



The Turán number of book graphs

Jingru Yan · Xingzhi Zhan

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Abstract Given a graph H and a positive integer n , the Turán number of H for the order n , denoted by $\text{ex}(n, H)$, is the maximum size of a simple graph of order n not containing H as a subgraph. The book with p pages, denoted by B_p , is the graph that consists of p triangles sharing a common edge. Bollobás and Erdős initiated the research on the Turán number of book graphs in 1975. The two numbers $\text{ex}(p + 2, B_p)$ and $\text{ex}(p + 3, B_p)$ have been determined by Qiao and Zhan. In this paper we determine the numbers $\text{ex}(p + 4, B_p)$, $\text{ex}(p + 5, B_p)$ and $\text{ex}(p + 6, B_p)$, and characterize the corresponding extremal graphs for the numbers $\text{ex}(n, B_p)$ with $n = p + 2, p + 3, p + 4, p + 5$.

Keywords Turán number · Book · Triangle · Extremal graph

Mathematics Subject Classification 05C35 · 05C75

1 Introduction

We consider finite simple graphs. The *order* of a graph G , denoted by $|G|$, is its number of vertices, and the *size* is its number of edges.

Definition 1 Given a graph H and a positive integer n , the *Turán number of H for the order n* , denoted by $\text{ex}(n, H)$, is the maximum size of a simple graph of order n not containing H as a subgraph.

Determining the Turán number of various graphs H is one of the main topics in extremal graph theory [1].

Definition 2 The *book* with p pages, denoted by B_p , is the graph that consists of p triangles sharing a common edge.

B_5 is depicted in Fig. 1. Note that the graph B_p has order $p + 2$.

We denote by K_p the complete graph of order p . B_1 is just K_3 . Denote by $K_{s,t}$ the complete bipartite graph whose partite sets have cardinalities s and t . A classic result of Mantel [4] from 1907 states that $\text{ex}(n, K_3) = \lfloor n^2/4 \rfloor$ and the balanced complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is the unique extremal graph.

In 1975 Bollobás and Erdős [2] posed the conjecture that $\text{ex}(n, B_{\lceil n/6 \rceil}) \leq n^2/4$, which was proved by Edwards [3].

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J. Yan · X. Zhan (✉)
Department of Mathematics, East China Normal University, Shanghai 200241, China
E-mail: zhan@math.ecnu.edu.cn

J. Yan
E-mail: mathyjr@163.com

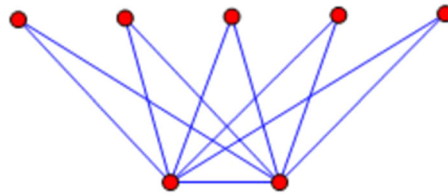


Fig. 1 The book graph B_5

The two numbers $\text{ex}(p + 2, B_p)$ and $\text{ex}(p + 3, B_p)$ have been determined by Qiao and Zhan [5] in solving a problem of Erdős and they posed the problem [5, Problem 3] of determining $\text{ex}(n, B_p)$ for a general order n . In this paper we determine the numbers $\text{ex}(p + 4, B_p)$, $\text{ex}(p + 5, B_p)$ and $\text{ex}(p + 6, B_p)$ and characterize the corresponding extremal graphs for the numbers $\text{ex}(n, B_p)$ with $n = p + 2, p + 3, p + 4, p + 5$.

For graphs we will use equality up to isomorphism, so $G = H$ means that G and H are isomorphic. Given graphs G and H , the notation $G + H$ means the disjoint union of G and H , and $G \vee H$ denotes the join of G and H , which is obtained from $G + H$ by adding edges joining every vertex of G to every vertex of H . mH denotes the disjoint union of m copies of a graph H . We denote by $V(G)$ and $E(G)$ the vertex set and edge set of a graph G respectively. Let \overline{G} , $\delta(G)$ and $\Delta(G)$ denote the complement, minimum degree and maximum degree of a graph G respectively. For $A \subseteq V(G)$, we denote by $G[A]$ the subgraph of G induced by A .

For a vertex $v \in V(G)$, let $N_G(v)$ and $\text{deg}_G(v)$ denote the neighborhood and degree of v in G respectively. If the graph G is clear from the context, we will omit it as the subscript. A vertex of degree 0 is called an *isolated vertex* and a vertex of degree 1 a *leaf*.

We denote by C_s and P_t the cycle of length s and path of order t respectively.

Notation. For an even positive integer n , the notation $K_n - PM$ denotes the graph obtained from the complete graph K_n by deleting all the edges in a perfect matching of K_n ; i.e., it is the complement of $(n/2)K_2$.

2 Main Results

We will repeatedly use the first two lemmas below.

Lemma 1 *Let n and p be positive integers with $n \geq p + 2$. Then a graph G of order n does not contain B_p as a subgraph if and only if for every edge $xy \in E(G)$, $|N_{\overline{G}}(x) \cup N_{\overline{G}}(y)| \geq n - p - 1$. Consequently, if G is such a graph then \overline{G} has at most one isolated vertex.*

Proof For $xy \in E(G)$, $V(G) \setminus \{x, y\} = (N_G(x) \cap N_G(y)) \cup (N_{\overline{G}}(x) \cup N_{\overline{G}}(y))$ and $(N_G(x) \cap N_G(y)) \cap (N_{\overline{G}}(x) \cup N_{\overline{G}}(y)) = \emptyset$. Hence

$$|N_{\overline{G}}(x) \cup N_{\overline{G}}(y)| = n - 2 - |N_G(x) \cap N_G(y)|.$$

G does not contain B_p if and only if for every edge $xy \in E(G)$, $|N_G(x) \cap N_G(y)| \leq p - 1$ if and only if $|N_{\overline{G}}(x) \cup N_{\overline{G}}(y)| \geq n - p - 1$. This shows the first conclusion. The second conclusion is obvious and it also follows from the first immediately. \square

Lemma 2 *Let n and p be positive integers with $n \geq p + 5$ and let G be a graph of order n . If \overline{G} has a component which is a cycle of length at least 4, then G contains B_p .*

Proof Let C be a component of \overline{G} which is a cycle of length at least 4, and let x, z, y be three consecutive vertices of C . Then $xy \in E(G)$ and $|N_{\overline{G}}(x) \cup N_{\overline{G}}(y)| \leq 3 < n - p - 1$, since $n \geq p + 5$. By Lemma 1, G contains B_p . \square

Lemma 3 (Qiao and Zhan [5]) *If p is an even positive integer, then $\text{ex}(p + 2, B_p) = p(p + 2)/2$ and $\text{ex}(p + 3, B_p) = p(p + 4)/2$; if p is an odd positive integer, then $\text{ex}(p + 2, B_p) = (p + 1)^2/2$ and $\text{ex}(p + 3, B_p) = (p + 1)(p + 3)/2$.*

We first characterize the extremal graphs for the two numbers $\text{ex}(p + 2, B_p)$ and $\text{ex}(p + 3, B_p)$ determined in Lemma 3.

Theorem 4 *Suppose G is a graph of order $p + 2$ not containing B_p .*



- (1) If p is even, then G has size $\text{ex}(p+2, B_p)$ if and only if $G = K_{p+2} - PM$ or $G = \overline{K_1 + ((p-2)/2)K_2 + P_3}$.
 (2) If p is odd, then G has size $\text{ex}(p+2, B_p)$ if and only if $G = K_1 \vee (K_{p+1} - PM)$.

Proof It is easy to check that the three graphs in the theorem satisfy the requirements. Next we prove that they are the only extremal graphs. Suppose G has size $\text{ex}(p+2, B_p)$ and its vertices are v_1, \dots, v_{p+2} with $\text{deg}_{\overline{G}}(v_1) \leq \text{deg}_{\overline{G}}(v_2) \leq \dots \leq \text{deg}_{\overline{G}}(v_{p+2})$.

(1) p is even. By Lemma 3, $|E(G)| = p(p+2)/2$ which implies that $|E(\overline{G})| = (p+2)/2$. By the degree-sum formula [6, p.35], $\sum_{i=1}^{p+2} \text{deg}_{\overline{G}}(v_i) = 2|E(\overline{G})| = p+2$, implying $\delta(\overline{G}) \leq 1$.

Case 1. $\delta(\overline{G}) = 1$. The degree sequence of \overline{G} is $1, 1, \dots, 1$. Thus $\overline{G} = ((p+2)/2)K_2$, implying that $G = K_{p+2} - PM$.

Case 2. $\delta(\overline{G}) = 0$. By Lemma 1, v_1 is the only isolated vertex of \overline{G} . Since $\sum_{i=1}^{p+2} \text{deg}_{\overline{G}}(v_i) = p+2$ and $\text{deg}_{\overline{G}}(v_i) \geq 1$ for every $i \geq 2$, we deduce that $\Delta(\overline{G}) = 2$ and \overline{G} has a unique vertex of degree 2. Thus the degree sequence of \overline{G} is $2, 1, \dots, 1, 0$, implying that $\overline{G} = K_1 + ((p-2)/2)K_2 + P_3$; i.e., $G = \overline{K_1 + ((p-2)/2)K_2 + P_3}$.

(2) p is odd. By Lemma 3, $|E(G)| = (p+1)^2/2$, implying $|E(\overline{G})| = (p+1)/2$. Hence $\sum_{i=1}^{p+2} \text{deg}_{\overline{G}}(v_i) = 2|E(\overline{G})| = p+1$, implying $\delta(\overline{G}) = 0$. By Lemma 1, \overline{G} has a unique isolated vertex. Thus, the degree sequence of \overline{G} is $1, 1, \dots, 1, 0$. It follows that $\overline{G} = K_1 + ((p+1)/2)K_2$; i.e., $G = K_1 \vee (K_{p+1} - PM)$. \square

Theorem 5 Suppose G is a graph of order $p+3$ not containing B_p .

- (1) If $p = 2$, then G has size $\text{ex}(p+3, B_p)$ if and only if $G = K_1 \vee (2P_2)$, or $G = K_{2,3}$, or $G = \overline{P_5}$; if p is even and $p \geq 4$, then G has size $\text{ex}(p+3, B_p)$ if and only if $G = (3K_1) \vee (K_p - PM)$ or $G = \overline{P_5} \vee (K_{p-2} - PM)$.
 (2) If p is odd, then G has size $\text{ex}(p+3, B_p)$ if and only if $G = K_{p+3} - PM$.

Proof It is easy to check that the graphs in the theorem satisfy the requirements. Next we prove that they are the only extremal graphs. Suppose G has size $\text{ex}(p+3, B_p)$ and its vertices are v_1, \dots, v_{p+3} with $\text{deg}_{\overline{G}}(v_1) \leq \text{deg}_{\overline{G}}(v_2) \leq \dots \leq \text{deg}_{\overline{G}}(v_{p+3})$.

(1) p is even. By Lemma 3, $|E(G)| = p(p+4)/2$ which implies that $|E(\overline{G})| = (p+6)/2$. Since $\sum_{i=1}^{p+3} \text{deg}_{\overline{G}}(v_i) = 2|E(\overline{G})| = p+6$, we have $\delta(\overline{G}) \leq 1$.

Case 1. $\delta(\overline{G}) = 0$. By Lemma 1, v_1 is the only isolated vertex of \overline{G} and hence by Lemma 1 we deduce that $\text{deg}_{\overline{G}}(v_i) \geq 2$ for every $i \geq 2$. We have

$$p+6 = \sum_{i=1}^{p+3} \text{deg}_{\overline{G}}(v_i) \geq 0 + (p+2) \times 2 = 2p+4,$$

implying $p = 2$. The degree sequence of \overline{G} is $2, 2, 2, 2, 0$. Thus $\overline{G} = K_1 + C_4$; i.e., $G = K_1 \vee (2P_2)$.

Case 2. $\delta(\overline{G}) = 1$. Since $\sum_{i=1}^{p+3} \text{deg}_{\overline{G}}(v_i) = p+6$, we have $2 \leq \Delta(\overline{G}) \leq 4$. We assert that $\Delta(\overline{G}) = 2$. If $\Delta(\overline{G}) = 4$ or $\Delta(\overline{G}) = 3$, then the vertex v_{p+3} has at least two neighbors which are leaves in \overline{G} , contradicting the fact that for any $xy \in E(G)$, $|N_{\overline{G}}(x) \cup N_{\overline{G}}(y)| \geq 2$. Thus the degree sequence of \overline{G} is $2, 2, 2, 1, 1, \dots, 1$. Note that the two neighbors of a vertex of degree 2 in \overline{G} cannot be both leaves. It follows that $\overline{G} = (p/2)K_2 + K_3$ or $\overline{G} = ((p-2)/2)K_2 + P_5$; i.e., $G = (3K_1) \vee (K_p - PM)$ or $G = \overline{P_5} \vee (K_{p-2} - PM)$. Note that when $p = 2$, these two graphs are $K_{2,3}$ and $\overline{P_5}$.

(2) p is odd. By Lemma 3, $|E(G)| = (p+1)(p+3)/2$ which implies that $|E(\overline{G})| = (p+3)/2$. Since $\sum_{i=1}^{p+3} \text{deg}_{\overline{G}}(v_i) = 2|E(\overline{G})| = p+3$, we have $\delta(\overline{G}) \leq 1$. If $\delta(\overline{G}) = 0$, then by Lemma 1 we deduce that for each $i \geq 2$, $\text{deg}_{\overline{G}}(v_i) \geq 2$, which implies $\sum_{i=1}^{p+3} \text{deg}_{\overline{G}}(v_i) \geq 0 + (p+2) \times 2 > p+3$, a contradiction. Hence $\delta(\overline{G}) = 1$, which, together with the fact that $\sum_{i=1}^{p+3} \text{deg}_{\overline{G}}(v_i) = p+3$, yields that the degree sequence of \overline{G} is $1, 1, \dots, 1$. It follows that $\overline{G} = ((p+3)/2)K_2$; i.e., $G = K_{p+3} - PM$. This completes the proof. \square

In the next two results we determine the Turán number $\text{ex}(p+4, B_p)$ and characterize the corresponding extremal graphs.

Theorem 6 If p is an integer with $p \geq 3$, then $\text{ex}(p+4, B_p) = (p+2)(p+3)/2$.



Proof Let G be a graph of order $p + 4$ and size e with vertices v_1, \dots, v_{p+4} such that $\deg(v_i) = d_i, i = 1, \dots, p + 4$ and $d_1 \geq d_2 \geq \dots \geq d_{p+4}$. Suppose $e \geq ((p + 2)(p + 3)/2) + 1$. We will show that G contains B_p . We have $\sum_{i=1}^{p+4} d_i = 2e \geq p^2 + 5p + 8$. We distinguish two cases.

Case 1. $d_1 = p + 3$.

If $d_2 \leq p$, then $\sum_{i=1}^{p+4} d_i \leq p + 3 + (p + 3)p = p^2 + 4p + 3 < p^2 + 5p + 8$, a contradiction. Hence $d_2 \geq p + 1$. Then G has p triangles sharing the common edge v_1v_2 .

Case 2. $d_1 \leq p + 2$.

If $d_3 \leq p + 1$, then $\sum_{i=1}^{p+4} d_i \leq 2(p + 2) + (p + 2)(p + 1) = p^2 + 5p + 6 < p^2 + 5p + 8$, a contradiction. Hence $d_1 = d_2 = d_3 = p + 2$. Consequently v_1 is adjacent to at least one of v_2 and v_3 , say v_2 . Let $v_s \notin N(v_1)$ and $v_t \notin N(v_2)$. Then $\{v_3, v_4, \dots, v_{p+4}\} \setminus \{v_s, v_t\} = N(v_1) \cap N(v_2)$. We have $|N(v_1) \cap N(v_2)| \geq p$. Thus G has p triangles sharing the common edge v_1v_2 . This shows $\text{ex}(p + 4, B_p) \leq (p + 2)(p + 3)/2$.

On the other hand, the graph $G_1 = \overline{K_2} + C_{p+2}$ has order $p + 4$ and size $(p + 2)(p + 3)/2$, and using the criterion in Lemma 1 we check that G_1 does not contain B_p . Thus we conclude that $\text{ex}(p + 4, B_p) = (p + 2)(p + 3)/2$. \square

Theorem 6 excludes the two small values $p = 1, 2$. We remark that $\text{ex}(5, B_1) = 6$ agrees with the formula in Theorem 6, but $\text{ex}(6, B_2) = 9$ does not.

Theorem 7 *Let p be an integer with $p \geq 3$ and let H be a graph of order $p + 4$ not containing B_p . Then H has size $\text{ex}(p + 4, B_p)$ if and only if $H = \overline{K_2} + C_{i_1} + C_{i_2} + \dots + C_{i_t}$ where $i_j \neq 4$ for each j and $i_1 + i_2 + \dots + i_t = p + 2$.*

Proof It is easy to check that every graph of the form in the theorem satisfies all the requirements; i.e., it is an extremal graph. We use the criterion in Lemma 1 to check that such a graph does not contain B_p .

Conversely suppose that H is a graph of order $p + 4$ not containing B_p and H has size $\text{ex}(p + 4, B_p)$. Denote $n = p + 4$. Since $|E(H)| = (p + 2)(p + 3)/2$ by Theorem 6, $|E(\overline{H})| = n - 1$. By Lemma 1, for any two non-adjacent vertices in \overline{H} , we have

$$|N_{\overline{H}}(x) \cup N_{\overline{H}}(y)| \geq p + 4 - p - 1 = 3. \tag{1}$$

Claim 1 \overline{H} is disconnected.

To the contrary, suppose \overline{H} is connected. Since \overline{H} has order n and size $n - 1$, it is a tree [6, p.68]. Then \overline{H} has two distinct nonadjacent leaves u and v , since $n \geq 7$. Now $|N_{\overline{H}}(u) \cup N_{\overline{H}}(v)| = 2 < 3$, contradicting (1). This proves Claim 1.

Claim 2 Among the components of \overline{H} , there is exactly one which is a tree. The tree is K_2 .

Since \overline{H} has order n and size $n - 1$, it has at least one component which is a tree. We assert that \overline{H} has exactly one such component. To the contrary, suppose T_1 and T_2 are two distinct components which are trees. Then T_1 has a vertex s and T_2 has a vertex t such that both s and t have degree at most 1. We have $|N_{\overline{H}}(s) \cup N_{\overline{H}}(t)| \leq 2$, contradicting (1). Let T be the unique component of \overline{H} which is a tree. If the order of T is at least 3, then as in the above proof of Claim 1 we would obtain a contradiction. Hence T has order at most 2. On the other hand, T cannot have order 1. Otherwise T is an isolated vertex and \overline{H} has a vertex w other than T which has degree at most 2, since the degree sum of \overline{H} is $2(n - 1)$. Then we have $|N_{\overline{H}}(T) \cup N_{\overline{H}}(w)| \leq 2$, contradicting (1). It follows that $T = K_2$.

Claim 3 $\overline{H} = K_2 + C_{i_1} + C_{i_2} + \dots + C_{i_t}$ where $i_j \neq 4$ for each j .

By Claim 2, $\overline{H} = K_2 + H_1 + H_2 + \dots + H_t$ where H_j is a component of \overline{H} with order i_j . Let $x \in V(K_2)$ and let $y \in V(H_j)$. Then the inequality (1) implies that $\deg_{\overline{H}}(y) \geq 2$. Let v_1, v_2, \dots, v_n be the vertices of \overline{H} with $\deg_{\overline{H}}(v_1) = \deg_{\overline{H}}(v_2) = 1$ and $2 \leq \deg_{\overline{H}}(v_3) \leq \dots \leq \deg_{\overline{H}}(v_n)$. We have $2(n - 1) = \sum_{i=1}^n \deg_{\overline{H}}(v_i) \geq 2 \times 1 + (n - 2) \times 2 = 2(n - 1)$, implying that $\deg_{\overline{H}}(v_i) = 2$ for each $i = 3, \dots, n$. Hence H_j is a cycle for every $j = 1, 2, \dots, t$. But none of these cycles can be C_4 . To the contrary, suppose $H_s = C_4 = abcd$. Then $|N_{\overline{H}}(a) \cup N_{\overline{H}}(c)| = 2$, contradicting (1). This shows Claim 3 and completes the proof. \square

We determine the Turán number $\text{ex}(p + 5, B_p)$ in the following result.



Theorem 8 *Let p be a positive integer. Then*

$$\text{ex}(p + 5, B_p) = \begin{cases} \frac{(p+2)(p+5)}{2} & \text{if } p \equiv 1 \pmod{3} \\ \frac{(p+1)(p+6)}{2} & \text{if } p \equiv 0 \text{ or } 2 \pmod{3}. \end{cases}$$

Proof Let G be a graph of order $p + 5$ and size e with vertices v_1, \dots, v_{p+5} such that $\deg(v_i) = d_i, i = 1, \dots, p + 5$ and $d_1 \geq d_2 \geq \dots \geq d_{p+5}$.

(1) $p \equiv 1 \pmod{3}$. Suppose $e \geq ((p + 2)(p + 5)/2) + 1$. We will show that G contains B_p . We have $\sum_{i=1}^{p+5} d_i = 2e \geq p^2 + 7p + 12$. We distinguish two cases.

Case 1. $d_1 = p + 4$.

If $d_2 \leq p$, then $\sum_{i=1}^{p+5} d_i \leq p + 4 + (p + 4)p = p^2 + 5p + 4 < p^2 + 7p + 12$, a contradiction. Hence $d_2 \geq p + 1$ and consequently G contains p triangles sharing the common edge $v_1 v_2$.

Case 2. $d_1 \leq p + 3$.

If $d_2 \leq p + 2$, then $\sum_{i=1}^{p+5} d_i \leq p + 3 + (p + 4)(p + 2) = p^2 + 7p + 11 < p^2 + 7p + 12$, a contradiction. Thus $d_1 = d_2 = p + 3$. If $d_3 \leq p + 1$, then $\sum_{i=1}^{p+5} d_i \leq 2(p + 3) + (p + 3)(p + 1) = p^2 + 6p + 9 < p^2 + 7p + 12$, a contradiction. Hence $d_3 \geq p + 2$. If v_1 and v_2 are adjacent, then G contains at least $p + 1$ triangles sharing the common edge $v_1 v_2$. If v_1 and v_2 are nonadjacent, then G contains at least p triangles sharing the common edge $v_1 v_3$. This shows that $\text{ex}(p + 5, B_p) \leq (p + 2)(p + 5)/2$.

On the other hand, the graph $G_2 = ((p + 5)/3)K_3$ has order $p + 5$ and size $(p + 2)(p + 5)/2$, and using Lemma 1 it is easy to verify that G_2 does not contain B_p . Thus, $\text{ex}(p + 5, B_p) = (p + 2)(p + 5)/2$.

(2) $p \equiv 0$ or $2 \pmod{3}$. Suppose $e \geq ((p + 1)(p + 6)/2) + 1$. We will show that G contains B_p . We have $\sum_{i=1}^{p+5} d_i = 2e \geq p^2 + 7p + 8$. We distinguish three cases.

Case 1. $d_1 = p + 4$.

The proof for this case is similar to Case 1 of (1) above.

Case 2. $d_1 = p + 3$.

If $d_3 \leq p + 1$, then $\sum_{i=1}^{p+5} d_i \leq 2(p + 3) + (p + 3)(p + 1) = p^2 + 6p + 9 < p^2 + 7p + 8$, a contradiction. Hence $d_3 \geq p + 2$. There are two possible values of d_2 . If $d_2 = p + 3$, as in Case 2 of (1) above we can show that G contains B_p . If $d_2 = p + 2$, we have $d_2 = d_3 = p + 2$. Since $d_1 = p + 3$, v_1 is adjacent to at least one of v_2 and v_3 . Without loss of generality, suppose v_1 is adjacent to v_2 . Now G contains p triangles sharing the common edge $v_1 v_2$.

Case 3. $d_1 \leq p + 2$.

If $d_{p+3} \leq p + 1$, then $\sum_{i=1}^{p+5} d_i \leq (p + 2)(p + 2) + 3(p + 1) = p^2 + 7p + 7 < p^2 + 7p + 8$, a contradiction. Hence $d_1 = d_2 = \dots = d_{p+3} = p + 2$.

Subcase 3.1. $d_{p+5} = p + 2$. Now $\deg_{\overline{G}}(v_1) = \dots = \deg_{\overline{G}}(v_{p+5}) = 2$. Thus \overline{G} is a cycle or a union of vertex-disjoint cycles. The condition $p \equiv 0$ or $2 \pmod{3}$ implies that the order $p + 5$ is not divisible by 3. It follows that \overline{G} has a component which is a cycle of length at least 4. By Lemma 2, G contains B_p .

Subcase 3.2. $d_{p+5} \leq p + 1$. The condition $\sum_{i=1}^{p+5} d_i \geq p^2 + 7p + 8$ yields that $d_{p+4} \geq p + 1$ and that if $d_{p+4} = p + 1$ then $d_{p+5} = p + 1$. First suppose in G , every non-neighbor of v_{p+5} has degree $p + 2$. Since $\deg_{\overline{G}}(v_{p+5}) \geq 3$, we deduce that v_{p+5} has two adjacent non-neighbors, say v_1 and v_2 , in G . Then $|N_{\overline{G}}(v_1) \cup N_{\overline{G}}(v_2)| \leq 3 < 4 = n - p - 1$. By Lemma 1, G contains B_p .

It remains to consider the case when in G , v_{p+5} has at least one non-neighbor whose degree is not $p + 2$. Such a non-neighbor can only be v_{p+4} , since $d_1 = d_2 = \dots = d_{p+3} = p + 2$. Then the degree sum condition implies that $d_{p+4} = d_{p+5} = p + 1$. Let x and y be the two non-neighbors of v_{p+5} besides v_{p+4} in G . If x and y are adjacent in G , then $|N_{\overline{G}}(x) \cup N_{\overline{G}}(y)| \leq 3 < 4 = n - p - 1$. By Lemma 1, G contains B_p . Similarly, if the two non-neighbors w and z of v_{p+4} besides v_{p+5} are adjacent in G , then G contains B_p . Next suppose in G , x and y are non-adjacent and w and z are non-adjacent. Note that $|\{x, y, w, z\}| = 4$, since each of these four vertices has degree 2 in \overline{G} . Using the conditions that in \overline{G} , v_{p+4} and v_{p+5} have degree 3, and all other vertices have degree 2, we deduce that \overline{G} contains a component F of order 6 depicted in Figure 2 where the two cut-vertices are v_{p+4} and v_{p+5} .

The graph $\overline{G} - V(F)$ is a regular graph of degree 2 and has order $p - 1$ which is not divisible by 3. It follows that $\overline{G} - V(F)$ and hence \overline{G} contains a component which is a cycle of length at least 4. By Lemma 2 we deduce that G contains B_p .

So far we have proved that $\text{ex}(p + 5, B_p) \leq (p + 1)(p + 6)/2$. Next we show that this upper bound can be attained.



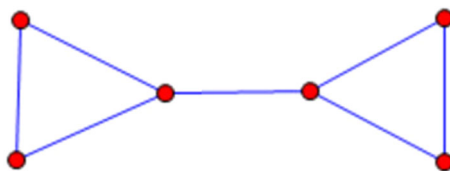


Fig. 2 The component F

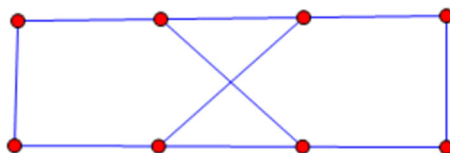


Fig. 3 The graph W

First suppose $p \equiv 0 \pmod{3}$. Let W denote the graph of order 8 in Figure 3.

Denote $G_3 = ((p - 3)/3)K_3 + W$. Then G_3 has order $p + 5$ and size $(p + 1)(p + 6)/2$, and by using Lemma 1 we can verify that G_3 does not contain B_p .

Next suppose $p \equiv 2 \pmod{3}$. Denote $G_4 = ((p + 1)/3)K_3 + K_4$. Then G_4 has order $p + 5$ and size $(p + 1)(p + 6)/2$, and G_4 does not contain B_p . This completes the proof. \square

We need the following three lemmas to characterize the extremal graphs for the Turán number $\text{ex}(p + 5, B_p)$.

Lemma 9 *Let G be a graph of order $p + 5$ not containing B_p . If x and y are two distinct vertices with $\deg_{\overline{G}}(x) = \deg_{\overline{G}}(y) = 2$ and the distance between them in \overline{G} is at most 2, then x and y are adjacent in \overline{G} .*

Proof To the contrary, suppose that x and y are nonadjacent in \overline{G} . Then by Lemma 1, $|N_{\overline{G}}(x) \cup N_{\overline{G}}(y)| \geq (p + 5) - p - 1 = 4$. Also, the condition that the distance between x and y in \overline{G} is at most 2 implies that $N_{\overline{G}}(x) \cap N_{\overline{G}}(y) \neq \emptyset$. We have

$$|N_{\overline{G}}(x) \cup N_{\overline{G}}(y)| = |N_{\overline{G}}(x)| + |N_{\overline{G}}(y)| - |N_{\overline{G}}(x) \cap N_{\overline{G}}(y)| \leq 2 + 2 - 1 = 3,$$

a contradiction. \square

Lemma 10 *Let G be a graph of order $p + 5$ not containing B_p . If P is a path in \overline{G} and every internal vertex of P has degree 2 in \overline{G} , then the length of P is at most 3.*

Proof To the contrary, suppose that $P = v_1v_2 \dots v_k$ where $k \geq 5$ and in \overline{G} , each v_i has degree 2 for $i = 2, 3, \dots, k - 1$. But then

$$|N_{\overline{G}}(v_2) \cup N_{\overline{G}}(v_4)| = 3 < 4 = (p + 5) - p - 1,$$

contradicting Lemma 1. \square

Lemma 11 *Let G be a graph of order $p + 5$ and size $\text{ex}(p + 5, B_p)$ not containing B_p where $p \geq 4$.*

- (1) *If $p \equiv 1 \pmod{3}$, then \overline{G} is 2-regular;*
- (2) *if $p \equiv 0$ or $2 \pmod{3}$, then the degree sequence of \overline{G} is $3, 3, 3, 3, 2, \dots, 2$.*

Proof Let v_1, \dots, v_{p+5} be the vertices of G with $\deg_{\overline{G}}(v_i) = d_i$, $i = 1, \dots, p + 5$ such that $d_1 \geq d_2 \geq \dots \geq d_{p+5}$.

- (1) $p \equiv 1 \pmod{3}$. By Theorem 8, $|E(G)| = (p + 2)(p + 5)/2$, implying that \overline{G} has size $p + 5$. If \overline{G} has an isolated vertex x , then by Lemma 1 we deduce that for any vertex $u \in V(G) \setminus \{x\}$, $\deg_{\overline{G}}(u) \geq 4$. Hence $2p + 10 = 2|E(\overline{G})| = \sum_{i=1}^{p+5} \deg_{\overline{G}}(v_i) \geq 0 + (p + 4) \times 4 = 4p + 16$, a contradiction. If \overline{G} has a leaf y , let v be the neighbor of y in \overline{G} . Then by Lemma 1, for any vertex $w \in V(G) \setminus \{y, v\}$, $\deg_{\overline{G}}(w) \geq 3$. We have $2p + 10 = 2|E(\overline{G})| = \sum_{i=1}^{p+5} \deg_{\overline{G}}(v_i) \geq 1 + 1 + (p + 3) \times 3 = 3p + 11$, a contradiction. Thus $\delta(\overline{G}) \geq 2$. Now the inequality $2p + 10 = 2|E(\overline{G})| = \sum_{i=1}^{p+5} \deg_{\overline{G}}(v_i) \geq (p + 5) \times 2 = 2p + 10$ forces $\deg_{\overline{G}}(v_i) = 2$ for each $i = 1, \dots, p + 5$; i.e., \overline{G} is 2-regular.



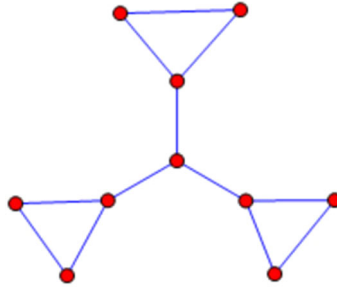


Fig. 4 The graph Q

(2) $p \equiv 0$ or $2 \pmod{3}$. By Theorem 8, \bar{G} has size $p + 7$. As in the above proof of part (1), we deduce that $\delta(\bar{G}) \geq 2$. If $\delta(\bar{G}) \geq 3$, then $2p + 14 = 2|E(\bar{G})| = \sum_{i=1}^{p+5} \deg_{\bar{G}}(v_i) \geq (p + 5) \times 3 = 3p + 15$, a contradiction. Hence $\delta(\bar{G}) = 2$. From $2p + 14 = 2|E(\bar{G})| = \sum_{i=1}^{p+5} \deg_{\bar{G}}(v_i) \geq (p + 4) \times 2 + \Delta(\bar{G})$ we obtain $\Delta(\bar{G}) \leq 6$.

If $\Delta(\bar{G}) = 6$, then the degree sequence of \bar{G} is $6, 2, \dots, 2$; if $\Delta(\bar{G}) = 5$, then the degree sequence of \bar{G} is $5, 3, 2, \dots, 2$; if $\Delta(\bar{G}) = 4$, then the degree sequence of \bar{G} is $4, 4, 2, \dots, 2$ or $4, 3, 3, 2, \dots, 2$. In the first three cases, i.e., \bar{G} has degree sequence $6, 2, \dots, 2$, or $5, 3, 2, \dots, 2$, or $4, 4, 2, \dots, 2$, v_1 has at least three neighbors of degree 2 in \bar{G} which are pairwise adjacent by Lemma 9, a contradiction. Now suppose \bar{G} has degree sequence $4, 3, 3, 2, \dots, 2$. We will show that this cannot occur.

By the above argument, v_1 has at most two neighbors of degree 2 in \bar{G} . Hence $v_2, v_3 \in N_{\bar{G}}(v_1)$. Without loss of generality, suppose $N_{\bar{G}}(v_1) = \{v_2, v_3, v_4, v_5\}$. By Lemma 9, v_4 and v_5 are adjacent in \bar{G} . We distinguish two cases.

Case 1. v_2 and v_3 are adjacent in \bar{G} .

Subcase 1.1. $|N_{\bar{G}}(v_2) \cap N_{\bar{G}}(v_3)| \geq 2$. Since $d_2 = d_3 = 3$, we have $|N_{\bar{G}}(v_2) \cap N_{\bar{G}}(v_3)| = 2$. Without loss of generality, suppose $N_{\bar{G}}(v_2) \cap N_{\bar{G}}(v_3) = \{v_1, v_6\}$. Then $H = \bar{G}[v_1, v_2, v_3, v_4, v_5, v_6]$ is a component of \bar{G} and $\bar{G} - V(H)$ is a 2-regular graph of order $p - 1$. Since 3 does not divide $p - 1$, \bar{G} contains a component which is a cycle of length at least 4. By Lemma 2, G contains B_p , a contradiction.

Subcase 1.2. $|N_{\bar{G}}(v_2) \cap N_{\bar{G}}(v_3)| = 1$. Without loss of generality, suppose $N_{\bar{G}}(v_2) = \{v_1, v_3, v_6\}$ and $N_{\bar{G}}(v_3) = \{v_1, v_2, v_7\}$. By Lemma 10, v_6 and v_7 are adjacent in \bar{G} . Now v_3 and v_6 are adjacent in G , and $|N_{\bar{G}}(v_3) \cup N_{\bar{G}}(v_6)| = 3 < 4 = (p + 5) - p - 1$. By Lemma 1, G contains B_p , a contradiction.

Case 2. v_2 and v_3 are nonadjacent in \bar{G} .

Without loss of generality, suppose $N_{\bar{G}}(v_2) = \{v_1, v_6, v_7\}$ and $N_{\bar{G}}(v_3) = \{v_1, v_8, v_9\}$. By Lemma 9, v_6 and v_7 are adjacent in \bar{G} , and v_8 and v_9 are adjacent in \bar{G} . Thus, $R = \bar{G}[v_1, v_2, \dots, v_9]$ is a component of \bar{G} , and $\bar{G} - V(R)$ is a 2-regular graph of order $p - 4$. Since $p - 4$ is not divisible by 3, \bar{G} has a component which is a cycle of length at least 4. By Lemma 2, G contains B_p , a contradiction.

We have proved that $\Delta(\bar{G}) \leq 3$. Since $\delta(\bar{G}) = 2$ and \bar{G} has size $p + 7$, we deduce that the degree sequence of \bar{G} is $3, 3, 3, 3, 2, \dots, 2$.

We denote by Q the graph of order 10 in Figure 4.

We characterize the extremal graphs for the Turán number $\text{ex}(p + 5, B_p)$ in the following result.

Theorem 12 *Let p be an integer with $p \geq 4$ and let G be a graph of order $p + 5$ not containing B_p . Then G has size $\text{ex}(p + 5, B_p)$ if and only if*

- (1) $G = \overline{((p - 3)/3)K_3} + W$ where W is the graph depicted in Figure 3 if $p \equiv 0 \pmod{3}$;
- (2) $G = \overline{((p + 5)/3)K_3}$ if $p \equiv 1 \pmod{3}$;
- (3) $G = \overline{((p + 1)/3)K_3} + K_4$ or $G = \overline{((p - 5)/3)K_3} + Q$ where Q is the graph depicted in Figure 4 if $p \equiv 2 \pmod{3}$.

Proof The sufficiency can be easily verified by using Lemma 1. Now we prove the necessity. Suppose G has size $\text{ex}(p + 5, B_p)$. We will consider the complement graph \bar{G} .

We first prove part (2) where $p \equiv 1 \pmod{3}$. By Lemma 11, \bar{G} is 2-regular, or equivalently \bar{G} is a union of vertex-disjoint cycles. By Lemma 2, every cycle must be a triangle. Thus $\bar{G} = ((p + 5)/3)K_3$.



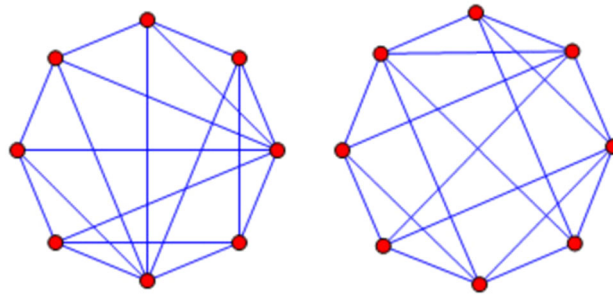


Fig. 5 The two extremal graphs for $\text{ex}(8, B_3) = 18$

Next we treat parts (1) and (3). Suppose $p \equiv 0$ or $2 \pmod{3}$. By Lemma 11, the degree sequence of \overline{G} is $3, 3, 3, 3, 2, \dots, 2$. By Lemma 9, in \overline{G} each vertex of degree 3 must have at least one neighbor of degree 3.

Claim The four vertices of degree 3 lie in one component of \overline{G} and there is a vertex of degree 3 with at least two neighbors of degree 3.

To the contrary, suppose the four vertices of degree 3 do not lie in one component of \overline{G} . Then they lie in two components and by Lemma 9 we deduce that each of these two components is the graph F of order 6 in Fig. 2. Every component of \overline{G} other than these two components is a cycle and at least one of these cycles has length at least 4, since the order $p + 5$ of \overline{G} is not divisible by 3. But this contradicts Lemma 2. Hence the first conclusion in the Claim is proved. Now we prove the second conclusion. To the contrary, suppose in \overline{G} every vertex of degree 3 has at most one and hence exactly one neighbor of degree 3. In \overline{G} let x be a vertex of degree 3 with exactly one neighbor y of degree 3. Let $N_{\overline{G}}(x) = \{y, u, v\}$. Note that both u and v have degree 2 and the distance between u and v is at most 2 in \overline{G} . By Lemma 9, u and v are adjacent in \overline{G} . Similarly, the two neighbors of y in \overline{G} besides x are adjacent in \overline{G} . Thus \overline{G} contains a component which is the graph F in Figure 2, contradicting the first conclusion of the Claim. This proves the second conclusion.

Denote by Z the component of \overline{G} in which the four vertices of degree 3 lie. By the above properties, Z contains either a claw or a path of order 4 each of whose vertices has degree 3. In both cases, by using Lemmas 9, 10 and 2 it can be verified that Z has order at most 10. By Lemma 2, the graph $\overline{G} - V(Z)$ is a union of vertex-disjoint triangles. Hence $p + 5 = |\overline{G}| \equiv |Z| \pmod{3}$.

If $p \equiv 0 \pmod{3}$, $|Z| \in \{5, 8\}$; if $p \equiv 2 \pmod{3}$, $|Z| \in \{4, 7, 10\}$.

Suppose $p \equiv 0 \pmod{3}$. By Lemma 1, $|Z| \neq 5$. If $|Z| = 8$, then Z is the graph W in Fig. 3. It follows that $\overline{G} = ((p - 3)/3)K_3 + W$. This proves part (1).

Suppose $p \equiv 2 \pmod{3}$. By Lemma 1, $|Z| \neq 7$. If $|Z| = 4$, then $Z = K_4$ and if $|Z| = 10$, then $Z = Q$. This proves part (3). The proof is complete. \square

Theorem 12 does not include the small values $p = 1, 2, 3$. The information about them is as follows. $K_{3,3}$ is the unique extremal graph for $\text{ex}(6, B_1) = 9$. $K_{3,4}$ is the unique extremal graph for $\text{ex}(7, B_2) = 12$. There are two extremal graphs for $\text{ex}(8, B_3) = 18$ which are depicted in Figure 5.

We determine the Turán number $\text{ex}(p + 6, B_p)$ in the following result.

Theorem 13 *Let p be a positive integer.*

- (1) *If p is odd and $p \geq 5$, then $\text{ex}(p + 6, B_p) = (p + 3)(p + 5)/2$;*
- (2) *if p is even and $p \neq 2, 6, 10$, then $\text{ex}(p + 6, B_p) = 1 + (p + 2)(p + 6)/2$.*

Proof Let G be a graph of order $p + 6$ and size e with vertices v_1, \dots, v_{p+6} such that $\text{deg}(v_i) = d_i$, $i = 1, \dots, p + 6$ and $d_1 \geq d_2 \geq \dots \geq d_{p+6}$.

(1) p is odd and $p \geq 5$. Suppose $e \geq ((p + 3)(p + 5)/2) + 1$. We will show that G contains B_p . We have $\sum_{i=1}^{p+6} d_i = 2e \geq p^2 + 8p + 17$. We distinguish three cases.

Case 1. $d_1 = p + 5$.

If $d_2 \leq p$, then $\sum_{i=1}^{p+6} d_i \leq p + 5 + (p + 5) \times p = p^2 + 6p + 5 < p^2 + 8p + 17$, a contradiction. Hence $d_2 \geq p + 1$ and G contains p triangles sharing the common edge $v_1 v_2$.

Case 2. $d_1 = p + 4$.

If $d_3 \leq p + 2$, then $\sum_{i=1}^{p+6} d_i \leq 2(p + 4) + (p + 4)(p + 2) = p^2 + 8p + 16 < p^2 + 8p + 17$, a contradiction. Hence $d_3 \geq p + 3$. v_1 has at least one neighbor in $\{v_2, v_3\}$. If v_1 is adjacent to v_2 , then $|N(v_1) \cap N(v_2)| \geq p + 1$



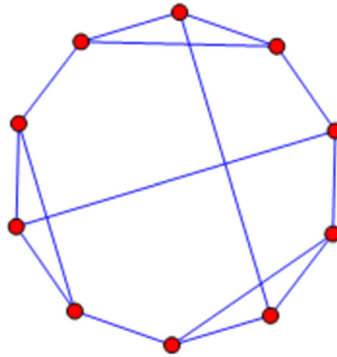


Fig. 6 The graph Y

and thus G contains $p + 1$ triangles sharing the common edge $v_1 v_2$; if v_1 is adjacent to v_3 , then similarly we see that G contains p triangles sharing the common edge $v_1 v_3$.

Case 3. $d_1 \leq p + 3$.

If $d_4 \leq p + 2$, then $\sum_{i=1}^{p+6} d_i \leq 3(p + 3) + (p + 3)(p + 2) = p^2 + 8p + 15 < p^2 + 8p + 17$, a contradiction. Hence $d_4 \geq p + 3$ and $d_1 = d_2 = d_3 = d_4 = p + 3$. v_1 has at least one neighbor in $\{v_2, v_3, v_4\}$. Without loss of generality, suppose v_1 and v_2 are adjacent. Then $|N(v_1) \cap N(v_2)| \geq p$ and G contains p triangles sharing the common edge $v_1 v_2$.

We have proved that $\text{ex}(p + 6, B_p) \leq (p + 3)(p + 5)/2$. Next we construct graphs to show that this upper bound can be attained. First suppose $p \equiv 1 \pmod{4}$. The graph $G_5 = \overline{K_3 + ((p + 3)/4)K_4}$ has order $p + 6$, size $(p + 3)(p + 5)/2$ and by using Lemma 1 we can verify that G_5 does not contain B_p . Now suppose $p \equiv 3 \pmod{4}$. Denote by Y the graph of order 10 in Figure 6.

Then the graph $G_6 = \overline{K_3 + ((p - 7)/4)K_4 + Y}$ has order $p + 6$ and size $(p + 3)(p + 5)/2$, and G_6 does not contain B_p . This proves $\text{ex}(p + 6, B_p) = (p + 3)(p + 5)/2$.

(2) p is even and $p \neq 2, 6, 10$. Suppose $e \geq 1 + ((p + 2)(p + 6)/2) + 1$. We will show that G contains B_p . We have $\sum_{i=1}^{p+6} d_i = 2e \geq p^2 + 8p + 16$. We distinguish three cases.

Case 1. $d_1 = p + 5$.

If $d_2 \leq p$, then $\sum_{i=1}^{p+6} d_i \leq p + 5 + (p + 5)p = p^2 + 6p + 5 < p^2 + 8p + 16$, a contradiction. Thus $d_2 \geq p + 1$ and G contains p triangles sharing the common edge $v_1 v_2$.

Case 2. $d_1 = p + 4$.

If $d_3 \leq p + 1$, then $\sum_{i=1}^{p+6} d_i \leq 2(p + 4) + (p + 4)(p + 1) = p^2 + 7p + 12 < p^2 + 8p + 16$, a contradiction. Hence $d_2 \geq d_3 \geq p + 2$. v_1 has at least one neighbor in $\{v_2, v_3\}$ and consequently G contains p triangles sharing the common edge $v_1 v_2$ or $v_1 v_3$.

Case 3. $d_1 \leq p + 3$.

If $d_4 \leq p + 2$, then $\sum_{i=1}^{p+6} d_i \leq 3(p + 3) + (p + 3)(p + 2) = p^2 + 8p + 15 < p^2 + 8p + 16$, a contradiction. Hence $d_4 \geq p + 3$. In fact, $d_1 = d_2 = d_3 = d_4 = p + 3$. It follows that G contains B_p .

We have proved that $\text{ex}(p + 6, B_p) \leq 1 + (p + 2)(p + 6)/2$. Next we show that this upper bound can be attained.

First suppose $p \equiv 0 \pmod{4}$. Denote by S the graph of order 7 in Fig. 7.

Then the graph $G_7 = \overline{K_3 + ((p - 4)/4)K_4 + S}$ has order $p + 6$ and size $1 + (p + 2)(p + 6)/2$, and by using Lemma 1 we can verify that G_7 does not contain B_p .

Now suppose $p \equiv 2 \pmod{4}$. We have $p \geq 14$. The graph

$$G_8 = \overline{K_3 + ((p - 14)/4)K_4 + S + Y}$$

has order $p + 6$ and size $1 + (p + 2)(p + 6)/2$, and G_8 does not contain B_p . This completes the proof. \square

Theorem 13 does not include the small values $p = 2, 3, 6, 10$. The information about them is as follows. $\text{ex}(8, B_2) = 16$, $\text{ex}(9, B_3) = 21$, $\text{ex}(12, B_6) = 48$ and $\text{ex}(16, B_{10}) = 96$.

It seems difficult to characterize the extremal graphs for $\text{ex}(p + 6, B_p)$. A computer search shows that there are 16 extremal graphs for $\text{ex}(9, B_3) = 21$ and there are 20 extremal graphs for $\text{ex}(12, B_6) = 48$.



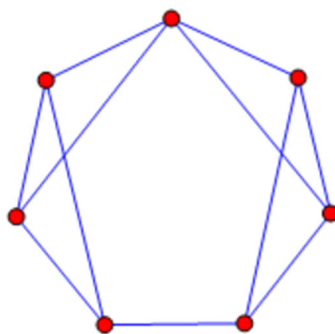


Fig. 7 The graph S

The results in this paper suggest that perhaps there is no uniform formula for the Turán number $ex(n, B_p)$ for all pairs (n, p) .

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