Sparse graphs with an independent or foresty minimum vertex cut

Xingzhi Zhan (詹兴致)

zhan@math.ecnu.edu.cn

East China Normal University

Joint work

with Kun Cheng (成锟) and Yurui Tang (唐雨睿)

We consider finite simple graphs. The order of a graph is its number of vertices, and the size is its number of edges.

A connected graph is called fragile if it contains an independent vertex cut.

In 2002 Chen and Yu proved a conjecture of Caro that every connected graph of order n and size at most $2n-4$ is fragile. The size bound $2n-4$ is sharp.

Recently Chernyshev, Rauch and Rautenbach has initiated the study of foresty vertex cuts of graphs.

It is natural to consider minimum vertex cuts.

We will determine the largest possible size of a graph of order n that ensures the existence of an independent minimum vertex cut or a foresty minimum vertex cut.

Theorem 1 Every connected graph of order n with $n \geq 7$ and size at most $\lfloor 3n/2 \rfloor$ has an independent minimum vertex cut.

To show that the size bound $\lfloor 3n/2 \rfloor$ is best possible, we construct a graph G_n of order n and size $\lfloor 3n/2 \rfloor + 1$ such that G_n has no independent minimum vertex cut.

G_n has vertex set $\{v_1, v_2, \ldots, v_n\}$.

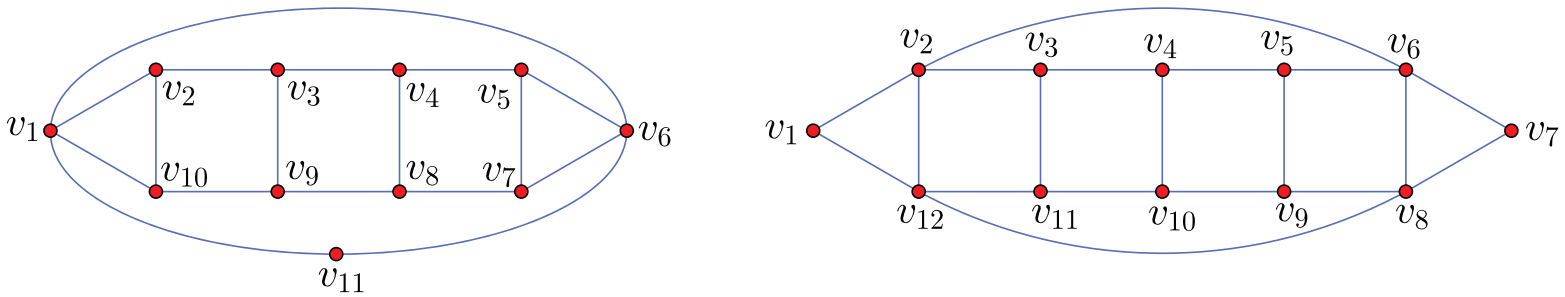

图: G_{11} and G_{12}

If n is odd, $\{v_1, v_{(n+1)/2}\}$ is the unique minimum vertex cut of G_n, which induces an edge. If n is even, G_n has exactly two minimum vertex cuts: $\{v_2, v_n\}$ and $\{v_{n/2}, v_{(n+4)/2}\}$, each of which induces an edge.

Theorem 2 Every connected graph of order n with $n \geq 7$ and size at most $2n$ has a foresty minimum vertex cut.

To show that the size bound $2n$ is best possible, we construct a graph F_n of order n and size $2n+1$ such that F_n has no foresty minimum vertex cut.

F_n has vertex set $\{v_1, v_2, \ldots, v_n\}$.

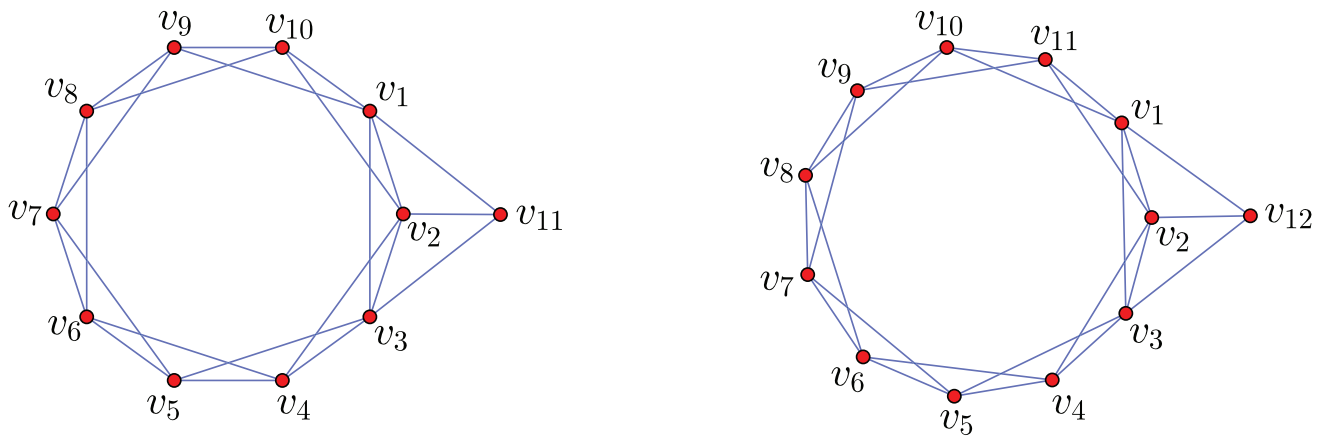

图: F_{11} and F_{12}

$\kappa(F_n) = 3$ and $\{v_1, v_2, v_3\}$ is the unique minimum vertex cut, which induces a triangle.

References

1, G. Chen, R.J. Faudree and M.S. Jacobson, Fragile graphs with small independent cuts, J. Graph Theory, 41(2002), no.4, 327–341.

2, G. Chen and X. Yu, A note on fragile graphs, Discrete Math., 249(2002), no 1-3, 41–43.

3, K. Cheng, Y. Tang and X. Zhan, Sparse graphs with an independent or foresty minimum vertex cut, arXiv: 2412.03869, 5 December 2024.

4, V. Chernyshev, J. Rauch and D. Rautenbach, Forest cuts in sparse graphs, arXiv: 2409.17724, 26 September 2024.

THANK YOU