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ON SEMILOCAL CONVERGENCE OF INEXACT NEWTON METHODS *1)

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Abstract

Inexact Newton methods are constructed by combining Newton's method with another iterative method that is used to solve the Newton equations inexactly. In this paper, we establish two semilocal convergence theorems for the inexact Newton methods. When these two theorems are specified to Newton's method, we obtain a different Newton-Kantorovich theorem about Newton's method. When the iterative method for solving the Newton equations is specified to be the splitting method, we get two estimates about the iteration steps for the special inexact Newton methods.

Mathematics subject classification: 65H10.
Key words: Banach space, Systems of nonlinear equations, Newton's method, The splitting method, Inexact Newton methods.

1. Introduction

Consider the system of nonlinear equations

$$F(x) = 0, \tag{1.1}$$

where F: $\mathcal{D} \subset \mathcal{X} \to \mathcal{Y}$ is a continuously differentiable operator mapping an open convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$. Newton's method,

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n = 0, 1, \cdots, \tag{1.2}$$

is the most classical method among a great deal of iterative methods used to solve (1.1). There are many research works on the existence and uniqueness of the solution of (1.1) (see [23, 24, 27]), and the convergence of Newton's method (see [12, 15, 16, 17, 20, 23, 24, 26, 27, 28] and the references therein) in addition to the classic Newton-Kantorovich theorem (see [19, 22]). These results are distinguished into two classes. One is about local convergence discussing the properties of Newton's method as x is sufficiently close to the solution x^* of (1.1) (see [24, 26, 27]), and another is about semilocal convergence which only deals with the initial point x_0 (see [15, 16, 17, 20, 23, 28]).

In order to get the $(n+1)$ iteration x_{n+1} in Newton's method, we need solve the Newton equation

$$F'(x_n)s_n = -F(x_n), \tag{1.3}$$

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and then obtain $x_{n+1} = x_n + s_n$, see (1.2). In fact, (1.3) is a system of linear equations in the form

$$Ax = b. \tag{1.4}$$

In principle, there are two groups of methods for the solution of linear systems (1.4).

One group of methods are the so-called direct methods, or elimination methods, that is, the exact solution is determined through a finite number of arithmetic operations (in real arithmetic without considering the roundoff errors) (see [21]). It is not efficient to obtain the exact solution of (1.4) by using a direct method such as Gaussian elimination, if the coefficient matrix A is large and sparse; and when the iterate x_n is far from x^* the iteration sequence $\{x_n\}$ may not converge to x^*. We should mention that for the large sparse system of nonlinear equations, Bai and Wang [10, 11] established the following sparse factorization update algorithm based on matrix triangular factorization:

$$\begin{cases} given \quad x_0, \\ x_{n+1} = x_n + s_n, \\ U_n s_n = -H_n F(x_n), \\ H_0^{-1} U_0 \quad is \quad an \quad approximation \quad to \quad F'(x_0), \end{cases}$$

where U_n is an upper triangular matrix and H_n a unit low triangular matrix generated by two recursion formulas. As now the matrix A in (1.4) is automatically factorized into the form $A = H_n^{-1} U_n$, we can directly solve the Newton equation (1.3) by solving an upper triangular linear system of the coefficient matrix U_k at each iteration step of Newton's method.

Another group is the iterative methods, which results in the two-stage method, or sometimes termed as inner/outer iterations, for solving the system of nonlinear equations (1.1). In the two-stage method, Newton's method is the outer iteration, while an iterative method which is used to solve the Newton equations is the inner iteration. Particularly, in the two cases described below, two classes of two-stage iterative methods have been established and analyzed.

One case is that the nonlinear mapping $F(x)$ is a mildly nonlinear mapping, i.e.,

$$F(x) = Ax - \phi(x),$$

where $A \in \mathcal{R}^{n\times n}$ is nonsingular and $\phi : \mathcal{R}^n \to \mathcal{R}^n$ is a nonlinear diagonal function with certain local smoothness properties [5, 6, 8, 22]. Another case is that $F(x)$ is a linear mapping, i.e.,

$$F(x) = Ax - b,$$

where $A \in \mathcal{R}^{n\times n}$ is a large sparse and, possibly, a very ill-conditioned symmetric positive definite matrix, $x \in \mathcal{R}^n$ and $b \in \mathcal{R}^n$ (see [13]). See also [7] for efficient splitting method and its inexact variant for non-Hermitian positive definite linear systems.

When we solve the Newton equations with an iterative method, some residuals will be given. This is the reason we call this process the inexact Newton methods. To sum up, inexact Newton methods have the form

$$x_{n+1} = x_n + s_n, \quad n = 0, 1, \cdots , \tag{1.5}$$

where the step s_n satisfies

$$F'(x_n) s_n = -F(x_n) + r_n, \quad n = 0, 1, \cdots , \tag{1.6}$$

for some residual sequence $\{r_n\} \subseteq \mathcal{Y}$ (see [14]).

It is necessary to study when the iterations can be stopped and how to choose a forcing sequence in the above process (see [1, 2, 3, 7, 9, 14, 18, 29]). Most of the results gave the local convergence (see [1, 3, 14, 18, 29]), among them a fundamental result giving below can be found in [14].

Theorem 1.1 (see [14], Th.2.3). *Let F: $\mathcal{R}^N \to \mathcal{R}^N$ be a nonlinear mapping with the following properties:*

(i) $\dfrac{\|r_n\|}{\|F(x_n)\|} \leq \eta_n$; (1.7)

(ii) *there exists an* $x^* \in \mathcal{R}^N$ *such that* $F(x^*) = 0$;

(iii) F *is continuously differentiable in a neighborhood of* x^*, *and* $F'(x^*)$ *is nonsingular.*

Assume that $\eta_n \leq \eta_{max} < t < 1$. *Then there exists* $\epsilon > 0$ *such that if* $\|x_0 - x^*\| \leq \epsilon$ *then the sequence* $\{x_n\}$ *of inexact Newton methods converges to* x^*. *Moreover, the convergence is linear in the sense that*

$$\|x_{n+1} - x^*\|_* \leq t\|x_n - x^*\|_*, \quad n \geq 0,$$

where $\|y\|_* \equiv \|F'(x^*)y\|$.

In this theorem, we do not know the exact value of ϵ. Argyros [1, 3] suppose that $F''(x)$ satisfies a Lipschitz condition, and Huang [18] suppose $F''(x)$ satisfies the first order-γ condition (see [25]). In [29], the affine invariant condition

$$\frac{\|F'(x_i)^{-1}r_i\|}{\|F'(x_i)^{-1}F(x_i)\|} \leq \nu_i \leq \nu, \quad i = 0, 1, \cdots,$$

is satisfied instead of the condition $\frac{\|r_n\|}{\|F(x_n)\|} \leq \eta_n$ used in [1, 3, 18], which makes the method become an affine invariant one.

The above results concerning local convergence are not applicable in practical computations. To be more practical, a semilocal convergence theorem with Lipschitz-type conditions on the second Fréchet derivative was given in [2]. In [9], the assumptions

$$\begin{cases} \| F'(x) - F'(y) \| \leq L \| x - y \|, \\ \| F'(x)^{-1} \| \leq \beta, \\ \frac{\|r_n\|}{\|F(x_n)\|} \leq \eta_n, \end{cases}$$

are presented for (1.5) and (1.6), and the assumptions

$$\begin{cases} \|F'(y)^{-1}(F'(x + t(y - x)) - F'(x))\| \leq L \| y - x \|, \\ \| F'(x_0)^{-1}F(x_0) \| \leq \eta, \\ \frac{\|r_n\|}{\|F'(x_n)^{-1}F(x_n)\|} \leq \nu_n, \end{cases}$$

combined with the affine invariant property are provided for the improved inexact Newton methods as follows:

$$\begin{cases} x_{n+1} = x_n + s_n, \\ F'(x_n)s_n = -F(x_n) + F'(x_n)r_n, \quad n = 0, 1, \cdots. \end{cases}$$

We try to establish a new semilocal convergence theorem in a way as brief as that in Newton-Kantorovich theorem. In other words, the new hypotheses are based on Lipschitz-continuity on the first Fréchet derivative F' of the operator F and on the condition $\|F'(x_0)^{-1}F(x_0)\| \leq \beta$. We can not establish this new semilocal convergence theorem using the technique that was used to prove the local convergence theorems as they heavily depend on the assumption $F(x^*) = 0$.

The technique used in the proof of Theorem 1 in [9], however, is very useful and can help us to obtain our new semilocal convergence theorem.

In this paper, we first demonstrate this new semilocal convergence theorem. Then, we replace the original assumption on the residual sequence by

$$\|F'(x_0)^{-1}(r_n - r_{n-1})\| \le \eta_n \|x_n - x_{n-1}\|,$$

and give another new semilocal convergence theorem. In particular, the well-known Newton-Kantorovich theorem is a special case of our second theorem when the latter is applied to Newton's method. The reduction of our first theorem to Newton's method is different from the Newton-Kantorovich theorem. Moreover, by combining Newton's method with the splitting method for the Newton equations, we get a special class of two-stage methods of the inexact Newton methods, called as the Newton-splitting method, for which we provide two types of convergence theorems in which the number of inner iteration steps for the Newton equations is shown to be different from that in [4, 30].

2. Semilocal Convergence Theorems

In this section, we establish two new Kantorovich-type semilocal convergence theorems under the assumptions

$$\frac{\|F'(x_0)^{-1}r_n\|}{\|F'(x_0)^{-1}F(x_n)\|} \le \eta_n$$

and

$$\|F'(x_0)^{-1}(r_n - r_{n-1})\| \le \eta_n \|x_n - x_{n-1}\|,$$

imposed on the residuals, respectively.

2.1. Semilocal Convergence Theorem: Type I

It is well-known that the Newton-Kantorovich hypothesis is concise. But the hypotheses in [2] are on the second F-derivative and a little intricate. The hypotheses in [9] are on the first F-derivative but dealing with the bound of $\|F'(x)^{-1}\|$. So, it is necessary to establish a new Kantorovich-type semilocal convergence theorem by utilizing the advantages of those two kinds of hypotheses. The technique used in [9] (in the proof of Theorems 1 or 2) can be used to establish this theorem.

For $x \in \mathcal{D}$ and a positive number r, let $B(x,r)$ denote an open ball with radius r and center x, and $\overline{B(x,r)}$ denotes its closure. Suppose that $F'(x_0)^{-1}$ exists and

$$\begin{cases} \|F'(x_0)^{-1}F(x_0)\| \le \beta, \\ \|F'(x_0)^{-1}(F'(x) - F'(y))\| \le \gamma \|x - y\|, \quad \forall x, y \in \mathcal{D}, \\ \frac{\|F'(x_0)^{-1}r_n\|}{\|F'(x_0)^{-1}F(x_n)\|} \le \eta_n, \quad \eta = \max\limits_n \{\eta_n\} < 1. \end{cases} \tag{2.1}$$

Then the following lemmas can be established, which are useful for proving the new semilocal convergence theorems of our inexact Newton methods.

Lemma 2.1. *Let*

$$h(s) = \gamma s^3 + \beta\gamma(1-\eta)s^2 - 2\beta(1-\eta)s + 2\beta^2(1-\eta^2).$$

If

$$\beta\gamma \le -\frac{2\eta^2 + 14\eta + 11 - \sqrt{(4\eta+5)^3}}{(1+\eta)(1-\eta)^2},$$

then $h(s)$ has two positive zeros s_1 and s_2 that satisfy

$$0 < \beta(1-\eta) \le s_1 \le \overline{s} \le s_2 < \frac{1}{\gamma},$$

where

$$\overline{s} = \frac{-\beta\gamma(1-\eta) + \sqrt{(\beta\gamma)^2(1-\eta)^2 + 6\beta\gamma(1-\eta)}}{3\gamma}.$$

Proof. Consider

$$h'(s) = 3\gamma s^2 + 2\beta\gamma(1-\eta)s - 2\beta(1-\eta).$$

We can deduce that the function $h(s)$ is nonincreasing for $0 \le s \le \overline{s}$ and nondecreasing for $s \ge \overline{s}$. In addition, we know that

$$h(0) > 0, \quad h(\frac{2}{\gamma}) > 0, \quad h(\overline{s}) \le 0,$$

by computing with Maple. Hence, it follows that

$$0 < s_1 \le \overline{s} \le s_2 < \frac{1}{\gamma}.$$

On the other hand, from $h(s_1) = 0$, we have

$$s_1 \ge \beta(1+\eta) \ge \beta(1-\eta).$$

Therefore the proof is complete.

By using some basic arithmetic operation, we have the following results.

Lemma 2.2. *Let*

$$\sigma = \frac{-\beta + \beta\eta + s_1}{\beta - \beta\eta + s_1}.$$

Then we have

$$0 \le \sigma < 1, \quad s_1 < \frac{1-\sigma}{\gamma}.$$

Lemma 2.3. *Let*

$$\phi_n(t) = \frac{1-\sigma}{1-\sigma-\gamma s_1}[\frac{\gamma}{2}t^2 + (\sigma+1)\sigma^{n-1}\beta\eta].$$

Define a sequence $\{t_n\}$ as follows:

$$\begin{cases} t_{n+1} - t_n = \phi_n(t_n - t_{n-1}), \\ t_0 = 0, \qquad t_1 = s_1. \end{cases} \tag{2.2}$$

Then $\{t_n\}$ converges to t^* monotonically increasingly, and

$$0 < t^* \le \frac{s_1}{1-\sigma}, \quad t^* - t_n \le \frac{\sigma^n}{1-\sigma}s_1.$$

Proof. Since $t_1 - t_0 = s_1$, if we suppose

$$t_k - t_{k-1} \le \sigma^{k-1}s_1, \quad k = 1, 2, \cdots n,$$

then

$$\begin{aligned} t_{n+1}-t_n &= \phi_n(t_n-t_{n-1}) \le \phi_n(\sigma^{n-1}s_1) \\ &= \frac{1-\sigma}{1-\sigma-\gamma s_1}[\frac{\gamma}{2}(\sigma^{n-1}s_1)^2+(\sigma+1)\sigma^{n-1}\beta\eta]. \end{aligned}$$

It follows from Lemma 2.2 that

$$t_{n+1}-t_n \le \sigma^{n-1}\frac{1-\sigma}{1-\sigma-\gamma s_1}[\frac{\gamma}{2}(s_1)^2+(\sigma+1)\beta\eta] \le \sigma^n s_1.$$

Hence

$$t_{n+1} = \sum_{k=0}^{n}(t_{k+1}-t_k) \le \sum_{k=0}^{n}\sigma^k s_1 \le \frac{s_1}{1-\sigma}.$$

That is, $\{t_n\}$ is monotonically increasing in $[0, \frac{s_1}{1-\sigma}]$. Thus, there exists t^* such that $\lim\limits_{n\to\infty} t_n = t^*$ and

$$t_n \le t^* \le \frac{s_1}{1-\sigma}.$$

Because

$$t_{n+m}-t_n \le (\sigma^{n+m-1}+\sigma^{n+m-2}+\cdots+\sigma^n)s_1,$$

letting $m\to+\infty$, we have

$$t^*-t_n \le \frac{\sigma^n}{1-\sigma}s_1.$$

This completes the proof of this lemma.

By Banach's theorem (see Theorem V.4.3 in [19]), the following result can be obtained directly.

Lemma 2.4. *For all $x \in B(x_0, \frac{1}{\gamma})$, $F'(x)^{-1}$ exists and satisfies*

$$\|F'(x)^{-1}F'(x_0)\| \le \frac{1}{1-\gamma\|x-x_0\|}.$$

Lemma 2.5. *For all $x_n \in B(x_0, \frac{s_1}{1-\sigma})$, it holds that*

$$\|F'(x_0)^{-1}F(x_n)\| \le \sigma^n\beta, \quad n=0,1,\cdots. \tag{2.3}$$

Proof. The result can be proved by induction. Indeed, the inequality is true for $n=0$.

Assume that (2.3) holds true for some nonnegative integer n. Then we have

$$\begin{aligned} \|F'(x_0)^{-1}F(x_{n+1})\| &= \|F'(x_0)^{-1}[F(x_{n+1})-F(x_n)-F'(x_n)(x_{n+1}-x_n)]+F'(x_0)^{-1}r_n\| \\ &\le \frac{\gamma}{2}\|x_{n+1}-x_n\|^2+\|F'(x_0)^{-1}r_n\| \\ &= \frac{\gamma}{2}\|F'(x_n)^{-1}[F(x_n)-r_n]\|^2+\|F'(x_0)^{-1}r_n\| \\ &\le \frac{\gamma}{2}\frac{1}{(1-\gamma\|x_n-x_0\|)^2}(1+\eta)^2\|F'(x_0)^{-1}F(x_n)\|^2+\eta\|F'(x_0)^{-1}F(x_n)\| \\ &\le [\frac{\gamma}{2}(\frac{1-\sigma}{1-\sigma-\gamma s_1})^2(1+\eta)^2\|F'(x_0)^{-1}F(x_n)\|+\eta]\|F'(x_0)^{-1}F(x_n)\| \\ &\le [\frac{\gamma}{2}(\frac{1-\sigma}{1-\sigma-\gamma s_1})^2(1+\eta)^2\beta+\eta]\|F'(x_0)^{-1}F(x_n)\| \\ &\le \sigma^{n+1}\beta. \end{aligned}$$

Thus (2.3) follows.

From the above lemmas, we have the following theorem.

Theorem 2.6. *Under the assumption* (2.1), *if*

$$\overline{B(x_0, \frac{s_1}{1-\sigma})} \subset \mathcal{D} \quad and \quad \beta\gamma \le -\frac{2\eta^2+14\eta+11-\sqrt{(4\eta+5)^3}}{(1+\eta)(1-\eta)^2},$$

where s_1 is the smallest positive zero of the function $h(s)$ defined in Lemma 2.1, *then the inexact Newton sequence $\{x_n\}$ converges to a solution x^* of Eq.* (1.1). *Moreover, we have the error estimate*

$$\|x_n - x^*\| \le t^* - t_n \le \frac{\sigma^n}{1-\sigma} s_1, \quad n = 0, 1, \cdots,$$

where t^* and $\{t_n\}$ are defined in Lemma 2.3.

Proof. First, we prove the following conclusions by induction:

$$\begin{cases} \|x_{n+1} - x_n\| \le t_{n+1} - t_n, \\ \|x_n - x_0\| \le \dfrac{s_1}{1-\sigma}, \quad n = 0, 1, \cdots. \end{cases} \tag{2.4}$$

For $n = 0$, we have

$$\begin{aligned} \|x_1 - x_0\| &= \|F'(x_0)^{-1}[F(x_0) - r_0]\| \le (1+\eta)\beta \\ &\le \frac{1-\sigma}{1-\sigma-\gamma s_1}(1+\eta)\beta = t_1 - t_0 \le \frac{s_1}{1-\sigma}. \end{aligned}$$

Suppose that for all $n \le k$ the conclusions hold true. Then, we have

$$\begin{aligned} &\|x_{n+1} - x_n\| = \|F'(x_n)^{-1}[F(x_n) - r_n]\| \\ &\le \|F'(x_n)^{-1}F'(x_0)\|\{\|F'(x_0)^{-1}[F(x_n) - F(x_{n-1}) - F'(x_{n-1})(x_n - x_{n-1})]\| + \|F'(x_0)^{-1}(r_{n-1} - r_n)\|\} \\ &\le \frac{1}{1-\gamma\|x_n - x_0\|}[\frac{\gamma}{2}\|x_n - x_{n-1}\|^2 + \eta\|F'(x_0)^{-1}F'(x_{n-1})\| + \eta\|F'(x_0)^{-1}F'(x_n)\|] \\ &\le \frac{1-\sigma}{1-\sigma-\gamma s_1}[\frac{\gamma}{2}\|x_n - x_{n-1}\|^2 + (\sigma+1)\sigma^{n-1}\beta\eta] \\ &= \phi_n(\|x_n - x_{n-1}\|) \le \phi_n(t_n - t_{n-1}) = t_{n+1} - t_n \end{aligned}$$

and

$$\|x_{n+1} - x_0\| \le t_{n+1} - t_0 \le \frac{s_1}{1-\sigma}.$$

Therefore, (2.4) is true.

Hence, by Lemma 2.3 and the inequalities (2.4), the theorem is proved.

2.2. Semilocal Convergence Theorem: Type II

When we use an iterative method to solve a system of linear equations, we can get the relation between the residuals. So an analogous assumption to the Lipschitz-continuity on the residual

$$\|F'(x_0)^{-1}(r_n - r_{n-1})\| \le \eta_n \|x_n - x_{n-1}\|$$

is reasonable. To this end, we use the assumption

$$\|F'(x_0)^{-1}[F(x_0) - r_0]\| \le \beta,$$

instead of

$$\|F'(x_0)^{-1}F(x_0)\| \le \beta,$$

in the following discussions.

The following theorem is another semilocal convergence theorem.

Theorem 2.7. *Suppose that $F'(x_0)^{-1}$ exists and*

$$\begin{cases} \|F'(x_0)^{-1}[F(x_0)-r_0]\| \le \beta, \\ \|F'(x_0)^{-1}[F'(x)-F'(y)]\| \le \gamma\|x-y\|, \quad \forall x,y \in \mathcal{D}, \\ \|F'(x_0)^{-1}(r_n - r_{n-1})\| \le \eta_n \|x_n - x_{n-1}\|, \quad \eta = \max_n\{\eta_n\} < 1. \end{cases}$$

If $(1-\eta)^2 - 2\beta\gamma \ge 0$ and $\overline{B(x_0,t^*)} \subset \mathcal{D}$, then the inexact Newton sequence $\{x_n\}$ converges to a solution x^* of Eq. (1.1). *Moreover,*

$$\|x_n - x^*\| \le t^* - t_n, \quad n = 0,1,\cdots,$$

where $\{t_n\}$ is defined by

$$\begin{cases} t_{n+1} = t_n + \dfrac{\frac{\gamma}{2}t_n^2 - (1-\eta)t_n + \beta}{1-\gamma t_n}, \\ t_0 = 0 \end{cases}$$

and

$$t^* = \frac{1-\eta-\sqrt{(1-\eta)^2 - 2\beta\gamma}}{\gamma}.$$

Proof. First, let us prove

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n, \quad n = 0,1,\cdots.$$

Since when $\|x_n - x_{n-1}\| \le t_n - t_{n-1}$, we have

$$\begin{aligned} &\|x_{n+1} - x_n\| = \|F'(x_n)^{-1}[F(x_n) - r_n]\| \\ &\le \|F'(x_n)^{-1}F'(x_0)\|\{\|F'(x_0)^{-1}[F(x_n) - F(x_{n-1}) - F'(x_{n-1})(x_n - x_{n-1})]\| + \|F'(x_0)^{-1}(r_{n-1} - r_n)\|\} \\ &\le \frac{1}{1-\gamma\|x_n - x_0\|}(\frac{\gamma}{2}\|x_n - x_{n-1}\|^2 + \eta_n\|x_n - x_{n-1}\|) \\ &\le \frac{1}{1-\gamma t_n}[\frac{\gamma}{2}(t_n - t_{n-1})^2 + \eta(t_n - t_{n-1})] = t_{n+1} - t_n, \end{aligned}$$

the inequality holds for all n by induction.

Furthermore, we can obtain the fact

$$0 < t_n \le t_{n+1} \le t^*$$

from Theorem 12.6.3 in [22]. This completes the proof of this theorem.

3. Applications

Inexact Newton methods are reduced to Newton's method in the case where we solve the Newton equations exactly. Hence the convergence theorem for inexact Newton methods should include the Newton-Kantorovich theorem as a special case. See also [6, 8, 13] and [7]. In this section, by letting $r_n \equiv 0$ and $\eta_n \equiv 0$, we reduce Theorem 2.7 and Theorem 2.6 to the Newton-Kantorovich theorem and a different Newton-Kantorovich theorem, respectively. Then, for the special inexact Newton methods which are combinations of Newton's method with the splitting methods, we can give two estimates about the inner iteration steps, see [7, 13].

3.1. Type-I Semilocal Convergence Theorem

For $r_n \equiv 0$ in (1.6) and $\eta_n \equiv 0$ in (2.1), we get Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n = 0, 1, \cdots , \tag{3.1}$$

and its convergence. Of course, the function $h(s)$ in Lemma 2.1 is reduced to the form

$$h(s) = \gamma s^3 + \beta\gamma s^2 - 2\beta s + 2\beta^2. \tag{3.2}$$

The convergence of Newton's method can be described as follows.

Corollary 3.1. *Suppose that* $F'(x_0)^{-1}$ *exists and*

$$\begin{aligned} &\|F'(x_0)^{-1}F(x_0)\| \le \beta, \\ &\|F'(x_0)^{-1}(F'(x) - F'(y))\| \le \gamma\|x - y\|, \quad \forall x, y \in \mathcal{D}. \end{aligned}$$

If $\beta\gamma \le 5\sqrt{5} - 11$ *and* $\overline{B(x_0, \frac{s_1}{1-\sigma})} \subset \mathcal{D}$*, where* s_1 *is the smallest positive zero of the function* (3.2)*, then the Newton sequence* $\{x_n\}$ *defined by* (3.1) *converges to the solution of the system of nonlinear equations* (1.1) *in* $\overline{B(x_0, \frac{s_1}{1-\sigma})}$*, and*

$$\|x_n - x^*\| \le \frac{\sigma^n}{1-\sigma}s_1, \quad n = 0, 1, \cdots .$$

By splitting the matrix $F'(x_k)$ in the Newton equations into

$$F'(x_k) = B_k - C_k,$$

we obtain the inner/outer iteration (see [4, 6, 7, 8, 13, 22, 30])

$$\begin{cases} x_{k+1} = x_k - (H_k^{m_k-1} + \cdots + H_k + I)B_k^{-1}F(x_k), \\ H_k = B_k^{-1}C_k, \quad k = 0, 1, \cdots , \end{cases} \tag{3.3}$$

where m_k is the number of the inner iteration. We can set $m_k \equiv m$, or choose any sequence in advance. For example, $m_k = k + 1, k = 0, 1, \cdots$. An appropriate sequence m_k is important for the convergence of $\{x_k\}$. Now a choice m_k is as follows.

Corollary 3.2. *Suppose that* $F'(x_0)^{-1}$ *exists and*

$$\begin{cases} \|F'(x_0)^{-1}F(x_0)\| \le \beta, \\ \|F'(x_0)^{-1}(F'(x) - F'(y))\| \le \gamma\|x - y\|, \quad \forall x, y \in \mathcal{D}, \\ m_k \ge \dfrac{\ln(l_k)}{\ln(\|H_k\|)}, \quad \eta = \max\limits_k\{\eta_k\} < 1, \quad l_k = \dfrac{1 - \sigma - \gamma s_1}{1 - \sigma + \gamma s_1}\eta_k, \end{cases}$$

where s_1 *and* σ *are defined in Lemmas* 2.1 *and* 2.2*, respectively. If*

$$\overline{B(x_0, \frac{s_1}{1-\sigma})} \subset \mathcal{D} \quad \text{and} \quad \beta\gamma \le -\frac{2\eta^2 + 14\eta + 11 - \sqrt{(4\eta + 5)^3}}{(1 + \eta)(1 - \eta)^2},$$

then the conclusion of Theorem 2.6 *is true.*

Remark. The following two conditions are always required such that the above-mentioned inner/outer iteration is well defined.

(i) B_k^{-1} exists for all nonnegative k;

(ii) the spectral radius of the matrix H_k is less than one, i.e., $\rho(H_k) < 1$, for all nonnegative k.

Proof. Since

$$x_{k+1} = x_k - F'(x_k)^{-1}[F(x_k) - r_k],$$

it follows from (3.3) that

$$\begin{aligned} r_k &= F(x_k) - F'(x_k)[(H_k^{m_k-1} + \cdots + H_k + I)B_k^{-1}F(x_k)] \\ &= F(x_k) - F'(x_k)[(I - H_k^{m_k})(I - H_k)^{-1}B_k^{-1}F(x_k)] \\ &= F(x_k) - F'(x_k)(I - H_k^{m_k})F'(x_k)^{-1}F(x_k) \\ &= F(x_k) - F'(x_k)F'(x_k)^{-1}F(x_k) + F'(x_k)H_k^{m_k}F'(x_k)^{-1}F(x_k) \\ &= F'(x_k)H_k^{m_k}F'(x_k)^{-1}F(x_k). \end{aligned}$$

By Lemma 2.4 and the above two conditions, the proof is complete.

3.2. Type-II Semilocal Convergence Theorem

Theorem 2.7 also includes the Newton-Kantorovich theorem as a special case with respect to $\eta_n \equiv 0$.

We also want to estimate the number of inner iteration under the condition in Theorem 2.7. But the estimate is so hard that the simplified hypothesis $m_k \equiv m$ should be imposed. The following result is parallel to Corollary 3.2.

Corollary 3.3. *Suppose that $F'(x_0)^{-1}$ exists and*

$$\begin{cases} \|F'(x_0)^{-1}F(x_0)\| \le \beta, \\ \|F'(x_0)^{-1}(F'(x) - F'(y))\| \le \gamma\|x - y\|, \quad \forall x, y \in \mathcal{D}, \\ \|B_0^{-1}F'(x_0)\| \le \delta, \\ ah^m + mbh^{m-1} \le \eta_k, \quad \sup_k \|H_k\| \le h < 1, \quad \eta = \max_k\{\eta_k\} < 1, \\ a = \frac{3-2\eta+2\beta\gamma}{\eta^2}, \\ b = \frac{2-\eta}{\eta}\frac{\delta(\delta+1)\gamma}{[1-(1-\eta)\delta]^2}[\frac{(1-\eta)^2}{2\gamma} + \frac{1-\eta}{\gamma} + \beta], \end{cases}$$

where the matrix norm has the property: $\|F'(x_0)^{-1}E\| \le \|F'(x_0)^{-1}D\|$ if E is a submatrix of D. If $(1-\eta)^2 - 2(1+h^m)\beta\gamma \ge 0$ and $\overline{B(x_0, t^*)} \subset \mathcal{D}$, then the conclusion of Theorem 2.7 *holds true.*

Proof. From

$$r_k = F'(x_k)H_k^mF'(x_k)^{-1}F(x_k),$$

we have

$$\begin{aligned} r_k - r_{k-1} &= F'(x_k)H_k^mF'(x_k)^{-1}F(x_k) - F'(x_{k-1})H_{k-1}^mF'(x_{k-1})^{-1}F(x_{k-1}) \\ &= F'(x_k)H_k^mF'(x_k)^{-1}[F(x_k) - F(x_{k-1})] \\ &\quad +[F'(x_k)H_k^mF'(x_k)^{-1} - F'(x_{k-1})H_{k-1}^mF'(x_{k-1})^{-1}]F(x_{k-1}) \\ &= F'(x_k)H_k^mF'(x_k)^{-1}[F(x_k) - F(x_{k-1})] \\ &\quad +[F'(x_k) - F'(x_{k-1})]H_k^mF'(x_k)^{-1}F(x_{k-1}) \\ &\quad +F'(x_{k-1})[H_k^mF'(x_k)^{-1} - H_{k-1}^mF'(x_{k-1})^{-1}]F(x_{k-1}) \\ &= F'(x_k)H_k^mF'(x_k)^{-1}[F(x_k) - F(x_{k-1})] \\ &\quad +[F'(x_k) - F'(x_{k-1})]H_k^mF'(x_k)^{-1}F(x_{k-1}) \\ &\quad +F'(x_{k-1})H_k^mF'(x_k)^{-1}[F'(x_{k-1}) - F'(x_k)]F'(x_{k-1})^{-1}F(x_{k-1}) \\ &\quad +F'(x_{k-1})(H_k - H_{k-1})(H_k^{m-1} + H_k^{m-2}H_{k-1} + \cdots + H_{k-1}^{m-1})F'(x_{k-1})^{-1}F(x_{k-1}). \end{aligned}$$

From the assumptions, we can also get

$$\begin{aligned}
&\|F'(x_0)^{-1}F'(x_k)\| \le \gamma\|x_k - x_0\| + 1;\\
&\|F'(x_k)^{-1}F'(x_0)\| \le \tfrac{1}{1-\gamma\|x_k - x_0\|};\\
&\|F'(x_0)^{-1}F(x_k)\| \le \tfrac{\gamma}{2}\|x_k - x_0\|^2 + \|x_k - x_0\| + \beta;\\
&\|F'(x_0)^{-1}(B_k - B_{k-1})\| \le \gamma\|x_k - x_{k-1}\|;\\
&\|B_k^{-1}F'(x_0)^{-1}\| \le \tfrac{\delta}{1-\gamma\|x_k - x_0\|\delta};
\end{aligned}$$

by induction. Hence, by Theorem 2.7 and some concrete estimates, we can complete the proof.

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