{-t\Delta}\right) \backsim_{t \searrow 0} \sum_{j \geq 0} V_j(f, \Delta) t^{(j-d)/2}, \quad d = \dim M, \ f \in C^\infty(M).$$

Definition

If we take the Laplacian operator Δ as the definition of a "Riemannian metric". The we will call the functional density $\mathcal{R} \in C^\infty(M)$ of the second heat coefficient functional

$$V_j(f, \Delta) = \int_M f\mathcal{R}, \ f \in C^\infty(M)$$

as the associated scalar curvature.

Guassian Equations

Conformal change of metric $g' = e^{-h} g$,

- the Laplacian operators are linked by

$$\Delta_{g'} = e^{-h} \Delta_g .$$

- the Gaussian curvatures are related by the Guassian equation

$$(2\Delta_g(h) + K_g)e^h = K_{g'} .$$

Yamabe Equations

Let $n = \dim M \geq 3$. Conformal change of metric: $g_u = u^{\frac{4}{n-2}} g$ for some positve function u. The scalar curvatures R_{g_u} and R_g are related by the Yamabe equation

$$L_g u = \frac{n-2}{4(n-1)} R_{g_u} u^{\frac{n+2}{n-2}},$$

where

$$L_g = -\Delta_g + \frac{n-2}{4(n-1)} R_g$$

is the conformal Laplacian operator defined on (M, g) with $n \geq 3$.

In dimension four: under the conformal change of metric: $g_u = u^2 g$, the scalar curvatures are related as follows:

$$R_{g_u} = -6u^{1/3}(\Delta_g u) + u^{1/3} R_g.$$

Conformal change of metric in noncommutative setting

- In Riemannian geometry, the Hilbert spaces of $L^2(M, g)$ of L^2-functions depends on the metric g.
- When a family of metrics is considered, for instance, when studying variation problems, we often choose to fix the Hilbert space.
- The price to pay is a purturbation of the Laplacian operator.

Conformal perturbation of the Laplacian operator

Now on our noncommutative spaces, the conformal factor $k = e^h$ is implemented by exponentiate a self-adjoint operator h. The resulting operator k is invertible and positive. The new Laplacian, upto a conjugation by k is of the form:

$$\Delta_k = k\Delta + \text{lower order terms.}$$

Toric manifolds

Let M be a smooth manifold and $\mathbb{T}^n \subset \mathrm{Diff}(M)$. Then $C^\infty(M)$ be come a smooth $\mathbb{T}^n$-module via the pull-back action:

$$(U_t(f))(x) \triangleq f(t^{-1} \cdot x), \; x \in M, f(x) \in C^\infty(M), t \in \mathbb{T}^n. \tag{1}$$

The notation U_t stands for "unitary" because later we will assume that the torus acts on M as isometries, then U_t admits a unitary extension to $L^2(M)$. The smoothness means that for any fixed $f \in C^\infty(M)$, the function $t \mapsto U_t(f)$ belongs to $C^\infty(\mathbb{T}^n, C^\infty(M))$. By Fourier theory on $\mathbb{T}^n$, any elements in $C^\infty(M)$ has a isotypical decomposition: let $\mathbb{T}^n \cong \mathbb{R}^n/2\pi\mathbb{Z}^n$,

$$f = \sum_{r \in \mathbb{Z}^n} f_r, \;\; f_r = \int_{\mathbb{T}^n} e^{-2\pi i r \cdot t} U_t(f) dt \tag{2}$$

Deformation of $C^\infty(M)$

- Given a $n \times n$ skew symmetric matrix Θ, we denote a bicharacter: $\chi_\Theta : \mathbb{Z}^n \times \mathbb{Z}^n \to S^1$: $\chi_\Theta(r, l) = \langle r, \Theta l \rangle$. We deformed $C^\infty(M)$ with respect to Θ, the resulting new algebra is denoted by

$$C^\infty(M_\Theta) = (C^\infty(M), \times_\Theta)$$

which is identical to $C^\infty(M)$ as a topological vector space with the pointwise multiplication replaced via a twisted convolution:

$$f \times_\Theta g \triangleq \sum_{r,l \in \mathbb{Z}^n} \chi_\Theta(r, l) f_r g_l, \quad f, g \in C^\infty(M), \tag{3}$$

where f_r, g_l are isotypical components of f and g.

- Since the torus action can be quickly extends to the cotangent bundle T^*M, the deformed algebra is defined in a similar way:

$$C^\infty(T^*M_\Theta) = (C^\infty(T^*M), \times_\Theta)$$

Noncommutative two tori

- Let $\Theta = \begin{pmatrix} 0 & -\theta/2 \\ \theta/2 & 0 \end{pmatrix}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$.
- Consider $\mathbb{T}^2$ acts on itself via translations:

$$t \cdot (e^{2\pi i s_1}, e^{2\pi i s_2}) = (e^{2\pi i (s_1 - t_1)}, e^{2\pi i (s_2 - t_2)}), \quad t = (t_1, t_2) \in \mathbb{R}^2.$$

- Take $u = e^{2\pi i s_1}$ and $v = e^{2\pi i s_1}$ in $C^\infty(\mathbb{T}^2)$, thus

$$U_t(u) = e^{2\pi i t_1} u, \quad U_t(v) = e^{2\pi i t_2} v.$$

- We recover the noncommutative relation which defines the commutative two torus:

$$u \times_\Theta v = e^{2\pi i \theta} v \times_\Theta u.$$

- The deformed algebra $C^\infty(T^2_\Theta)$ is called a smooth noncommutative two torus.

Noncommtutative four spheres

- Let $\Theta = \begin{pmatrix} 0 & -\theta/2 \\ \theta/2 & 0 \end{pmatrix}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$.
- Let $\mathbb{T}^2$ act on $\mathbb{R}^5$ via rotations on the first four components, namely,

$$t = (t_1, t_2) \mapsto \begin{pmatrix} e^{2\pi i t_1} & & \\ & e^{2\pi i t_2} & \\ & & 1 \end{pmatrix} \in SO(5),$$

  the induced action on S^4 gives rise to the noncommutative four sphere $C^\infty(S^4_\theta)$.

Deforming operators

Now we assume that the torus acts on M as isometries: $\mathbb{T}^n \subset \mathrm{Iso}(M)$ and M is a closed Riemannian manifold. Let $\mathcal{H} = L^2(M)$. $\forall t \in \mathbb{T}^n$, U_t extends to a unitary operator on $\mathcal{H}$

- Observation: the representation $C^\infty(M) \subset B(\mathcal{H})$ of left-multiplication is equivariant:

$$L_{U_t(f)} = U_t L_f U_t^{-1}, \;\; f \in C^\infty(M).$$

- We impose a $\mathbb{T}^n$-module on $B(\mathcal{H})$ via the adjoint action:

$$\mathrm{Ad}_t : B(\mathcal{H}) \to B(\mathcal{H}) : P \mapsto U_t P U_t^{-1}, \;\; t \in \mathbb{T}^n.$$

- For any $g \in C^\infty(M)$, we define the deformed operator $\pi^\Theta(L_f)$

$$\pi^\Theta(L_f)(g) \triangleq \sum_{r,l \in \mathbb{Z}^n} \chi_\Theta(r,l)(L_f)_r g_l,$$

  which recovers the left $\times_\Theta$-multiplication.

Deformation of tensor calculus

- Take f and g in the previous page to be vector fields or one-forms, we can deform the tensor product $X \otimes Y$ and the contraction $X \cdot \omega$ into $X \otimes_\Theta Y$ and $X \cdot_\Theta \omega$ with mixed assocativity: $(X \otimes_\Theta Y) \cdot_\Theta \omega = X \otimes_\Theta (Y \cdot_\Theta \omega)$.
- Assume $\mathbb{T}^n \subset \mathrm{Iso}(M)$, let $\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ be the Levi-Civita connection, one can check that ∇ is $\mathbb{T}^n$-equivariant, as a consequence, we gain the Leibniz property in the deformed setting:

$$\nabla(X \otimes_\Theta Y) = (\nabla X) \otimes_\Theta Y + X \otimes_\Theta (\nabla Y)$$
$$\nabla(X \cdot_\Theta \omega) = (\nabla X) \cdot_\Theta \omega + X \cdot_\Theta (\nabla \omega).$$

Modular scalar curvature

Let $\triangle(\cdot) = k^{-1}(\cdot)k$ be the modular operator. Then the modular scalar curvature is of the form (upto a constant factor $2\mathrm{Vol}(S^{m-1})$):

$$k^{-m/2}\mathcal{K}_m(\triangle)(\Delta k) + k^{-m/2-1}\mathcal{G}_m(\triangle_{(1)}, \triangle_{(2)})\left((\nabla k)(\nabla k)\right) g^{-1} + C_m k^{-m/2+1} S_\Delta$$

(1) $m = \dim M$;

(2) S_Δ is the scalar curvature function associated to the Riemannian metric g^{-1} on T^*M.

(3) C_m is a constant:

(4) the modular curvature functions $\mathcal{K}_m(s)$, $\mathcal{G}_m(s,t)$ are the new ingredients coming from noncommutative geometry;

(5) $\triangle_{(j)}$, $j = 1, 2$ indicates that the operator acts on the j-th factor; Say if $\mathcal{G}(s,t) = st$, then the action becomes $\triangle(\nabla k) \triangle (\nabla k)$.

Known features about those modular curvature functions

In the Gaussian curvature of NC two tori (Connes and Moscovici 2014):

- $\tilde{K}_0$ is (upto a factor $1/8$) the generating function of Bernoulli numbers:

$$\frac{1}{8}\tilde{K}_0(s) = \frac{t}{e^s - 1}.$$

- $\tilde{H}_0(s, t)$ can be expressed via $\tilde{K}_0(s)$ as follows:

$$-\frac{1}{2}\tilde{H}_0(s_1, s_2) = \frac{\tilde{K}_0(s_2) - \tilde{K}_0(s_1)}{s_1 + s_2} + \frac{\tilde{K}_0(s_1 + s_2) - \tilde{K}_0(s_2)}{s_1} - \frac{\tilde{K}_0(s_1 + s_2) - \tilde{K}_0(s_1)}{s_2}$$

Some literature

- Connes, A., *C^*-algebres et géométrie différentielle*, CR Acad. Sci. Paris Sr. AB, 1980;
- Cohen, PB and Connes, Alain, *Conformal geometry of the irrational rotation algebra*, preprint, 1992;
- Connes, Alain and Tretkoff, Paula, *The Gauss-Bonnet theorem for the noncommutative two torus*, incollection: *Noncommutative geometry, arithmetic, and related topics*, 2011;
- Alain Connes and Henri Moscovici, *Modular curvature for noncommutative two-tori*, J. Amer. Math. Soc. 27 (2014).

Modular geometry on NC two tori with coefficients (Heisenberg modules):

- Matthias Lesch and Henri Moscovici, *Modular curvature and Morita equivalence*, arXiv:1505.00964.

The compuatation for the Gauss-Bonnet theorem and explicity expression of the modular curvature were carried out independently by aother team Farzad Fathizadeh and Masoud Khalkhali via a different CAS (computer algebra system).

- , *The Gauss-Bonnet theorem for noncommutative two tori with a general conformal structure*, J. Noncommut. Geom. 6 (2012);
- , *Scalar curvature for the noncommutative two torus*, J. Noncommut. Geom. 7 (2013);

Work on noncommutative four tori:

- , *Scalar curvature for noncommutative four-tori*, J. Noncommut. Geom. 9 (2015);
- Farzad Fathizadeh, *On the scalar curvature for the noncommutative four torus*, J. Math. Phys. 56 (2015);

Magnus expansion, Volterra series

Let us parametrize the standard k-simplex $\blacktriangle^k = 0 \leq s_k \leq \cdots \leq s_1$, denote $ds = ds_1 \cdots ds_k$,

$$\exp(a+b) = e^a + \sum_{n=1}^{\infty} \int_{\blacktriangle^n} e^{(1-s_1)a} b e^{(s_1-s_2)a} b \cdot \cdots \cdot e^{(s_n)a} ds.$$

We shall need only first three terms:

$$\begin{aligned} \exp(a+b) &= e^a + \int_0^1 e^{(1-u)a} b e^u a du \\ &+ \int_0^1 \int_0^u e^{(1-u)a} b e^{(u-v)a} b e^{va} dv du \\ &+ \cdots \end{aligned}$$

We would like to express $[D, e^h]$ in terms of $[D, h]$ using modular operators. Consider

$$\alpha_t(x) = e^{itD} x e^{-itD}$$

so that

$$\delta(x) \triangleq \frac{d}{dt}\Big|_{t=0} \alpha_t(x) = -i[D, x].$$

Let $B_t = \alpha_t(h) - h$. For small $t > 0$, apply the Taylor expansion:

$$B_t = \alpha_t(h) - h = \sum_{j=1}^{\infty} \frac{1}{j!} \delta^j(h) t^j = \delta(h)t + \frac{1}{2}\delta^2(h)t^2 + \cdots .$$

Duhamel's formula

Consider

$$\begin{aligned}\alpha_t(k) = \alpha_t(e^h) = e^{\alpha_t(h)} = e^{h+B_t} \\ = e^h + \int_0^1 e^{uh} B_t e^{(1-u)h} du + o(t^2) \\ = e^h + \int_0^1 e^{uh} t\delta(h) e^{(1-u)h} du + o(t^2).\end{aligned}$$

Differentiate in t, we obtain the following Duhamel's formula:

$$\delta(e^h) = \int_0^1 e^{(1-u)h} \delta(h) e^{uh} du$$

Recall the modular operator $\triangle(x) = k^{-2}xk^2$ and its logarithm $\triangledown = -2[h, \cdot]$.

$$
\begin{aligned}
\delta(e^h) &= \int_0^1 e^{(1-u)h}\delta(h)e^{uh}du \\
&= e^h \int_0^1 e^{u\triangledown/2}du(\delta h) = kF(\triangledown)(\delta h),
\end{aligned}
$$

with

$$
F(s) = \frac{e^{s/2} - 1}{(s/2)}, \;\; s \in \mathbb{R}.
$$

$\delta^2(e^h)$ can be treated in a similar way. Because

$$
\begin{aligned}
\alpha_t(k) = \alpha_t(e^h) = e^{\alpha_t(h)} &= e^{h+B_t} \\
&= e^h + \int_0^1 e^{uh}B_t e^{(1-u)h}du \\
&+ \int_{0\leq v\leq u\leq 1} e^{(1-u)h}B_t e^{(u-v)h}B_t e^{(v)h}dvdu + \cdots
\end{aligned}
$$

The coefficient for t^2 in the expansion of $\alpha_t(k)$ is given by

$$\int_0^1 e^{(1-u)h}\frac{1}{2}\delta^2(h)t^2e^{uh}du+\int_{0\le v\le u\le 1}e^{(1-u)h}t\delta(h)e^{(u-v)h}t\delta(h)e^{(v)h}dvdu,$$

diffrenciate in t twice:

$$e^hF(\bigtriangledown)(\delta(h))+2e^hG(\bigtriangledown_{(1)},\bigtriangledown_{(2)})(\delta(h)\delta(h))$$

with

$$F(s)=\int_0^1 e^{us/2}du=2\frac{e^{s/2}-1}{s}$$

and

$$G(s_1,s_2)=\int_0^1\int_0^u e^{us_1/2}e^{vs_2/2}dvdu=\frac{4\left(se^{\frac{s+t}{2}}-e^{s/2}(s+t)+t\right)}{st(s+t)}$$