UNIFORM LOCAL AMENABILITY

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Outline

1. Coarse Geometry
2. Uniform local amenability and Property A

Outline

Objects of Interest

Idea: Study spaces from a large–scale perspective.

Notation: $X, Y \dots$ metric spaces; d metric. Γ discrete group.

Examples:

- Finitely generated groups with word metric: $\Gamma = \langle S \rangle$, $S = S^{-1}$, $|S| < \infty$. Define a metric by $d_S(g,h) =$ the length of a shortest word in alphabet S representing $g^{-1}h$.
  E.g. if $\mathbb{Z} = \langle 1, -1 \rangle$, then the metric is $d(m,n) = |n-m|$.
- Graphs (finite or infinite), endowed with the path metric.
- Complete Riemannian manifolds.
- "Coarse disjoint union" $X = \sqcup_n G_n$ of a sequence of finite graphs (G_n). Metric: on each G_n the path metric, $d(G_n, G_m) = m + n + |G_n| + |G_m|$.
- "Box space": $X = \sqcup_n \mathrm{Cayley}(\Gamma/\Gamma_n; S/\Gamma_n)$, where $\Gamma = \langle S \rangle$, $|S| < \infty$, $\Gamma_n \leq \Gamma$ normal, $[\Gamma : \Gamma_n] < \infty$.

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Expanders

For a finite graph G, the Cheeger constant is

$$h(G) = \min\left\{ \frac{|\partial S|}{|S|} : S \subset V(G), 0 < |S| \le \frac{|G_n|}{2} \right\}.$$

A *sequence of expanders* (expander) is a sequence of finite graphs G_n, such that

- the degrees of vertices are uniformly bounded,
- $|G_n| \nearrow \infty$ and
- $\inf_n h(G_n) > 0$.

Think of it as a metric space $X = \sqcup_n G_n$.

First examples: Box spaces of residually finite groups with property (T), e.g. $\sqcup_p SL_n(\mathbb{Z})/SL_n(\mathbb{Z}/p\mathbb{Z})$. [Margulis]

In fact, a group Γ has property (τ) with respect to a family of finite index subgroups $(\Gamma_n)_{n\in\mathbb{N}}$ iff the box space $X = \sqcup_n(\Gamma/\Gamma_n)$ is an expander.

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Coarse Definitions

A map $f : X \to Z$ is a $(\star)$ *embedding*, if there exist $\rho_+, \rho_- : [0,\infty) \to [0,\infty)$ with $(\star\star)$, such that

$$\rho_-(d(x,y)) \le d(f(x),f(y)) \le \rho_+(d(x,y)).$$

$(\star)$	$(\star\star)$
coarse (CE)	$\rho_- \nearrow \infty$
quasi–isometric (QI)	ρ_+ and ρ_- are affine $(t \mapsto At + B)$
bilipschitz	ρ_+ and ρ_- are linear $(t \mapsto At)$

A $(\star)$ embedding f is a $(\star)$ *equivalence*, if $\sup\{d(z,f(X)) \mid z \in Z\} < \infty$.

Examples

Equivalences:

- Bounded space $\sim_c$ {pt}.
- $\mathbb{Z}^n \sim_c \mathbb{R}^n$, but $\mathbb{Z}^m \sim_c \mathbb{Z}^n \implies m = n$ (asymptotic dimension).
- Free groups F_r (with the free generating sets), $2 \le r < \infty$ are all bilipschitz equivalent [Papasoglu '95].
- $X = \{2^{2^n} \mid n \in \mathbb{N}\} \subset \mathbb{N}$, metric from $\mathbb{N}$. Any bijection is a coarse equivalence, but any QI $X \to X$ is eventually constant.

Basic Geometric Group Theory Lemma:
If $\Gamma = \langle S \rangle = \langle S' \rangle$, $|S|, |S'| < \infty$, then $(\Gamma, d_S) \sim_{QI} (\Gamma, d_{S'})$. So any QI invariant is actually an invariant of the underlying group.

Švarc–Milnor Theorem:
If Γ acts properly and cocompactly on a length space X, then $\Gamma \sim_{QI} X$. [So $\pi_1(M) \sim_{QI} \widetilde{M}$ for a compact Riemannian manifold M.]

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Coarse Properties and Some Theorems

…amenability, asymptotic dimension, coarse embeddability into ___ (e.g. a Hilbert space)

Theorem (G. Yu '97)

If M is a uniformly contractible complete Riemannian manifold with bounded geometry and finite asymptotic dimension, then it admits no metric of uniformly positive scalar curvature, within the class of CE metrics.

Theorem (G. Yu '00)

If Γ admits a coarse embedding into a Hilbert space, then the Novikov conjecture holds for Γ.

Proved using operator algebras (Roe C*-algebras: algs encoding coarse structure of a space) and K-theory. "Coarse Baum–Connes conjecture."

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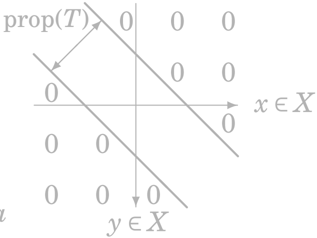
Roe algebras ("Encode coarse structure")

Let X be a uniformly discrete [$\exists c > 0$ with $x \neq y \implies d(x,y) \geq c$] metric space with bounded geometry [$\forall R > 0$, $\sup_{x\in X} |B(x,R)| < \infty$].

The *translation algebra of* X, $\mathbb{C}[X]$, is the $*$-algebra of X-by-X matrices $(t_{yx})_{x,y\in X}$, $t_{yx} \in \mathbb{C}$, with *finite propagation* [there exists $R \geq 0$, so that $d(x,y) \geq R$ implies $t_{yx} = 0$] with uniformly bounded entries [$\sup_{x,y} |t_{xy}| < \infty$].



There is a $*$-representation $\lambda : \mathbb{C}[X] \to \mathscr{B}(\ell^2 X)$ "by multiplication". The *uniform Roe C^*-algebra* $C^*_{\mathrm{u}}X$ is the norm–closure of $\lambda(\mathbb{C}[X]) \subset \mathscr{B}(\ell^2 X)$.

Use $\mathscr{K}(H)$ instead of $\mathbb{C}$ $\rightsquigarrow$ *Roe algebra* C^*X. Coarse Baum–Connes conjecture: "Compute $K_*(C^*X)$".

Theorem (Willett–S '11)

If X has property A, then $C^*_{\mathrm{u}}X \cong C^*_{\mathrm{u}}Y$ implies $X \sim_c Y$.

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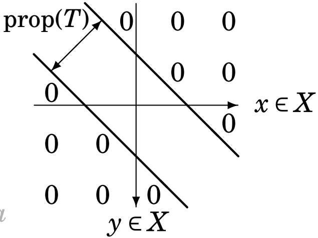

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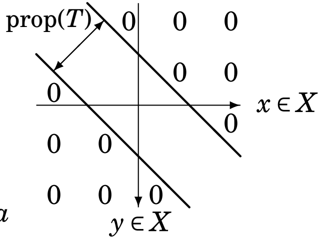

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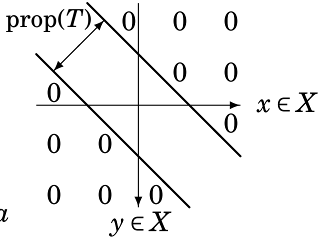

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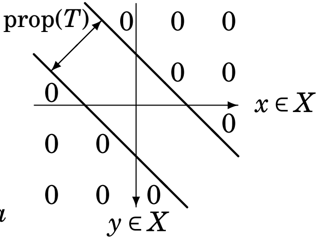

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Amenability

Definition (Følner)

A space X is *amenable*, if for all $R, \varepsilon > 0$ there exists finite $E \subset X$ with $|\partial_R E| < \varepsilon |E|$. $[\partial_R E = B_R(E) \setminus E]$

- Good for groups (they are very homogeneous).
- Bad for dicrete metric spaces (e.g. every $\sqcup_n X_n$ with X_n finite, is amenable)
- Fix for spaces? Need to "look at all the places within a space with uniform parameters".

Property A

Definition (Yu '00)

X is said to have *property A*, if for every $R, \varepsilon > 0$ there exists $S \geq 0$ and finite subsets $A_x \subset X \times \mathbb{N}$ for each $x \in X$, such that

- $(x,1) \in A_x$ for every $x \in X$,
- $|A_x \triangle A_y| < \varepsilon |A_x \cap A_y|$ if $d(x,y) \leq R$ and
- the projection of A_x to X is contained in $B(x,S)$ for every $x \in X$.

- Implies CE(HSp) (a criterion).
- Classes of discrete groups having A: amenable, hyperbolic, linear [Guentner–Higson–Weinberger '05], mapping class groups [Bestvina–Bromberg–Fujiwara '10].
- finite dim'l $CAT(0)$-cube complexes have A [Campbell–Niblo '04].
- Not known: Thompson's group F.
- What about **not** having A?

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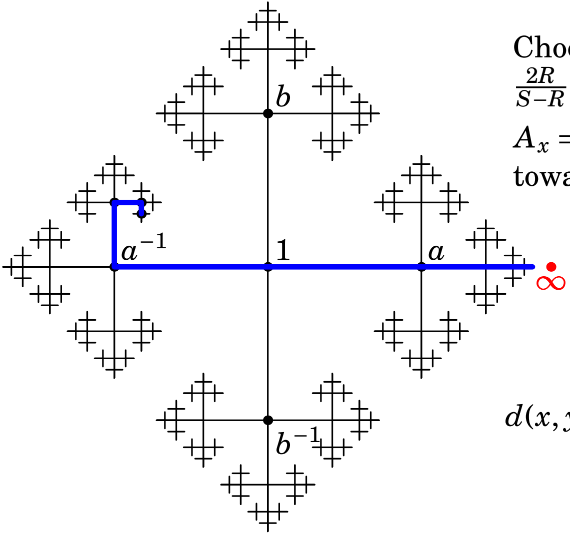
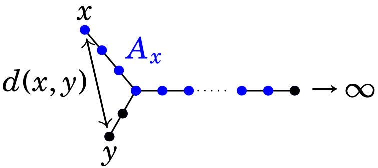
Free groups have A



Choose S>0 so that $\frac{2R}{S-R} < \varepsilon$.

$A_x = \{\, S$ points from x towards $\infty\,\}$.



Property A and Others

equivariant side:		coarse side:
amenability	$\Longrightarrow$	property A
$\Downarrow$		$\Downarrow$
Haagerup prop.	$\Longrightarrow$	CE(HSp)

Theorem

For a finitely generated group Γ*, the following are equivalent:*

- Γ *has property A*
- Γ *acts amenably on some compact space [Higson–Roe '00]*
- $C^*_{\mathrm{red}}\Gamma$ *is an exact C*-algebra [Guentner–Kaminker, Ozawa '00]*
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Uniform Local Amenability (ULA)

Definition (Følner condition)

A space X is *amenable*, if for all $R, \varepsilon > 0$ there exists finite $E \subset X$ with $|\partial_R E| < \varepsilon |E|$. $[\partial_R E = B_R(E) \setminus E]$

- "Localizing with finite measures instead of sets" or a "weighted version": ULA$_\mu$: ...for any finitely supported probability measure μ on X ...and $\mu(\partial_R E) < \varepsilon \mu(E)$.
- ULA and ULA$_\mu$ are coarse invariants.
- Why?
  - "Easy" to check that it fails.
  - For showing that a couple of known coarse properties are all equivalent.

Uniform Local Amenability (ULA)

Definition (Brodzki–Niblo–S–Willett–Wright)

A space X is *uniformly locally amenable*, if for all $R, \varepsilon > 0$ there exists $S > 0$, such that for any finite $F \subset X$ there exists finite $E \subset X$ with $\operatorname{diam}(E) \leq S$ and $|\partial_R E \cap F| < \varepsilon |E \cap F|$. $[\partial_R E = B_R(E) \setminus E]$

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- "Localizing with finite measures instead of sets" or a "weighted version": ULA_μ: ...for any finitely supported probability measure μ on X ...and $\mu(\partial_R E) < \varepsilon \mu(E)$.
- ULA and ULA_μ are coarse invariants.
- Why?
  - "Easy" to check that it fails.
  - For showing that a couple of known coarse properties are all equivalent.

Equivalences

- [Chen–Tessera–Wang–Yu] Metric Sparsification Property (MSP) $\Longrightarrow$ Operator Norm Localization Property (ONL); for K-theory purposes
- [Elek–Timár] weighted hyperfiniteness $\Longleftrightarrow$ MSP

Theorem (Brodzki–Niblo–S–Willett–Wright)

For spaces with bounded geometry:

$$\begin{array}{ccccccc} & & CE(HSp) & & & & ULA \\ & & \Uparrow & & & & \Uparrow \\ A & \Longrightarrow & ULA_\mu & \Longleftrightarrow & MSP & \Longrightarrow & ONL. \end{array}$$

Theorem (H. Sako)

For spaces with bounded geometry: $A \Longleftrightarrow ONL$.

Question: Relation between CE(HSp) and ULA? (Known CE: $\not\Rightarrow$ ULA: [AGS] example.)

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Problem: Non–exact groups?

- Gromov's Idea: take and expander $X = \sqcup X_n$ and find a group with X in its Cayley graph. Details (tough) [Arzhantseva–Delzant '09-12].
- ToDo: Elementary construction. Find a non-A group which CE(HSp)

Spaces of the sort $X = \sqcup_n X_n$; X_n finite graphs

- Expanders do not CE(HSp) [Gromov], so they do not have A.
- X does not have A if girth$(X_n) \to \infty$ and degrees of vertices are between 3 and some $N < \infty$. [Willett '11]
- $\sqcup_n (\mathbb{Z}/2\mathbb{Z})^n$: not A, but CE(HSp). Not bounded geometry. [Nowak '07]
- A bdd. geom. example of X without A, but CE(HSp). [Arzhantseva–Guentner–S '10]; extended by [Khukhro '13]

In all these examples, showing "not ULA" is easy.

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NOT ULA: $\exists R, \varepsilon > 0\ \forall S > 0\ \exists$finite $F \subset X$ such that for any $E \subset X$ with $\operatorname{diam}(E) \le S$, $|\partial_R E \cap F| \ge \varepsilon |E \cap F|$.

EXPANDERS: Let $X = \sqcup_n X_n$ be an expander. Denote $N_S = \max_{x \in X} |B(x,S)|$. Let n be so large, that $|X_n| > 2N_S$ and take $F = X_n$. Then any $E \subset X_n$ with $\operatorname{diam}(E) \le S$ satisfies $|\partial_1 E| \ge h(X_n)|E|$.

GRAPHS WITH LARGE GIRTH: Let $X = \sqcup_n X_n$ satisfy $\operatorname{girth}(X_n) \nearrow \infty$ and the degrees of all vertices are between 3 and some $D < \infty$. For $S > 0$, let n be so large, that a subset E of diameter S in X_n is isometric to a subset of a tree. Then any such E satisfies $|\partial_1 E| \ge \frac{1}{D-1}|E|$.

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