The minimum size of a 3-connected locally nonforesty graph

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Joint work

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We consider finite simple graphs. The order of a graph is its number of vertices, and the size its number of edges.

Definition 1. A local subgraph of a graph G is the subgraph of G induced by the neighborhood of a vertex.

Thus a graph of order n has n local subgraphs. It has been a traditional topic to deduce properties of a graph by its local subgraphs.

Definition 2. A graph G is called locally nonforesty if every local subgraph of G contains a cycle.

Recently, in studying forest cuts of a graph, Chernyshev, Rauch and Rautenbach posed the following

Conjecture 1. If n and m are the order and size of a 3-connected locally nonforesty graph respectively, then

$$m \geq \frac{7}{3}(n-1).$$

We will determine the minimum size of a 3-connected locally nonforesty graph of order n.

It turns out that Conjecture 1 does not hold.

Theorem. Given an integer $n \geq 8$, define

$$f(n) = \begin{cases} 2n - \lfloor n/8 \rfloor & \text{if } n \equiv 0,\ 4,\ 7 \pmod 8, \\ 2n + 1 - \lfloor n/8 \rfloor & \text{otherwise.} \end{cases}$$

Then the minimum size of a 3-connected locally nonforesty graph of order n is $f(n)$.

Denote $b(n) = 7(n-1)/3$, the lower bound in Conjecture 1. For any $n \geq 8$,

$$b(n) > 2n \geq f(n).$$

Their difference can be large. For example,

$$b(100) - f(100) = 43.$$

Thus Conjecture 1 does not hold.

Ideas in the proof

We estimate the size of a 3-connected locally nonforesty graph G of order n in two ways.

Let s be the number of vertices of degree 3, and

denote $\varphi(s) = \max\{\lceil 2n - s/2 \rceil, \lceil 3(n+s)/2 \rceil\}$.

Then

$$e(G) \geq \min_{1 \leq s \leq n} \varphi(s).$$

Extremal graphs

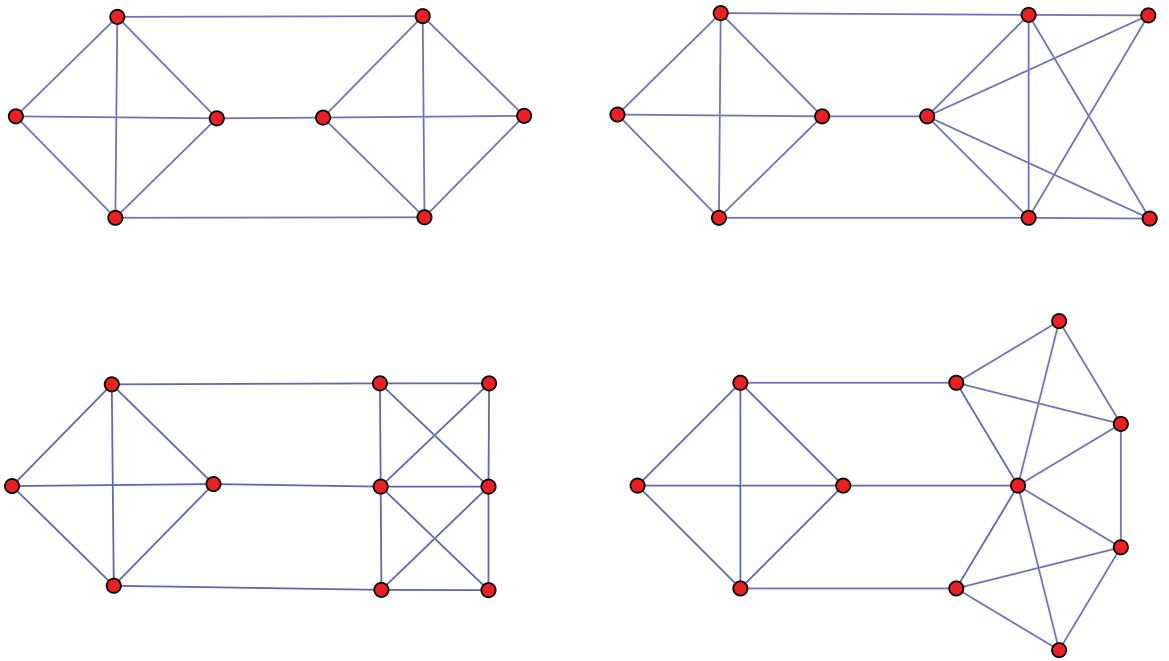

图: G_8, G_9, G_{10} and G_{11}

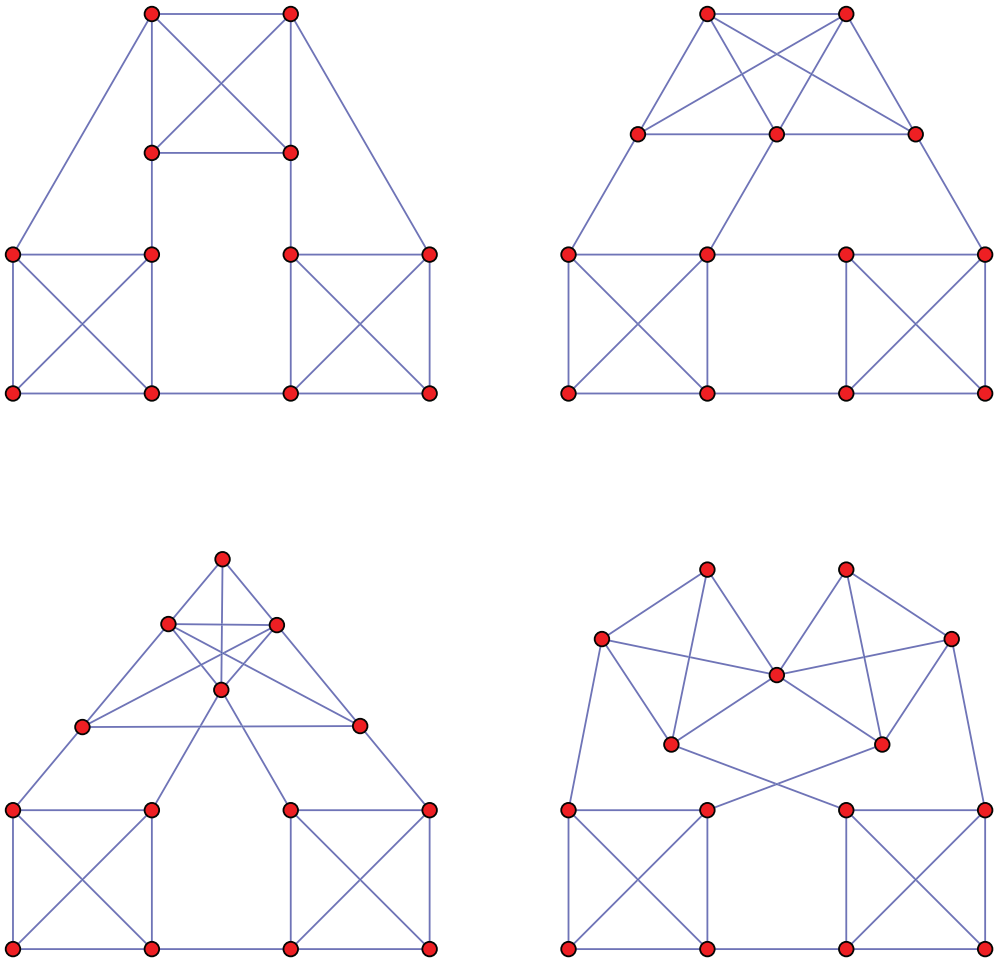

图: G_{12}, G_{13}, G_{14} and G_{15}

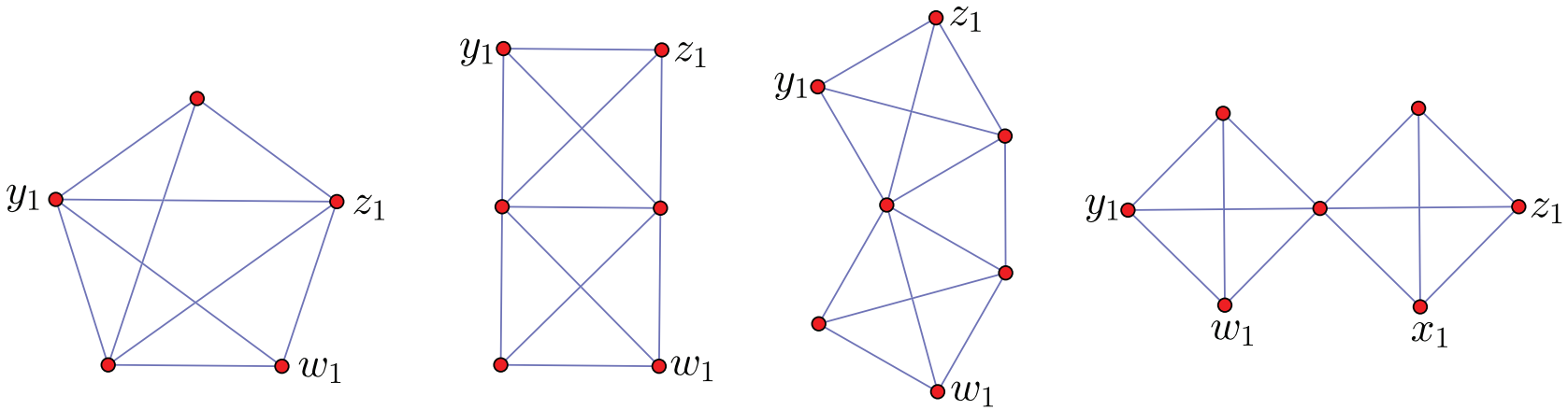

图: B_1, C_1, D_1 and D_2

References

1, V. Chernyshev, J. Rauch and D. Rautenbach, Forest cuts in sparse graphs, arXiv: 2409.17724, 26 September 2024.

2, C. Li, Y. Tang and X. Zhan, The minimum size of a 3-connected locally nonforesty graph, arXiv: 2410.23702, 31 October 2024.

THANK YOU