K-Theory for Banach algebras

This course will be about K-theory for Banach and other topological algebras. In the first two lectures I shall review the fundamental features of K-theory, and then I will present more specialized topics, including the Oka principle in several complex variables, and Fredholm index theory from the point of view of Lafforgue and others, especially as it applies to Fréchet algebras, not just Banach algebras. These topics are of current interest in connection with the Baum-Connes conjecture in noncommutative geometry, among other things.

Lecture outlines

My tentative plans for the individual lectures are as follows:

Invertible elements in a Banach algebra

Groups of invertible elements, connectivity properties, fibrations, homotopy sequences, matrices, examples.

Projective modules and Bott periodicity

Basic properties of projective modules, pullback construction, K-theory group, long exact sequence, periodicity map, the elementary proof of periodicity, examples.

Oka principle

Multi-variable functional calculus, Banach algebras of holomorphic functions, localizations of Banach algebras, formulation and proof of an abstract Oka principle, examples, including the Gelfand transform of a commutative Banach algebra.

Fredholm operators

Review of elementary Fredholm index theory, Fredholm operators on Hilbert modules, Fredholm operators in the Banach algebra context, homotopy classes of Fredholm operators, examples.

K-theory for Fréchet algebras

Fréchet algebras, multiplicative convexity, nuclear operators in the Fréchet algebra context, coherence, Fredholm complexes, K-theory, examples.

Preliminary Reading

Prerequisites for the course include a familiarity with basic Hilbert space theory, Banach space theory, Banach algebra theory and C*-algebra theory, as presented for example in Douglas's book Banach algebra techniques in operator theory. See the first five chapters.

The first principles of K-theory for Banach algebras are the same as the first principles of algebraic K-theory, and it may be helpful to study these before the mini-course begins, although they will be covered again in the lectures. See Sections 1–4 of Milnor's book Introduction to algebraic K-theory.

It will be helpful to be familiar with the concepts of group convolution algebra and crossed product algebra. These are introduced in Chapters 7 and 8 Davidson's book C*-algebras by example.

Our treatment of the Oka principle will require some basic facts from several complex variables. See for example Chapters 1 and 2 of Taylor's book Several complex variables with connections to algebraic geometry and Lie groups.

Finally, the theory of Fredholm operators over a Banach algebra is particularly simple in the C*-algebra context, where it is part of the theory of Hilbert modules. The first four chapters of Lance's book Hilbert C*-modules give the basic definitions in Hilbert module theory.

References

Here are some references to ideas that will be developed during the lectures. I will add to this list later on.

Atiyah, Bott - On the periodicity theorem for complex vector bundles

Baum, Connes, Higson - Classifying space for proper actions and K-theory of group C*-algebras

Lafforgue - K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes

Lafforgue - Banach KK-theory and the Baum-Connes conjecture

Phillips - K-theory for Fréchet algebras

Taylor - Banach algebras and topology

Taylor - Topological invariants of the maximal ideal space of a Banach algebra

Taylor - Homotopy invariants for Banach algebras